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Research article

Stieltjes integral boundary value problem involving a nonlinear multi-term Caputo-type sequential fractional integro-differential equation

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Abstract: In this article, we analyze the existence and uniqueness of mild solution to the Stieltjes integral boundary value problem involving a nonlinear multi-term, Caputo-type sequential fractional integro-differential equation. Krasnoselskii's fixed-point theorem and the Banach contraction principle are utilized to obtain the existence and uniqueness of the mild solution of the aforementioned problem. Furthermore, the Hyers-Ulam stability is obtained with the help of established methods. Our proposed model contains both the integer order and fractional order derivatives. As a result, the exponential function appears in the solution of the model, which is a fundamental and naturally important function for integer order differential equations and its many properties. Finally, two examples are provided to illustrate the key findings.

Keywords: existence; mild solution; fixed point; Hyers-Ulam stability; Caputo derivative; Stieltjes boundary conditions
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1. Introduction

In the previous few decades, fractional differential equations have become of great interest for researchers due to its high accuracy and usability in numerous subjects of science and technology. A lot of physical and natural phenomena can be modeled through fractional differential equations, which provides better results compared to integer order differential equations. Due to this, fractional differential equations are counted as a special tool for modeling. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, electrochemistry, aerodynamics, viscoelasticity, polymer rheology, economics, biology, electrodynamics of complex medium, etc. For details, see [2, 6–9, 12, 14–17, 19, 20, 22, 24, 28, 29, 31, 37, 39].

Additionally, fractional differential equations serve as an excellent tool for the description of the hereditary properties of various materials and processes. Consequently, the subject of the aforementioned equations is gaining great importance and attention from the researchers. Additionally, researchers are attracted to the enriched material on theoretical aspects and analytic/numerical methods for solving fractional order models. Furthermore, the mathematical models involving fractional order derivatives are more realistic and practical compared to the classical models. In the most recent years, many researchers have focused on the existence of solutions for fractional differential equations, for instance, see [3, 4, 10] and the references therein.

In 1940, while interacting with the mathematical community at the University of Wisconsin, Ulam expressed his concern regarding the stability of group homomorphisms [25]. The broader form of his views about the stability of functional equations is, "Impose constraints which converges the solutions of an inequality to the exact solutions of the corresponding equations". In 1941, by considering Banach spaces, Hyers gave half an answer to Ulam's question about the stability of functional equations [18]. Due to this contribution of Hyers, Ulam's problem was refereed to as Hyers-Ulam stability of functional equations. For the first time, Hyers-Ulam stability of linear differential equations were introduced by Obloza [21]. Along with generalization, the work of Obloza has been enhanced with different features by using new approaches as time progressed. For more details regarding Ulam's stability with different approaches, we recommend [26, 27, 33–36].

In [32], the authors studied a new class of impulsive implicit sequential fractional differential equations of the following form:

$$\begin{cases} {}^{c}D^{\beta}(D+\lambda)u(x) := f(x, u(x), {}^{c}D^{\beta}u(x)), \quad x \in (x_k, w_k],\ k = 0, 1, \ldots, m,\ \beta \in (0, 1], \\ u(x) := G_k(x, u(x)), \quad x \in (w_{k-1}, x_k],\ \ k = 1, 2, \ldots, m, \\ u(0) := 0,\ u(w_k) := 0, \quad k = 0, 1, 2, \ldots, m, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\beta}$ denotes the Caputo fractional derivative of order β, D denotes an ordinary derivative, with the lower limit 0, $0 = x_0 < w_0 < x_1 < w_1 < \cdots < x_m < w_m = T$, and T is a pre-fixed number and $\beta \in \mathrm{R}^+$. The function $f : [0, T] \times \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ is continuous and $G_k : [w_{k-1}, x_k] \times \mathrm{R} \to \mathrm{R}$ is also continuous for all $k = 1, 2, \ldots, m$,

Binlin et al. [38] studied the existence and uniqueness (EU) of the solution, as well as the stability in the form of Ulam's problem, for the following FDEs with Stieltjes integral condition:

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}(\mathscr{D}+\lambda)x(\xi) = \phi(\xi), \ \ \xi \in [0, 1] \\ x(0) = 0, {}^{C}\mathscr{D}^{\beta_0}_{0,\xi}x(1) = \sum_{i=1}^{p} \int_0^1 \mathscr{D}^{\beta_i}_{0,\xi}\xi(s)dx_i(s). \end{cases}$$

Bashir et al. [5] studied the EU of the solution of the nonlinear multi-term fractional integro-DE with anti-periodic conditions:

$$\begin{cases}(\lambda_1^C\mathscr{D}^v x(\xi) + \lambda_2^C\mathscr{D}^\varrho x(\xi)) = f(\xi, x(\xi)) + I^\omega g(\xi, x(\xi)), & \xi \in [0,\tau]\\ x(0) = -x(\tau), x'(0) = -x'(\tau), \end{cases}$$

where, $\lambda_1,\ \lambda_2\ \in \mathcal{R},\ \lambda_1 \neq 0,\ v \in [1,2),\ \varrho \in (1,v),\ \omega > 0,\ f,g : [0,1]\times\mathcal{R} \to \mathcal{R}$ are appropriate functions.

In this article, we analyze the multi-term, nonlinear, sequential fractional differential equation with Stieltjes integral conditions of the following form:

$$\begin{cases}(\lambda_1^C\mathscr{D}^v + \lambda_2^C\mathscr{D}^\varrho)(\mathscr{D} + \lambda_3)x(\xi) = f(\xi, x(\xi)) + I^\omega g(\xi, x(\xi)), & \xi \in [0,1]\\ x(0) = 0, \quad x'(0) = 0,^C\mathscr{D}_{0,\xi}^{\beta_0}x(1) = \sum_{i=1}^{p}\int_0^1 \mathscr{D}_{0,\xi}^{\beta_i}x(s)dx_i(s), \end{cases} \tag{1.2}$$

where, $v \in (1,2],\ \varrho \in (1,v), \lambda_1, \lambda_2, \in \mathcal{R}, \lambda_3 \in \mathbb{R}^+, \lambda_1 \neq 0, p \in \mathbb{N}, \beta_i \in \mathcal{R}$ for all $i = 0,1,\cdots,p, 0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_p < v, \beta_0 \in [1,v),\ f,g : [0,1]\times\mathcal{R} \to \mathcal{R}$ are appropriate functions, and the integrals presented in Boundary Condions (BCs) are Riemann-Stieltjes integrals with $x_i(i = 1,2,\cdots,p)$ functions of a bounded variation. In addition to general FDEs, the multi–point boundary conditions are more valuable than the classical initial/boundary conditions, because these conditions describe the characteristics of chemical, physical or others processes happening inside the domain.

2. Preliminaries

Here, we present necessary preliminaries so that the paper will be self contained.

Let $C = C([0,1],\mathcal{R})$ be the Banach space of all continuous functions endowed with the norm denoted by $\|.\|$.

Definition 2.1. *[1] The fractional integral of order v from 0 to ξ for the function x is defined by the following:*

$$I_{0,\xi}^v x(\xi) = \frac{1}{\Gamma(v)}\int_0^\xi (\xi - s)^{v-1}x(s)ds, \quad \xi > 0,\ v > 0$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2. *[1] The Caputo derivative of fractional order **v** from **0** to ξ for a function **x** can be defined as follows:*

$${}^C\mathscr{D}_{0,\xi}^v x(\xi) = \frac{1}{\Gamma(n-v)}\int_0^\xi (\xi - s)^{n-v-1}x^n(s)ds, \quad \textit{where} \quad n = \lfloor v \rfloor + 1.$$

Lemma 2.1. *[1] The FDE ${}^C\mathscr{D}_{0,\xi}^v x(\xi) = 0$ with $v > 0$, involving Caputo differential operator ${}^C\mathscr{D}_{0,\xi}^v$ has a solution in the following form:*

$$x(\xi) = c_0 + c_1\xi + c_2\xi^2 + \cdots + c_{m-1}\xi^{m-1},$$

where $c_k \in \mathcal{R}, k = 0,\ 1,\cdots,m-1 =: \overline{0,m-1}$ and $m = \lfloor v \rfloor + 1$.

