Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 28558–28568.
DOI:10.3934/math.20231461
Received: 03 August 2023
Revised: 29 September 2023
Accepted: 09 October 2023
Published: 20 October 2023

Research article

The inverses of tails of the generalized Riemann zeta function within the range of integers

Zhenjiang Pan and Zhengang Wu*

School of Mathematics, Northwest University, Xi'an 710127, China

* **Correspondence:** Email: 20144743@nwu.edu.cn.

Abstract: In recent years, many mathematicians researched infinite reciprocal sums of various sequences and evaluated their value by the asymptotic formulas. We study the asymptotic formulas of the infinite reciprocal sums formed as $\left(\sum_{k=n}^{\infty} \frac{1}{k^r(k+t)^s}\right)^{-1}$ for $r, s, t \in \mathbb{N}^+$, where the asymptotic formulas are polynomials.

Keywords: Riemann zeta function; Hurwitz zeta function; reciprocal sums; asymptotic formulas; infinite sums
Mathematics Subject Classification: 11B83, 11M06

1. Introduction

Throughout the years, many mathematicians have been working on partial infinite sums of reciprocal linear recurrence sequences.

In 2011, Takao Komatsu [8] researched the nearest integer of the sum of reciprocal Fibonacci numbers and derived

$$\begin{aligned} \left\| \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\| &= F_n - F_{n-1}, \\ \left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^n}{F_k} \right)^{-1} \right\| &= (-1)^n (F_n + F_{n+1}), \end{aligned} \tag{1.1}$$

where $\|\cdot\|$ denoted the nearest integer; in other words, $\|x\| = \lfloor x - \frac{1}{2} \rfloor$.

In 2020, Ho-Hyeong Lee and Jong-Do Park [9] gave the concept of asymptotic formulas, which were more accurate. The conclusions were as follows:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+2l}} \right)^{-1} \right\rfloor \sim F_{n+l-1}F_{n+l} - (F_l^2 + (-1)^l)\frac{(-1)^n}{3},$$
$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+2l-1}} \right)^{-1} \right\rfloor \sim F_{n+l-1}^2 - (F_{l-1}F_l + (-1)^l)\frac{(-1)^n}{3}, \tag{1.2}$$

where $a_n \sim b_n$ meant $\lim_{n\to\infty}(a_n - b_n) = 0$. For more results related to the infinite reciprocal sums of linear recurrence sequences, see [1, 13, 15] and references therein.

The zeta function $\zeta(z)$ is undoubtedly the most famous function in analytic number theory. Initially studied by Euler and achieved prominence with Riemann, it abstracted the attention of many mathematicians. Another well-known sequence harmonic number H_n is the sum of the first n terms of $\zeta(z)$ when $z = 1$, and the generating function of harmonic numbers $\sum_{n=1}^{\infty} H_n x^n$ is an important tool to study the property of H_n. Kim [4–7] derived many worthy and interesting results associated with the zeta function, harmonic number and its generating function, which inspired us deeply.

At the same time, many researchers began to study the tails of well-known functions such as the Riemann zeta function and the Hurwitz zeta function in [2, 3, 10, 12, 14].

For example, Kim Donggyun and Song Kyunghwan [3] studied the inverses of tails of the Riemann zeta function. Derived for s on the critical strip $0 < s < 1$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{k^s} \right)^{-1} \right\rfloor \sim \begin{cases} 2(1-2^{1-s})(n-\frac{1}{2})^s, & \text{if } n \text{ is even,} \\ -2(1-2^{1-s})(n-\frac{1}{2})^s, & \text{if } n \text{ is odd.} \end{cases} \tag{1.3}$$

Ho-Hyeong Lee and Jong-Do Park [10] dealt with the inverses of tails of Hurwitz zeta function when $s \geq 2$, $s \in \mathbb{N}$ and $0 \leq a < 1$, and derived

$$\left(\sum_{k=n}^{\infty} \frac{1}{(k+a)^s} \right)^{-1} \sim \sum_{j=0}^{s-1} A_j^*(n+a)^j, \tag{1.4}$$

where $A_{s-1}^* = s-1$, $A_l^s = -\sum_{j=1}^{s-l-1} x_j^s A_{l+j}^s$, $x_j^s = \binom{s-2+j}{j} B_j$ and B_j are Bernoulli numbers.

