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Research article

A faster iterative scheme for solving nonlinear fractional differential equations of the Caputo type

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Abstract: In this paper, we introduce a new fixed point iterative scheme called the AG iterative scheme that is used to approximate the fixed point of a contraction mapping in a uniformly convex Banach space. The iterative scheme is used to prove some convergence result. The stability of the new scheme is shown. Furthermore, weak convergence of Suzuki's generalized non-expansive mapping satisfying condition (C) is shown. The rate of convergence result is proved and it is demonstrated via an illustrative example which shows that our iterative scheme converges faster than the Picard, Mann, Noor, Picard-Mann, M and Thakur iterative schemes. Data dependence results for the iterative scheme are shown. Finally, our result is used to approximate the solution of a nonlinear fractional differential equation of Caputo type.

Keywords: fixed point; rate of convergence; AG iterative scheme; Caputo fractional differential equation; weak convergence; $\mathcal{J}$-stability
Mathematics Subject Classification: 34A08, 47J25, 47J26

1. Introduction

Fixed point theory is widely becoming an indispensable area of mathematics in its right and the tools involved are used to solve nonlinear problems that sometimes appear unsolvable with the traditional analytical methods [1]. Practically, invoking some tools in fixed point theory have helped to circumvent the challenge encountered while trying to obtain the analytical solution of certain nonlinear problems. The reader can refer to [2]. Ways to address the challenge include the transformation of the nonlinear problem into a fixed point operator equation that is subsequently solved via approximation of the fixed point operator equation by using any suitable fixed point iterative scheme.

Many physical problems are usually formulated as differential equations (which could be ordinary or partial) and subsequently transformed into an integral equation of any type or kind. It is in this manner that the majority of physical problems have been formulated and represented as fractional differential equations. It is common knowledge through research, that fractional differential equations tend to have wider a range of application to real life situations (see, e.g., [3–11] and the references therein). As mentioned in the previous paragraph, it has been observed in several studies that obtaining the analytical solutions of quite a large number of nonlinear problems has been difficult and sometimes, impossible. Therefore, as a measure to circumvent this challenge, many methods including the fixed point method have been adopted by many researchers in an effort to obtain solutions to nonlinear fractional differential equations (NFDEs). Specifically, fixed point iterative schemes have been applied to solve nonlinear differential equations (see, e.g., [3,12–22]

It is our purpose in this paper to introduce a new fixed point iterative scheme called the AG iterative scheme that approximates the fixed point of a contraction mapping in a uniformly convex Banach space. We use the new scheme to prove some convergence, stability and data dependence results. Also, we show that our scheme converges faster than some existing schemes in literature, and we use a numerical example to substantiate our result. We show that our scheme converges weakly to a fixed point of Suzuki's generalized nonexpansive mapping that satisfies condition (C). As an application, we use the new scheme (2.9) to approximate the solution of an NFDEs of the Caputo type. Our result generalizes and extends many existing results in literature.

2. Preliminaries

Let $\mathscr{X}$ be a Banach space and D be a nonempty, closed and convex subset of $\mathscr{X}$. Assume that $\mathbb{N}$, in this section and elsewhere, is the set of natural numbers and $\mathbb{R}$ represents the set of real numbers. The mapping $\mathcal{J} : D \to D$ is called a contraction mapping if it satisfies the following condition:

$$\|\mathcal{J}\omega - \mathcal{J}\nu\| \leq \delta\|\omega - \nu\| \tag{2.1}$$

for $\delta \in [0, 1)$. If condition (2.1) reduces to

$$\|\mathcal{J}\omega - \mathcal{J}\nu\| \leq \|\omega - \nu\|,$$

then the mapping $\mathcal{J}$ is said to be a nonexpansive mapping having a fixed point $p^* \in \mathscr{F}(\mathcal{J}) \neq \emptyset$.

Alternatively, Suzuki in 2008 (see [23]), gave the following definition for a generalized nonexpansive mapping.

Definition 2.1. *[23] Let D be a nonempty closed convex subset of a Banach space $\mathscr{X}$. Let $\mathcal{J} : D \to D$ be a mapping. Then, $\mathcal{J}$ is said to satisfy condition (C) if the following condition holds*

$$\frac{1}{2}\|\omega - \mathcal{J}\omega\| \leq \|\omega - \nu\| \Rightarrow \|\mathcal{J}\omega - \mathcal{J}\nu\| \leq \|\omega - \nu\| \tag{C}$$

for all $\omega, \nu \in D$.

Let D be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and $\{u_n\}$ be a bounded sequence in $\mathscr{X}$. For each $u \in \mathscr{X}$, we define the following (see, for example, [24])

(a) asymptotic radius of $\{u_n\}$ at u according to $\mathscr{R}(u, \{u_n\}) = \limsup_{n\to\infty} \|u - u_n\|$,

(b) asymptotic radius of $\{u_n\}$ relative to the set D according to

$$\mathscr{R}(D, \{u_n\}) = \inf\{\mathscr{R}(u, \{u_n\}) : u \in D\} \text{ and}$$

(c) asymptotic center of $\{u_n\}$ relative to the set $\mathscr{X}$ according to

$$\mathscr{A}(D, \{u_n\}) = \{u \in \mathscr{X} : \mathscr{R}(u, \{u_n\}) = \mathscr{R}(D, \{u_n\})\}.$$

Remark 2.1. *[25] It is obvious that the set $\mathscr{A}(D, \{u_n\})$ is a singleton in a uniformly convex Banach space.*

Definition 2.2. *[24] A Banach space $\mathscr{X}$ is said to satisfy the Opial condition [26] if for each sequence $\{u_n\}$ in $\mathscr{X}$, converging weakly to $u \in \mathscr{X}$, we have*

$$\limsup_{n\to\infty} \|u_n - u\| < \limsup_{n\to\infty} \|u_n - w\| \tag{2.2}$$

for all $w \in \mathscr{X}$ such that $u \neq w$.

Definition 2.3. *[27] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two iterative schemes converging respectively to a and b. Suppose that there exists*

$$\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0;$$

then, $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.

Definition 2.4. *[27] Suppose that for two fixed point iterations $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ converging to the same fixed point y^*, the error estimates*

$$\|u_n - y^*\| \leq a_n, \ n = 0, 1, 2, \ldots,$$

$$\|v_n - y^*\| \leq b_n, \ n = 0, 1, 2, \ldots,$$

hold, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers converging to zero. Furthermore, if $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to a fixed point p^*.

Definition 2.5. *[28] Let $\{s_n\}$ be any arbitrary sequence in $C[0, 1]$. Then an iterative scheme $u_{n+1} = f(\mathcal{J}, u_n)$, converging to a fixed point p^*, is said to be $\mathcal{J}$-stable, or stable with respect to $\mathcal{J}$, if, for $\epsilon_n = \|s_{n+1} - f(\mathcal{J}, s_n)\|$, $\forall\, n \in \mathbb{N}$, $\lim_{n\to\infty} \epsilon_n = 0$ iff $\lim_{n\to\infty} s_n = p^*$.*

Lemma 2.1. *[29] If $\rho \in [0, 1)$ is a real number and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then, for any sequence of positive numbers, $\{s_n\}_{n=0}^{\infty}$ satisfies that $s_{n+1} \leq \rho s_n + \epsilon_n$ ($n = 0, 1, 2, \ldots$) such that $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.2. *[30] Let $\mathscr{X}$ be a uniformly convex Banach space and $\{\gamma_n\}_{n=0}^{\infty}$ be any sequence of numbers such that $0 < a \leq \gamma_n \leq b < 1$, $n \geq 1$, for $a, b \in \mathbb{R}$. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be sequences in $\mathscr{X}$ such that $\limsup_{n\to\infty} \|u_n\| \leq \varphi$, $\limsup_{n\to\infty} \|r_n\| \leq \varphi$ and $\limsup_{n\to\infty} \|\gamma_n u_n + (1 - \gamma_n) r_n\| = \varphi$ for some $\varphi \geq 0$. Then, $\lim_{n\to\infty} \|u_n - r_n\| = 0$.*

Lemma 2.3. *[23] Let $\mathcal{J}$ be a self mapping on a uniformly convex subset D of a Banach space $\mathscr{X}$. Suppose that $\mathcal{J}$ satisfies condition (C). Then*

$$\|\omega - \mathcal{J}\nu\| \leq 3\|\mathcal{J}\omega - \omega\| + \|\omega - \nu\|$$

holds for $\omega, \nu \in D$.

Lemma 2.4. *[23] Let $\mathcal{J}$ be a mapping on a subset D of a Banach space $\mathscr{X}$ with the Opial condition satisfying (2.2). Suppose that $\mathcal{J}$ is a Suzuki generalized nonexpansive mapping satisfying condition (C). If $\{u_n\}$ converges weakly to p^* and $\lim_{n\to\infty} \|\mathcal{J}u_n - u_n\| = 0$, then $\mathcal{J}p^* = p^*$. That is, $I - \mathcal{J}$ is demiclosed at zero.*

Lemma 2.5. *[31] Let $\{\sigma_n\}$ be a nonnegative sequence for which one assumes that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we can suppose that the following inequality is satisfied:*

$$\sigma_{n+1} \leq (1 - \varpi_n)\sigma_n + \varpi_n\eta_n$$

where $\varpi_n \in (0, 1)$, $\forall n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \varpi_n = \infty$ and $\eta_n \geq 0$ $\forall n \in \mathbb{N}$. Then,

$$0 \leq \limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \eta_n.$$

Lemma 2.6. *[32] Let σ_n be a nonnegative sequence satisfying the inequality*

$$\sigma_{n+1} \leq (1 - \eta_n)\sigma_n + \lambda_n$$

with $\eta_n \in [0, 1]$, $\sum_{j=0}^{\infty} \eta_j = \infty$ and $\lambda_n = o(\eta_n)$. Then $\lim_{n\to\infty} \sigma_n = 0$.

Lemma 2.7. *[23] Let D be a nonempty subset of a Banach space $\mathscr{X}$ and $\mathcal{J} : D \to D$. If $\mathcal{J}$ is a Suzuki generalized nonexpansive mapping, then for all $x \in D$ and $p^* \in \mathscr{F}(\mathcal{J})$, $\|\mathcal{J}x - \mathcal{J}p^*\| \leq \|x - p^*\|$ holds.*

Approximation via a fixed point iterative scheme has been adopted by several researchers as a method to approximate several classes of operators. For example, Mann [33], in 1953, introduced the following iterative scheme:

$$\begin{cases} t_0 \in D \\ t_{n+1} = (1 - \alpha_n)t_n + \alpha_n\mathcal{J}t_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.3}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Khan [34] and Thakur et al. [35], in 2013 and 2016, respectively constructed the following schemes, called the Picard-Mann hybrid and Thakur iterative schemes:

$$\begin{cases} s_0 \in D \\ s_{n+1} = \mathcal{J}t_n \\ t_n = (1 - \alpha_n)s_n + \alpha_n\mathcal{J}s_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.4}$$

$$\begin{cases} p_0 \in D \\ p_{n+1} = \mathcal{J}q_n \\ q_n = \mathcal{J}[(1 - \alpha_n)p_n + \alpha_n r_n] \\ r_n = (1 - \beta_n)p_n + \beta_n\mathcal{J}p_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$. Khan proved that (2.4) converges faster than the Picard, Mann and Ishikawa iterative schemes.