Lemma 2.2. *[1] For each $v > 0$, we have the following:*

$$I^{v}_{0,\xi}({}^{C}\mathscr{D}^{v}_{0,\xi}x(\xi)) = c_0 + c_1\xi + c_2\xi^2 + \cdots + c_{m-1}\xi^{m-1},$$

where $c_k \in \mathcal{R}$, $k = \overline{0, m-1}$ and $m = \lfloor v \rfloor + 1$.

Lemma 2.3. *[38] If $\Re(v) > 0$ and $\lambda > 0$, then*

$${}^{C}\mathscr{D}^{v}_{+}e^{\lambda\xi} = \lambda^{v}e^{\lambda\xi} \text{ and } {}^{C}\mathscr{D}^{v}_{-}e^{-\lambda\xi} = \lambda^{v}e^{-\lambda\xi}.$$

Theorem 2.1. *(Krasnoselskii's fixed point theorem [11]) If K is a closed, convex, non-empty subset of a Banach space M such that P and Q map K into M and,*
(i) $Px + Qy \in K \quad (\forall x, y \in K)$,
(ii) P is compact and continuous,
(iii) Q is a contraction mapping,
then $\exists$ y in K such that, $Py + Qy = y$.

Theorem 2.2. *(Banach fixed point theorem) Every contraction mapping δ from B to B has a fixed point (unique), where B is a non-empty closed set in a Banach space X.*

Definition 2.3. *Consider a Cauchy problem $\frac{d}{dt}x(t) = f(t)$ with $x(t_0) = x_0$, then, a continuous function u is called its mild solution if*

$$u(t) = x(t_0) + \int_{t_0}^{t} f(s)ds.$$

Lemma 2.4. *Suppose that $f, g \in C([0,1] \times \mathcal{R}, \mathcal{R})$, then, the mild solution of (1.2) has the following form:*

$$\begin{aligned}
x(\xi) =& -\frac{\lambda_2}{\lambda_1}\int_0^{\xi} e^{-\lambda_3(\xi-s)}I^{v-\varrho-1}x(s)ds - \frac{\lambda_1\lambda_3}{\lambda_1}\int_0^{\xi} e^{-\lambda_3(\xi-s)}I^{v-\varrho}x(s)ds + \frac{1}{\lambda_1}\int_0^{\xi} e^{-\lambda_3(\xi-s)}I^{v}f(s,x(s))ds\\
&+ \frac{1}{\lambda_1}\int_0^{\xi} e^{-\lambda_3(\xi-s)}I^{v+\omega}g(s,x(s))ds + \frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{\lambda_2}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi} e^{-\lambda_3(\xi-s)}I^{v-\varrho-\beta_i-1}x(s)dsdx_i(s)\\
&+ \frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi} e^{-\lambda_3(\xi-s)}I^{v-\beta_i-\varrho}x(s)dsdx_i(s) - \frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi} e^{-\lambda_3(\xi-s)}I^{v-\beta_i}f(s,x(s))dsdx_i(s)\\
&- \frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi} e^{-\lambda_3(\xi-s)}I^{v-\beta_i+\omega}g(s,x(s))dsdx_i(s)\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}] + \frac{\nabla}{\lambda_1\lambda_3}(1 - e^{-\lambda_3\xi})\\
&+ \frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{-\lambda_2}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho-1}x(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho}x(s)ds\\
&+ \frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0}f(s,x(s))ds + \frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0+\omega}g(s,x(s))ds\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}]\\
&+ \frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3} - \sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}dx_i(s)\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}],
\end{aligned}$$

where, (2.1)

$$\nabla = \lambda_2 I^{v-\varrho-1}x(0) + \lambda_2 I^{v-\varrho}x(0) - I^v f(0, x(0)) - I^{v+\omega}g(0, x(0)).$$

$$\Delta = \frac{1}{\lambda_1\lambda_3^2}\Big[\sum_{i=1}^{p}\int_0^1 (-\lambda_3)^{\beta_i}e^{-\lambda_3\xi} - (-\lambda_3)^{\beta_0}e^{-\lambda_3}\Big] \neq 0.$$

Proof. Consider problem (1.2) and apply the fractional integral of order v to obtain the following:

$$(\lambda_1 + \lambda_2 I^{v-\varrho})(\mathscr{D} + \lambda_3)x(\xi) = I^v f(\xi, x(\xi)) + I^{v+\omega}g(\xi, x(\xi)) + C_0 + C_1\xi.$$

This implies the following:

$$\lambda_1\mathscr{D}x(\xi) + \lambda_1\lambda_3 x(\xi) + \lambda_2 I^{v-\varrho-1}x(\xi) + \lambda_2\lambda_3 I^{v-\varrho}x(\xi) = I^v f(\xi, x(\xi)) + I^{v+\omega}g(\xi, x(\xi)) + C_0 + C_1\xi.$$

Equivalently, we obtain te following:

$$\mathscr{D}x(\xi) + \lambda_3 x(\xi) = \frac{-\lambda_2}{\lambda_1}I^{v-\varrho-1}x(\xi) - \frac{\lambda_2\lambda_3}{\lambda_1}I^{v-\varrho}x(\xi) + \frac{1}{\lambda_1}I^v f(\xi, x(\xi)) + \frac{1}{\lambda_1}I^{v+\omega}g(\xi, x(\xi)) + \frac{1}{\lambda_1}C_0 + \frac{1}{\lambda_1}C_1\xi.$$

Now, multiplying by the integrating factor $e^{\lambda_3\xi}$ and then integrating from **0** to ξ, we obtain the following:

$$\begin{aligned} x(\xi) =& \frac{-\lambda_2}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho-1}x(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho}x(s)ds + \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^v f(s, x(s))ds \\ &+ \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v+\omega}g(s, x(s))ds + \frac{c_0}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}ds + \frac{c_1}{\lambda_1}\int_0^\xi se^{-\lambda_3(\xi-s)}ds + c_2 e^{-\lambda_3\xi}. \end{aligned} \tag{2.2}$$

Therefore,

$$\int_0^\xi e^{-\lambda_3(\xi-s)}ds = \frac{1}{\lambda_3}(1 - e^{-\lambda_3\xi}) \quad \text{and,} \quad \int_0^\xi se^{-\lambda_3(\xi-s)}ds = \frac{1}{\lambda_3^2}[\lambda_3\xi - 1 + e^{-\lambda_3\xi}].$$

Consequently, Eq (2.2) becomes the following:

$$\begin{aligned} x(\xi) =& \frac{-\lambda_2}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho-1}x(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho}x(s)ds + \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^v f(s, x(s))ds \\ &+ \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v+\omega}g(s, x(s))ds + \frac{c_0}{\lambda_1\lambda_3}[1 - e^{-\lambda_3\xi}] + \frac{c_1}{\lambda_1\lambda_3^2}[\lambda_3\xi - 1 + e^{-\lambda_3\xi}] + c_2 e^{-\lambda_3\xi}. \end{aligned} \tag{2.3}$$

Now, the boundary condition $x(0) = 0$ implies that $c_2 = 0$. Thus, (2.3) implies the following:

$$\begin{aligned} x(\xi) =& \frac{-\lambda_2}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho-1}x(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho}x(s)ds + \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^v f(s, x(s))ds \\ &+ \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v+\omega}g(s, x(s))ds + \frac{c_0}{\lambda_1\lambda_3}[1 - e^{-\lambda_3\xi}] + \frac{c_1}{\lambda_1\lambda_3^2}[\lambda_3\xi - 1 + e^{-\lambda_3\xi}]. \end{aligned} \tag{2.4}$$

Differentiating Eq (2.4) w.r.t ξ and then applying the boundary condition, we obtain the following:

$$c_0 = \lambda_2 I^{v-\varrho-1}x(0) + \lambda_2\lambda_3 I^{v-\varrho}x(0) - I^v f(0, x(0)) - I^{v+\omega}g(0, x(0)).$$

Now, let $\beta_0 \in [1, \nu)$, then, from [38], we obtain the following:

$$
\begin{aligned}
{}^{C}\mathscr{D}^{\beta_0}x(1) =& \frac{-\lambda_2}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\varrho-\beta_0-1}x(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\varrho-\beta_0}x(s)ds \\
&+ \frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\beta_0}f(s,x(s))ds + \frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu+\omega-\beta_0}g(s,x(s))ds \\
&+ \frac{c_0}{\lambda_1\lambda_3}(-\lambda_3)^{\beta_0}e^{-\lambda_3} + \frac{c_1}{\lambda_1\lambda_3^2}(-\lambda_3)^{\beta_0}e^{-\lambda_3}. \qquad (2.5)
\end{aligned}
$$