In this paper, we extend their asymptotic formulas for the methods and results by considering the tails of $\left(\sum_{k=n}^{\infty} \frac{1}{k(k+t)}\right)^{-1}$, $\left(\sum_{k=n}^{\infty} \frac{1}{[k(k+t)]^2}\right)^{-1}$, $\left(\sum_{k=n}^{\infty} \frac{1}{k^r(k+t)^s}\right)^{-1}$ and further revealing the property of reciprocal sums of the various sequences.

2. Main results

Before our conclusion, we define $\binom{i}{-1} = 0$ for all $i \in \mathbb{N}^+$, which will take effect in expressing the asymptotic formulas in Theorem 3.

Theorem 1. *For all $m \in \mathbb{N}$, we have*

$$\left(\sum_{k=n}^{\infty} \frac{1}{k(k+t)} \right)^{-1} \sim n + \frac{t}{2} - \frac{1}{2}.$$

Theorem 2. *For all $m \in \mathbb{N}$, we have*

$$\left(\sum_{k=n}^{\infty} \frac{1}{[k(k+t)]^2}\right)^{-1} \sim 3n^3 + an^2 + bn + c,$$

where

$$\begin{aligned} a &= \frac{9}{2}t - \frac{9}{2}, \\ b &= \frac{27}{20}t^2 - \frac{9}{2}t + \frac{15}{4}, \\ c &= -\frac{3}{40}t^3 - \frac{27}{40}t^2 + \frac{15}{8}t - \frac{9}{8}. \end{aligned}$$

Theorem 3. *If $r + s - 1 > 0$ and $r, s, t \in \mathbb{N}$, then there exists the unique polynomial*

$$B(n, r, s, t) = b_{r+s-1}n^{r+s-1} + b_{r+s-2}n^{r+s-2} + \cdots + b_1 n + b_0,$$

subject to

$$\left(\sum_{k=n}^{\infty} \frac{1}{n^r(n+t)^s}\right)^{-1} \sim B(n, r, s, t),$$

where

$$\begin{aligned} b_{r+s-1} =& r + s - 1, \\ b_{r+s-2} =& \frac{(r+s-1)^2}{2(r+s)}[(2st - (r+s)], \\ b_{r+s-3} =& \frac{(r+s-1)^2}{12(r+s)^2(r+s+1)}[6s(r^2 s - r^2 + 2rs^2 - 2rs + s^3 - s^2 - 2s)t^2 \\ & - 6s(r + 3r^2 s - r^2 + 3rs^2 - 2rs - 2r + s^3 - s^2 - 2s)t \\ & + (2r^4 + 8r^3 s - r^3 + 12r^2 s^2 - 3r^2 s - 3r^2 + 8rs^3 - 3rs^2 - 6rs + 2s^4 - s^3 - 3s^2)], \\ & \cdots\cdots \\ & \cdots\cdots \\ b_{i-r-s+1} =& \frac{\sum\limits_{\substack{k_1+k_2=i \\ i-r-s+1 \le k_1 \le r+s-2 \\ r \le k_2 \le r+s}} \binom{s}{k_2-r} t^{r+s-k_2} \sum\limits_{j=k_1+1}^{r+s-1} b_j \binom{j}{k_1}}{3r+3s-i-3} - \frac{\sum\limits_{\substack{k_3+k_4=i \\ i-r-s+2 \le k_3, k_4 \le r+s-1}} b_{k_3} \sum\limits_{j=k_4}^{r+s-1} b_j \binom{j}{k_4}}{3r+3s-i-3} \\ & + \frac{\sum\limits_{j=i-r-s+2}^{r+s-1} b_j \left[\binom{j}{i-r-s} - (r+s-1)\binom{j}{i-r-s+1}\right]}{3r+3s-i-3}, \\ & \cdots\cdots \\ & \cdots\cdots \\ b_0 =& \frac{\sum\limits_{\substack{k_1+k_2=i \\ 0 \le k_1 \le r+s-2 \\ r \le k_2 \le r+s}} \binom{s}{k_2-s} t^{r+s-k_2} \sum\limits_{j=k_1+1}^{r+s-1} b_j \binom{j}{k_1}}{2(r+s-1)} - \frac{\sum\limits_{\substack{k_3+k_4=i \\ 1 \le k_3, k_4 \le r+s-1}} b_{k_3} \sum\limits_{j=k_4}^{r+s-1} b_j \binom{j}{k_4} + (r+s-1) \sum\limits_{j=1}^{r+s-1} b_j}{2(r+s-1)}. \end{aligned}$$