In 2018, Ullah and Arshad [36] introduced the M iterative scheme as follows:

$$\begin{cases} m_0 \in D \\ m_{n+1} = \mathcal{J} d_n \\ d_n = \mathcal{J} c_n \\ c_n = (1-\alpha_n) m_n + \alpha_n \mathcal{J} m_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The scheme was used to prove weak and strong convergence theorems for Suzuki generalized nonexpansive mappings in the framework of uniformly convex Banach spaces.

Noor [37], in 2000, introduced an iterative scheme that included both the Mann and the Ishikawa iterative schemes as special cases. The scheme was defined as follows:

$$\begin{cases} u_0 = u \in D \\ u_{n+1} = (1-\alpha_n) u_n + \alpha_n \mathcal{J} w_n \\ w_n = (1-\beta_n) u_n + \beta_n \mathcal{J} y_n \\ y_n = (1-\gamma_n) u_n + \gamma_n \mathcal{J} u_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$.

Recently in 2019, Okeke [38] introduced the following iterative scheme:

$$\begin{cases} x_0 = x \in D \\ x_{n+1} = \mathcal{J} v_n \\ v_n = (1-\alpha_n) x_n + \mathcal{J} u_n \\ u_n = (1-\beta_n) x_n + \beta_n \mathcal{J} x_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and it was shown that the scheme converges faster than the Picard, Krasnoselskii [39], Mann, Ishikawa [40], Noor, Picard-Mann and Picard-Krasnoselkii [41] iterative schemes.

Motivated by the aforementioned developments, it is our aim in this paper to introduce a new fixed point iterative scheme that is more efficient than the ones highlighted above and others in literature. To achieve this, the AG fixed point iterative scheme is defined by the sequence $\{u_n\}$ as follows:

$$\begin{cases} u_0 = u \in D \\ u_{n+1} = \mathcal{J} v_n \\ v_n = \mathcal{J}[(1-\alpha_n) w_n + \alpha_n \mathcal{J} w_n] \\ w_n = (1-\beta_n) \mathcal{J} u_n + \beta_n \mathcal{J} x_n \\ x_n = (1-\gamma_n) u_n + \gamma_n \mathcal{J} u_n, \; n \in \mathbb{N}, \end{cases} \tag{2.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of real numbers in $[0, 1]$.

3. Main results

In this section, we consider and prove the main results of this paper.

3.1. Convergence and stability results

Theorem 3.1. *Let D be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and $\mathcal{J} : D \to D$ be a contraction mapping satisfying condition (2.1) such that $\mathscr{F}(\mathcal{J}) \neq \emptyset$. Suppose that $\{u_n\}_{n=0}^{\infty}$ is an iterative sequence generated by the AG iterative scheme (2.9) satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{u_n\}_{n=0}^{\infty}$ converges to a unique fixed point p^* of $\mathcal{J}$.*

Proof. Let $p^* \in \mathscr{F}(\mathcal{J})$ be a fixed point of a contraction mapping $\mathcal{J}$. The Banach contraction mapping principle guarantees the existence and uniqueness of the fixed point p^*. So, we want to show that $u_n \to p^*$ as $n \to \infty$.
From (2.1) and (2.9), we have

$$\begin{aligned} \|x_n - p^*\| &= \|(1-\gamma_n)u_n + \gamma_n \mathcal{J}u_n - p^*\| \\ &\leq (1-\gamma_n)\|u_n - p^*\| + \gamma_n\|\mathcal{J}u_n - p^*\| \\ &\leq (1-\gamma_n)\|u_n - p^*\| + \delta\gamma_n\|u_n - p^*\| \\ &= [(1-\gamma_n) + \delta\gamma_n]\|u_n - p^*\| \\ &= [1-(1-\delta)\gamma_n]\|u_n - p^*\|. \end{aligned} \tag{3.1}$$

Using (2.1), (2.9) and (3.1), we have

$$\begin{aligned} \|w_n - p^*\| &= \|(1-\beta_n)\mathcal{J}u_n + \beta_n\mathcal{J}x_n - p^*\| \\ &\leq (1-\beta_n)\|\mathcal{J}u_n - p^*\| + \beta_n\|\mathcal{J}x_n - p^*\| \\ &\leq \delta(1-\beta_n)\|u_n - p^*\| + \delta\beta_n\|x_n - p^*\| \\ &= \delta(1-\beta_n)\|u_n - p^*\| + \delta\beta_n[1-(1-\delta)\gamma_n]\|u_n - p^*\| \\ &= \Big\{\delta(1-\beta_n) + \delta\beta_n[1-(1-\delta)\gamma_n(1-\delta)]\Big\}\|u_n - p^*\| \\ &\leq \delta[1-\beta_n\gamma_n(1-\delta)]\|u_n - p^*\|. \end{aligned} \tag{3.2}$$

Again, using (2.1), (2.9) and (3.2), we have

$$\begin{aligned} \|v_n - p^*\| &= \|\mathcal{J}[(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n] - p^*\| \\ &\leq \delta\|[(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n] - p^*\| \\ &\leq \delta\{(1-\alpha_n)\|w_n - p^*\| + \alpha_n\|\mathcal{J}w_n - p^*\|\} \\ &\leq \delta\{(1-\alpha_n)\|w_n - p^*\| + \delta\alpha_n\|w_n - p^*\|\} \\ &= \delta\{[(1-\alpha_n) + \delta\alpha_n]\|w_n - p^*\|\} \\ &= \delta[(1-\alpha_n) + \delta\alpha_n]\|w_n - p^*\| \\ &\leq \delta^2[1-(1-\delta)\alpha_n][1-(1-\delta)\beta_n\gamma_n]\|u_n - p^*\|. \end{aligned} \tag{3.3}$$

Using (2.1), (2.9) and (3.3), we have

$$\begin{aligned} \|u_{n+1} - p^*\| &= \|\mathcal{J}v_n - p^*\| \\ &\leq \delta\|v_n - p^*\| \\ &\leq \delta^3[1-(1-\delta)\alpha_n][1-(1-\delta)\beta_n\gamma_n]\|u_n - p^*\|. \end{aligned}$$

Since $\beta_n,\ \gamma_n \in [0,1]$ and $\delta \in (0,1)$, then

$$\|u_{n+1} - p^*\| \leq \delta^3[1 - (1-\delta)\alpha_n]\|u_n - p^*\|.$$

Repeating the process, we have

$$\begin{aligned}
\|u_n - p^*\| &\leq \delta^3[1 - (1-\delta)\alpha_{n-1}]\|u_{n-1} - p^*\| \\
\|u_{n-1} - p^*\| &\leq \delta^3[1 - (1-\delta)\alpha_{n-2}]\|u_{n-2} - p^*\| \\
\|u_{n-2} - p^*\| &\leq \delta^3[1 - (1-\delta)\alpha_{n-3}]\|u_{n-3} - p^*\| \\
&\vdots \quad \vdots \\
\|u_1 - p^*\| &\leq \delta^3[1 - (1-\delta)\alpha_0]\|u_0 - p^*\|
\end{aligned}$$

Hence,

$$\|u_{n+1} - p^*\| \leq \delta^{3(n+1)}\|u_0 - p^*\| \prod_{k=0}^{n}[1 - \alpha_k(1-\delta)] \tag{3.4}$$

$\delta \in [0,1)$ and $\alpha_n \in [0,1]$; thus, $1 - \alpha_n(1-\delta) < 1$ for all $n \in \mathbb{N}$.
It is obvious from classical analysis that $1 - x = e^{-x}$ for $x \in (0,1)$. Thus,

$$\begin{aligned}
\|u_{n+1} - p^*\| &\leq \delta^{3(n+1)}\|u_0 - p^*\| \prod_{k=0}^{n} e^{-(1-\delta)\alpha_k} \\
&\leq \delta^{3(n+1)}\|u_0 - p^*\|^{n+1} e^{-(1-\delta)\sum_{k=0}^{\infty}\alpha_k}.
\end{aligned}$$

From the hypothesis of the theorem, $\sum\limits_{k=0}^{\infty} \alpha_k = \infty$ such that $e^{-(1-\delta)\sum_{k=0}^{\infty}\alpha_k} \to 0$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \|u_n - p^*\| = 0.$$

This completes the proof. □

Theorem 3.2. *Let D be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and $\mathcal{J} : D \to D$ be a contraction mapping satisfying condition (2.1) with the fixed point $p^* \in \mathscr{F}(\mathcal{J}) \neq \emptyset$. Let $x_0 \in D$ generate the sequence $\{x_n\}_{n=0}^{\infty} \subset D$, as defined in (2.3), and $u_0 \in D$, given $\{u_n\}_{n=0}^{\infty} \subset D$ as defined by (2.9) with real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma\}_{n=0}^{\infty} \in [0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then, the following statements are equivalent:*

(i) The Mann iterative scheme (2.3) converges to the fixed point $p^* \in \mathscr{F}(\mathcal{J})$.
(ii) The AG iterative scheme (2.9) converges to the fixed point $p^* \in \mathscr{F}(\mathcal{J})$.

Proof. We shall show that (i) ⇒ (ii), that is, if the Mann iterative scheme (2.3) converges to a fixed point p^*, then the AG iterative scheme (2.9) also converges.
Using (2.3), (2.9) and condition (2.1), we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \|(1-\alpha_n)x_n + \alpha_n\mathcal{J}x_n - \mathcal{J}v_n\| \\
&\leq (1-\alpha_n)\|x_n - \mathcal{J}v_n\| + \alpha_n\|\mathcal{J}x_n - \mathcal{J}v_n\| \\
&\leq (1-\alpha_n)\|x_n - \mathcal{J}x_n + \mathcal{J}x_n - \mathcal{J}v_n\| + \alpha_n\|\mathcal{J}x_n - \mathcal{J}v_n\| \\
&\leq (1-\alpha_n)\{\|x_n - \mathcal{J}x_n\| + \|\mathcal{J}x_n - \mathcal{J}v_n\|\} + \alpha_n\|\mathcal{J}x_n - \mathcal{J}v_n\| \\
&\leq (1-\alpha_n)\|x_n - \mathcal{J}x_n\| + \delta(1-\alpha_n)\|x_n - v_n\| + \alpha_n\delta\|x_n - v_n\| \\
&= (1-\alpha_n)\|x_n - \mathcal{J}x_n\| + \delta\|x_n - v_n\|,
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
\|x_n - v_n\| &= \|x_n - \mathcal{J}[(1-\alpha_n)w_n + \alpha_n \mathcal{J} w_n]\| \\
&\le \|x_n - \mathcal{J} x_n + \mathcal{J} x_n - \mathcal{J}[(1-\alpha_n)w_n + \alpha_n \mathcal{J} w_n]\| \\
&\le \|x_n - \mathcal{J} x_n\| + \|\mathcal{J} x_n - \mathcal{J}[(1-\alpha_n)w_n + \alpha_n \mathcal{J} w_n]\| \\
&\le \|x_n - \mathcal{J} x_n\| + \delta\|x_n - (1-\alpha_n)w_n - \alpha_n \mathcal{J} w_n\| \\
&\le \|x_n - \mathcal{J} x_n\| + \delta\{(1-\alpha_n)\|x_n - w_n\| + \alpha_n\|x_n - \mathcal{J} w_n\|\} \\
&\le \|x_n - \mathcal{J} x_n\| + \delta\{(1-\alpha_n)\|x_n - w_n\| + \alpha_n\|x_n - \mathcal{J} x_n + \mathcal{J} x_n - \mathcal{J} w_n\|\} \\
&\le \|x_n - \mathcal{J} x_n\| + \delta\{(1-\alpha_n)\|x_n - w_n\| + \alpha_n\|x_n - \mathcal{J} x_n\| + \alpha_n\delta\|x_n - w_n\|\} \\
&\le \|x_n - \mathcal{J} x_n\| + \delta(1-\alpha_n)\|x_n - w_n\| + \delta\alpha_n\|x_n - \mathcal{J} w_n\| + \alpha_n\delta^2\|x_n - w_n\| \\
&= \|x_n - \mathcal{J} x_n\| + \delta\alpha_n\|x_n - \mathcal{J} x_n\| + [\delta(1-\alpha_n) + \delta^2\alpha_n]\|x_n - w_n\| \\
&= \|x_n - \mathcal{J} x_n\| + \delta\alpha_n\|x_n - \mathcal{J} x_n\| + \delta[1 - (1-\delta)\alpha_n]\|x_n - w_n\|,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\|x_n - w_n\| &= \|x_n - (1-\beta_n)\mathcal{J} u_n - \beta_n \mathcal{J} x_n\| \\
&\le (1-\beta_n)\|x_n - \mathcal{J} u_n\| + \beta_n\|x_n - \mathcal{J} x_n\| \\
&\le (1-\beta_n)\|x_n - \mathcal{J} x_n + \mathcal{J} x_n - \mathcal{J} u_n\| + \beta_n\|x_n - \mathcal{J} x_n\| \\
&\le (1-\beta_n)\{\|x_n - \mathcal{J} x_n\| + \|\mathcal{J} x_n - \mathcal{J} u_n\|\} + \beta_n\|x_n - \mathcal{J} x_n\| \\
&\le \|x_n - \mathcal{J} x_n\| + (1-\beta_n)\|\mathcal{J} x_n - \mathcal{J} u_n\| \\
&\le \|x_n - \mathcal{J} x_n\| + \delta(1-\beta_n)\|x_n - u_n\|.
\end{aligned} \tag{3.7}
$$