Similarly, for $1 \le \beta_1 \le \beta_2 \le \cdots \le \beta_p < \nu$, we obtain the following:

$$
\begin{aligned}
{}^{C}\mathscr{D}^{\beta_i}x(\xi) =& \frac{-\lambda_2}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i-1}x(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i}x(s)ds \\
&+ \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i}f(s,x(s))ds + \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu+\omega-\beta_i}g(s,x(s))ds \\
&+ \frac{c_0}{\lambda_1\lambda_3}(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi} + \frac{c_1}{\lambda_1\lambda_3^2}(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}. \qquad (2.6)
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\sum_{i=1}^{p}\int_0^1 \mathscr{D}^{\beta_i}x(\xi)dx_i(s) =& \frac{-\lambda_2}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i-1}x(s)dsdx_i(s) \\
&+ \frac{c_0}{\lambda_1\lambda_3}\sum_{i=1}^{p}\int_0^1 (-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}dx_i(s) \\
&- \frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i}x(s)dsdx_i(s) \\
&+ \frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i}f(s,x(s))dsdx_i(s) \\
&+ \frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu+\omega-\beta_i}g(s,x(s))dsdx_i(s) \\
&+ \frac{c_1}{\lambda_1\lambda_3^2}\sum_{i=1}^{p}\int_0^1 (-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}dx_i(s). \qquad (2.7)
\end{aligned}
$$

From Eqs (2.5) and (2.7), we have the following:

$$
\begin{aligned}
c_1 =& \frac{-\lambda_2}{\lambda_1\Delta}\int_0^1 e^{(1-s)}I^{\nu-\varrho-\beta_0-1}x(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1\Delta}\int_0^1 e^{(1-s)}I^{\nu-\varrho-\beta_0}x(s)ds + \frac{1}{\lambda_1\Delta}\int_0^1 e^{(1-s)}I^{\nu-\beta_0}f(s,x(s))ds \\
&+ \frac{1}{\lambda_1\Delta}\int_0^1 e^{(1-s)}I^{\nu-\beta_0+\omega}g(s,x(s))ds + \frac{\nabla}{\lambda_1\lambda_3\Delta}[(-\lambda_3)^{\beta_0}e^{-\lambda_3} - \sum_{i=1}^{p}\int_0^1 (-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}dx_i(s)] \\
&+ \frac{\lambda_2}{\lambda_1\Delta}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i-1}x(s)dsdx_i(s) + \frac{\lambda_2\lambda_3}{\lambda_1\Delta}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i}x(s)dsdx_i(s)
\end{aligned}
$$

$$-\frac{1}{\lambda_1\Delta}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_0}f(s,x(s))dsdx_i(s)-\frac{1}{\lambda_1\Delta}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_0+\omega}g(s,x(s))dsdx_i(s).$$

Now, Eq (2.4) becomes the following:

$$\begin{aligned}
x(\xi)=&-\frac{\lambda_2}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-1}x(s)ds-\frac{\lambda_1\lambda_3}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho}x(s)ds+\frac{1}{\lambda_1}\int_0^{\xi}I^{\nu}e^{-\lambda_3(\xi-s)}f(s,x(s))ds\\
&+\frac{1}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu+\omega}g(s,x(s))ds+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{\lambda_2}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i-1}x(s)dsdx_i(s)\\
&+\frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i-1}x(s)dsdx_i(s)-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_0}f(s,x(s))dsdx_i(s)\\
&-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_0+\omega}g(s,x(s))dsdx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]+\frac{\nabla}{\lambda_1\lambda_3}(1-e^{-\lambda_3\xi})\\
&+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{-\lambda_2}{\lambda_1}\int_0^1e^{-\lambda_3(1-s)}I^{\nu-\beta_0-\varrho-1}x(s)ds-\frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1e^{-\lambda_3(1-s)}I^{\nu-\beta_0-\varrho}x(s)ds\\
&+\frac{1}{\lambda_1}\int_0^1e^{-\lambda_3(1-s)}I^{\nu-\beta_0}f(s,x(s))ds+\frac{1}{\lambda_1}\int_0^1e^{-\lambda_3(1-s)}I^{\nu-\beta_0+\omega}g(s,x(s))ds\Big][\lambda_3\xi-1e^{-\lambda_3\xi}]\\
&+\frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3}-\sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}\mathscr{D}x_i(s)\Big][\lambda_3\xi-1e^{-\lambda_3\xi}],
\end{aligned}$$

which is the required proof.

Remark 2.1. *According to Lemma 2, and the Counter-Example 1 in [13], Example 3.1 from [23] and Fact 2, from [23], the existence of continuous (even Holderian) solutions of the fractional-type integral forms is not sufficient to ensure the existence of solutions to the corresponding Caputo-type fractional differential problems. For this reason, we only obtain the mild solution.*

3. Existence and uniqueness result

Throughout the paper, in the case of the Stieltjes integral, we consider the same monotonic functions, which are functions of t and also functions of bounded variation.

The given problem can be converted into a fixed point problem as follows: $\delta(x)=x$, where $\delta: C\to C$ is defined by

$$\begin{aligned}
\delta x(\xi)=&-\frac{\lambda_2}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-1}x(s)ds-\frac{\lambda_1\lambda_3}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho}x(s)ds+\frac{1}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu}f(s,x(s))ds\\
&+\frac{1}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu+\omega}g(s,x(s))ds+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{\lambda_2}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i-1}x(s)dsdx_i(s)\\
&+\frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i-1}x(s)dsdx_i(s)-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i}f(s,x(s))dsdx_i(s)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\beta_i+\omega}g(s,x(s))dsdx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]+\frac{\nabla}{\lambda_1\lambda_3}(1-e^{-\lambda_3\xi})\\
&+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{-\lambda_2}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho-1}x(s)ds-\frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho}x(s)ds\\
&+\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0}f(s,x(s))ds+\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0+\omega}g(s,x(s))ds\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]\\
&+\frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3}-\sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}dx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]. \qquad (3.1)
\end{aligned}
$$

Theorem 3.1. H_1 : *Suppose* $f, g \in C([0,1],\mathcal{R}) \to \mathcal{R}$ *and*

$$|f(\xi,0)| \le N_1 < \infty \ \ \textit{and} \ \ |g(\xi,0)| \le N_2 < \infty,$$

where $N = \max\{N_1, N_2\}$, *and*

$$|f(\xi,x)-f(\xi,y)| \le L_1|x-y|, \quad |g(\xi,x)-g(\xi,y)| \le L_2|x-y|,$$

with $L_1, L_2 > 0 \ \forall\ \xi \in [0,1]$ *and* $x,\ y \in \mathcal{R}$.

H_2 : $L = \max\{L_1, L_2\}$ *and* $\Omega_1 + L\,\Omega_2 < 1$, *where*

$$
\begin{aligned}
\Omega_1 =\ &\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\varrho)}(1-e^{-\lambda_3})+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho+1)}(1-e^{-\lambda_3})+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{|\lambda_2|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\rho})\rho^{v-\varrho-\beta_i-1}\mathfrak{R}_i}{\Gamma(v-\beta_i-\varrho)}+\\
&\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\rho})\rho^{v-\varrho-\beta_i}\mathfrak{R}_i}{\Gamma(v-\beta_i-\varrho+1)}\Big][\lambda_3-1+e^{-\lambda_3}]+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\varrho-\beta_0)}(1-e^{-\lambda_3})+\\
&\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho-\beta_0+1)}(1-e^{-\lambda_3})\Big][\lambda_3-1+e^{-\lambda_3}].
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2 =\ &\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3}\Big(\frac{1}{\Gamma(v+1)}+\frac{1}{\Gamma(v+1+\omega)}\Big)+\frac{(1-e^{-\lambda_3})}{|\Delta\lambda_1\lambda_3^2|\lambda_3}\Big[\frac{1}{\Gamma(v-\beta_0+1)}+\frac{1}{\Gamma(v-\beta_0+1+\omega)}\Big][\lambda_3-1+e^{-\lambda_3}]+\\
&\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\rho})\rho^{v-\beta_i}\mathfrak{R}_i}{\Gamma(v-\beta_i+1)}+\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\rho})\rho^{v-\beta_i+\omega}\mathfrak{R}_i}{\Gamma(v-\beta_i+1+\omega)}\Big][\lambda_3-1+e^{-\lambda_3}].
\end{aligned}
$$

$$\Omega_3 = \frac{|\nabla|(1-e^{-\lambda_3})}{|\lambda_1\lambda_3|}+\frac{|\nabla|}{|\Delta\lambda_1^2\lambda_3^3|}\Big[(\lambda_3)^{\beta_0}e^{-\lambda_3}+\sum_{i=1}^{p}(-\lambda_3)^{\beta_i}e^{-\lambda_3\rho}\mathfrak{R}_i\Big][\lambda_3-1+e^{-\lambda_3}].$$

Then, Eq (1.2) has a unique mild solution in C.

