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Research article

Conservation laws and symmetry analysis of a generalized Drinfeld-Sokolov system

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Abstract: The generalized Drinfeld-Sokolov system is a widely-used model that describes wave phenomena in various contexts. Many properties of this system, such as Hamiltonian formulations and integrability, have been extensively studied and exact solutions have been derived for specific cases. In this paper we applied the direct method of multipliers to obtain all low-order local conservation laws of the system. These conservation laws correspond to physical quantities that remain constant over time, such as energy and momentum, and we provided a physical interpretation for each of them. Additionally, we investigated the Lie point symmetries and first-order symmetries of the system. Through the point symmetries and constructing the optimal systems of one-dimensional subalgebras, we were able to reduce the system of partial differential equations to ordinary differential systems and obtain new solutions for the system.

Keywords: Drinfeld-Sokolov system; conservation laws; conserved quantities; Lie symmetries; reductions
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1. Introduction

One of the most important, and surely one of the most studied, nonlinear evolution partial differential equations (PDEs) is the Korteweg-de Vries equation (KdV)

$$u_t + \lambda u u_x + \mu u_{xxx} = 0, \tag{1.1}$$

where u has the physical meaning of the wave amplitude and also the wave speed. The KdV equation arises to model shallow water waves with weak nonlinearities and it has applications in a wide variety of physical phenomena, such as surface gravity waves [1], propagation of nonlinear acoustic waves in liquids with small volume concentrations of gas bubbles [2, 3], magma flow and conduit waves [4] and many other applications [5].

Drinfeld and Sokolov [6, 7] obtained all the simplest generalized KdV equations corresponding to the classical Kac-Moody algebras of ranks one and two. Among these generalizations is the system of equations given as follows:

$$\begin{aligned} u_t &= u_{xxx} + uu_x - vv_x, \\ v_t &= -2v_{xxx} - uv_x. \end{aligned} \tag{1.2}$$

This system from Eq (1.2) is an evolutionary system for two dependent variables that possesses a Lax pair and, after linear changes and scale transformations, yields a Hamiltonian form with the Hamiltonian given by

$$H = u^2 + v^2 - 6uv. \tag{1.3}$$

In the present paper, we have considered a generalization of (1.2). It is called the generalized Drinfeld-Sokolov (gDS) system

$$\begin{aligned} & u_t + \alpha_1 uu_x + \beta_1 u_{xxx} + \gamma v^{\delta} v_x = 0, \\ & v_t + \alpha_2 uv_x + \beta_2 v_{xxx} = 0, \end{aligned} \tag{1.4}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma \neq 0$ and $\delta \neq -1$ are arbitrary constants.

There are several papers in which different kinds of generalizations of system (1.2) have been studied. However, in many of these papers, the gDS system is named in different ways. We summarized them, giving the background and the results obtained for some of these systems.

In [8], the so-called Drinfeld-Sokolov system was studied

$$\begin{aligned} & u_t + u_{xxx} - 6uu_x - 6v_x = 0, \\ & v_t - 2v_{xxx} + 6uv_x = 0. \end{aligned} \tag{1.5}$$

The authors concluded that the Drinfeld-Sokolov system (1.5) admitted a compatible bi-Hamiltonian structure and therefore, was an integrable system. This system can be obtained from (1.4) by substituting $\alpha_1 = -6, \alpha_2 = 6, \beta_1 = 1, \beta_2 = -2, \delta = 0$, and $\gamma = -6$.

Nevertheless, system (1.5) is known as the Satsuma-Hirota system [9]. In this paper, system (1.5) was transformed into the scalar sixth-order Drinfeld-Sokolov-Satsuma-Hirota equation by using a Miura-type transformation. Moreover, three different methods were applied to obtain solutions to the equation.

The same pair of equations from (1.5) were called Bogoyavlenskii coupled KdV equations in [10–12]. Soliton solutions were found by means of the standard and non-standard truncation Painlevé expansion and a family of exact solutions was obtained by a Bäcklund transformation.

In [13], system (1.5) was considered but named as a Drinfeld-Sokolov-Satsuma-Hirota. Exact solutions were obtained using the (G'/G)-expansion method and the generalized tanh-coth method.

System (1.5) was also studied in [14]. Their non-local symmetries were studied and exact solutions like solitons, cnoidal waves and Painlevé waves were derived through similarity reductions.

Through the Weiss method of truncated singular expansions, authors of [15] constructed an explicit Bäcklund transformation of the Drinfeld-Sokolov-Satsuma-Hirota system (1.5) into itself. They also obtained special solutions generated by this transformation.

In [16], solitary wave, kink (anti-kink) and periodic wave solutions were obtained by using the theory of planar dynamical systems for the so-called generalized Drinfeld-Sokolov equations

$$\begin{aligned} & u_t + u_{xxx} - 6uu_x - 6\left(v^{\alpha}\right)_x = 0, \\ & v_t - 2v_{xxx} + 6uv_x = 0. \end{aligned} \tag{1.6}$$

The previous system (1.6) can be obtained from (1.4) by considering $\alpha_1 = -6, \alpha_2 = 6, \beta_1 = 6, \beta_2 = -2, \delta = \alpha - 1$, and $\gamma = -6\alpha$.

Some exact solutions of system (1.6) are obtained in [17]; first analytically by using the tanh function method, and second, numerically through the Adomian decomposition method.

Moreover, in [18] traveling wave solutions for system (1.6) were studied. In particular, the authors obtained solutions with physical interest like kink and anti-kink wave solutions and periodic wave solutions.

Therefore, the gDS system presented in this paper embraces all the previous equations and unifies all the cases under the study of the gDS system (1.4). Note that the named generalized Drinfeld-Sokolov (1.4) does not have an infinite number of conservation laws, which will be shown in Section 2, so this system is not completely integrable.