Remark. From Theorem 3, we derive that coefficients b_j $(0 \le j \le r+s-1)$ are determined by r, s and t. At the same time, if we calculate b_0 by using the representation of $b_{i-r-s+1}$, there will appear $\binom{r+s-1}{-1}, \binom{r+s-2}{-1}, \cdots, \binom{1}{-1}$. In order to make b_0 the representation of b_j, we give the definition $\binom{i}{-1} = 0$ for $i \in \mathbb{N}^+$ in this paper. Undoubtedly, it satisfies $\binom{n+1}{0} = \binom{n}{0} + \binom{n}{-1}$ for $n \in \mathbb{N}^+$.

Corollary 1. *If $s = 1$ and $m = 0$, we have*

$$\left(\sum_{k=n}^{\infty} \frac{1}{n^2}\right)^{-1} \sim n - \frac{1}{2}.$$

Corollary 2. *If $s = 2$ and $t = 0$, we have*

$$\left(\sum_{k=n}^{\infty} \frac{1}{n^4}\right)^{-1} \sim 3n^3 - \frac{9}{2}n^2 + \frac{15}{4}n - \frac{9}{8}.$$

Corollary 3. *If $s = 2$, we have*

$$\left(\sum_{k=n}^{\infty} \frac{1}{n^2(n+1)^2}\right)^{-1} \sim 3n^3 + \frac{3}{5}n,$$

$$\left(\sum_{k=n}^{\infty} \frac{1}{n^2(n+2)^2}\right)^{-1} \sim 3n^3 + \frac{9}{2}n^2 + \frac{3}{20}n - \frac{5}{8},$$

$$\left(\sum_{k=n}^{\infty} \frac{1}{n^2(n+3)^2}\right)^{-1} \sim 3n^3 + 9n^2 + \frac{12}{5}n - \frac{18}{5}.$$

3. Proof of theorem

3.1. Proof of Theorem 1

We need to solve several lemmas for the proof.

Lemma 1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of a positive real number with $\lim_{n\mapsto\infty} a_n = \lim_{n\mapsto\infty} b_n = 0$. If $a_n < b_n + a_{n+1}$ hold for any $n \in N^+$, then we have*

$$a_n < \sum_{k=n}^{\infty} b_n \quad \text{for } n \in N^+.$$

Proof. See Lemma 2.1 [11]. □

Lemma 2. *For all $t \ge 2$ and $t \in \mathbb{N}$, we have*

$$\frac{1}{n + \frac{t}{2} - \frac{1}{2}} < \frac{1}{n(n+t)} + \frac{1}{n + \frac{t}{2} + \frac{1}{2}}.$$

Proof. It is equivalent with

$$\frac{1}{n + \frac{t}{2} - \frac{1}{2}} - \frac{1}{n(n+t)} - \frac{1}{n + \frac{t}{2} + \frac{1}{2}} < 0. \tag{3.1}$$

$$\text{The left side} = \frac{n(n+t) - (n+\frac{t}{2}-\frac{1}{2})(n+\frac{t}{2}+\frac{1}{2})}{(n+\frac{t}{2}-\frac{1}{2})(n+\frac{t}{2}+\frac{1}{2})n(n+t)}$$
$$= \frac{-\frac{t^2}{4}+\frac{1}{4}}{(n+\frac{t}{2}-\frac{1}{2})(n+\frac{t}{2}+\frac{1}{2})n(n+t)}.$$