Proof. Considering a closed ball $B_r = \{x \in C : \|x\| \le r\}$, we show that $\delta B_r \subset B_r$, where δ is defined by Eq (3.1) and $r \ge (N\Omega_2 + \Omega_3)(1 - \Omega_1 - L\Omega_2)^{-1}$. For any $x \in B_r$, it follows from condition (H_1) that

$$|f(\xi,x)| \le L_1\|x\| + N_1 \le L_1 r + N_1 \quad \textit{and} \quad g(\xi,x) \le L_2 r + N_2.$$

Now, as $\|\delta(x)\| = \sup_{\xi\in[0,1]} |\delta x(\xi)|$, the following holds true:

$$\begin{aligned}
\|\delta(x)\| = \sup_{\xi\in[0,1]} \Big| &-\frac{\lambda_2}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)} I^{v-\varrho-1}x(s)ds - \frac{\lambda_1\lambda_3}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)} I^{v-\varrho}x(s)ds + \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)} I^{v} f(s,x(s))ds \\
&+ \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)} I^{v+\omega} g(s,x(s))ds + \frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{\lambda_2}{\lambda_1}\sum_{i=1}^p \int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)} I^{v-\varrho-\beta_i-1}x(s)dsdx_i(s) \\
&+ \frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^p \int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)} I^{v-\beta_i-\varrho}x(s)dsdx_i(s) - \frac{1}{\lambda_1}\sum_{i=1}^p \int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)} I^{v-\beta_i} f(s,x(s))dsdx_i(s) \\
&- \frac{1}{\lambda_1}\sum_{i=1}^p \int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)} I^{v-\beta_i+\omega} g(s,x(s))dsdx_i(s)\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}] + \frac{\nabla}{\lambda_1\lambda_3}(1-e^{-\lambda_3\xi}) \\
&+ \frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{-\lambda_2}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)} I^{v-\beta_0-\varrho-1}x(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)} I^{v-\beta_0-\varrho}x(s)ds \\
&+ \frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)} I^{v-\beta_0} f(s,x(s))ds + \frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)} I^{v-\beta_0+\omega} g(s,x(s))ds\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}] \\
&+ \frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3} - \sum_{i=1}^p\int_0^1 (-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}dx_i(s)\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}]\Big|.
\end{aligned}$$

Now, using the mean value Theorem [30] for Stieltjes integral with $\chi \in [0,1]$ and $x_i(1) = \Re_i > 0$ and from (H_1), we obtain the following:

$$\begin{aligned}
\|\delta(x)\| \le &\frac{r|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\varrho)}(1-e^{-\lambda_3}) + \frac{r|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho+1)}(1-e^{-\lambda_3}) + \frac{(L_1r+N_1)(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(v+1)} \\
&+ \frac{(L_2r+N_2)(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(v+1+\omega)} + \frac{|\nabla|}{|\lambda_1\lambda_2|}(1-e^{-\lambda_3}) + \frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{r|\lambda_2|}{|\lambda_1|}\sum_{i=1}^p \frac{(1-e^{-\lambda_3\chi})\chi^{v-\varrho-\beta_i-1}\Re_i}{\Gamma(v-\beta_i-\varrho)} \\
&+ \frac{r|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3}\sum_{i=1}^p \frac{(1-e^{-\lambda_3\chi})\chi^{v-\varrho-\beta_i}\Re_i}{\Gamma(v-\beta_i-\varrho+1)} + \frac{(L_1r+N_1)}{|\lambda_1|\lambda_3}\sum_{i=1}^p \frac{(1-e^{-\lambda_3\chi})\chi^{v-\beta_i}\Re_i}{\Gamma(v-\beta_i+1)} \\
&+ \frac{(L_2r+N_2)}{|\lambda_1|\lambda_3}\sum_{i=1}^p \frac{(1-e^{-\lambda_3\chi})\chi^{v-\beta_i+\omega}\Re_i}{\Gamma(v-\beta_i+1+\omega)}\Big][\lambda_3 - 1 + e^{-\lambda_3}] + \frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{r|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\beta_0-\varrho)}(1-e^{-\lambda_3}) \\
&+ \frac{r|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho-\beta_0+1)}(1-e^{-\lambda_3}) + \frac{(L_1r+N_1)}{|\lambda_1|\lambda_3\Gamma(v-\beta_0+1)}(1-e^{-\lambda_3}) \\
&+ \frac{(L_2r+N_2)}{|\lambda_1|\lambda_3\Gamma(v-\beta_0+\omega+1)}(1-e^{-\lambda_3})\Big][\lambda_3 - 1 + e^{-\lambda_3}] \\
&+ \frac{|\nabla|}{|\lambda_1^2\lambda_3^3\Delta|}\Big|\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3} - \sum_{i=1}^p (-\lambda_3)^{\beta_i}e^{-\lambda_3\chi}\Re_i\Big]\Big|[\lambda_3 - 1 + e^{-\lambda_3}] \\
\le r\Big[&\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\varrho)}(1-e^{-\lambda_3}) + \frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho+1)}(1-e^{-\lambda_3}) + \frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\beta_0-\varrho)}(1-e^{-\lambda_3}) \\
&+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho-\beta_0+1)}(1-e^{-\lambda_3})\Big][\lambda_3 - 1 + e^{-\lambda_3}] + \frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{|\lambda_2|}{|\lambda_1|}\sum_{i=1}^p \frac{(1-e^{-\lambda_3\chi})\chi^{v-\varrho-\beta_i-1}\Re_i}{\Gamma(v-\beta_i-\varrho)}
\end{aligned}$$

$$+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{v-\varrho-\beta_i}\Re_i}{\Gamma(v-\beta_i-\varrho+1)}\Big][\lambda_3-1+e^{-\lambda_3}]+(Lr+N)\Big[\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(v+1)}+\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(v+1+\omega)}$$

$$+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{v-\beta_i}\Re_i}{\Gamma(v-\beta_i+1)}+\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{v-\beta_i+\omega}\Re_i}{\Gamma(v-\beta_i+1+\omega)}\Big][\lambda_3-1+e^{-\lambda_3}]\Big]$$

$$+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{1}{|\lambda_1|\lambda_3\Gamma(v-\beta_0+1)}(1-e^{-\lambda_3})+\frac{1}{|\lambda_1|\lambda_3\Gamma(v-\beta_0+\omega+1)}(1-e^{-\lambda_3})\Big][\lambda_3-1+e^{-\lambda_3}]\Big]$$

$$+\frac{|\nabla|}{|\lambda_1^2\lambda_3^3\Delta|}\Big|\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3}-\sum_{i=1}^{p}(-\lambda_3)^{\beta_i}e^{-\lambda_3\chi}\Re_i\Big]\Big|[\lambda_3-1+e^{-\lambda_3}]+\frac{|\nabla|}{|\lambda_1\lambda_2|}(1-e^{-\lambda_3}).$$