On the other hand, in order to simplify the number of arbitrary constants involved in Eq (1.4), we have applied a scaling equivalence transformation. It is a change of variables that acts on independent and dependent variables

$$\begin{aligned} \tilde{t} &= \lambda_1 t, \\ \tilde{x} &= \lambda_2 x, \\ \tilde{u} &= \lambda_3 u, \\ \tilde{v} &= \lambda_4 v. \end{aligned} \tag{1.7}$$

From (1.7), by taking $\beta = \frac{\beta_1}{\beta_2}$ and $\alpha = \frac{\alpha_1}{\alpha_2}$, system (1.4) can be invertibly reduced into a system involving four parameters instead of six given by

$$\begin{aligned} & u_t + uu_x + u_{xxx} - \gamma v^{\delta} v_x = 0, \\ & v_t + \alpha u v_x + \beta v_{xxx} = 0, \end{aligned} \tag{1.8}$$

where $\alpha, \beta \neq 0$, $\gamma^2 = 1$ and $\delta \neq -1$. Hereafter, without loss of generality, we can restrict the study of system (1.4) to system (1.8). Moreover, to avoid cases already studied, we consider the extra condition $\delta \neq 0$.

This paper has two main objectives. First, it seeks to investigate the conservation laws of the gDS system (1.8) in order to determine physical properties that remain constant over time. These conservation laws are crucial in demonstrating whether quantities such as charge, momentum, angular momentum, and energy are conserved. Second, the paper aims to study the Lie point symmetries and first-order generalized symmetries of system (1.8), which can be used to obtain reductions and exact group-invariant solutions.

To achieve the first objective, we used the direct method of multipliers [19–21]. This method updates the classical method in which Noether's theorem was applied to the variational symmetries of the equation [22]. For the second objective, we determined all symmetries by using the Lie method [21, 23, 24], a widely used technique for deriving analytical solutions of nonlinear partial differential equations [25, 26].

In a previous paper [27], Lie point symmetries were obtained for system (1.4) with different parameters and conditions than those of system (1.8). However, the present paper extends this work by updating the Lie point symmetry classification for the gDS system, studying the first-order generalized symmetries, constructing the optimal systems of one-dimensional subalgebras, and presenting all the reduced ordinary differential systems.

The paper is organized as follows: Section 2 presents the classification of low-order local conservation laws and interprets the associated conserved quantities. In Section 3, we update the Lie point symmetry classification of the gDS system (1.8) and study the first-order generalized symmetries. In Section 4, we construct the optimal systems of one-dimensional subalgebras using the Lie point symmetries. Finally, in Section 5, we show all the reduced ordinary differential systems as an application of the Lie point symmetries.

2. Conservation laws

For the generalized Drinfeld-Sokolov system (1.8), a local conservation law is a divergence expression of the form

$$D_tT(t, x, u, v, u_t, v_t, \dots) + D_xX(t, x, u, v, u_t, v_t, \dots) = 0 \tag{2.1}$$

that holds for the whole set of solutions $(u(t, x), v(t, x))$, where D_t and D_x denote the total derivative operators with respect to t and x, respectively.

T and X are functions depending on t, x, u, v and a finite number of derivatives of u and v. T is called conserved density, and X is called spatial flux. The expression (T, X) is the conserved current.

A conservation law is called locally trivial if there is a function $\Theta(t, x, u, v, u_t, v_t, \dots)$ so that the conserved current is

$$(T, X) = (\Phi + D_x\Theta, \Psi - D_t\Theta)\,,$$

where $\Phi = \Psi = 0$ holds on all solutions $(u(t, x), v(t, x))$. Hence, conservation laws are considered to be locally equivalent if they differ by a locally trivial conservation law.

Notice that the system is formed by a couple of evolution equations, so through use of the system (1.8) and its differential consequences, it is possible to eliminate t-derivatives of u and v in (T, X).

Every non-trivial local conservation law (2.1) for Eq (1.8) can be expressed in its characteristic form

$$D_tT + D_xX = \Big(u_t + uu_x + u_{xxx} - \gamma v^{\delta} v_x\Big)Q^u + \Big(v_t + \alpha uv_x + \beta v_{xxx}\Big)Q^v, \tag{2.2}$$

where the pair of functions

$$Q = \Big(Q^u, Q^v\Big) = \left(\frac{\delta T}{\delta u}, \frac{\delta T}{\delta v}\right) \tag{2.3}$$

is called a multiplier and $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta v}$ are the Euler operators with respect to the variables u and v, respectively [21, 23, 28].

Since the Euler operator has the property of annihilating total derivatives, all multipliers (Q^u, Q^v) can be found by solving the system of determining equations that is given by

$$\begin{aligned} &\frac{\delta}{\delta u}\Big((u_t + uu_x + u_{xxx} - \gamma v^{\delta} v_x)Q^u + (v_t + \alpha uv_x + \beta v_{xxx})Q^v\Big) = 0, \\ &\frac{\delta}{\delta v}\Big((u_t + uu_x + u_{xxx} - \gamma v^{\delta} v_x)Q^u + (v_t + \alpha uv_x + \beta v_{xxx})Q^v\Big) = 0, \end{aligned} \tag{2.4}$$

holding for the set of solutions of Eq (1.8).

All non-trivial conservation laws arise from multipliers (2.3), and conservation laws of physical importance come from low-order multipliers [29–31] of the form

$$Q = (Q^u(t, x, u, v, u_x, v_x, u_{xx}, v_{xx}), Q^v(t, x, u, v, u_x, v_x, u_{xx}, v_{xx}))\,. \tag{2.5}$$

The term low-order is related to the dependencies of the multiplier function (2.5). In general, a low-order multiplier depends on the dependent variables u, v and those variables that can be differentiated up to the leading derivatives of the system. The leading derivatives of system (1.8) are u_t, u_{xxx}, v_t, and v_{xxx}. Hence, u_t, v_t can be obtained by differentiation of u, v respectively, u_{xxx} by differentiation of u, u_x, u_{xx} and v_{xxx} by differentiation of v, v_x, and v_{xx}.