Putting (3.7) in (3.6), we have

$$
\begin{aligned}
\|x_n - v_n\| \le& \|x_n - \mathcal{J} x_n\| + \delta\alpha_n\|x_n - \mathcal{J} x_n\| \\
&+ \delta[1 - (1-\delta)\alpha_n]\{\|x_n - \mathcal{J} x_n\| + \delta(1-\beta_n)\|x_n - u_n\|\} \\
\le& \|x_n - \mathcal{J} x_n\| + \delta\alpha_n\|x_n - \mathcal{J} x_n\| + \delta[1 - (1-\delta)\alpha_n]\|x_n - \mathcal{J} x_n\| \\
&+ \delta^2[1 - (1-\delta)\alpha_n](1-\beta_n)\|x_n - u_n\| \\
=& \delta^2(1-\beta_n)[1 - (1-\delta)\alpha_n]\|x_n - u_n\| \\
&+ \{(1 + \delta\alpha_n) + \delta[1 - (1-\delta)\alpha_n]\}\|x_n - \mathcal{J} x_n\|.
\end{aligned} \tag{3.8}
$$

Since $\delta \in (0, 1)$ and $\beta_n \in [0, 1]$, then for each $n \in \mathbb{N}$, $\delta^2(1 - \beta_n) < 1$ such that (3.8) reduces to

$$
\|x_n - v_n\| \le [1 - (1-\delta)\alpha_n]\|x_n - u_n\| + \{(1 + \delta\alpha_n) + \delta[1 - (1-\delta)\alpha_n]\}\|x_n - \mathcal{J} x_n\|. \tag{3.9}
$$

Putting (3.9) in (3.5), we have

$$
\begin{aligned}
\|x_{n+1} - u_{n+1}\| \le& (1-\alpha_n)\|x_n - \mathcal{J} x_n\| + \delta\{[1 - (1-\delta)\alpha_n]\|x_n - u_n\| \\
&+ \{(1 + \delta\alpha_n) + \delta[1 - (1-\delta)\alpha_n]\}\|x_n - \mathcal{J} x_n\|\} \\
=& (1-\alpha_n)\|x_n - \mathcal{J} x_n\| + \delta[1 - (1-\delta)\alpha_n]\|x_n - u_n\| \\
&+ [\delta(1 + \delta\alpha_n) + \delta^2[1 - (1-\delta)\alpha_n]]\|x_n - \mathcal{J} x_n\| \\
\le& [1 - (1-\delta)\alpha_n]\|x_n - u_n\| + [(1-\alpha_n) + \delta(1 + \delta\alpha_n) \\
&+ \delta^2[1 - (1-\delta)\alpha_n]]\|x_n - \mathcal{J} x_n\|.
\end{aligned} \tag{3.10}
$$

Let $\sigma_n := \|x_n - u_n\|$, $\eta_n := \alpha_n(1-\delta) \in (0,1)$ and $\lambda_n := [(1-\alpha_n)+\delta(1+\delta\alpha_n)+\delta^2[1-(1-\delta)\alpha_n]]\|x_n - \mathcal{J}x_n\|$. Using the fact that $\mathcal{J}p^* = p^*$ and $\|x_n - p^*\| \to 0$ as $n \to \infty$, we have that

$$\begin{aligned}\|x_n - \mathcal{J}x_n\| &= \|x_n - \mathcal{J}p^* + \mathcal{J}p^* - \mathcal{J}x_n\| \\ &\le \|x_n - \mathcal{J}p^*\| + \|\mathcal{J}p^* - \mathcal{J}x_n\| \\ &\le \|x_n - \mathcal{J}p^*\| + \delta\|x_n - p^*\| \\ &= (1+\delta)\|x_n - p^*\|;\end{aligned}$$

thus,

$$\|x_n - \mathcal{J}x_n\| \le (1+\delta)\|x_n - p^*\| \to 0 \text{ as } n \to \infty.$$

This implies that $\lambda_n \to 0$ as $n \to \infty$. By Lemma 2.6, we have that $\lim_{n\to\infty} \|x_n - u_n\| = 0$.
Since

$$\begin{aligned}\|u_n - p^*\| &= \|u_n - x_n + x_n - p^*\| \\ &\le \|u_n - x_n\| + \|x_n - p^*\| \\ &\le \|x_n - u_n\| + \|x_n - p^*\|,\end{aligned}$$

we have that $\|u_n - p^*\| \to 0$ as $n \to \infty$.
Next, we shall show that (ii)⇒ (i). Using (2.3), (2.9) and condition (2.1), we have

$$\begin{aligned}\|u_{n+1} - x_{n+1} &= \|\mathcal{J}v_n - (1-\alpha_n)x_n - \alpha_n\mathcal{J}x_n\| \\ &\le (1-\alpha_n)\|\mathcal{J}v_n - x_n\| + \alpha_n\|\mathcal{J}v_n - \mathcal{J}x_n\| \\ &\le (1-\alpha_n)\|\mathcal{J}v_n - \mathcal{J}x_n + \mathcal{J}x_n - x_n\| + \alpha_n\|\mathcal{J}v_n - \mathcal{J}x_n\| \\ &\le (1-\alpha_n)\|\mathcal{J}v_n - \mathcal{J}x_n\| + (1-\alpha_n)\|\mathcal{J}x_n - x_n\| + \alpha_n\|\mathcal{J}v_n - \mathcal{J}x_n\| \\ &\le \|\mathcal{J}v_n - \mathcal{J}x_n\| + (1-\alpha_n)\|\mathcal{J}x_n - x_n\| \\ &\le \delta\|v_n - x_n\| + (1-\alpha_n)\|\mathcal{J}x_n - x_n\|,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\|v_n - x_n\| =&\|\mathcal{J}[(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n] - x_n\| \\ \le&\|\mathcal{J}[(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n] - \mathcal{J}u_n + \mathcal{J}u_n - x_n\| \\ \le&\|\mathcal{J}[(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n] - \mathcal{J}u_n\| + \|\mathcal{J}u_n - x_n\| \\ \le&\delta\|(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n - u_n\| + \|\mathcal{J}u_n - x_n\| \\ \le&\delta[1-(1-\delta)\alpha_n]\{\|\mathcal{J}u_n - u_n\| \\ &+ \delta\beta_n\|u_n - x_n\|\} + \delta\alpha_n\|\mathcal{J}u_n - u_n\| + \|\mathcal{J}u_n - x_n\|.\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| \le&(1-\alpha_n)\|\mathcal{J}x_n - x_n\| + \delta^2[1-(1-\delta)\alpha_n]\|\mathcal{J}u_n - u_n\| \\ &+ \delta^3[1-(1-\delta)\alpha_n]\beta_n\|u_n - x_n\| + \delta^2\alpha_n\|\mathcal{J}u_n - u_n\| + \delta\|\mathcal{J}u_n - x_n\| \\ =&\delta^2[1-(1-\delta)\alpha_n + \alpha_n]\|\mathcal{J}u_n - u_n\| + (1-\alpha_n)\|\mathcal{J}x_n - x_n\| \\ &+ \delta\|\mathcal{J}u_n - x_n\| + [1-(1-\delta)\alpha_n]\beta_n\delta^3\|u_n - x_n\| \\ =&[1-(1-\delta)\alpha_n]\beta_n\delta^3\|u_n - x_n\| + [1-\delta\alpha_n]\delta^2\|\mathcal{J}u_n - u_n\| \\ &+ (1-\alpha_n)\|\mathcal{J}x_n - x_n\| + \delta\|\mathcal{J}u_n - x_n\| \\ \le&[1-(1-\delta)\alpha_n]\|u_n - x_n\| + [1-\delta\alpha_n]\delta^2\|\mathcal{J}u_n - u_n\| \\ &+ (1-\alpha_n)\|\mathcal{J}x_n - x_n\| + \delta\|\mathcal{J}u_n - x_n\|.\end{aligned}$$

Let $\sigma_n := \|u_n - x_n\|$, $\eta_n := (1-\delta)\alpha_n \in (0,1)$ and $\lambda_n := [\delta^2 - \delta^3\alpha_n]\|\mathcal{J}u_n - u_n\| + (1-\alpha_n)\|\mathcal{J}x_n - x_n\| + \delta\|\mathcal{J}u_n - x_n\|$.
Since $\mathcal{J}p^* = p^*$ and $\|u_n - p^*\| \to 0$, as $n \to \infty$, then

$$\begin{aligned}\|\mathcal{J}u_n - u_n\| &= \|\mathcal{J}u_n - \mathcal{J}p^*\| + \|\mathcal{J}p^* - u_n\| \\ &\leq \delta\|u_n - p^*\| + \|p^* - u_n\| \\ &\leq (1+\delta)\|u_n - p^*\|;\end{aligned}$$

thus, $\|\mathcal{J}u_n - u_n\| \to 0$ as $n \to \infty$.
Also, if

$$\begin{aligned}\|\mathcal{J}x_n - x_n\| &\leq \|\mathcal{J}x_n - \mathcal{J}p^*\| + \|\mathcal{J}p^* - x_n\| \\ &\leq (1+\delta)\|x_n - p^*\|\end{aligned}$$

then $\|x_n - p^*\| \to 0$ as $n \to \infty$; thus, $\|\mathcal{J}x_n - x_n\| \to 0$ as $n \to \infty$. Similarly, $\|\mathcal{J}u_n - x_n\| \to 0$ as $n \to \infty$, which is consequent to the fact that

$$\begin{aligned}\|\mathcal{J}u_n - x_n\| &\leq \|\mathcal{J}u_n - \mathcal{J}p^*\| + \|\mathcal{J}p^* - x_n\| \\ &\leq \delta\|u_n - p^*\| + \|x_n - p^*\| \\ &\leq \delta\|u_n - \mathcal{J}u_n + \mathcal{J}u_n - p^*\| + \|x_n - \mathcal{J}x_n + \mathcal{J}x_n - p^*\| \\ &\leq \delta\{\|u_n - \mathcal{J}u_n\| + \delta\|u_n - p^*\|\} + \|x_n - \mathcal{J}x_n\| + \delta\|x_n - p^*\|.\end{aligned}$$