This implies the following:

$$\|\delta(x)\|\le r\Omega_1+(Lr+N)\Omega_2+\Omega_3\le r.$$

Next, to prove the contraction, consider the following:

$$\begin{aligned}
\|\delta x-\delta y\| &= \sup_{\xi\in[0,1]}|\delta x(\xi)-\delta y(\xi)|\\
&= \sup_{\xi\in[0,1]}\Big|-\frac{\lambda_2}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\varrho-1}(x(s)-y(s))ds-\frac{\lambda_1\lambda_3}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\varrho}(x(s)-y(s))ds\\
&\quad+\frac{1}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v}(f(s,x(s))-f(s,y(s))ds+\frac{1}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v+\omega}(g(s,x(s))-g(s,y(s))ds\\
&\quad+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{\lambda_2}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\varrho-\beta_i-1}(x(s)-y(s))dsdx_i(s)\\
&\quad+\frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\beta_i-\varrho}(x(s)-y(s))dsdx_i(s)\\
&\quad-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\beta_0}(f(s,x(s))-f(s,y(s)))dsdx_i(s)\\
&\quad-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\beta_0+\omega}(g(s,x(s))-g(s,y(s))dsdx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]\\
&\quad+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{-\lambda_2}{\lambda_1}\int_0^1e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho-1}(x(s)-y(s))ds-\frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho}(x(s)-y(s))ds\\
&\quad+\frac{1}{\lambda_1}\int_0^1e^{-\lambda_3}I^{v-\beta_0}(f(s,x(s))-f(s,y(s)))ds\\
&\quad+\frac{1}{\lambda_1}\int_0^1e^{-\lambda_3(1-s)}I^{v-\beta_0+\omega}(g(s,x(s))-g(s,y(s))ds\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]\Big|.
\end{aligned}$$

Now, using the mean value theorem for the Stieltjes integral with $\chi\in[0,1]$ and $x_i(1)=\Re_i>0$ and from (H_1), we obtain the following:

$$\|\delta x-\delta y\|\le\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\varrho)}(1-e^{-\lambda_3})\|x-y\|+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho+1)}(1-e^{-\lambda_3})\|x-y\| \tag{3.2}$$

$$+\frac{L_1(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(\nu+1)}\|x-y\|+\frac{L_2(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(\nu+1+\omega)}\|x-y\|$$
$$+\frac{\|x-y\|}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{|\lambda_2|}{|\lambda_1|}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\varrho-\beta_i-1}\Re_i}{\Gamma(\nu-\beta_i-\varrho)}+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\varrho-\beta_i}\Re_i}{\Gamma(\nu-\beta_i-\varrho+1)}$$
$$+\frac{L_1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\beta_i}\Re_i}{\Gamma(\nu-\beta_i+1)}+\frac{L_2}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\beta_i+\omega}\Re_i}{\Gamma(\nu-\beta_i+1+\omega)}\Big][\lambda_3-1+e^{-\lambda_3}]$$
$$+\frac{\|x-y\|}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(\nu-\beta_0-\varrho)}(1-e^{-\lambda_3})+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(\nu-\varrho-\beta_0+1)}(1-e^{-\lambda_3})$$
$$+\frac{L_1}{|\lambda_1|\lambda_3\Gamma(\nu-\beta_0+1)}(1-e^{-\lambda_3})+\frac{L_2}{|\lambda_1|\lambda_3\Gamma(\nu-\beta_0+\omega+1)}(1-e^{-\lambda_3})\Big][\lambda_3-1+e^{-\lambda_3}].$$
$$\leq\|x-y\|\Big[\Big[\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(\nu-\varrho)}(1-e^{-\lambda_3})+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(\nu-\varrho+1)}(1-e^{-\lambda_3})$$
$$+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{|\lambda_2|}{|\lambda_1|}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\varrho-\beta_i-1}\Re_i}{\Gamma(\nu-\beta_i-\varrho)}+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\varrho-\beta_i}\Re_i}{\Gamma(\nu-\beta_i-\varrho+1)}\Big][\lambda_3-1+e^{-\lambda_3}]$$
$$+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\Big[\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(\nu-\beta_0-\varrho)}(1-e^{-\lambda_3})+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(\nu-\varrho-\beta_0+1)}(1-e^{-\lambda_3})\Big][\lambda_3-1+e^{-\lambda_3}]\Big]$$
$$+L\|x-y\|\Big[\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(\nu+1)}+\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(\nu+1+\omega)}$$
$$+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\beta_i}\Re_i}{\Gamma(\nu-\beta_i+1)}+\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\beta_i+\omega}\Re_i}{\Gamma(\nu-\beta_i+1+\omega)}\Big][\lambda_3-1+e^{-\lambda_3}]$$
$$+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\Big[\frac{1}{|\lambda_1|\lambda_3\Gamma(\nu-\beta_0+1)}(1-e^{-\lambda_3})+\frac{1}{|\lambda_1|\lambda_3\Gamma(\nu-\beta_0+\omega+1)}(1-e^{-\lambda_3})\Big][\lambda_3-1+e^{-\lambda_3}]\Big]$$
$$\leq\|x-y\|[\Omega_1+L\Omega_2].$$

Thus,

$$\|\delta x-\delta y\|\leq[\Omega_1+L\Omega_2]\|x-y\|.$$

From (H_2), we know that $(\Omega_1+L\Omega_2)<1$, which demonstrates that the operator δ is contractive. Therefore, according to the Banach contraction principle, there exists a fixed point, which is the mild solution of the problem (1.2).

Theorem 3.2. *Suppose that (H_1) and the following condition are satisfied.*
(H_3)*: $\exists\,\mu_1,\mu_2\in C([0,1],\mathbb{R}^+)$ such that $|f(\xi,x)|\leq\mu_1(\xi)$ and $|g(\xi,x)|\leq\mu_2$, $\Omega_1<1$, then, $\exists$ at least one mild solution of problem* (1.2) *in C.*

Proof. Suppose $K_\xi=\{x\in C:\|x\|\leq\xi\}$ be a closed subset of B_r and define δ_1 and δ_2 on $K_\xi\to C$ as follows:

$$\delta_1x(\xi)=\frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^\nu f(s,x(s))ds+\frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu+\omega}g(s,x(s))ds+\frac{\nabla}{\lambda_1\lambda_3}(1-e^{-\lambda_3\xi})$$
$$-\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i}f(s,x(s))dsdx_i(s)$$

$$+\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i+\omega}g(s,x(s))dsdx_i(s)\Big]$$
$$\times[\lambda_3\xi-1+e^{-\lambda_3\xi}]+\frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3}-\sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}dx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]$$
$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\beta_0}f(s,x(s))ds$$
$$+\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\beta_0+\omega}g(s,x(s))ds\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}], \tag{3.3}$$

and

$$\delta_2x(\xi)=-\frac{\lambda_2}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-1}x(s)ds-\frac{\lambda_1\lambda_3}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho}x(s)ds$$
$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{\lambda_2}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i-1}x(s)dsdx_i(s)$$
$$+\frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-\beta_i}x(s)dsdx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]$$
$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{-\lambda_2}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\beta_0-\varrho-1}x(s)ds \tag{3.4}$$
$$-\frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\beta_0-\varrho}x(s)ds\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}].$$

Clearly, $\delta=\delta_1+\delta_2$. Now, we verify the hypothesis of Krasnoselskii's fixed point theorem for $x,y\in K_\xi$ with $\xi\geq(\mu\Omega_2+\Omega_3)(1-\Omega)^{-1}$. Presently,

$$\|\delta_1x(\xi)+\delta_2y(\xi)\|=\sup_{\xi\in[0,1]}|\delta_1x(\xi)+\delta_2y(\xi)|$$
$$=\sup_{\xi\in[0,1]}\Big|\frac{1}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu}f(s,x(s))ds+\frac{1}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu+\omega}g(s,x(s))ds+\frac{\nabla}{\lambda_1\lambda_3}(1-e^{-\lambda_3\xi})$$
$$-\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i}f(s,x(s))dsdx_i(s)$$
$$+\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\beta_i+\omega}g(s,x(s))dsdx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]$$
$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\beta_0}f(s,x(s))ds+\frac{1}{\lambda}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\beta_0+\omega}g(s,x(s))ds\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]$$
$$+\frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3}-\sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}dx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]$$
$$-\frac{\lambda_2}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho-1}x(s)ds-\frac{\lambda_1\lambda_3}{\lambda_1}\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{\nu-\varrho}x(s)ds$$