Similar to the determining equations for Lie symmetries, here the determining Eq (2.4) for low-order multipliers split with respect to $u_t, v_t, u_{xxx}, v_{xxx}$ and differential consequences.

We have obtained a general classification for low-order multipliers (2.5), depending on the parameters that system (1.8) involves, by solving the determining Eq (2.4). For this purpose we have used Maple, "rifsimp" and "pdsolve" commands.

Proposition 2.1. *The general classification of low-order multipliers (2.5) for the generalized Drinfeld-Sokolov system (1.8) is:*

(i) For $\alpha,\beta \neq 0$, $\gamma^2 = 1$, *and* $\delta \neq 0, -1$,

$$Q_1 = (1, 0). \tag{2.6}$$

(ii) There are additional multipliers for the following special α,β,δ*:*

a) For $\delta = 1$,

$$Q_2 = (\alpha u, \gamma v)\,. \tag{2.7}$$

b) For $\delta = 1$ *and* $2\alpha = \beta$,

$$Q_3 = \Big((2\beta - 2)u_{xx} + (\beta - 1)u^2 + \gamma v^2, 2\gamma(uv + 6v_{xx})\Big)\,. \tag{2.8}$$

c) For $\delta = 1$ *and* $\alpha = -1$,

$$Q_4 = (-tu + x, \gamma t v)\,. \tag{2.9}$$

d) For $\alpha = \frac{1}{2}$ *and* $\beta = 1$,

$$Q_{5a} = (v, u)\,, \tag{2.10a}$$

$$Q_{5b} = \Big(v^2, 2uv + 12v_{xx}\Big)\,. \tag{2.10b}$$

Proof. The multipliers determining system (2.4) leads to an overdetermined linear system of 57 equations. By solving the easiest equations, we obtain that

$$\begin{aligned}
Q^u = \ & A(t, x) + B(t, v) + C(t, x)u + D(t, x)v + E(t, x)uv + F(t)u^2 \\
& + \Big(2\tfrac{\partial I(t,x)}{\partial x} - G(t, x)\Big)v_x + H(t)u_{xx} + I(t, x)v_{xx}, \\
Q^v = \ & K(t, x, v) + \Big(\tfrac{\partial B(t,v)}{\partial v} + L(t, x)\Big)u + \tfrac{1}{2}E(t, x)u^2 + G(t, x)u_x + I(t, x)u_{xx} + J(t)v_{xx},
\end{aligned}$$

related by the following conditions:

$$
\begin{aligned}
(1-\beta)\, I &= 0,\\
(1-\beta)\, G - 3\beta I_x &= 0,\\
L - D - G_x + I_{xx} &= 0,\\
(1-\beta)\, G + (2\beta - 5)\, I_x &= 0,\\
D - \beta L + ((1-\beta)E + (1-\alpha)I)\, u + \gamma v^{\delta} H &\\
+(1-\beta)B_v + (1-3\beta)G_x - (1+3\beta)I_{xx} &= 0,\\
D - \beta L + ((1-\beta)E + (1-\alpha)I)\, u + \gamma v^{\delta} H &\\
+(1-\beta)B_v + (\beta-3)G_x + (9-\beta)I_{xx} &= 0,\\
\alpha v^2 \Big(4uv_xB_v + 3u^2v_xE + 4(uv_x)_xI_x + 2uv_xI_{xx} + 4(u_xv_x)_xI &\\
-2uv_xG_x - 2uv_{xx}G + 2v_xv_{xx}J + 2v_xK + 4uv_xL) &\\
-2\gamma v^{\delta} v_x \Big(v^2C + v^3E + 2uv^2F + (3\delta v v_{xx} - \delta {v_x}^2 + \delta^2 {v_x}^2)H\Big) &\\
2uv^2\,(-A_x - v_xB_v - C_t - C_{xxx} - v_xD - 3(v_xE_x)_x + (v_xG_x)_x &\\
-(2v_xI_{xx} + 3v_{xx}I_x)) - uv^3\,(D_x + E_t + E_{xxx}) &\\
-2u^2v^2\,(C_x + F' + (vE)_x) - 2v^3\,(D_t + D_{xxx} + 3(u_xE_x)_x) &\\
+2v^2 \Big(-A_t - A_{xxx} - B_t - 3v_xv_{xx}B_{vv} - {v_x}^3B_{vvv} - 3(u_xC_x)_x - 3(v_xD_x)_x &\\
-3(u_xv_x)_xE - 6u_xv_xE_x - 6u_xu_{xx}F + v_xG_t + 3v_{xx}G_{xx} + v_xG_{xxx} &\\
+3u_xu_{xx}H - u_{xx}H_t - 2v_xI_{tx} - 7v_{xx}I_{xxx} - 2v_xI_{xxxx} - v_{xx}I_t) &= 0,\\
-\alpha v^2 \Big(4uu_xB_v + 3u^2u_xE + u^3E_x + 2uu_xG_x + 2(uu_x)_xG &\\
+2u_{xx}(uI)_x - 2(u_xv_x)_xJ + 2(uK)_x + 4uu_xL + 2u^2L_x\Big) &\\
-\beta v^2 \Big(6(u_xv_x)_xB_{vv} + 6v_x(uv_x)_xB_{vvv} + 2u{v_x}^3B_{vvvv} + 6u_x(uE_x)_x &\\
+6u_{xx}(uE)_x + u^2E_{xxx} + 2u_xG_{xxx} + 6u_{xx}G_{xx} + 2u_{xx}I_{xxx} &\\
+6(v_xK_{xv})_x + 6v_x(v_xK_{vv})_x + 2K_{xxx} + 2{v_x}^3K_{vvv} + 6(u_xL_x)_x + 2uL_{xxx}\Big) &\\
+2\gamma v^{\delta} \Big(v^2A_x + v^2(uC)_x + v^3D_x + v^3(uE)_x + 2uv^2u_xF &\\
-2v^2(v_xG)_x - \delta v {v_x}^2 G + \delta(1-\delta){v_x}^3 I + 3v^2(v_xI_x)_x - 3\delta v v_x v_{xx} I\Big) &\\
+2uv^2\,(-B_{vt} + u_xB_v + u_xD + (u_xG)_x - u_xI_{xx} - L_t) + u^2v^2\,(-E_t + 2u_xE) &\\
+2v^2 \Big({u_x}^2G - u_xG_t + 3u_xu_{xx}I - u_{xx}I_t - v_{xx}J' - K_t + u_t\,(D + G_x - I_{xx} - L)\Big) &= 0.
\end{aligned}
\tag{2.11}
$$