Thus, from Lemma 2.6, $\sigma_n = \|u_n - x_n\|$ and $\lim_{n\to\infty} \|u_n - x_n\| = 0$.
Since

$$\|x_n - p^*\| \leq \|u_n - x_n\| + \|u_n - p^*\| \to 0 \text{ as } n \to \infty,$$

we have that $\lim_{n\to\infty} \|x_n - p^*\| = 0$. Hence, we have completed the proof. □

Theorem 3.3. *Let $\mathscr{X}$ be a Banach space and $\mathcal{J} : D \to D$ be a contraction mapping satisfying condition (2.1) with $\delta \in [0,1)$. Assume that $\mathcal{J}$ has a fixed point $p^* \in \mathscr{F}(\mathcal{J}) \neq \emptyset$. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated by the AG iterative scheme (2.9) satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$, $n \in \mathbb{N}$, and that converges to p^*. Then, the AG iterative scheme is $\mathcal{J}$-stable.*

Proof. Suppose that $\{s_n\}_{n=0}^{\infty} \subset \mathscr{X}$ is an arbitrary sequence in D and suppose that the sequence generated by (2.9) is $u_{n+1} = f(\mathcal{J}, u_n)$ converging to a unique fixed point p^*.
Let $\epsilon_n = \|s_{n+1} - f(\mathcal{J}, s_n)\|$. We want to show that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} \|s_n - p^*\| = 0$.
Suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Using the triangle inequality, we have

$$\begin{aligned}\|s_{n+1} - p^*\| &= \|s_{n+1} - f(\mathcal{J}, s_n) + f(\mathcal{J}, s_n) - p^*\| \\ &\leq \|s_{n+1} - f(\mathcal{J}, s_n)\| + \|f(\mathcal{J}, s_n) - p^*\| \\ &\leq \epsilon_n + \|f(\mathcal{J}, s_n) - p^*\| \\ &\leq \epsilon_n + \|\mathcal{J}v_n - p^*\| \\ &\leq \epsilon_n + \delta\|v_n - p^*\|, \qquad (3.13)\end{aligned}$$

$$\begin{aligned}\|v_n - p^*\| &= \|\mathcal{J}[(1-\alpha_n)w_n + \alpha_n \mathcal{J} w_n] - p^*\| \\ &\leq \delta\|(1-\alpha_n)w_n + \alpha_n \mathcal{J} w_n - p^*\| \\ &\leq \delta\{(1-\alpha_n)\|w_n - p^*\| + \alpha_n\|\mathcal{J} w_n - p^*\|\} \\ &\leq \delta(1-\alpha_n)\|w_n - p^*\| + \alpha_n\delta^2\|w_n - p^*\| \\ &= [\delta(1-\alpha_n) + \alpha_n\delta^2]\|w_n - p^*\|, \end{aligned} \tag{3.14}$$

$$\begin{aligned}\|w_n - p^*\| &= \|(1-\beta_n)\mathcal{J} s_n + \beta_n \mathcal{J} x_n - p^*\| \\ &\leq (1-\beta_n)\|\mathcal{J} s_n - p^*\| + \beta_n\|\mathcal{J} x_n - p^*\| \\ &\leq (1-\beta_n)\delta\|s_n - p^*\| + \beta_n\delta\|x_n - p^*\|, \end{aligned} \tag{3.15}$$

$$\begin{aligned}\|x_n - p^*\| &= \|(1-\gamma_n)s_n + \gamma_n \mathcal{J} s_n - p^*\| \\ &\leq (1-\gamma_n)\|s_n - p^*\| + \gamma_n\|\mathcal{J} s_n - p^*\| \\ &\leq (1-\gamma_n)\|s_n - p^*\| + \gamma_n\delta\|s_n - p^*\| \\ &= [(1-\gamma_n) + \gamma_n\delta]\|s_n - p^*\| \\ &\leq [1 - (1-\delta)\gamma_n]\|s_n - p^*\|. \end{aligned} \tag{3.16}$$

Combining (3.13)–(3.16), we have

$$\|s_{n+1} - p^*\| \leq \epsilon_n + \{(1-\beta_n)\delta + \beta_n\delta[1-(1-\delta)\gamma_n]\}[\delta^2(1-\alpha_n) + \alpha_n\delta^3]\|s_n - p^*\|.$$

Since $\delta \in (0,1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0,1]$, then $\{(1-\beta_n)\delta + \beta_n\delta[1-(1-\delta)\gamma_n]\}[\delta^2(1-\alpha_n) + \alpha_n\delta^3] < 1$; thus,

$$\|s_{n+1} - p^*\| \leq \epsilon_n + \|s_n - p^*\|.$$

By Lemma 2.1, we have that $\lim_{n\to\infty} \|s_n - p^*\| = 0$, that is, $\lim_{n\to\infty} s_n = p^*$.
Conversely, suppose that $\lim_{n\to\infty} s_n = p^*$; then,

$$\begin{aligned}\epsilon_n &= \|s_{n+1} - f(\mathcal{J}, s_n)\| \\ &\leq \|s_{n+1} - p^*\| + \|p^* - f(\mathcal{J}, s_n)\| \\ &\leq \|s_{n+1} - p^*\| + \|p^* - \mathcal{J} v_n\| \\ &\leq \|s_{n+1} - p^*\| + \delta\|v_n - p^*\| \\ &\leq \|s_{n+1} - p^*\| + \delta[\delta(1-\alpha_n) + \alpha_n\delta^2]\|w_n - p^*\| \\ &\leq \|s_{n+1} - p^*\| + \delta[\delta(1-\alpha_n) + \alpha_n\delta^2]\{(1-\beta_n)\delta\|s_n - p^*\| + \beta_n\delta\|x_n - p^*\|\} \\ &\leq \|s_{n+1} - p^*\| + \delta[\delta(1-\alpha_n) + \alpha_n\delta^2]\{(1-\beta_n)\delta\|s_n - p^*\| \\ &\quad + \beta_n\delta[1-(1-\delta)\gamma_n]\|s_n - p^*\|\} \\ &\leq \|s_{n+1} - p^*\| + \delta[\delta(1-\alpha_n) + \alpha_n\delta^2]\{(1-\beta_n)\delta + \beta_n\delta[1-(1-\delta)\gamma_n]\}\|s_n - p^*\| \\ &\leq \|s_{n+1} - p^*\| + \{\delta^2[\delta(1-\alpha_n) + \alpha_n\delta^2](1-\beta_n) \\ &\quad + \beta_n\delta^2[\delta(1-\alpha_n) + \alpha_n\delta^2][1-(1-\delta\gamma_n)]\}\|s_n - p^*\|. \end{aligned}$$

By assumption, we have that $\lim_{n\to\infty} \|s_n - p^*\| = 0$. On taking the limit as $n \to \infty$, $\lim_{n\to\infty} \epsilon_n = 0$. Hence, the AG iterative scheme (2.9) is $\mathcal{J}$-stable. □

3.2. Weak convergence for Suzuki's generalized nonexpansive mapping

Before we continue with the next result in this section, it would be necessary to outline the following lemmas, as they will be important in proving subsequent results.

Lemma 3.1. *Let D be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and $\mathcal{J} : D \to D$ be a Suzuki generalized nonexpansive mapping satisfying condition (C) with $\mathscr{F}(\mathcal{J}) \neq \emptyset$. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated by the AG iterative scheme (2.9) for $u_0 \in D$; then, $\lim_{n\to\infty} \|u_n - p^*\|$ exists for all $p^* \in \mathscr{F}(\mathcal{J})$.*

Proof. Let $p^* \in \mathscr{F}(\mathcal{J})$ and $\{u_n\}_{n=0}^{\infty}$ for all $n \in \mathbb{N}$. Since $\mathcal{J}$ is a Suzuki generalized nonexpansive mapping, by Lemma 2.7, we have that for $x \in D$ and $p^* \in \mathscr{F}(\mathcal{J})$, $\|\mathcal{J}x - \mathcal{J}p^*\| \leq \|x - p^*\|$.
Using (2.9), we have

$$\begin{aligned} \|x_n - p^*\| &= \|(1-\gamma_n)u_n + \gamma_n \mathcal{J}u_n - p^*\| \\ &\leq (1-\gamma_n)\|u_n - p^*\| + \gamma_n\|\mathcal{J}u_n - p^*\| \\ &\leq (1-\gamma_n)\|u_n - p^*\| + \gamma_n\|u_n - p^*\| \\ &= \|u_n - p^*\|; \end{aligned} \tag{3.17}$$

using (2.9) and (3.17), we have

$$\begin{aligned} \|w_n - p^*\| &= \|(1-\beta_n)\mathcal{J}u_n + \beta_n\mathcal{J}x_n - p^*\| \\ &\leq (1-\beta_n)\|\mathcal{J}u_n - p^*\| + \beta_n\|\mathcal{J}x_n - p^*\| \\ &\leq (1-\beta_n)\|u_n - p^*\| + \beta_n\|x_n - p^*\| \\ &= \|u_n - p^*\| \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \|v_n - p^*\| &= \|\mathcal{J}[(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n] - p^*\| \\ &\leq \|[(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n] - p^*\| \\ &\leq (1-\alpha_n)\|w_n - p^*\| + \alpha_n\|\mathcal{J}w_n - p^*\| \\ &= \|w_n - p^*\| \\ &\leq \|u_n - p^*\|. \end{aligned} \tag{3.19}$$

And using (3.19) and (2.9), we have

$$\begin{aligned} \|u_{n+1} - p^*\| &= \|\mathcal{J}v_n - p^*\| \\ &\leq \|v_n - p^*\| \\ &\leq \|u_n - p^*\|. \end{aligned}$$

Hence, $\{\|u_n - p^*\|\}$ is bounded and a non-increasing sequence for $p^* \in \mathscr{F}(\mathcal{J})$. Therefore, $\lim_{n\to\infty} \|u_n - p^*\|$ exists. □

Lemma 3.2. *Let D be a nonempty closed convex subset of a Banach space $\mathscr{X}$. Assume that $\mathcal{J} : D \to D$ is a Suzuki generalized nonexpansive mapping satisfying condition (C). Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated by the AG iterative scheme (2.9). Then, $\mathscr{F}(\mathcal{J}) \neq \emptyset$ if and only if $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|\mathcal{J}u_n - u_n\| = 0$.*

Proof. Suppose that $\mathscr{F}(\mathcal{J}) \neq \emptyset$ and $p^* \in \mathscr{F}(\mathcal{J})$. Then, by Lemma 3.1, we have that $\lim_{n\to\infty} \|u_n - p^*\|$ exists and $\{u_n\}_{n=0}^{\infty}$ is bounded.
Let

$$\lim_{n\to\infty} \|u_n - p^*\| = \varphi. \tag{3.20}$$

From (3.17), (3.18) and (3.19),

$$\limsup_{n\to\infty} \|x_n - p^*\| \leq \limsup_{n\to\infty} \|u_n - p^*\| \leq \varphi, \tag{3.21}$$

$$\limsup_{n\to\infty} \|w_n - p^*\| \leq \limsup_{n\to\infty} \|u_n - p^*\| \leq \varphi \tag{3.22}$$

and

$$\limsup_{n\to\infty} \|v_n - p^*\| \leq \limsup_{n\to\infty} \|u_n - p^*\| \leq \varphi. \tag{3.23}$$