$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{\lambda_2}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\varrho-\beta_i-1}x(s)dsdx_i(s)+\frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi}e^{-\lambda_3(\xi-s)}I^{v-\beta_i-1}x(s)dsdx_i(s)\Big]$$
$$\times[\lambda_3\xi-1+e^{-\lambda_3\xi}]+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{-\lambda_2}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho-1}x(s)ds$$
$$-\frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho}x(s)ds\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]\Big|$$

$$\leq\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(v+1)}\|\mu_1\|+\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(v+1+\omega)}\|\mu_2\|+\frac{|\nabla|}{|\lambda_1\lambda_2|}(1-e^{-\lambda_3})+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{\|\mu_1\|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{v-\beta_i}\Re_i}{\Gamma(v-\beta_i+1)}$$
$$+\frac{\|\mu_2\|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{v-\beta_i+\omega}\Re_i}{\Gamma(v-\beta_i+1+\omega)}\Big][\lambda_3-1+e^{-\lambda_3}]+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{\|\mu_1\|}{|\lambda_1|\lambda_3\Gamma(v-\beta_0+1)}(1-e^{-\lambda_3})$$
$$+\frac{\|\mu_2\|}{|\lambda_1|\lambda_3\Gamma(v-\beta_0+\omega+1)}(1-e^{-\lambda_3})\Big][\lambda_3-1+e^{-\lambda_3}]+\frac{|\nabla|}{|\lambda_1^2\lambda_3^3\Delta|}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3}-\sum_{i=1}^{p}(-\lambda_3)^{\beta_i}e^{-\lambda_3\chi}\Re_i\Big][\lambda_3-1+e^{-\lambda_3}]$$
$$\frac{|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\varrho)}(1-e^{-\lambda_3})\xi+\frac{|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho+1)}(1-e^{-\lambda_3})\xi+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{\xi|\lambda_2|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{v-\varrho-\beta_i-1}\Re_i}{\Gamma(v-\beta_i-\varrho)}$$
$$+\frac{\xi|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{v-\varrho-\beta_i}\Re_i}{\Gamma(v-\beta_i-\varrho+1)}\Big][\lambda_3-1+e^{-\lambda_3}]+\frac{1}{|\Delta\lambda_1\lambda_3^2|}\Big[\frac{\xi|\lambda_2|}{|\lambda_1|\lambda_3\Gamma(v-\beta_0-\varrho)}(1-e^{-\lambda_3})+$$
$$\frac{\xi|\lambda_2\lambda_3|}{|\lambda_1|\lambda_3\Gamma(v-\varrho-\beta_0+1)}(1-e^{-\lambda_3})\Big][\lambda_3-1+e^{-\lambda_3}].$$
$$\leq\mu\Omega_2+\Omega_3+\xi\Omega_1\leq\xi.$$

Thus, $\delta_1 x+\delta_2 y\in K_\xi$.
(*ii*) Next, we show that δ_1 is continuous and compact. Since f and g are continuous thus δ_1 is also continuous. Additionally, δ_1 is uniformly bounded, i.e.,

$$\|\delta_1 x\|\leq\|\mu\|\Omega_2+\Omega_3.$$

For compactness, assume that

$$\sup_{(\xi,x)\in[0,1]\times K_\xi}|f(\xi,x)|=f_1,\qquad \sup_{(\xi,x)\in[0,1]\times K_\xi}|g(\xi,x)|=g_1$$

and

$$[\lambda_3\xi-1+e^{-\lambda_3\xi}]=\max\{[\lambda_3\xi_1-1+e^{-\lambda_3\xi_1}],[\lambda_3\xi_2-1+e^{-\lambda_3\xi_2}]\}.$$

Then, for $\xi_1<\xi_2$, we have the following:

$$|\delta_1 x(\xi_2)-\delta_1 x(\xi_1)|=\Big|\Big[\frac{1}{\lambda_1}\int_0^{\xi_2}e^{-\lambda_3(\xi_2-s)}I^{v}f(s,x(s))ds+\frac{1}{\lambda_1}\int_0^{\xi_2}e^{-\lambda_3(\xi_2-s)}I^{v+\omega}g(s,x(s))ds$$
$$+\frac{\nabla}{\lambda_1\lambda_3}(1-e^{-\lambda_3\xi_2})-\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi_2}e^{-\lambda_3(\xi_2-s)}I^{v-\beta_i}f(s,x(s))dsdx_i(s)$$

$$+\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi_2} e^{-\lambda_3(\xi_2-s)}I^{v-\beta_i+\omega}g(s,x(s))dsdx_i(s)\Big]\times[\lambda_3\xi_2-1+e^{-\lambda_3\xi_2}]$$
$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0}f(s,x(s))ds+\frac{1}{\lambda}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0+\omega}g(s,x(s))ds\Big]$$
$$\times[\lambda_3\xi_2-1+e^{-\lambda_3\xi_2}]+\frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3}-\sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi_2}dx_i(s)\Big][\lambda_3\xi_2-1+e^{-\lambda_3\xi_2}]\Big]$$
$$-\Big[\frac{1}{\lambda_1}\int_0^{\xi_1} e^{-\lambda_3(\xi_1-s)}I^{v}f(s,x(s))ds+\frac{1}{\lambda_1}\int_0^{\xi_1} e^{-\lambda_3(\xi_1-s)}I^{v+\omega}g(s,x(s))ds+\frac{\nabla}{\lambda_1\lambda_3}(1-e^{-\lambda_3\xi_1})$$
$$-\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi_1} e^{-\lambda_3(\xi_1-s)}I^{v-\beta_i}f(s,x(s))dsdx_i(s)$$
$$-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^{\xi_1} e^{-\lambda_3(\xi_1-s)}I^{v-\beta_i+\omega}g(s,x(s))dsdx_i(s)\Big]$$
$$\times[\lambda_3\xi_1-1+e^{-\lambda_3\xi_1}]+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0}f(s,x(s))ds$$
$$+\frac{1}{\lambda}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0+\omega}g(s,x(s))ds\Big]$$
$$\times[\lambda_3\xi_1-1+e^{-\lambda_3\xi_1}]+\frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3}$$
$$-\sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi_1}dx_i(s)\Big][\lambda_3\xi_1-1+e^{-\lambda_3\xi_1}]\Big]\Big|.$$

$$\leq\Big|\frac{f_1}{\lambda_1\Gamma(v+1)}\Big(\int_0^{\xi_1}(\xi_2^{v}e^{-\lambda_3(\xi_2-s)}-\xi_1^{v}e^{-\lambda_3(\xi_1-s)})ds+\int_{\xi_1}^{\xi_2}\xi_2^{v}e^{-\lambda_3(\xi_2-s)}ds\Big)$$
$$+\frac{g_1}{\lambda_1\Gamma(v+1+\omega)}\Big(\int_0^{\xi_1}(\xi_2^{v+\omega}e^{-\lambda_3(\xi_2-s)}-\xi_1^{v+\omega}e^{-\lambda_3(\xi_1-s)})ds+\int_{\xi_1}^{\xi_2}\xi_2^{v+\omega}e^{-\lambda_3(\xi_2-s)}ds\Big)$$
$$-\frac{f_1}{\Delta\lambda_1\lambda_3^2}\Big[\frac{1}{\lambda_1}\sum_{i=1}^{p}\frac{1}{\Gamma(v-\beta_i+1)}\int_0^1\Big(\int_0^{\xi_1}(\xi_2^{v-\beta_i}e^{-\lambda_3(\xi_2-s)}$$
$$-\xi_1^{v-\beta_i}e^{-\lambda_3(\xi_1-s)})ds+\int_{\xi_1}^{\xi_2}\xi_2^{v-\beta_i}e^{-\lambda_3(\xi_2-s)}ds\Big)dx_i(s)\Big]$$
$$\times[\lambda_3\xi-1+e^{-\lambda_3\xi}]-\frac{g_1}{\Delta\lambda_1\lambda_3^2}\Big[\frac{1}{\lambda_1}\sum_{i=1}^{p}\frac{1}{\Gamma(v-\beta_i+1+\omega)}\int_0^1$$
$$\Big(\int_0^{\xi_1}(\xi_2^{v-\beta_i+\omega}e^{-\lambda_3(\xi_2-s)}-\xi_1^{v-\beta_i+\omega}e^{-\lambda_3(\xi_1-s)})ds$$
$$+\int_{\xi_1}^{\xi_2}\xi_2^{v-\beta_i+\omega}e^{-\lambda_3(\xi_2-s)}ds\Big)dx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]+\frac{\nabla}{\lambda_1\lambda_3}(e^{-\lambda_3\xi_1}-e^{-\lambda_3\xi_2})$$