The solutions of system (2.11) lead to the different cases listed in Proposition 2.1. □

Once the multipliers are found, the corresponding non-trivial conserved currents can be obtained by integrating the characteristic equation (2.2).

Theorem 2.2. *All non-trivial low-order local conservation laws of the generalized Drinfeld-Sokolov system (1.8) are:*

(i) For $\alpha, \beta \neq 0$, $\gamma^2 = 1$, *and* $\delta \neq 0, -1$,

$$
\begin{aligned}
T_1 &= u,\\
X_1 &= u_{xx} + \frac{1}{2}u^2 - \frac{\gamma}{\delta+1}v^{\delta+1}.
\end{aligned}
\tag{2.12}
$$

(ii) For special values of the parameters α, β, δ:

a) For $\delta = 1$,

$$
\begin{aligned}
T_2 &= \frac{1}{2\gamma}\left(\alpha u^2 + \gamma v^2\right),\\
X_2 &= \frac{1}{6\gamma}\left(2\alpha u^3 + 6\alpha u u_{xx} - 3\alpha u_x^2 + 6\beta\gamma v v_{xx} - 3\beta\gamma v_x^2\right).
\end{aligned}
\tag{2.13}
$$

b) For $\delta = 1$ and $2\alpha = \beta$,

$$
\begin{aligned}
T_3 =& \frac{1}{36\gamma}\left[(\beta - 1)\left(u^3 - 3u_x^2\right) + 3\gamma u v^2 - 18\gamma v_x^2\right],\\
X_3 =& \frac{1}{48\gamma}\left[(\beta - 1)\left(u^4 + 4u^2 u_{xx} + 8u_t u_x + 4u_{xx}^2\right) + 2\gamma\left(u^2 v^2 + 2u_{xx} v^2 + 24 v_t v_x\right) - \gamma^2 v^4\right.\\
&\left. + 8\beta\gamma\left(u v v_{xx} + u v_x^2 - u_x v v_x + 3v_{xx}^2\right)\right].
\end{aligned}
\tag{2.14}
$$

c) For $\delta = 1$ and $\alpha = -1$,

$$
\begin{aligned}
T_4 &= \frac{1}{2\gamma}\left(\gamma t v^2 - t u^2 + 2xu\right),\\
X_4 &= \frac{1}{6\gamma}\left(-2tu^3 + 3xu^2 - 6tuu_{xx} + 3tu_x^2 - 6u_x + 6xu_{xx} - 3\gamma x v^2 - 3\beta\gamma t v_x^2 + 6\beta\gamma t v v_{xx}\right).
\end{aligned}
\tag{2.15}
$$

d) For $\alpha = \frac{1}{2}$ and $\beta = 1$,

$$
\begin{aligned}
T_{5a} &= uv,\\
X_{5a} &= \frac{1}{2}u^2 v + u_{xx} v + u v_{xx} - u_x v_x - \frac{\gamma}{\delta + 2} v^{\delta+2};
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
T_{5b}7 &= \frac{1}{12}uv^2 - \frac{1}{2}6v_x^2,\\
X_{5b} &= \frac{1}{24}\left(u^2 v^2 + 2u_{xx} v^2 + 4uvv_{xx} - 4u_x v v_x + 4uv_x^2 + 24 v_t v_x + 12 v_{xx}^2\right) - \frac{\gamma}{12(\delta + 3)} v^{\delta+3}.
\end{aligned}
\tag{2.17}
$$

With regards to the physical meaning of these conservation laws, every conservation law has associated a conserved quantity of the form

$$
C[u, v] = \int_\Omega T\, dx, \tag{2.18}
$$

where Ω is the domain of solutions $u(t, x), v(t, x)$.

In the general case, where $\alpha, \beta \neq 0$, $\gamma^2 = 1$ and $\delta \neq 0, -1$, the admitted conservation law (2.12) yields the conserved quantity

$$
\mathcal{M}[u, v] = \int_\Omega u\, dx \tag{2.19}
$$

that is the mass density.

Conservation law (2.13), in the case $\delta = 1$, leads to the conserved quantity

$$
\mathcal{P}[u, v] = \int_\Omega \frac{1}{2\gamma}\left(\alpha u^2 + \gamma v^2\right)\, dx, \tag{2.20}
$$

which is the momentum density.

In the case where $\delta = 1$ and $2\alpha = \beta$, the conservation law (2.14) yields the conserved quantity

$$\mathcal{E}[u, v] = \int_\Omega \frac{1}{36\gamma}\left[(\beta - 1)\left(u^3 - 3u_x^2\right) + 3\gamma uv^2 - 18\gamma v_x^2\right] dx, \tag{2.21}$$

which represents the total energy for solutions $u(t, x), v(t, x)$.

For $\delta = 1$ and $\alpha = -1$, the conserved quantity associated to conservation law (2.15) is

$$\mathcal{G}[u, v] = \int_\Omega \frac{1}{2\gamma}\left(\gamma tv^2 - tu^2 + 2xu\right) dx, \tag{2.22}$$

which is the Galilean momentum.