Hence, we have

$$-\frac{t^2}{4}+\frac{1}{4} < 0 \quad \text{for } n > 2,$$

so

$$\frac{1}{n+\frac{t}{2}-\frac{1}{2}} < \frac{1}{n(n+t)} + \frac{1}{n+\frac{t}{2}+\frac{1}{2}} \qquad \text{for } n > 2.$$

This completes the proof. □

Lemma 3. *For all $\varepsilon > 0$, there exists $N_0 > 2$, subject to*

$$\frac{1}{n+\frac{t}{2}-\frac{1}{2}-\varepsilon} > \frac{1}{n+\frac{t}{2}+\frac{1}{2}-\varepsilon} + \frac{1}{n(n+t)} \qquad \textit{for } n > N_0.$$

Proof. It is equivalent with

$$\frac{1}{n+\frac{t}{2}-\frac{1}{2}-\varepsilon} - \frac{1}{n+\frac{t}{2}+\frac{1}{2}-\varepsilon} - \frac{1}{n(n+t)} > 0. \tag{3.2}$$

$$\text{The left side} = \frac{n^2+tn-(n^2+tn+\frac{t^2}{4})-\frac{1}{4}-2\varepsilon(n+\frac{t}{2})+\varepsilon^2}{(n+\frac{t}{2}-\frac{1}{2}-\varepsilon)(n-\frac{t}{2}+\frac{1}{2}-\varepsilon)n(n+t)}$$
$$= \frac{2\varepsilon n+t\varepsilon+\frac{1}{4}-\frac{t^2}{4}-\varepsilon^2}{(n+\frac{t}{2}-\frac{1}{2}-\varepsilon)(n-\frac{t}{2}+\frac{1}{2}-\varepsilon)n(n+t)}.$$

We can restrict $\varepsilon < 1$ and fix t, then we have

$$t\varepsilon + \frac{1}{4} - \frac{t^2}{4} - \varepsilon^2 = O(1),$$

so there exists $N_0 > 2$, subject to

$$2\varepsilon n + O(1) > 0 \quad \text{for } n > N_0,$$

then

$$\frac{1}{n+\frac{t}{2}-\frac{1}{2}-\varepsilon} > \frac{1}{n+\frac{t}{2}+\frac{1}{2}-\varepsilon} + \frac{1}{n(n+t)},$$

which proves (3.2) and completes the proof. □

Proof of Theorem 1.

Case 1. *When $t \geq 2$.*

By Lemmas 1–3, we have for all $\varepsilon > 0$, there exists $N_0 > 2$, subject to

$$n+\frac{t}{2}-\frac{1}{2}-\varepsilon<\left(\sum_{k=n}^{\infty}\frac{1}{n(n+t)}\right)^{-1}<n+\frac{t}{2}-\frac{1}{2}\quad \textit{for } n>N_0, \tag{3.3}$$

hence

$$\left|\left(\sum_{k=t}^{\infty}\frac{1}{k(k+t)}\right)^{-1}-(n+\frac{t}{2}-\frac{1}{2})\right|<\varepsilon;$$

in other words,

$$\left(\sum_{k=n}^{\infty}\frac{1}{k(k+t)}\right)^{-1}\sim(n+\frac{t}{2}-\frac{1}{2}).$$

Case 2. *When $t = 1$,*

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{k(k+1)}&=\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}+\cdots\\&=(\frac{1}{n}-\frac{1}{n+1})+(\frac{1}{n+1}-\frac{1}{n+2})+(\frac{1}{n+2}-\frac{1}{n+3})+\cdots\\&=\frac{1}{n},\end{aligned}$$

hence

$$\left(\sum_{k=n}^{\infty}\frac{1}{k(k+1)}\right)^{-1}\sim n.$$

Case 3. *When $t = 0$, the proof is similar with Case 1, and we can easily deduce the result.*