Since $\mathcal{J}$ satisfies condition (C), we have that

$$\|\mathcal{J}u_n - p^*\| = \|\mathcal{J}u_n - \mathcal{J}p^*\| \leq \|u_n - p^*\|$$

and

$$\limsup_{n\to\infty} \|\mathcal{J}u_n - p^*\| \leq \limsup \|u_n - p^*\| \leq \varphi. \tag{3.24}$$

Now,

$$\begin{aligned}\|u_{n+1} - p^*\| &= \|\mathcal{J}v_n - p^*\| \\ &\leq \|v_n - p^*\|.\end{aligned}$$

Taking the lim inf on both sides, we have

$$\varphi = \liminf_{n\to\infty} \|u_{n+1} - p^*\| \leq \liminf_{n\to\infty} \|v_n - p^*\|. \tag{3.25}$$

Thus, (3.23) and (3.25) will give

$$\varphi \leq \liminf_{n\to\infty} \|v_n - p^*\| \leq \limsup_{n\to\infty} \|v_n - p^*\| \leq \varphi,$$

$$\lim_{n\to\infty} \|v_n - p^*\| = \varphi; \tag{3.26}$$

again,

$$\begin{aligned}\|v_n - p^*\| &= \|\mathcal{J}[(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n] - p^*\| \\ &\leq \|(1-\alpha_n)w_n + \alpha_n\mathcal{J}w_n - p^*\| \\ &\leq (1-\alpha_n)\|w_n - p^*\| + \alpha_n\|\mathcal{J}w_n - p^*\| \\ &= \|w_n - p^*\|.\end{aligned}$$

Taking the lim inf on both sides, we have

$$\varphi = \liminf_{n\to\infty} \|v_n - p^*\| \leq \liminf_{n\to\infty} \|w_n - p^*\|. \tag{3.27}$$

Thus, (3.22) and (3.27) will give

$$\varphi \leq \liminf_{n\to\infty} \|w_n - p^*\| \leq \limsup \|w_n - p^*\| \leq \varphi,$$

$$\lim_{n\to\infty} \|w_n - p^*\| = \varphi \tag{3.28}$$

and

$$\begin{aligned} \|w_n - p^*\| &= \|(1-\beta_n)\mathcal{J}u_n + \beta_n \mathcal{J}x_n - p^*\| \\ &\le (1-\beta_n)\|\mathcal{J}u_n - p^*\| + \beta_n\|\mathcal{J}x_n - p^*\| \\ &\le (1-\beta_n)\|u_n - p^*\| + \beta_n\|x_n - p^*\| \\ &= \|u_n - p^*\| + \beta_n(\|x_n - p^*\| - \|u_n - p^*\|). \end{aligned} \tag{3.29}$$

Obviously,

$$\|w_n - p^*\| - \|u_n - p^*\| \le \beta_n(\|x_n - p^*\| - \|u_n - p^*\|).$$

Given that $\{\beta_n\} \in (0, 1)$ and considering (3.29), it is convenient to have that

$$\|w_n - p^*\| - \|u_n - p^*\| \le \frac{\|w_n - p^*\| - \|u_n - p^*\|}{\beta_n} \le \|x_n - p^*\| - \|u_n - p^*\|,$$

which results in

$$\|w_n - p^*\| \le \|x_n - p^*\|.$$

Taking the lim inf on both sides, we have

$$\varphi \le \liminf_{n\to\infty} \|w_n - p^*\| \le \liminf_{n\to\infty} \|x_n - p^*\|. \tag{3.30}$$

Thus, using (3.21) and (3.30), we have

$$\varphi \le \liminf_{n\to\infty} \|x_n - p^*\| \le \limsup_{n\to\infty} \|x_n - p^*\| \le \varphi$$

and

$$\lim_{n\to\infty} \|x_n - p^*\| = \varphi. \tag{3.31}$$

From (3.31),

$$\begin{aligned} \varphi &= \lim_{n\to\infty} \|x_n - p^*\| \\ &= \lim_{n\to\infty} \|(1-\gamma_n)u_n + \gamma_n \mathcal{J}u_n - p^*\| \\ &\lim_{n\to\infty} \|(1-\gamma_n)(u_n - p^*) + \gamma_n(\mathcal{J}u_n - p^*)\|. \end{aligned}$$

Therefore,

$$\varphi = \lim_{n\to\infty} \|(1-\gamma_n)(u_n - p^*) + \gamma_n(\mathcal{J}u_n - p^*)\|. \tag{3.32}$$

Using (3.20), (3.24), (3.32) and Lemma 2.2, we end by stating that $\lim_{n\to\infty} \|\mathcal{J}u_n - u_n\| = 0$. Conversely, suppose that $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|u_n - \mathcal{J}u_n\| = 0$. We want to show that $\mathscr{F}(\mathcal{J}) \neq \emptyset$. Let $p^* \in \mathscr{A}(D, \{u_n\})$. By Lemma 2.3, we have that

$$\begin{aligned} \mathscr{R}(\mathcal{J}p^*, \{u_n\}) &= \limsup_{n\to\infty} \|u_n - \mathcal{J}p^*\| \\ &\le 3\limsup_{n\to\infty} \|\mathcal{J}u_n - u_n\| + \limsup_{n\to\infty} \|u_n - p^*\| \\ &= \limsup_{n\to\infty} \|u_n - p^*\| \\ &= \mathscr{R}(p^*, \{u_n\}). \end{aligned}$$

It follows that $\mathcal{J}p^* \in \mathscr{A}(D, \{u_n\})$. By Remark 2.1, we have that $\mathcal{J}p^* = p^*$. Hence the fixed point set $\mathscr{F}(\mathcal{J})$ is nonempty. □

At this point, we now consider the weak convergence result for a Suzuki generalized nonexpansive mapping satisfying condition (C).

Theorem 3.4. *Let D be a nonempty closed convex subset of a uniformly convex Banach space $\mathscr{X}$. Let $\mathcal{J} : D \to D$ be a mapping satisfying condition (C). For any arbitrary $u_0 \in D$, the sequence $\{u_n\}_{n=0}^{\infty}$ is generated by the AG iterative scheme (2.9) for $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of real numbers in $[0, 1]$ such that $\mathscr{F}(\mathcal{J}) \neq \emptyset$. Assume that $\mathscr{X}$ satisfies the Opial condition (2.2). Then, $\{u_n\}$ converges weakly to the fixed point $p^* \in \mathscr{F}(\mathcal{J})$.*

Proof. From Lemma 3.2, we have that $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|\mathcal{J}u_n - u_n\| = 0$ is subject to the fact that $\mathscr{F}(\mathcal{J}) \neq \emptyset$. Since $\mathscr{X}$ is uniformly convex, we can say that it is reflexive. By Eberlin's theorem there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup p_1$ for some $p_1 \in D$.
By Lemma 2.4, $p_1 \in \mathscr{F}(\mathcal{J})$. We want to prove that p_1 is a weak limit of $\{u_n\}$, that is, $\{u_n\}$ converges weakly to p_1. On the contrary, suppose that $\{u_n\}$ does not converge weakly to p_1; then, we can construct another subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup p_2$ for some $p_2 \in D$ and $p_1 \neq p_2$.
Again by Lemma 2.4, $p_2 \in \mathscr{F}(\mathcal{J})$. Since $\lim_{n\to\infty} \|u_n - p^*\|$ exists for all $p^* \in \mathscr{F}(\mathcal{J})$, by Lemma 3.2 and Opial condition (2.2), we have

$$\begin{aligned}
\lim_{n\to\infty} \|u_n - p_1\| &= \lim_{i\to\infty} \|u_{n_i} - p_1\| \\
&< \lim_{i\to\infty} \|u_{n_i} - p_2\| \\
&= \lim_{n\to\infty} \|u_n - p_2\| \\
&= \lim_{j\to\infty} \|u_{n_j} - p_2\| \\
&< \lim_{j\to\infty} \|u_{n_j} - p_1\| \\
&= \lim_{n\to\infty} \|u_n - p_1\|,
\end{aligned}$$

which is a contradiction. So $p_1 = p_2$. This implies that $\{u_n\}$ converges weakly to a fixed point of $\mathcal{J}$, thereby completing the proof. □

3.3. Rate of convergence and data dependence result

Theorem 3.5. *Let D be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and $\mathcal{J} : D \to D$ be a contraction mapping satisfying (2.1) with $\delta \in (0, 1)$ such that $\mathscr{F}(\mathcal{J}) \neq \emptyset$. If $\{s_n\}$, $\{p_n\}$, $\{m_n\}$ and $\{u_n\}$ are sequences respectively defined by the Picard-Mann, Thakur, M and AG iterative schemes converging to a fixed point $p^* \in \mathscr{F}(\mathcal{J})$. Then, the AG iterative scheme is faster than (2.4)–(2.6).*

Proof. From (3.4) in Theorem 3.1, we have that

$$\|u_{n+1} - p^*\| \leq \|u_0 - p^*\|\delta^{3(n+1)} \prod_{k=0}^{n} [1 - \alpha_k(1 - \delta)]. \tag{3.33}$$

From Picard-Mann iterative scheme (2.4), we have

$$\begin{aligned}
\|s_{n+1} - p^*\| &= \|\mathcal{J}t_n - p^*\| \\
&\leq \delta\|t_n - p^*\|,
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
\|t_n - p^*\| &= \|(1-\alpha_n)s_n + \alpha_n \mathcal{J} s_n - p^*\| \\
&\leq (1-\alpha_n)\|s_n - p^*\| + \alpha_n \|\mathcal{J} s_n - p^*\| \\
&\leq (1-\alpha_n)\|s_n - p^*\| + \alpha_n \delta \|s_n - p^*\| \\
&= [1 - \alpha_n + \alpha_n \delta]\|s_n - p^*\| \\
&\leq [1 - (1-\delta)\alpha_n]\|s_n - p^*\|. \qquad (3.35)
\end{aligned}$$

Now, by combining (3.34) and (3.35), we have

$$\|s_{n+1} - p^*\| \leq \delta[1 - (1-\delta)\alpha_n]\|s_n - p^*\|$$

such that, by induction, we have

$$\begin{aligned}
\|s_{n+1} - p^*\| &\leq \delta^{n+1} \prod_{k=0}^{n} [1 - (1-\delta)\alpha_k]\|s_0 - p^*\| \\
&= \|s_0 - p^*\|\delta^{n+1}[1 - (1-\delta)\alpha_k]^{n+1}.
\end{aligned}$$

This implies that

$$\|s_{n+1} - p^*\| \leq \|s_0 - p^*\|\delta^{n+1}[1 - (1-\delta)\alpha]^{n+1}. \qquad (3.36)$$