In the case where $\alpha = \frac{1}{2}$ and $\beta = 1$, the two admitted conservation laws (2.16) and (2.17) yield the conserved quantities

$$\mathcal{A}[u, v] = \int_\Omega uv\, dx, \tag{2.23}$$

$$\mathcal{T}[u, v] = \int_\Omega \frac{1}{12}uv^2 - \frac{1}{2}6v_x^2\, dx. \tag{2.24}$$

The first one represents the angular momentum, and the second one the energy-momentum.

3. Symmetries

3.1. Lie point symmetries

Now, we consider a one-parameter Lie group of point transformations acting on independent and dependent variables

$$\begin{aligned}
\tilde{t} &= t + \varepsilon\tau(t, x, u, v) + O(\varepsilon^2), \\
\tilde{x} &= x + \varepsilon\xi(t, x, u, v) + O(\varepsilon^2), \\
\tilde{u} &= u + \varepsilon\eta_u(t, x, u, v) + O(\varepsilon^2), \\
\tilde{v} &= v + \varepsilon\eta_v(t, x, u, v) + O(\varepsilon^2),
\end{aligned} \tag{3.1}$$

where ε is the group parameter. The associated vector field has the form

$$\mathbf{X} = \tau(t, x, u, v)\partial_t + \xi(t, x, u, v)\partial_x + \eta_u(t, x, u, v)\partial_u + \eta_v(t, x, u, v)\partial_v. \tag{3.2}$$

The Lie group of transformations (3.1) are the point symmetries admitted by the system if and only if it leaves invariant the solution space of system (1.8) . It is known as the infinitesimal criterion of invariance:

$$\begin{aligned}
&\mathbf{X}^{(3)}\left(u_t + uu_x + u_{xxx} - \gamma v^\delta v_x\right) = 0, \\
&\mathbf{X}^{(3)}\left(v_t + \alpha uv_x + \beta v_{xxx}\right) = 0,
\end{aligned} \tag{3.3}$$

when the gDS system (1.8) holds, where $\mathbf{X}^{(3)}$ is the third-order prolongation of the vector field (3.2).

Equation (3.3) is known as the determining equation. These equations can be split by considering u, v and its derivatives as independent variables. Consequently, it leads to an overdetermined linear system of equations for the infinitesimals τ, ξ, η_u, η_v together with the arbitrary parameters, $\alpha, \beta, \gamma, \delta$ subject to the classification conditions $\alpha,\beta \neq 0$, $\gamma^2 = 1, \delta \neq 0, -1$. We obtain the following result.

Theorem 3.1. *The classification of point symmetries admitted by the generalized Drinfeld-Sokolov system (1.8) is given by:*

(i) For the non-zero arbitrary parameters $\alpha, \beta, \gamma, \delta$, there are three infinitesimal generators

$$\mathbf{X}_1 = \partial_t, \tag{3.4a}$$

$$\mathbf{X}_2 = \partial_x, \tag{3.4b}$$

$$\mathbf{X}_3 = 3t\partial_t + x\partial_x - 2u\partial_u - \frac{4v}{\delta+1}\partial_v. \tag{3.4c}$$

(ii) For $\alpha = 1$, an additional point symmetry is admitted

$$\mathbf{X}_4 = t\partial_x + \partial_u. \tag{3.5}$$

Proof. The symmetry determining system (3.3) leads to an overdetermined linear system of 33 equations. By solving the easiest equations for the infinitesimals, we obtain that $\tau = \tau(t)$, $\xi = \xi(t, x)$, $\eta_u = A(t)v + B(t, x, u)$ and $\eta_v = C(t)u + D(t, x, v)$, which must verify

$$\begin{aligned}
-\xi_{xx} + u_x B_{uu} + B_{xu} &= 0,\\
-\xi_{xx} + v_x D_{vv} + D_{xv} &= 0,\\
-v_t\tau' - \beta v_x\xi_{xxx} + (2\alpha u v_x + 3v_t)\,\xi_x - v_x\xi_t + \alpha v v_x A + \alpha v_x B&\\
+ \left(\beta\gamma v^\delta v_x - \beta(u_t + uu_x) + u_t + \alpha u u_x\right) C + uC'&\\
+D_t + \alpha u D_x + 3\beta v_x^2 D_{vv} + \beta v_x^3 D_{vvv} + 3\beta v_x D_{xxv} + \beta D_{xxx} &= 0,\\
-\beta u_t v\tau' + \beta v\left(-u_x\xi_t + (2uu_x + 3u_t - 2\gamma v^\delta v_x)\xi_x - u_x\xi_{xxx}\right)&\\
+ \left((\beta-\alpha)u v_x + \beta u_x v + (\beta - 1)v_t\right) vA + \beta v^2 A'&\\
+\beta v\left(u_x B + B_t + uB_x + B_{xxx} + u_x^3 B_{uuu} + 3u_x^2 B_{xuu} + 3u_x^2 B_{xxu}\right)&\\
-\beta\gamma v^\delta\left(\delta u v_x C + u_x v C + \delta v_x D + vD_x - v v_x D_u + v v_x D_v\right) &= 0.
\end{aligned} \tag{3.6}$$