□

3.2. Proof of Theorem 2

Lemma 4. *Let $f(n,t,\varepsilon) = 3n^3 + an^2 + bn + c$ and a, b, c are defined in Theorem 2. Then for all $\varepsilon > 0$, there exists $N_1 > 0$, subject to*

$$\frac{1}{f(n,t,\varepsilon)}<\frac{1}{f(n+1,t,\varepsilon)}+\frac{1}{n^2(n+t)^2}\quad \textit{for } n>N_1.$$

Proof. It is equivalent with

$$\frac{1}{f(n,t,\varepsilon)}-\frac{1}{f(n+1,t,\varepsilon)}-\frac{1}{n^2(n+t)^2}<0\quad \text{for } n>N_1. \tag{3.4}$$

$$\begin{aligned}\text{The left side} &= \frac{[f(n+1,t,\varepsilon)-f(n,t,\varepsilon)]n^2(n+t)^2 - f(n,t,\varepsilon)f(n+1,t,\varepsilon)}{f(n,t,\varepsilon)f(n+1,t,\varepsilon)n^2(n+t)^2} \\ &= \frac{[9n^2+(2a+9)n+(a+b+3)]n^2(n+t)^2}{f(n,t,\varepsilon)f(n+1,t,\varepsilon)n^2(n+t)^2} \\ &\quad - \frac{(3n^3+an^2+bn+c)}{f(n,t,\varepsilon)f(n+1,t,\varepsilon)n^2(n+t)^2}[3n^3+(a+9)n^2+(2a+b+9)n+(a+b+c+3+\varepsilon)] \\ &= \frac{-6\varepsilon n^3 + A(t)O(n^2)}{f(n,t,\varepsilon)f(n+1,t,\varepsilon)n^2(n+t)^2},\end{aligned}$$

where $A(t)$ is a function with variable t, then we fix t for all $\varepsilon > 0$, and there exists $N_1 > 0$, subject to

$$-6\varepsilon n^3 + A(t)O(n^2) = -6\varepsilon n^3 + O(n^2) < 0, \qquad \text{for } n > N_1,$$

hence

$$\frac{1}{f(n,t,\varepsilon)} - \frac{1}{f(n+1,t,\varepsilon)} - \frac{1}{n^2(n+t)^2} < 0,$$

and this completes the proof. □

Lemma 5. *Let $g(n,t,\varepsilon) = 3n^3+an^2+bn+c-\varepsilon$, then for all $\varepsilon > 0$ there exists $N_2 > 0$, subject to*

$$\frac{1}{g(n,t,\varepsilon)} > \frac{1}{g(n+1,t,\varepsilon)} + \frac{1}{n^2(n+t)^2} \qquad \text{for } n > N_2.$$

Proof. The proof is similar with Lemma 4, and we can easily deduce the result. □

Proof of Theorem 2. By Lemmas 1, 4 and 5, we have for all $\varepsilon > 0$. There exists $N_3 = max\{N_1, N_2\} > 0$, subject to

$$\frac{1}{f(n,t,\varepsilon)} < \sum_{k=n}^{\infty} \frac{1}{n^2(n+t)^2} < \frac{1}{g(n,t,\varepsilon)} \qquad \text{for } n > N_3, \tag{3.5}$$

hence

$$3n^3+an^2+bn+c-\varepsilon < \left(\sum_{k=n}^{\infty} \frac{1}{n^2(n+t)^2}\right)^{-1} < 3n^3+an^2+bn+c+\varepsilon,$$

which is equivalent to

$$\left(\sum_{k=n}^{\infty} \frac{1}{n^2(n+t)^2}\right)^{-1} \sim 3n^3+an^2+bn+c.$$

□

3.3. Proof of Theorem 3

Proof of Theorem 3. According to the method of proving Theorem 2, it is enough to prove that there exists polynomial

$$B(n) = b_l n^l + b_{l-1} n^{l-1} + \cdots + b_1 n + b_0,$$

subject to

$$\frac{1}{B(n)} - \frac{1}{B(n+1)} - \frac{1}{n^r(n+t)^s} = \frac{O(n^{l-1})}{B(n)B(n+1)n^r(n+t)^s}. \tag{3.6}$$

$$\text{The left side } = \frac{B(n+1)-B(n)}{B(n)B(n+1)} - \frac{1}{n^r(n+t)^s} = \frac{[B(n+1)-B(n)]n^r(n+t)^s - B(n)B(n+1)}{B(n)B(n+1)n^r(n+t)^s}.$$