From Thakur iterative scheme (2.5), we have

$$\begin{aligned}
\|p_{n+1} - p^*\| &= \|\mathcal{J} q_n - p^*\| \\
&\leq \delta \|q_n - p^*\| \qquad (3.37)
\end{aligned}$$

$$\begin{aligned}
\|q_n - p^*\| &= \|\mathcal{J}[(1-\alpha_n)p_n + \alpha_n r_n] - p^*\| \\
&\leq \delta \|[(1-\alpha_n)p_n + \alpha_n r_n] - p^*\| \\
&\leq \delta\{(1-\alpha_n)\|p_n - p^*\| + \alpha_n \|r_n - p^*\|\} \qquad (3.38)
\end{aligned}$$

$$\begin{aligned}
\|r_n - p^*\| &= \|(1-\beta_n)p_n + \beta_n \mathcal{J} p_n - p^*\| \\
&\leq (1-\beta_n)\|p_n - p^*\| + \beta_n \|\mathcal{J} p_n - p^*\| \\
&\leq (1-\beta_n)\|p_n - p^*\| + \beta_n \delta \|\mathcal{J}\|p_n - p^*\| \\
&= \{(1-\beta_n) + \beta_n \delta\}\|p_n - p^*\| \\
&= [1 - (1-\delta)\beta_n]\|p_n - p^*\|. \qquad (3.39)
\end{aligned}$$

Combining (3.38) and (3.39), we have

$$\begin{aligned}
\|q_n - p^*\| &\leq \delta(1-\alpha_n)\|p_n - p^*\| + \delta\alpha_n[1 - (1-\delta)\beta_n]\|p_n - p^*\| \\
&= \{\delta(1-\alpha_n) + \delta\alpha_n[1 - (1-\delta)\beta_n]\}\|p_n - p^*\| \\
&\leq \delta[1 - (1-\delta)\alpha_n\beta_n]\|p_n - p^*\|. \qquad (3.40)
\end{aligned}$$

Again, combining (3.37) and (3.40), we have

$$\|p_{n+1} - p^*\| \leq \delta^2[1 - (1-\delta)\alpha_n\beta_n]\|p_n - p^*\|.$$

By induction,

$$\begin{aligned}\|p_{n+1} - p^*\| &\leq \|p_0 - p^*\|\delta^{2(n+1)} \prod_{k=0}^{n} [1 - (1 - \delta)\alpha_k\beta_k] \\ &= \|p_0 - p^*\|\delta^{2(n+1)}[1 - (1 - \delta)\alpha_k\beta_k]^{n+1} \\ &\leq \|p_0 - p^*\|\delta^{2(n+1)}[1 - (1 - \delta)\alpha\beta]^{n+1}. \end{aligned} \tag{3.41}$$

From M iterative scheme (2.6), using the same approach as in (3.34)–(3.40), we have

$$\begin{aligned}\|m_{n+1} - p^*\| &= \|\mathcal{J}d_n - p^*\| \\ &\leq \delta\|d_n - p^*\|, \end{aligned} \tag{3.42}$$

$$\begin{aligned}\|d_n - p^*\| &= \|\mathcal{J}c_n - p^*\| \\ &\leq \delta\|c_n - p^*\| \end{aligned} \tag{3.43}$$

and

$$\begin{aligned}\|c_n - p^*\| &= \|(1 - \alpha_n)m_n + \alpha_n\mathcal{J}m_n - p^*\| \\ &\leq (1 - \alpha_n)\|m_n - p^*\| + \alpha_n\|\mathcal{J}m_n - p^*\| \\ &\leq (1 - \alpha_n)\|m_n - p^*\| + \delta\alpha_n\|m_n - p^*\| \\ &= [(1 - \alpha_n) + \delta\alpha_n]\|m_n - p^*\| \\ &\leq [1 - (1 - \delta)\alpha_n]\|m_n - p^*\|. \end{aligned} \tag{3.44}$$

Combining (3.43) and (3.44), we have

$$\|d_n - p^*\| \leq \delta[1 - (1 - \delta)\alpha_n]\|m_n - p^*\|. \tag{3.45}$$

Combining (3.42) and (3.45), we have

$$\|m_{n+1} - p^*\| \leq \delta^2[1 - (1 - \delta)\alpha_n]\|m_n - p^*\|.$$

Inductively,

$$\|m_{n+1} - p^*\| = \delta^{2(n+1)} \prod_{k=0}^{n} [1 - (1 - \delta)\alpha_k]\|m_0 - p^*\|$$

such that

$$\begin{aligned}\|m_{n+1} - p^*\| &= \delta^{2(n+1)}[1 - (1 - \delta)\alpha_k]^{n+1}\|m_0 - p^*\| \\ &\leq \delta^{2(n+1)}[1 - (1 - \delta)\alpha]^{n+1}\|m_0 - p^*\|. \end{aligned} \tag{3.46}$$

From (3.33), (3.36) and (3.46), let

$$a_n = \delta^{3(n+1)}[1 - (1 - \delta)\alpha]^{n+1}\|u_0 - p^*\|$$

$$b_n = \delta^{n+1}[1 - (1 - \delta)\alpha]^{n+1}\|s_0 - p^*\|$$

$$c_n = \delta^{2(n+1)}[1 - (1 - \delta)\alpha]^{n+1}\|m_0 - p^*\|.$$

Hence,

$$\frac{a_n}{b_n} = \frac{\delta^{3(n+1)}[1 - (1 - \delta)\alpha]^{n+1}\|u_0 - p^*\|}{\delta^{n+1}[1 - (1 - \delta)\alpha]^{n+1}\|s_0 - p^*\|} \to 0 \text{ as } n \to \infty$$

and
$$\frac{a_n}{c_n} = \frac{\delta^{3(n+1)}[1-(1-\delta)\alpha]^{n+1}\|u_0 - p^*\|}{\delta^{2(n+1)}[1-(1-\delta)\alpha]^{n+1}\|m_0 - p^*\|} \to 0, \quad \text{as } n \to \infty.$$
It can be concluded that the AG iterative scheme (2.9) converges to the fixed point p^* faster than (2.4)–(2.6), thus completing the proof. □

Example 3.1. *Let $\mathscr{X} = \mathbb{R}$ and $D = [0, 20] \subseteq \mathscr{X}$. Let $\mathcal{J} : D \to D$ be a mapping defined by $\mathcal{J}u = \sqrt{u^2 - 9u + 54}$ for all $u \in D$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{3}{4}$ for each $n \in \mathbb{N}$ with the initial value $u_0 = 10$. $\mathcal{J}$ is a contraction mapping with contraction constant $\frac{9}{2\sqrt{54}}$ and $\mathscr{F}(\mathcal{J}) = \{6\}$. Tables 1 and 2 show that the AG fixed point iterative scheme (2.9) converges faster than the Picard-Mann, Mann, Thakur, Picard, Noor and M iterative schemes. Again, Figures 1 and 2 graphically display the fast convergence of the AG iterative scheme.*

Table 1. Comparison of speed of convergence of some iterative schemes for Example 3.1.

Step	AG	Picard-Mann	Mann	Thakur
1	10.0000000000	10.0000000000	10.0000000000	10.0000000000
2	6.0521589007	7.0533679898	8.5000000000	6.4371793563
3	6.0002097324	6.1515367954	7.4150259924	6.0180141243
4	6.0000008289	6.0172649142	6.7286051421	6.0006544493
5	6.0000000033	6.0018971898	6.3488560110	6.0000236518
6	6.0000000000	6.0002076117	6.1596478250	6.0000008546
7	6.0000000000	6.0000227088	6.0713292161	6.0000000309
8	6.0000000000	6.0000024838	6.0315037555	6.0000000011
9	6.0000000000	6.0000002717	6.0138409699	6.0000000000
10	6.0000000000	6.0000000297	6.0060666428	6.0000000000
11	6.0000000000	6.0000000032	6.0026563122	6.0000000000
12	6.0000000000	6.0000000004	6.0011625500	6.0000000000
13	6.0000000000	6.0000000000	6.0005086948	6.0000000000
14	6.0000000000	6.0000000000	6.0002225691	6.0000000000
15	6.0000000000	6.0000000000	6.0000973769	6.0000000000
16	6.0000000000	6.0000000000	6.0000426029	6.0000000000
17	6.0000000000	6.0000000000	6.0000186389	6.0000000000
18	6.0000000000	6.0000000000	6.0000081545	6.0000000000
19	6.0000000000	6.0000000000	6.0000035676	6.0000000000
20	6.0000000000	6.0000000000	6.0000015608	6.0000000000
21	6.0000000000	6.0000000000	6.0000006829	6.0000000000
22	6.0000000000	6.0000000000	6.0000002988	6.0000000000

Table 2. Comparison of speed of convergence of some iterative schemes for Example 3.1.

Step	AG	Picard	Noor	M
1	10.0000000000	10.0000000000	10.0000000000	10.0000000000
2	6.0521589007	8.0000000000	7.5072974202	6.3458402195
3	6.0002097324	6.7823299831	6.4993253300	6.0105078956
4	6.0000008289	6.2417169234	6.1587218874	6.0002882471
5	6.0000000033	6.0649466478	6.0498343531	6.0000078824
6	6.0000000000	6.0165653001	6.0155875999	6.0000002155
7	6.0000000000	6.0041627484	6.0048699050	6.0000000059
8	6.0000000000	6.0010420407	6.0015209084	6.0000000002
9	6.0000000000	6.0002605950	6.0004749371	6.0000000000
10	6.0000000000	6.0000651541	6.0001483043	6.0000000000
11	6.0000000000	6.0000162888	6.0000463091	6.0000000000
12	6.0000000000	6.0000040722	6.0000144603	6.0000000000
13	6.0000000000	6.0000010181	6.0000045153	6.0000000000
14	6.0000000000	6.0000002545	6.0000014099	6.0000000000
15	6.0000000000	6.0000000636	6.0000004403	6.0000000000
16	6.0000000000	6.0000000159	6.0000001375	6.0000000000
17	6.0000000000	6.0000000040	6.0000000429	6.0000000000
18	6.0000000000	6.0000000010	6.0000000134	6.0000000000
19	6.0000000000	6.0000000002	6.0000000042	6.0000000000
20	6.0000000000	6.0000000001	6.0000000013	6.0000000000
21	6.0000000000	6.0000000000	6.0000000004	6.0000000000
22	6.0000000000	6.0000000000	6.0000000001	6.0000000000

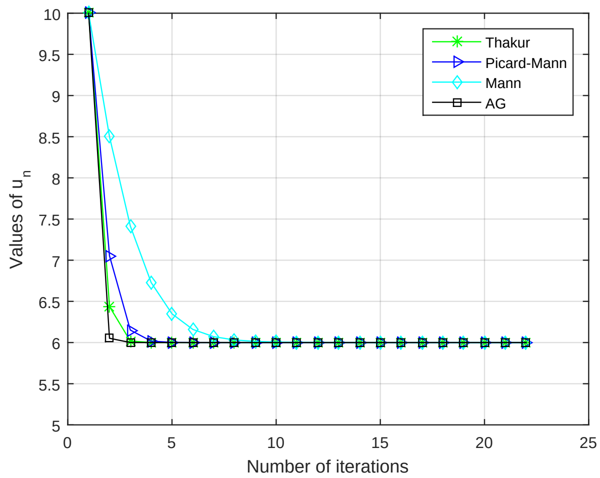

Figure 1. Graph corresponding to Table1 results.