From the first two equations of system (3.6), we obtain that $B(t, x, u) = E(t, x)u + F(t, x)$ and $D(t, x, v) = G(t, x)v + H(t, x)$. Substituting the functions in the previous system, we get

$$\begin{aligned}
-\xi_{xx} + E_x &= 0,\\
-\xi_{xx} + G_x &= 0,\\
-v_t\tau' - v_x\xi_t + (3v_t + 2\alpha u v_x)\,\xi_x - \beta v_x\xi_{xxx} + \alpha v v_x A&\\
+ \left((1-\beta)u_t + (\alpha-\beta)uu_x + \beta\gamma v^\delta v_x\right) C + uC' + \alpha u v_x E + \alpha v_x F&\\
+vG_t + \alpha u v G_x + 3\beta v_x G_{xx} + \beta v G_{xxx} + H_t + \alpha u H_x + \beta H_{xxx} &= 0,\\
-\beta u_t v\tau' + \beta v\left(-u_x\xi_t + 2uu_x\xi_x + 3u_t\xi_x - 2\gamma v^\delta v_x\xi_x - u_x\xi_{xxx}\right)&\\
+ \left((\beta-\alpha)u v_x + \beta u_x v + (\beta-1)v_t\right) vA + \beta v^2 A' - \beta\gamma v^\delta\left(\delta u v_x C + u_x v C\right)&\\
+\beta v\left(uu_x E + \gamma v^\delta v_x E + uE_t + u^2 E_x + 3u_x E_{xx} + uE_{xxx}\right)&\\
+\beta v\left(u_x F + F_t + uF_x + F_{xxx}\right)&\\
-\beta\gamma v^{\delta+1}\left((\delta+1)v_x G + vG_x\right) - \beta\gamma v^\delta\left(v_x H + vH_x\right) &= 0.
\end{aligned} \tag{3.7}$$

Now the functions involved in system (3.7) do not depend on u or v, so these equations split with respect to u, v and their derivatives. We derive and solve this overdetermined system with the help of Maple software. Using the commands “rifsimp” and “pdsolve”, we split the system and solve the equations with results leading to the different cases listed in Theorem 3.1. □

The transformation generated by a point symmetry is obtained by solving the initial value problem:

$$\begin{aligned}
\frac{\partial \tilde{t}}{\partial \varepsilon} &= \tau(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \quad \tilde{t}|_{\varepsilon=0} = t,\\
\frac{\partial \tilde{x}}{\partial \varepsilon} &= \xi(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \quad \tilde{x}|_{\varepsilon=0} = x,\\
\frac{\partial \tilde{u}}{\partial \varepsilon} &= \eta_u(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \quad \tilde{u}|_{\varepsilon=0} = u,\\
\frac{\partial \tilde{v}}{\partial \varepsilon} &= \eta_v(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \quad \tilde{v}|_{\varepsilon=0} = v.
\end{aligned}$$

Hence, for the symmetry generators (3.4a)–(3.4c) and (3.5) of the generalized Drinfeld-Sokolov system (1.8), the corresponding transformations are respectively given by:

$$(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v})_1 = (t + \varepsilon, x, u, v)\,, \tag{3.8a}$$

$$(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v})_2 = (t, x + \varepsilon, u, v)\,, \tag{3.8b}$$

$$(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v})_3 = \left(t\mathrm{e}^{3\varepsilon}, x\mathrm{e}^{\varepsilon}, u\mathrm{e}^{-2\varepsilon}, v\mathrm{e}^{-\frac{4\varepsilon}{\delta+1}}\right), \tag{3.8c}$$

$$(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v})_4 = (t, x + t\varepsilon, u + \varepsilon, v)\,. \tag{3.8d}$$

We interpret the symmetries by looking at the forms of the transformations. In case 1, transformation (3.8a) represents a time-translation, transformation (3.8b) represents a space-translation and transformation (3.8c) represents a scaling. For case 2, the additional symmetry (3.8d) is a Galilean boost.

3.2. First-order generalized symmetries

A first-order symmetry corresponds to a one-parameter Lie group of transformations in which the transformation of (t, x, u, v) necessarily depends on first-order derivatives of u and v. Hence, its transformations are given by

$$\begin{aligned}
\tilde{t} &= t + \varepsilon\tau(t, x, u, v, u_t, u_x, v_t, v_x) + O(\varepsilon^2),\\
\tilde{x} &= x + \varepsilon\xi(t, x, u, v, u_t, u_x, v_t, v_x) + O(\varepsilon^2),\\
\tilde{u} &= u + \varepsilon\eta_u(t, x, u, v, u_t, u_x, v_t, v_x) + O(\varepsilon^2),\\
\tilde{v} &= v + \varepsilon\eta_v(t, x, u, v, u_t, u_x, v_t, v_x) + O(\varepsilon^2),\\
\tilde{u}_t &= u_t + \varepsilon\zeta^t(t, x, u, v, u_t, u_x, v_t, v_x) + O(\varepsilon^2),\\
\tilde{u}_x &= u_x + \varepsilon\zeta^x(t, x, u, v, u_t, u_x, v_t, v_x) + O(\varepsilon^2),\\
\tilde{v}_t &= v_t + \varepsilon\sigma^t(t, x, u, v, u_t, u_x, v_t, v_x) + O(\varepsilon^2),\\
\tilde{v}_x &= v_x + \varepsilon\sigma^x(t, x, u, v, u_t, u_x, v_t, v_x) + O(\varepsilon^2),
\end{aligned} \tag{3.9}$$

where ε is the group parameter. Thus, the infinitesimal generator takes the form

$$\mathbf{X} = \tau\partial_t + \xi\partial_x + \eta_u\partial_u + \eta_v\partial_v + \zeta^t\partial_{u_t} + \zeta^x\partial_{u_x} + \sigma^t\partial_{v_t} + \sigma^x\partial_{v_x}. \tag{3.10}$$

Following a similar procedure to the one shown before for the point symmetries, applying the criterion of invariance and considering that $\zeta^t, \zeta^x, \sigma^t, \sigma^x$ do not cancel at the same time, we have obtained the following result.