Let

$$\begin{aligned} C(n) &= B(n+1)-B(n), \\ D(n) &= n^r(n+t)^s, \\ E(n) &= B(n), \\ F(n) &= B(n+1). \end{aligned}$$

Therefore, it is enough to prove

$$C(n)D(n) - E(n)F(n) = O(n^{l-1}), \tag{3.7}$$

hence we have the necessary condition (1):

$$\alpha\left(C(n)D(n)\right) = \alpha\left(E(n)F(n)\right),$$

the notation $\alpha(f(x))$ means the order of $f(x)$, then we have

$$l = r+s-1. \tag{3.8}$$

We note the number of coefficients is $r+s$, then we have

$$\begin{aligned} C(n)D(n) &= \left(\sum_{i=1}^{r+s-1}\left(b_i\sum_{j=0}^{i-1}\binom{i}{j}n^j\right)\right)\left(n^r\left(\sum_{p=0}^{s}\binom{s}{p}t^{s-p}n^p\right)\right) \\ &= \left(\sum_{i=0}^{r+s-2}\left(\sum_{j=i+1}^{r+s-1}b_j\binom{j}{i}\right)n^i\right)\left(\sum_{p=r}^{r+s}\left(\binom{s}{p-r}t^{r+s-p}\right)n^p\right) \\ &= \sum_{i=r}^{2r+2s-2}\left(\sum_{\substack{k_1+k_2=i,\\ 0\le k_1\le r+s-2, r\le k_2\le r+s}}\binom{s}{k_2-r}t^{r+s-k_2}\sum_{j=k_1+1}^{r+s-1}b_j\binom{j}{k_1}\right)n^i \\ &= \sum_{i=r}^{2r+2s-2}\sum_{\substack{k_1+k_2=i,\\ 0\le k_1\le r+s-2, r\le k_2\le r+s}}\binom{s}{k_2-r}t^{r+s-k_2}\sum_{j=k_1+1}^{r+s-1}b_j\binom{j}{k_1}n^i \end{aligned}$$

and

$$\begin{aligned} E(n)F(n) &= \left(\sum_{i=0}^{r+s-1}b_in^i\right)\left(\sum_{q=0}^{r+s-1}b_q\left(\sum_{j=0}^{q}\binom{q}{j}n^j\right)\right) = \left(\sum_{i=0}^{r+s-1}b_in^i\right)\left(\sum_{q=0}^{r+s-1}\left(\sum_{j=q}^{r+s-1}b_j\binom{j}{q}\right)n^q\right) \\ &= \sum_{i=0}^{2r+2s-2}\left(\sum_{\substack{k_3+k_4=i,\\ 0\le k_3,k_4\le r+s-1}}b_{k_3}\sum_{j=k_4}^{r+s-1}b_j\binom{j}{k_4}\right)n^i = \sum_{i=0}^{2r+2s-2}\sum_{\substack{k_3+k_4=i,\\ 0\le k_3,k_4\le r+s-1}}b_{k_3}\sum_{j=k_4}^{r+s-1}b_j\binom{j}{k_4}n^i. \end{aligned}$$

We get the necessary condition (2): If $r+s-1 \le i \le 2r+2s-2$, the coefficients of $C(n)D(n)$ and $E(n)F(n)$ are equal, which is equivalent to the system of equations as follows:

$$\sum_{\substack{k_1+k_2=i,\\ 0\le k_1\le r+s-2, r\le k_2\le r+s}}\binom{s}{k_2-r}t^{r+s-k_2}\sum_{j=k_1+1}^{r+s-1}b_j\binom{j}{k_1} = \sum_{\substack{k_3+k_4=i,\\ 0\le k_3,k_4\le r+s-1}}b_{k_3}\sum_{j=k_4}^{r+s-1}b_j\binom{j}{k_4}, \tag{3.9}$$

where $i = 2r + 2s - 2,\ 2r + 2s - 3,\ \cdots,\ r + s$ *and* $r + s - 1$.