$$+\frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[\sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi_1}dx_i(s)-\sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi_2}dx_i(s)\Big][\lambda_3\xi-1+e^{-\lambda_3\xi}]\Big]\Big\|.$$

$$\leq \frac{|f_1|}{|\lambda_1 \Gamma(v+1)|\lambda_3} \Big(\xi_2^v (e^{-\lambda_3(\xi_2-\xi_1)} - e^{-\lambda_3 \xi_2}) - \xi_1^v (1 - e^{-\lambda_3 \xi_1}) + \xi_2^v (1 - e^{-\lambda_3(\xi_2-\xi_1)}) \Big)$$
$$\frac{|g_1|}{|\lambda_1 \Gamma(v+1+\omega)|\lambda_3} \Big(\xi_2^{v+\omega} (e^{-\lambda_3(\xi_2-\xi_1)} - e^{-\lambda_3(\xi_2)}) - \xi_1^{v+\omega} (1 - e^{-\lambda_3 \xi_1}) + \xi_2^{v+\omega} (1 - e^{-\lambda_3(\xi_2-\xi_1)}) \Big)$$
$$\frac{|f_1|}{|\Delta \lambda_1 \lambda_3^2|} \Big[\frac{1}{|\lambda_1|\lambda_3} \sum_{i=1}^{p} \frac{1}{\Gamma(v-\beta_i+1)} \Big((\chi_2^{v-\beta_i} (e^{-\lambda_3(\chi_2-\chi_1)}$$
$$- e^{-\lambda_3 \chi_2}) - \chi_1^{v-\beta_i} (1 - e^{-\lambda_3 \chi_1}) + \chi_2^{v-\beta_i} (1 - e^{-\lambda_3(\chi_2-\chi_1)}) \Big) \mathfrak{R}_i \Big]$$
$$\times [\lambda_3 \xi - 1 + e^{-\lambda_3 \xi}] + \frac{|g_1|}{|\Delta \lambda_1 \lambda_3^2|} \Big[\frac{1}{|\lambda_1|\lambda_3} \sum_{i=1}^{p} \frac{1}{\Gamma(v-\beta_i+1+\omega)}$$
$$\Big((\chi_2^{v-\beta_i+\omega} (e^{-\lambda_3(\chi_2-\chi_1)} - e^{-\lambda_3 \chi_2}) - \chi_1^{v-\beta_i+\omega} (1 - e^{-\lambda_3 \chi_1})$$
$$+ \chi_2^{v-\beta_i+\omega} (1 - e^{-\lambda_3(\chi_2-\chi_1)}) \Big) \mathfrak{R}_i \Big] [\lambda_3 \xi - 1 + e^{-\lambda_3 \xi}] +$$
$$\frac{|\nabla|}{|\lambda_1^2 \lambda_3^3 \Delta|\lambda_3} \Big[\sum_{i=1}^{p} |(-\lambda_3)^{\beta_i} \Big(e^{-\lambda_3 \chi_1} - e^{-\lambda_3 \chi_2} \Big)| \mathfrak{R}_i \Big] [\lambda_3 \xi - 1 + e^{-\lambda_3 \xi}].$$
$$+ \frac{|\nabla|}{|\lambda_1 \lambda_3|} (e^{-\lambda_3 \xi_1} - e^{-\lambda_3 \xi_2}),$$

where $\chi_1, \ \chi_2 \in [0,1]$. If $\xi_1 \to \xi_2$, then $\chi_1 \to \chi_2$ and the right side approaches to 0 independent of x $\in K_\xi$. Thus, δ_1 is equicontinuous. Therefore, by the Arzela-Ascoli theorem, δ_1 is relatively compact.

(iii) As $\Omega_1 < 1$, therefore, δ_2 is contraction. Thus, there is at least one mild solution to the problem in C.

4. Stability analysis

Let $\epsilon > 0$, $f, \ g \in ([0,1] \times \mathcal{R}, \mathcal{R})$, and assume the following inequality:

$$|(\lambda_1 \mathscr{D}^v + \lambda_2 \mathscr{D}^\varrho)(\mathscr{D} + \lambda_3) u(\xi) - f(\xi, u(\xi)) - I^\omega g(\xi, u(\xi))| \leq \epsilon \quad \forall \quad \xi \in [0,1]. \tag{4.1}$$

Definition 4.1. *(1.2) is said to be Hyers-Ulam stable if there is a constant $k_0 \in \mathbb{R}^+$ such that for each $\epsilon > 0$ and each red mild solution $u \in C$ of (4.1), there is a mild solution $x \in C$ of (1.2) with,*
$|u(\xi) - x(\xi)| \leq k_0 \epsilon \quad \forall \quad \xi \in [0,1].$

Remark 4.1. *A function $u \in C$ is a mild solution of (4.1) if there is $\psi \in C$ depending on u, such that:*

(i) $(\lambda_1 \mathscr{D}^v + \lambda_2 \mathscr{D}^\varrho)(\mathscr{D} + \lambda_3) u(\xi) = f(\xi, u(\xi)) + I^\omega g(\xi, u(\xi)) + \psi(\xi) \quad \forall \ \xi \in [0,1].$

(ii) $|\psi(\xi)| \leq \epsilon \quad \forall \ \xi \in [0,1].$

Lemma 4.1. *If $u \in C$ is a mild solution of the Inequality (4.1), then u is also a mild solution of the following inequality:*

$$|u(\xi) - x(\xi)| \leq k_0 \epsilon.$$

where,

$$k_0 = \Big[\frac{1}{|\lambda_1|\lambda_3\Gamma(v+1)}(1-e^{-\lambda_3\xi}) + \frac{1}{|\lambda_1\lambda_3^2\Delta|}\Big[\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{v-\beta_i}\mathfrak{R}_i}{\Gamma(v-\beta_i+1)}\Big][\lambda_3 - 1 + e^{-\lambda_3}]$$
$$+ \frac{1}{|\lambda_1\lambda_3^2\Delta|}\Big(\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(v-\beta_0+1)}\Big)[\lambda_3 - 1 + e^{-\lambda_3}]\Big].$$

Proof. If u is the mild solution of Inequality (4.1), then u will also be the mild solution of the following:

$$(\lambda_1\mathscr{D}^v + \lambda_2\mathscr{D}^\varrho)(\mathscr{D}+\lambda_3)u(\xi) = f(\xi,u(\xi)) + I^\omega g(t,u(\xi)) + \psi(\xi), \quad t\in[0,1]$$
$$x(0)=0, \quad x'(0)=0, \quad {}^C\mathscr{D}^{\beta_0}_{0,\xi}u(1) = \sum_{i=1}^{p}\int_0^1 \mathscr{D}^{\beta_i}u(\xi)du_i(s). \tag{4.2}$$

The mild solution of (4.2) is given by the following:

$$u(\xi) = -\frac{\lambda_2}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho-1}u(s)ds - \frac{\lambda_1\lambda_3}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho}u(s)ds \tag{4.3}$$
$$+\frac{1}{\lambda_1}\int_0^\xi I^v e^{-\lambda_3(\xi-s)}f(s,u(s))ds + \frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v+\omega}g(s,u(s))ds$$
$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{\lambda_2}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\varrho-\beta_i-1}u(s)dsdu_i(s)$$
$$+\frac{\lambda_2\lambda_3}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\beta_i-1}u(s)dsdu_i(s) - \frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\beta_i}f(s,u(s))dsdu_i(s)$$
$$-\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\beta_i+\omega}g(s,u(s))dsdu_i(s)\Big][\lambda_3 - 1 + e^{-\lambda_3\xi}] + \frac{\nabla}{\lambda_1\lambda_3}(1-e^{-\lambda_3\xi})$$
$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{-\lambda_2}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho-1}u(s)ds - \frac{\lambda_2\lambda_3}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0-\varrho}u(s)ds$$
$$+\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0}f(s,u(s))ds + \frac{1}{\lambda}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0+\omega}g(s,u(s))ds\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}]$$
$$+\frac{\nabla}{\lambda_1^2\lambda_3^3\Delta}\Big[(-\lambda_3)^{\beta_0}e^{-\lambda_3} - \sum_{i=1}^{p}\int_0^1(-\lambda_3)^{\beta_i}e^{-\lambda_3\xi}du_i(\xi)\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}]$$
$$+\frac{1}{\lambda_1}\int_0^\xi e^{-\lambda_3(\xi-s)}I^v\psi(s))ds - \frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\beta_i}\psi(s)dsdu_i(s)\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}]$$
$$+\frac{1}{\lambda_1\lambda_3^2\Delta}\Big[\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{v-\beta_0}\psi(s)ds\Big][\lambda_3\xi - 1 + e^{-\lambda_3\xi}].$$