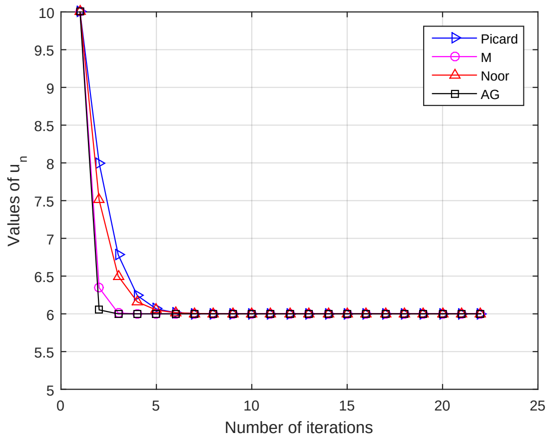

Figure 2. Graph corresponding to Table 2 results.

Remark 3.1. *(1) The graphs in Figures 1 and 2 compare the rate of convergence of various iterative schemes based on the values in Tables 1 and 2 for Example 3.1. The values in Tables 1 and 2 marked in blue indicate the fixed point at each step, and it can be seen that different iterative schemes converge at different steps. Moreover, where there is no such indication implies that the iterative scheme converges at a step beyond 22. Consequently, our iterative scheme converging at Step 6 which is faster than the Picard-Mann (Step 13), Mann (not visible within 22 steps), Picard (Step 21), Noor (not visible within 22 steps), Thakur and M (Step 9) schemes.*

Example 3.2. *Let* $C = [1, 6] \subseteq \mathscr{X} = \mathbb{R}$ *and* $\mathcal{J} : D \to D$ *be an operator defined by* $\mathcal{J}u = \frac{u}{2} + 1$ *for all* $u \in D$*. Choose* $\alpha_n = \frac{1}{2}$, $\beta_n = \frac{1}{3}$ *and* $\gamma_n = \frac{1}{4}$ *for each* $n \in \mathbb{N}$ *with the initial value* $u_0 = 2.5$*.* $\mathcal{J}$ *is a contraction mapping and the set of fixed points* $\mathscr{F}(\mathcal{J}) = \{2\}$*. The values obtained via computation of the mapping for various iterative schemes are shown in Tables 3 and 4. And, the corresponding plots for the values are shown in Figure 3, indicating that the AG iterative converges faster than the Picard-Mann, Noor, Mann, Picard, M and Thakur iterative schemes.*

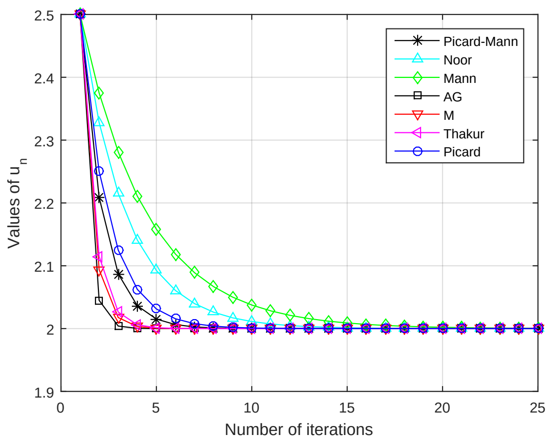

Figure 3. Graph corresponding to results listed in Tables 3 and 4.

Table 3. Comparison of the rate of convergence of several iteration processes for Example 2.

Steps	AG	Picard-Mann	Noor	Mann
1	2.5000000000	2.5000000000	2.5000000000	2.5000000000
2	2.0449414063	2.2087500000	2.3278906250	2.3750000000
3	2.0040394600	2.0871531250	2.2150245239	2.2812500000
4	2.0003630780	2.0363864297	2.1410090511	2.2109375000
5	2.0000326345	2.0151913344	2.0924710918	2.1582031250
6	2.0000029333	2.0063423821	2.0606408082	2.1186523438
7	2.0000002637	2.0026479445	2.0397671050	2.0889892578
8	2.0000000237	2.0011055168	2.0260785218	2.0667419434
9	2.0000000021	2.0004615533	2.0171018056	2.0500564575
10	2.0000000002	2.0001926985	2.0112150435	2.0375423431
11	2.0000000000	2.0000804516	2.0073546152	2.0281567574
12	2.0000000000	2.0000335886	2.0048230188	2.0211175680
13	2.0000000000	2.0000140232	2.0031628453	2.0158381760
14	2.0000000000	2.0000058547	2.0020741346	2.0118786320
15	2.0000000000	2.0000024443	2.0013601786	2.0089089740
16	2.0000000000	2.0000010205	2.0008919796	2.0066817305
17	2.0000000000	2.0000004261	2.0005849435	2.0050112979
18	2.0000000000	2.0000001779	2.0003835950	2.0037584734
19	2.0000000000	2.0000000743	2.0002515544	2.0028188551
20	2.0000000000	2.0000000310	2.0001649647	2.0021141413
21	2.0000000000	2.0000000129	2.0001081807	2.0015856060
22	2.0000000000	2.0000000054	2.0000709429	2.0011892045
23	2.0000000000	2.0000000023	2.0000465230	2.0008919034
24	2.0000000000	2.0000000009	2.0000305089	2.0006689275
25	2.0000000000	2.0000000004	2.0000200072	2.0005016956

Table 4. Comparison of the rate of convergence of several iteration processes for Example 2.

Steps	Thakur	Picard	M
1	2.5000000000	2.5000000000	2.5000000000
2	2.1146875000	2.2500000000	2.0937500000
3	2.0263064453	2.1250000000	2.0175781250
4	2.0060340409	2.0625000000	2.0032958984
5	2.0013840581	2.0312500000	2.0006179810
6	2.0003174683	2.0156250000	2.0001158714
7	2.0000728193	2.0078125000	2.0000217259
8	2.0000167029	2.0039062500	2.0000040736
9	2.0000038312	2.0019531250	2.0000007638
10	2.0000008788	2.0009765625	2.0000001432
11	2.0000002016	2.0004882813	2.0000000269
12	2.0000000462	2.0002441406	2.0000000050
13	2.0000000106	2.0001220703	2.0000000009
14	2.0000000024	2.0000610352	2.0000000002
15	2.0000000006	2.0000305176	2.0000000000
16	2.0000000001	2.0000152588	2.0000000000
17	2.0000000000	2.0000076294	2.0000000000
18	2.0000000000	2.0000038147	2.0000000000
19	2.0000000000	2.0000019073	2.0000000000
20	2.0000000000	2.0000009537	2.0000000000
21	2.0000000000	2.0000004768	2.0000000000
22	2.0000000000	2.0000002384	2.0000000000
23	2.0000000000	2.0000001192	2.0000000000
24	2.0000000000	2.0000000596	2.0000000000
25	2.0000000000	2.0000000298	2.0000000000

Theorem 3.6. *Let $\mathcal{T}$ be an approximate operator of $\mathcal{J}$ satisfying the contraction mapping condition (2.1). Let $\{u_n\}_{n=0}^{\infty}$ be an iterative sequence generated by the AG iterative scheme (2.9) for $\mathcal{J}$ and define an iterative sequence $\{\vartheta\}_{n=0}^{\infty}$ as follows*

$$\begin{cases} \vartheta_0 = \vartheta \in D \\ \vartheta_{n+1} = \mathcal{T}\mu_n \\ \mu_n = \mathcal{T}[(1-\alpha_n)\lambda_n + \alpha_n\mathcal{T}\lambda_n] \\ \lambda_n = (1-\beta_n)\mathcal{T}\vartheta_n + \beta_n\mathcal{T}\theta_n \\ \theta_n = (1-\gamma_n)\vartheta_n + \gamma_n\mathcal{T}\vartheta_n, \ \ n \in \mathbb{N}, \end{cases} \tag{3.47}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$ satisfying the following conditions: (a) $\frac{1}{2} \leq \alpha_n$ for all $n \in \mathbb{N}$, and (b) $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $\mathcal{J}p^* = p^*$ and $\mathcal{T}\tilde{p}^* = \tilde{p}^*$ such that $\lim_{n\to\infty} \vartheta_n = \tilde{p}^*$, then we have that $\|p^* - \tilde{p}^*\| \leq \frac{9\epsilon}{1-\delta}$ where $\epsilon > 0$ is a fixed constant.

Proof. Using (2.1), (2.9) and (3.47), we have

$$\begin{aligned}\|x_n-\theta_n\| &= \|(1-\gamma_n)u_n+\gamma_n\mathcal{J}u_n-(1-\gamma_n)\vartheta_n-\gamma_n\mathcal{T}\vartheta_n\|\\ &\le (1-\gamma_n)\|u_n-\vartheta_n\|+\gamma_n\|\mathcal{J}u_n-\mathcal{T}\vartheta_n\|\\ &\le (1-\gamma_n)\|u_n-\vartheta_n\|+\gamma_n\|\mathcal{J}u_n-\mathcal{J}\vartheta_n+\mathcal{J}\vartheta_n-\mathcal{T}\vartheta_n\|\\ &\le (1-\gamma_n)\|u_n-\vartheta_n\|+\gamma_n\|\mathcal{J}u_n-\mathcal{J}\vartheta_n\|+\gamma_n\|\mathcal{J}\vartheta_n-\mathcal{T}\vartheta_n\|\\ &\le (1-\gamma_n)\|u_n-\vartheta_n\|+\delta\gamma_n\|u_n-\vartheta_n\|+\gamma_n\epsilon\\ &= [1-(1-\delta)\gamma_n]\|u_n-\vartheta_n\|+\gamma_n\epsilon, \end{aligned} \tag{3.48}$$

$$\begin{aligned}\|w_n-\lambda_n\| &= \|(1-\beta_n)\mathcal{J}u_n+\beta_n\mathcal{J}x_n-(1-\beta_n)\mathcal{T}\vartheta_n-\beta_n\mathcal{T}\theta_n\|\\ &\le (1-\beta_n)\|\mathcal{J}u_n-\mathcal{T}\vartheta_n\|+\beta_n\|\mathcal{J}x_n-\mathcal{T}\theta_n\|\\ &\le (1-\beta_n)\|\mathcal{J}u_n-\mathcal{J}\vartheta_n+\mathcal{J}\vartheta_n-\mathcal{T}\vartheta_n\|+\beta_n\|\mathcal{J}x_n-\mathcal{J}\theta_n+\mathcal{J}\theta_n-\mathcal{T}\theta_n\|\\ &\le (1-\beta_n)\|\mathcal{J}u_n-\mathcal{J}\vartheta_n\|+(1-\beta_n)\|\mathcal{J}\vartheta_n-\mathcal{T}\vartheta_n\|+\beta_n\|\mathcal{J}x_n-\mathcal{J}\theta_n\|\\ &\quad+\beta_n\|\mathcal{J}\theta_n-\mathcal{T}\theta_n\|\\ &\le (1-\beta_n)\delta\|u_n-\vartheta_n\|+\beta_n\delta\|x_n-\theta_n\|+(1-\beta_n)\epsilon+\beta_n\epsilon; \end{aligned} \tag{3.49}$$

putting (3.48) in (3.49), we have

$$\begin{aligned}\|w_n-\lambda_n\| \le&(1-\beta_n)\delta\|u_n-\vartheta_n\|+\beta_n\delta\Big\{[1-(1-\delta)\gamma_n]\|u_n-\vartheta_n\|+\gamma_n\epsilon\Big\}\\ &+(1-\beta_n)\epsilon+\beta_n\epsilon\\ \le&(1-\beta_n)\delta\|u_n-\vartheta_n\|+\beta_n\delta[1-(1-\delta)\gamma_n]\|u_n-\vartheta_n\|+\beta_n\gamma_n\delta\epsilon+\epsilon\\ =&[\delta-\beta_n\gamma_n\delta(1-\delta)]\|u_n-\vartheta_n\|+\beta_n\gamma_n\delta\epsilon+\epsilon, \end{aligned} \tag{3.50}$$