We rewrite the system of (3.9) as

$$\begin{cases} (r+s-1)b_{r+s-1} = b_{r+s-1}^2, \\ [(r+s-2) - 2b_{r+s-1}]\, b_{r+s-2} = f_1(b_{r+s-1}), \\ [(r+s-3) - 2b_{r+s-1}]\, b_{r+s-3} = f_2(b_{r+s-1}, b_{r+s-2}), \\ \cdots \\ \cdots \\ -2b_{r+s-1}b_0 = f_{s+r}(b_{r+s-1}, b_{r+s-2}, \cdots, b_1). \end{cases} \tag{3.10}$$

Clearly the first equation of (3.10) has two solutions, $b_{r+s-1} = 0$ and $b_{r+s-1} = r + s - 1$, and combined with (3.8) $l = r + s - 1$, we have

$$b_{r+s-1} = r + s - 1. \tag{3.11}$$

Substitute (3.11) into the second equation of (3.10). We have the coefficient in the left side $[(r + s-2) - 2b_{r+s-1}]$ as not equal to zero, and the right side $f_1(b_{r+s-1})$ as a constant, then the second equation of (3.10) has unique solution.

Repeat the above process for every equation of (3.10). The coefficient in the left side is never equal to zero, and the right side is always a constant, which implies the system of equations has a unique solution, denoted it by $(\ b_{2s-1},\ b_{2s-2},\ b_{2s-3},\ \cdots,\ b_1,\ b_0\ )$ with

$$\begin{aligned} b_{r+s-1} &= r + s - 1, \\ b_{r+s-2} &= \frac{(r+s-1)^2}{2(r+s)}[(2st - (r+s)], \\ b_{r+s-3} &= \frac{(r+s-1)^2}{12(r+s)^2(r+s+1)}[6s(r^2s - r^2 + 2rs^2 - 2rs + s^3 - s^2 - 2s)t^2 \\ &\quad - 6s(r + 3r^2s - r^2 + 3rs^2 - 2rs - 2r + s^3 - s^2 - 2s)t \\ &\quad + (2r^4 + 8r^3s - r^3 + 12r^2s^2 - 3r^2s - 3r^2 + 8rs^3 - 3rs^2 - 6rs + 2s^4 - s^3 - 3s^2)], \\ &\quad \cdots\cdots \\ &\quad \cdots\cdots \end{aligned}$$

$$\begin{aligned} b_{i-r-s+1} &= \frac{\sum\limits_{\substack{k_1+k_2=i \\ i-r-s+1\le k_1\le r+s-2 \\ r\le k_2\le r+s}} \binom{s}{k_2-r} t^{r+s-k_2} \sum\limits_{j=k_1+1}^{r+s-1} b_j \binom{j}{k_1}}{3r+3s-i-3} - \frac{\sum\limits_{\substack{k_3+k_4=i \\ i-r-s+2\le k_3,k_4\le r+s-1}} b_{k_3} \sum\limits_{j=k_4}^{r+s-1} b_j \binom{j}{k_4}}{3r+3s-i-3} \\ &\quad + \frac{\sum\limits_{j=i-r-s+2}^{r+s-1} b_j \left[\binom{j}{i-r-s} - (r+s-1)\binom{j}{i-r-s+1}\right]}{3r+3s-i-3}, \\ &\quad \cdots\cdots \\ &\quad \cdots\cdots \end{aligned}$$

$$b_0 = \frac{\sum\limits_{\substack{k_1+k_2=i \\ 0\le k_1\le r+s-2 \\ r\le k_2\le r+s}} \binom{s}{k_2-s} t^{r+s-k_2} \sum\limits_{j=k_1+1}^{r+s-1} b_j \binom{j}{k_1}}{2(r+s-1)} - \frac{\sum\limits_{\substack{k_3+k_4=i \\ 1\le k_3,k_4\le r+s-1}} b_{k_3} \sum\limits_{j=k_4}^{r+s-1} b_j \binom{j}{k_4} + (r+s-1)\sum\limits_{j=1}^{r+s-1} b_j}{2(r+s-1)},$$

where we define $\binom{j}{-1} = 0$ for all $j \in \mathbb{N}^+$.