Proposition 3.2. *The generalized Drinfeld-Sokolov system (1.8) does not admit first-order generalized symmetries.*

Proof. By splitting and solving the system of determining equations

$$\mathbf{X}^{(3)}\Big(u_t + uu_x + u_{xxx} - \gamma v^\delta v_x\Big) = 0,$$
$$\mathbf{X}^{(3)}\Big(v_t + \alpha uv_x + \beta v_{xxx}\Big) = 0,$$

it is straightforward to prove that the solutions do not depend on first-order derivatives of the dependent variables. Therefore, all the first-order symmetries reduce to prolonged point symmetries of system (1.8). □

4. Optimal systems

The construction of the optimal system is an efficient way of classifying all possible group-invariant solutions. From these group-invariant solutions, every other solution can be derived.

For one-dimensional subalgebras, finding an optimal system is the same as classifying the orbits of the adjoint representation. Essentially this problem is attacked by taking a general element $\mathbf{X}$ in the symmetry algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible.

For further details on determining optimal systems, see [21, 32, 33].

Let us consider a general element from the Lie algebra and reduce it to its simplest equivalent form by using the chosen adjoint transformations

$$\mathrm{Ad}\left(\exp\left(\varepsilon\mathbf{X}_i\right)\right)\mathbf{X}_j = \mathbf{X}_j - \varepsilon\left[\mathbf{X}_i, \mathbf{X}_j\right] + \frac{\varepsilon^2}{2!}\left[\mathbf{X}_i, \left[\mathbf{X}_i, \mathbf{X}_j\right]\right] - \cdots,$$

where ε is a real number, and $\left[\mathbf{X}_i, \mathbf{X}_j\right]$ denotes the commutator operator defined by

$$\left[\mathbf{X}_i, \mathbf{X}_j\right] = \mathbf{X}_i\mathbf{X}_j - \mathbf{X}_j\mathbf{X}_i.$$

The commutator and adjoint tables of the generalized Drinfeld-Sokolov system (1.8) are given in Tables 1 and 2, corresponding to the general case and the $\alpha = 1$ case, respectively. By using these tables, we simplify as much as possible the maximal Lie algebra (made up from a linear combination of all its infinitesimal generators) through the adjoint table in order to get the optimal systems. It is important to point out paper [34], where all three and four-dimensional indecomposable real Lie algebras are classified.

Table 1. Commutator table.

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$
$\mathbf{X}_1$	0	0	$3\mathbf{X}_1$	$\mathbf{X}_2$
$\mathbf{X}_2$	0	0	$\mathbf{X}_2$	0
$\mathbf{X}_3$	$-3\mathbf{X}_1$	$-\mathbf{X}_2$	0	$2\mathbf{X}_4$
$\mathbf{X}_4$	$-\mathbf{X}_2$	0	$-2\mathbf{X}_4$	0

Table 2. Adjoint table.

Ad	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$
$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3 - 3\varepsilon\mathbf{X}_1$	$\mathbf{X}_4 - \varepsilon\mathbf{X}_2$
$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3 - \varepsilon\mathbf{X}_2$	$\mathbf{X}_4$
$\mathbf{X}_3$	$\mathrm{e}^{3\varepsilon}\mathbf{X}_1$	$\mathrm{e}^{\varepsilon}\mathbf{X}_2$	$\mathbf{X}_3$	$\mathrm{e}^{-2\varepsilon}\mathbf{X}_4$
$\mathbf{X}_4$	$\mathbf{X}_1 + \varepsilon\mathbf{X}_2$	$\mathbf{X}_2$	$\mathbf{X}_3 + 2\varepsilon\mathbf{X}_4$	$\mathbf{X}_4$

Theorem 4.1. *For the generalized Drinfeld-Sokolov system (1.8), an optimal system of one-dimensional subalgebras in each case is given by:*

(i) Case α, β, γ, δ non-zero arbitrary constants with $\alpha \neq 1$. An optimal system of one-dimensional subalgebras is given by

$$\{X_1 + c_1 X_2, X_2, X_3\}. \tag{4.1}$$

(ii) Case $\alpha = 1$ and β, γ, δ non-zero arbitrary constants. An optimal system of one-dimensional subalgebras is given by

$$\{X_1 + c_1 X_2, X_1 + c_2 X_4, X_2, X_3, X_4\}. \tag{4.2}$$

Proof. Consider a general element in the symmetry algebra $\mathbf{X} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3 + a_4\mathbf{X}_4$. The goal is to simplify as many of the a_i coefficients as possible through the application of adjoint maps to $\mathbf{X}$. Without loss of generality, suppose first that $a_1 \neq 0$. We can assume that $a_1 = 1$. Referring to Table 2, if we act on $\mathbf{X}$ by $\mathbf{X}_1$ we can make the coefficient of $\mathbf{X}_2$ vanish, obtaining a new $\mathbf{X}' = \mathbf{X}_1 + a_3'\mathbf{X}_3 + a_4'\mathbf{X}_4$. We continue acting on $\mathbf{X}'$ to cancel the remaining coefficients until no further simplifications are possible. Following a similar procedure for the rest of cases, we get the results of Theorem 4.1. □

5. Symmetry reductions

We reduce the generalized Drinfeld-Sokolov system (1.8) of PDEs to a system of two ordinary differential equations (ODEs) by using the symmetries shown in Theorem 3.1 and the optimal systems of one-dimensional subalgebras in Theorem 4.1.

These reductions are obtained by doing a change of variables with the solutions of the characteristic system

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta_u} = \frac{dv}{\eta_v}. \tag{5.1}$$

From (5.1), we obtain the similarity variable z and the similarity solutions $U(z)$, $V(z)$, which are substituted into system (1.8) to get the reduced one.

5.1. *Reduced system associated to $\boldsymbol{X}_1 + c_1\boldsymbol{X}_2$*

Considering α, β non-zero arbitrary parameters and by taking into account the generator $\mathbf{X}_1 + c_1\mathbf{X}_2$, we reduce the gDS system (1.8) into a nonlinear ODE system.