Let the terms free of ψ be denoted by $x(\xi)$; then,

$$|u(\xi) - x(\xi)| \le \frac{1}{|\lambda_1|}\int_0^\xi e^{-\lambda_3(\xi-s)}I^v|\psi(s))|ds + \frac{1}{|\lambda_1\lambda_3^2\Delta|}\Big[\frac{1}{\lambda_1}\sum_{i=1}^{p}\int_0^1\int_0^\xi e^{-\lambda_3(\xi-s)}I^{v-\beta_i}|\psi(s)|dsdu_i(s)\Big][\lambda_3\xi - 1e^{-\lambda_3\xi}]$$

$$+\frac{1}{|\lambda_1\lambda_3^2\Delta|}\Big[\frac{1}{\lambda_1}\int_0^1 e^{-\lambda_3(1-s)}I^{\nu-\beta_0}|\psi(s)|ds\Big][\lambda_3\xi-1e^{-\lambda_3\xi}].$$

$$\leq\epsilon\Big[\frac{1}{|\lambda_1|\lambda_3\Gamma(\nu+1)}(1-e^{-\lambda_3})+\frac{1}{|\lambda_1\lambda_3^2\Delta|}\Big[\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\beta_i}\Re_i}{\Gamma(\nu-\beta_i+1)}\Big][\lambda_3-1+e^{-\lambda_3}]$$
$$+\frac{1}{|\lambda_1\lambda_3^2\Delta|}\Big(\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(\nu-\beta_0+1)}\Big)[\lambda_3-1+e^{-\lambda_3}]\Big].$$

Assume that,

$$k_0=\Big[\frac{1}{|\lambda_1|\lambda_3\Gamma(\nu+1)}(1-e^{-\lambda_3})+\frac{1}{|\lambda_1\lambda_3^2\Delta|}\Big[\frac{1}{|\lambda_1|\lambda_3}\sum_{i=1}^{p}\frac{(1-e^{-\lambda_3\chi})\chi^{\nu-\beta_i}\Re_i}{\Gamma(\nu-\beta_i+1)}\Big][\lambda_3-1+e^{-\lambda_3}]$$
$$+\frac{1}{|\lambda_1\lambda_3^2\Delta|}\Big(\frac{(1-e^{-\lambda_3})}{|\lambda_1|\lambda_3\Gamma(\nu-\beta_0+1)}\Big)[\lambda_3-1+e^{-\lambda_3}]\Big].$$

In view of (ii) of Remark 3.2, we have the following:

$$|u(\xi)-x(\xi)|\leq k_0\epsilon,$$

which is the desired result.

5. Example

In this section, we solve some examples using the obtained theorems.

Example 5.1.

$$\begin{cases}(6\,{}^{C}\mathscr{D}^{1.35}+4\,{}^{C}\mathscr{D}^{1.1})(\mathscr{D}+1)=\frac{e^{-2\xi}|x(\xi)|}{(t+5)^2(1+|x(\xi)|)}+I^2\frac{sin\xi|x(\xi)|}{\sqrt{(}49+t)}, \quad \forall \quad \xi\in[0.1],\\ x(0)=0 \quad x'(0)=0 \quad {}^{C}\mathscr{D}^{\beta_0}_{0,\xi}x(1)=\sum_{i=1}^{2}\int_0^1\mathscr{D}^{\beta_i}x(s)dx_i(s),\end{cases}\tag{5.1}$$

where,

$$p=2,\nu=1.35,\xi=1.1,\omega=2,\lambda_1=6,\lambda_2=4,\lambda_3=1,\chi=0.1,$$
$$\beta_0=1.12,\beta_1=1.2,\beta_2=1.32,\Re_1=10,\Re_2=15.$$

Since,

$$f(\xi,x(\xi))=\frac{e^{-2\xi}|x(\xi)|}{(t+5)^2(1+|x(\xi)|)}\quad and\quad g(\xi,x(\xi))=\frac{sin\xi|x(\xi)|}{\sqrt{(}49+t)}.$$

From assumptions (H_1) and (H_2), we have the following:

$$|f(\xi,x(\xi))-f(\xi,y(\xi))|\leq\frac{e^{-2\xi}|x-y|}{(t+5)^2}\leq\frac{1}{25}|x-y|,$$

$$|g(\xi, x(\xi)) - g(\xi, y(\xi))| \leq \frac{sin\xi|x-y|}{\sqrt{(49+t)}} \leq \frac{1}{7}|x-y|,$$

where, $L_1 = \frac{1}{25}$, $L_2 = \frac{1}{7}$ *and* $L = \max\{L_1, L_2\} = \frac{1}{7}$. *Calculating* Ω_1 *and* Ω_2 *from the given data, we obtain* $\Omega_1 = 0.93461$ *and* $\Omega_2 = 0.136698$. *Additionally,* $\Omega_1 + L\Omega_2 \approx 0.95414 < 1$. *Hence, the given problem (5.1) has a unique mild solution in* C *for* $\xi \in [0, 1]$.

Example 5.2.

$$\begin{cases} (5^C\mathscr{D}^{1.75} -^C \mathscr{D}^{1.6})(\mathscr{D}+2)x(\xi) = \frac{\xi^2}{\xi^3+17}x(\xi) + I^2\frac{cos\xi}{e^t+15}x(\xi), & \forall\, \xi \in [0,1], \\ x(0) = 0 \;\; x'(0) = 0, \quad {}^C\mathscr{D}^{\beta_0}_{0,\xi}x(1) = \sum_{i=1}^{2}\int_0^1 \mathscr{D}^{\beta_i}x(s)dx_i(s), \end{cases} \tag{5.2}$$

where,

$$\begin{aligned} &\lambda_1 = 5, \lambda_2 = -1, \lambda_3 = 2, \chi = 0.5, p = 2, \mathfrak{R}_1 = 30, \mathfrak{R}_2 = 15, \\ &v = 1.75, \xi = 1.6, \beta_0 = 1.24, \beta_1 = 1.28, \beta_2 = 1.32. \end{aligned}$$

Therefore,

$$f(\xi, x(\xi)) = \frac{\xi^2}{\xi^3+17}x(\xi) \quad and \quad g(\xi, x(\xi)) = \frac{cos\xi}{e^t+15}x(\xi).$$

From assumption (H_1) *and* (H_2), *we have*

$$\begin{aligned} |f(\xi, x(\xi)) - f(\xi, y(\xi))| &\leq \frac{\xi^2}{\xi^3+17}|x-y| \leq \frac{1}{17}|x-y|, \\ |g(\xi, x(\xi)) - g(\xi, y(\xi))| &\leq \frac{cos\xi}{e^t+15}|x-y| \leq \frac{1}{16}|x-y|, \end{aligned}$$

where, $L_1 = \frac{1}{17}$, $L_2 = \frac{1}{16}$ *and* $L = max\{L_1, L_2\} = \frac{1}{16}$. *Calculating* Ω_1 *and* Ω_2 *from the given data, we obtain* $\Omega_1 = 0.2188677$, $\Omega_2 = 0.324496$. *Additionally,* $\Omega_1 + L\Omega_2 \approx 0.239415 < 1$. *Hence, the given problem (5.2) has a unique mild solution in* C *for* $t \in [0, 1]$.

6. Conclusions

The paper established sufficient conditions that showed the existence, uniqueness and Ulam's stability for the mild solutions of Problem (1.2). The conditions were obtained from the view of fixed point theorems. Furthermore, we demonstrated the obtained results using two examples. The multi-point boundary conditions can be used to describe the characteristics of chemical, physical or others processes occouring inside the domain. Thus, the obtained results can be fruitful in the mentioned processes. The Hyers-Ulam stability means that for any approximation in a specific region we will obtain an exact mild solution. Therefore, the obtained results of the Hyers-Ulam stability can be utilized in a numerical analysis and approximation theory of the related mentioned processes.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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