$$\begin{aligned}\|v_n-\mu_n\| =&\|\mathcal{J}[(1-\alpha_n)w_n+\alpha_n\mathcal{J}w_n]-\mathcal{T}[(1-\alpha_n)\lambda_n+\alpha_n\mathcal{T}\lambda_n]\|\\ \le&\|\mathcal{J}[(1-\alpha_n)w_n+\alpha_n\mathcal{J}w_n]-\mathcal{J}[(1-\alpha_n)\lambda_n+\alpha_n\mathcal{T}\lambda_n]\\ &+\mathcal{J}[(1-\alpha_n)\lambda_n+\alpha_n\mathcal{T}\lambda_n]-\mathcal{T}[(1-\alpha_n)\lambda_n+\alpha_n\mathcal{T}\lambda_n]\|\\ \le&\|\mathcal{J}[(1-\alpha_n)w_n+\alpha_n\mathcal{J}w_n]-\mathcal{J}[(1-\alpha_n)\lambda_n+\alpha_n\mathcal{T}\lambda_n]\|\\ &+\|\mathcal{J}[(1-\alpha_n)\lambda_n+\alpha_n\mathcal{T}\lambda_n]-\mathcal{T}[(1-\alpha_n)\lambda_n+\alpha_n\mathcal{T}\lambda_n]\|\\ \le&\delta\|(1-\alpha_n)w_n+\alpha_n\mathcal{J}w_n-(1-\alpha_n)\lambda_n-\alpha_n\mathcal{T}\lambda_n\|+\epsilon\\ \le&\delta\Big\{(1-\alpha_n)\|w_n-\lambda_n\|+\alpha_n\|\mathcal{J}w_n-\mathcal{T}\lambda_n\|\Big\}+\epsilon\\ \le&\delta(1-\alpha_n)\|w_n-\lambda_n\|+\delta\alpha_n\|\mathcal{J}w_n-\mathcal{T}\lambda_n\|+\epsilon\\ \le&\delta(1-\alpha_n)\|w_n-\lambda_n\|+\delta\alpha_n\|\mathcal{J}w_n-\mathcal{J}\lambda_n+\mathcal{J}\lambda_n-\mathcal{T}\lambda_n\|+\epsilon\\ \le&\delta(1-\alpha_n)\|w_n-\lambda_n\|+\delta^2\alpha_n\|w_n-\lambda_n\|+\delta\alpha_n\epsilon+\epsilon\\ =&[\delta(1-\alpha_n)+\delta^2\alpha_n]\|w_n-\lambda_n\|+\delta\alpha_n\epsilon+\epsilon; \end{aligned} \tag{3.51}$$

putting (3.50) into (3.51) yields

$$\begin{aligned}\|v_n-\mu_n\| \le&[\delta(1-\alpha_n)+\delta^2\alpha_n]\{[\delta-\beta_n\gamma_n\delta(1-\delta)]\|u_n-\vartheta_n\|\\ &+\beta_n\gamma_n\delta\epsilon+\epsilon\}+\delta\alpha_n\epsilon+\epsilon\\ \le&\delta[1-(1-\delta)\alpha_n][\delta-\beta_n\gamma_n\delta(1-\delta)]\|u_n-\vartheta_n\|\\ &+[1-(1-\delta)\alpha_n]\delta^2\beta_n\gamma_n\epsilon+[1-(1-\delta)\alpha_n]\delta^2\epsilon+\delta\alpha_n\epsilon+\epsilon. \end{aligned} \tag{3.52}$$

Again,

$$
\begin{aligned}
\|u_{n+1}-\vartheta_{n+1}\| =& \|\mathcal{J}v_n-\mathcal{T}\mu_n\| \\
\leq& \|\mathcal{J}v_n-\mathcal{J}\mu_n+\mathcal{J}\mu_n-\mathcal{T}\mu_n\| \\
\leq& \|\mathcal{J}v_n-\mathcal{J}\mu_n\|+\|\mathcal{J}\mu_n-\mathcal{T}\mu_n\| \\
\leq& \delta\|v_n-\mu_n\|+\epsilon \\
\leq& \delta^2[1-(1-\delta)\alpha_n][\delta-\beta_n\gamma_n\delta(1-\delta)]\|u_n-\vartheta_n\| \\
&+[1-(1-\delta)\alpha_n]\delta^3\beta_n\gamma_n\epsilon+[1-(1-\delta)\alpha_n]\delta^3\epsilon+\delta^2\alpha_n\epsilon+\delta\epsilon+\epsilon.
\end{aligned}
\tag{3.53}
$$

Since $\delta \in [0,1)$ and $\alpha_n, \beta_n, \gamma_n \in [0,1]$, $n \in \mathbb{N}$, then

$$
\begin{aligned}
&\delta < 1 \\
&\delta^2 < 1 \\
&\delta^3[1-(1-\delta)\beta_n\gamma_n] < 1 \\
&\delta^3[1-(1-\delta)\alpha_n]\beta_n\gamma_n < 1 \\
&\delta^3[1-(1-\delta)\alpha_n] < 1,
\end{aligned}
$$

and from assumption (a) where $1-\alpha_n \leq \alpha_n$, we have that

$$
\begin{aligned}
\|u_{n+1}-\vartheta_{n+1}\| &\leq [1-(1-\delta)\alpha_n]\|u_n-\vartheta_n\|+\alpha_n\epsilon+4\epsilon \\
&\leq [1-(1-\delta)\alpha_n]\|u_n-\vartheta_n\|+\alpha_n\epsilon+4(1-\alpha_n+\alpha_n)\epsilon \\
&\leq [1-(1-\delta)\alpha_n]\|u_n-\vartheta_n\|+\alpha_n(1-\delta)\frac{9\epsilon}{(1-\delta)}.
\end{aligned}
\tag{3.54}
$$

Let $\sigma_n := \|u_n-\vartheta_n\|$, $\varpi_n := \alpha_n(1-\delta) \in (0,1)$ and $\eta_n := \frac{9\epsilon}{(1-\delta)}$.
From Lemma 2.5, it follows that

$$
0 \leq \limsup_{n\to\infty}\|u_n-\vartheta_n\| \leq \limsup_{n\to\infty}\frac{9\epsilon}{1-\delta}.
$$

From Theorem 3.1, we know that $\lim_{n\to\infty} u_n = p^*$. Using this fact alongside the assumption that $\lim_{n\to\infty}\vartheta_n = \tilde{p}^*$, we obtain

$$
\|p^*-\tilde{p}^*\| \leq \frac{9\epsilon}{1-\delta}.
$$

This completes the proof. □

4. Application to nonlinear fractional differential equations of Caputo type

The evolution of research involving fractional differential equations has been expansive since its discovery and the relevant significant studies in that area have been attributed to the fact that fractional differential equations have a wide range of applications in different domains. The extent of application of fractional differential equations include, but are not limited to the following areas: fluid flow, signal processing, electronics, biology, robotics, telecommunication systems, electrical networks, diffusive transport, traffic flow, gas dynamics, generalized Casson fluid modeling with heat generation and chemical reaction (see for example, [4, 12, 42–44] and the references therein).

We want to consider approximation of the solution of an NFDE of the Caputo type by using the AG fixed point iterative scheme (2.9).

To achieve our aim in this section, we consider the following NFDE of Caputo type with initial conditions:

$$\begin{cases} {}^{C}\mathcal{D}^{\zeta}\, x(t) + f((t, x(t))) = 0, \\ x(0) = x(1) = 0, \\ 0 \leq t \leq 0,\ 1 < \zeta < 2, \end{cases} \tag{4.1}$$

where ${}^{C}\mathcal{D}^{\zeta}$ is a Caputo fractional derivative of order ζ and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X = C[0, 1]$ be a Banach space of continuous real functions from $[0, 1]$ into $\mathbb{R}$, endowed with the usual supremum norm. The corresponding Green function associated with the NFDE (4.1) is given by

$$G(t, s) = \begin{cases} \frac{1}{\Gamma(\zeta)}(t(1-s)^{\zeta-1} - (t-s)^{\zeta-1}) & \text{if } 0 \leq s \leq t \leq 1, \\ \frac{t(1-s)^{\zeta-1}}{\Gamma(\zeta)} & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 4.1. *Let $\mathscr{X} = C[0, 1]$ be a Banach space with the supremum norm $\|\cdot\|_\infty$. Suppose that $f : [0, 1] \times \mathscr{X} \to \mathscr{X}$ is a continuous function; also, for $\delta \in (0, 1)$, assume the following condition:*

$$(C_1): \qquad |f(t, g) - f(t, h)| \leq \delta|g - h| \text{ holds for all } t \in [0, 1] \text{ and } g, h \in \mathscr{X}.$$

Theorem 4.1. *Let $\mathscr{X} = C[0, 1]$ be a Banach space endowed with the supremum norm as in Lemma 4.1. Let $\{u_n\}$ be a sequence defined by AG iterative scheme (2.9) for the integral operator $\mathcal{J} : \mathscr{X} \to \mathscr{X}$ defined by*

$$\mathcal{J}(y(t)) = \int_0^1 G(t, s) f(s, y(s)) ds,$$

$\forall t \in [0, 1]$, $\forall y \in \mathscr{X}$. Suppose that condition (C_1) of Lemma 4.1 is satisfied. Then the sequence defined by the AG iterative scheme (2.9) converges to the solution of problem (4.1).

Proof. It is obvious to note that $y \in \mathscr{X}$ is a solution of (4.1) if and only if $y \in \mathscr{X}$ is a solution of the integral equation

$$y(t) = \int_0^1 G(t, s) f(t, y(s)) ds.$$

Let $x, y \in \mathscr{X}$ for all $t \in [0, 1]$. Invoking Lemma 4.1, we have

$$\begin{aligned} |\mathcal{J}y(t) - \mathcal{J}z(t)| &= |\int_0^1 G(t, s) f(s, y(s)) ds - \int_0^1 G(t, s) f(s, z(s)) ds| \\ &\leq \int_0^1 G(t, s)|f(s, y(s)) - f(s, z(s))| ds \\ &\leq \int_0^1 G(t, s)\{\delta|y(s) - z(s)|\} ds \\ &\leq (\sup_{t\in[0,1]} \int_0^1 G(t, s) ds)\delta\|y - z\| \\ &\leq \delta\|y - z\|. \end{aligned}$$

Consequently, $\|\mathcal{J}y - \mathcal{J}z\| \leq \delta\|y - z\|$. Therefore, $\mathcal{J}$ is a contraction mapping. By Theorem 3.1, the sequence $\{u_n\}_{n=0}^{\infty}$ generated by the AG iterative scheme converges to a fixed point of $\mathcal{J}$; hence, it converges to the solution of the NFDE (4.1). □

5. Conclusions

We have been able to show that the AG iterative scheme converges faster than the Picard, Mann, Picard-Mann, Thakur, Noor and M iterative schemes through the example given in Section 3, with the results presented in Tables 1 and 2 and Figures 1 and 2. Weak convergence result of AG iterative scheme for a Suzuki generalized nonexpansive mapping was presented. Moreover, the stability and data dependence results have been proved for the new scheme. Finally, the new scheme has been applied to approximate the solution of an NFDE of the Caputo type. Our result has generalized and extended other existing results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflict of interest.

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