It is clear that the coefficient b_i is determined by r, s and t, and the solution is corresponded to a polynomial with (r+s-1)-order denoted by $B(n, r, s, t)$. We can easily prove $B(n, r, s, t)$ satisfies (3.6), and then we have

$$\left(\sum_{k=n}^{\infty} \frac{1}{n^r (n+t)^s}\right)^{-1} \sim B(n, r, s, t).$$

□

4. Conclusions

In this paper we discussed the reciprocal sums of the generalized Riemann zeta function within the range of integers, and we also considered other functions within the range of integers.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors express their gratitude to the referee for very helpful and detailed comments. Supported by the National Natural Science Foundation of China (Grant No. 11701448).

Conflict of interest

The authors declare no conflict of interest.

References

1. G. Choi, Y. Choo, On the reciprocal sums of products of Fibonacci and Lucas number, *Filomat*, **32** (2018), 2911–2920. http://dx.doi.org/10.2298/FIL1808911C
2. W. Hwang, K. Song, A reciprocal sum related to the Riemann zeta function at s = 6, arXiv: 1709.07994.
3. D. Kim, K. Song, The inverses of tails of the Riemann zeta function, *J. Inequal. Appl.*, **2018** (2018), 157. http://dx.doi.org/10.1186/s13660-018-1743-6
4. D. Kim, H. Kim, T. Kim, Some identities on generalized harmonic numbers and generalized harmonic functions, *Demonstr. Math.*, **56** (2023), 20220229. http://dx.doi.org/10.1515/dema-2022-0229
5. T. Kim, D. Kim, H. Lee, J. Kwon, On some summation formulas, *Demonstr. Math.*, **55** (2022), 1–7. http://dx.doi.org/10.1515/dema-2022-0003
6. T. Kim, Euler numbers and polynomials associated with Zeta function, *Abstr. Appl. Anal.*, **2008** (2008), 581582. http://dx.doi.org/10.1155/2008/581582

7. T. Kim, Degenerate Euler zeta function, *Russ. J. Math. Phys.*, **22** (2015), 469–472. http://dx.doi.org/10.1134/S1061920815040068
8. T. Komatsu, On the nearest integer of the sum of reciprocal Fibonacci numbers, *Aportaciones Matematicas Investigacion*, **20** (2011), 171–184.
9. H. Lee, J. Park, Asymptotic behavior of reciprocal sum of two products of Fibonacci numbers, *J. Inequal. Appl.*, **2020** (2020), 91. http://dx.doi.org/10.1186/s13660-020-02359-z
10. H. Lee, J. Park, Asymptotic behavior of the inverse of tails of Hurwitz zeta function, *J. Korean Math. Soc.*, **57** (2020), 1535–1549. http://dx.doi.org/10.4134/JKMS.j190789
11. H. Lee, J. Park, The limit of reciprocal sum of some subsequential Fibonacci number, *AIMS Mathematics*, **11** (2021), 12379–12394. http://dx.doi.org/10.3934/math.2021716
12. X. Lin, Some identities related to Riemann zeta-function, *J. Inequal. Appl.*, **2016** (2016), 32. http://dx.doi.org/10.1186/s13660-016-0980-9
13. D. Marques, P. Trojovsky, The proof of a formula concerning the asymptotic behavior of the reciprocal sum of the square of multiple-angle Fibonacci numbers, *J. Inequal. Appl.*, **2022** (2022), 21. http://dx.doi.org/10.1186/s13660-022-02755-7
14. H. Xu, Some computational formulas related the Riemann zeta-function tails, *J. Inequal. Appl.*, **2016** (2016), 132. http://dx.doi.org/10.1186/s13660-016-1068-2
15. Z. Xu, T. Wang, The infinite sum of the cubes of reciprocal Pell numbers, *Adv. Differ. Equ.*, **2013** (2013), 184. http://dx.doi.org/10.1186/1687-1847-2013-184