From (5.1), we obtain the invariants

$$z = x - c_1 t, \qquad \begin{cases} u(t,x) = U(z), \\ v(t,x) = V(z). \end{cases} \tag{5.2}$$

The previous transformation (5.2) is called a traveling wave, where c is the wave speed. The group-invariant solution (5.2) reduces Eq (1.8) to the following third-order ODE system:

$$\begin{cases} (U - c_1)\, U' + U''' - \gamma V^{\delta} V' = 0, \\ (\alpha U - c_1)\, V' + \beta V''' = 0. \end{cases} \tag{5.3}$$

Let us assume that system (5.3) has a solution of the form

$$U(z) = H(z), \tag{5.4a}$$

$$V(z) = U(z)^n, \tag{5.4b}$$

where n is a parameter and $H(z)$ is a solution of Jacobi equation: $(H')^2 = r + pH^2 + qH^4$, where r, p and q are constants.

Substituting the Jacobi elliptic sine function, $U(z) = \mathrm{sn}(z,k)^m$ and (5.4b) in the reduced system (5.3), we have concluded that for $m = 2$, $n = -2, 1$, $c_1 = -(k+1)m^2$ and k as an arbitrary parameter, an exact solution of the gDS system (1.8) is

$$\begin{aligned} u(t,x) &= \mathrm{sn}(x - c_1 t, k)^m, \\ v(t,x) &= \mathrm{sn}(x - c_1 t, k)^{nm}. \end{aligned} \tag{5.5}$$

For example, considering $n = 1, k = 1$ and taking into account that $\mathrm{sn}(z,1) = \tanh(z)$, a special solution of the gDS system (1.8) is

$$u(t,x) = v(t,x) = \tanh^2(x + 8t), \tag{5.6}$$

a dark soliton represented in Figure 1.

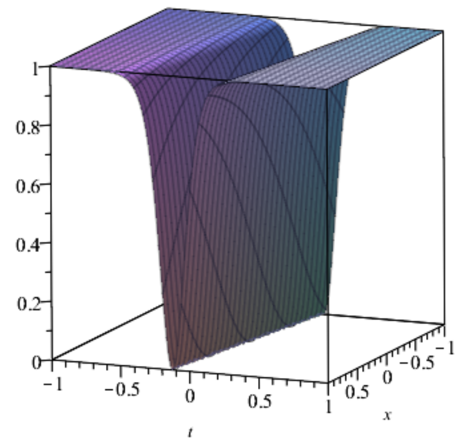

Figure 1. Dark soliton solution (5.6) of the gDS system (1.8).

5.2. Reduced system associated to $\boldsymbol{X}_3$

In the case of α, β non-zero arbitrary parameters and by using the generator $\mathbf{X}_3$, we obtain the similarity variable and solutions

$$z = xt^{\frac{-1}{3}}, \qquad \begin{cases} u(t,x) = U(z)t^{\frac{-2}{3}}, \\ v(t,x) = V(z)t^{\frac{-4}{3(\delta+1)}}. \end{cases} \tag{5.7}$$

Hence, the invariants (5.7) reduce system (1.8) to the third-order ODE system:

$$\begin{cases} (3U - z)\,U' - 2U + 3U''' - 3\gamma V^{\delta} V' = 0, \\ (\delta + 1)\,(-3\alpha U + z)\,V' + 4V - 3\beta\,(\delta + 1)\,V''' = 0. \end{cases} \tag{5.8}$$

5.3. Reduced system associated to $\boldsymbol{X}_1 + c_2\boldsymbol{X}_4$

In the case of $\alpha = 1, \beta$ a non-zero arbitrary parameter, and considering the generator $\mathbf{X}_1 + c_2\mathbf{X}_4$, we obtain the similarity variable and solutions

$$z = \frac{2x - c_2 t^2}{2}, \qquad \begin{cases} u(t,x) = c_2 t + U(z), \\ v(t,x) = V(z). \end{cases} \tag{5.9}$$

By substituting (5.9) into system (1.8), we reduce it to the third-order ODE system:

$$\begin{cases} c_2 + UU' + U''' - \gamma V^{\delta} V' = 0, \\ UV' - 3\beta V''' = 0. \end{cases} \tag{5.10}$$

5.4. *Reduced system associated to $\mathbf{X}_4$*

In the case of $\alpha = 1, \beta$ a non-zero arbitrary parameter, and considering the generator $\mathbf{X}_4$, we obtain the similarity variable and solutions

$$z = t, \qquad \begin{cases} u(t,x) = \dfrac{x}{t} + U(z), \\ v(t,x) = V(z). \end{cases} \tag{5.11}$$

By substituting (5.11) into system (1.8), we reduce it to the ODE system:

$$\begin{cases} U + zU' = 0, \\ V' = 0. \end{cases} \tag{5.12}$$

The trivial solution of the previous ODE system is $U(z) = \frac{k_1}{z}, V(z) = k_0$, where k_0, k_1 are arbitrary constants.

6. Conclusions

We have exhaustively classified all low-order local conservation laws that are admitted by the generalized Drinfeld-Sokolov system (1.8). This involved identifying all multipliers with a general low-order form (2.5) and determining their corresponding conserved densities and fluxes. Furthermore, we have associated each conserved quantity with its physical property, such as mass density, momentum density, total energy, Galilean momentum, angular momentum, and energy-momentum, which are all relevant to system (1.8).

Our analysis has revealed that first-order generalized symmetries can be reduced to Lie symmetries, and we have presented a classification of the Lie point symmetries. These symmetry transformations have been interpreted as translations, scalings, and Galilean boosts. Additionally, we have constructed optimal systems of one-dimensional subalgebras and used them to reduce the PDE system to ODE systems.

Finally, we have obtained new solutions for the traveling wave case using Jacobi elliptic functions. In particular, for special values of the parameters, we have shown a solution of physical interest: a dark soliton.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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