AIMS Mathematics

AIMS Mathematics, 8(12): 28376–28397.
DOI: 10.3934/math.20231452
Received: 24 July 2023
Revised: 27 September 2023
Accepted: 08 October 2023
Published: 18 October 2023

http://www.aimspress.com/journal/Math

Research article

Generalized triangular Pythagorean fuzzy weighted Bonferroni operators and their application in multi-attribute decision-making

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Abstract: The consolidation of evaluations from various decision-makers within a group, concerning multiple attributes of limited schemes, seeks to unify or compromise collective preferences according to specific rules. The superior characteristics of Possibility Fuzzy Sets (PFS) in membership endow it with enhanced capabilities in depicting ambiguous information. The Bonferroni operator proficiently mitigates the influences of interrelations between attributes in decision-making dilemmas. To address the Multi-Attribute Decision Making (MADM) conundrum wherein attribute values are associative Triangular Pythagorean Fuzzy Numbers (TPFNs), a novel methodology leveraging the Generalized Triangular Pythagorean Fuzzy Weighted Bonferroni Mean (GTPFWBM) operator and the Generalized Triangular Pythagorean Fuzzy Weighted Bonferroni Geometric Mean (GTPFWBGM) operator is advanced. Initiating with the foundational Triangular Pythagorean Fuzzy Set and the Generalized Bonferroni Mean (GBM) operator, both the GTPFWBM and GTPFWBGM operators are delineated. Subsequent exploration dives into the intrinsic properties of these pioneering operators, encompassing facets like reducibility, permutation invariance, idempotency, monotonicity and boundedness. Building upon this foundation, a MADM methodology predicated on the GTPFWBM and GTPFWBGM operators is conceptualized. The culmination of this research underscores the method's rationality and practicality, illustrated through a venture capital investment exemplar.

Keywords: triangular Pythagorean fuzzy set; generalized triangular Pythagorean fuzzy weighted Bonferroni mean operator; generalized triangular Pythagorean fuzzy weighted Bonferroni geometric mean operator; multi-attribute decision making
Mathematics Subject Classification: 47N10

Abbreviations: The following abbreviations are used in this manuscript:
MADM: multi-attribute decision making; PFS: Pythagorean fuzzy numbers; PFNs: Pythagorean fuzzy numbers; GBM: Bonferroni geometric mean

1. Introduction

In light of burgeoning societal informatization, the intricacies embedded in contemporary decision-making processes have intensified, amplifying the ambiguity surrounding individuals' comprehension of such dilemmas and thereby complicating the elicitation of precise evaluative metrics [1]. Consequently, to counteract the detrimental repercussions of informational uncertainty on decision outcomes, Zadeh [2] introduced the fuzzy set theory, elegantly encapsulating decisional data via the membership degrees of constituent elements within the set. Fuzzy multi-attribute decision-making, which converges evaluations of multiple attributes pertaining to finite schemes from an array of decision-makers based on specific criterion into a unified or compromised collective preference, has been extensively harnessed across a gamut of arenas, including economic benefit assessments [3], competitive evaluations [4], environmental quality appraisals [5] and project investments [6,7].

In the wake of foundational work in fuzzy set theory, Atanassov [8] elegantly posited the Intuitive Fuzzy Set (IFS) paradigm, thereby offering a sophisticated augmentation to the said theory. The IFS modality is adept at articulating both membership and non-membership gradations, with their cumulative value not exceeding unity. When juxtaposed against conventional fuzzy sets, IFS emerges as a more apt instrument for characterizing the intricacies and uncertainties inherent in real-world quandaries. Jana et al. [9] delved into the MADM paradigm where decision architects employed both two-scale intuitionistic fuzzy numerals and interval-valued intuitionistic fuzzy numerals to confer selection directives across varied temporal junctures, subsequently unveiling four dynamic weighted aggregation conduits to optimize information assimilation efficacy. Intriguingly, within tangible decision-making realms, the amalgamation of membership and non-membership degrees often surpasses unity, constricting the applicability expanse of the IFS framework. To illustrate, a decision strategist might attribute a membership gradation of 0.8 and a non-membership value of 0.4 whilst appraising a scheme's attributes. In response to this lacuna, Yager [10,11] conceptualized the Pythagorean Fuzzy Set (PFS). The PFS modality is tailored for scenarios wherein the summation of membership and non-membership values can eclipse unity, yet their squared aggregation remains firmly bounded by the same threshold. Concurrently, a coterie of scholars has embarked on rigorous inquiries into fuzzy multi-attribute dilemmas from diverse fuzzy data set perspectives, encompassing 2-tuple linguistic q-rung fuzzy [12], Hesitant Triangular Fuzzy [13], T-spherical fuzzy [14] and Pythagorean Fuzzy [15].

Scholarly pursuits into Pythagorean Fuzzy Set (PFS) have significantly augmented the theoretical and methodological dimensions of Multi-Attribute Decision Making (MADM) in nebulous environments. It warrants mention that the preponderance of extant methodologies postulates that memberships within the PFS realm can be ascertained with precise values. Yet, in manifold real-world scenarios, given the intricate nature of attributes coupled with the cognitive constraints of decision architects, a precise membership degree often proves elusive in mirroring genuine decision-making conundrums. In a bid to obviate the deleterious ramifications of such indeterminacy upon MADM outcomes, academic luminaries have channeled their energies into the exploration of information aggregation operators [16]. Owing to the pronounced membership degree of triangular fuzzy numbers at their median juxtaposed against a diminished membership at their boundaries, these boundary values seldom distort the quantitative encapsulation of data. Consequently, a consortium of researchers has melded the PFS doctrine with triangular fuzzy numbers to proffer a more nuanced portrayal of information's inherent vagueness. For instance, Fan et al. [17] synthesized triangular fuzzy numerals with PFS, unveiling the paradigm of Triangular Pythagorean Fuzzy Numbers (TPFNs). Their inquiry spanned the realms of Triangular Pythagorean Fuzzy Weighted Average (TPFWA) operator, Generalized Triangular Pythagorean Fuzzy Weighted Average (GTPFWA) operator, Triangular Pythagorean Fuzzy Weighted Geometry (TPFWG) operator and Generalized Triangular Pythagorean Fuzzy Weighted Geometry (GTPFWG) operator. Presently, aggregation conduits for TPFNs find their relevance solely in scenarios where attributes remain mutually exclusive. However, in tangible decision-making realms, attributes frequently exhibit interdependence, manifesting traits such as complementarity, redundancy and preference hierarchies. Regrettably, antecedent research has not adeptly addressed the intricate web of attribute interdependencies, occasionally culminating in decisional distortions in specific contexts.

Consequently, delving into the interrelations inherent in decision-making information becomes paramount in pragmatic contexts. The Bonferroni Mean (BM) operator [18] adeptly amalgamates multiple input variables into a singular cohesive entity, striking a balance between extremities—namely the apex and nadir. Expanding on the foundational tenets of the BM operator, the Generalized Bonferroni Mean (GBM) operator incorporates a tri-parametric perspective, offering a more holistic representation of inter-variable dynamics. Given the intrinsic merits of the GBM operator, it stands as a potent tool to navigate the challenges posed by inter-attribute correlations within the Triangular Pythagorean fuzzy milieu. Viewed through this lens, the BM operator adeptly addresses the conundrums of attribute interrelations within fuzzy multi-attribute decision-making paradigms. Consequently, this manuscript introduces both the GTPFWBM and the GTPFWBGM operators. The advent of these operators serves to enrich the multi-attribute decision-making framework within the Intuitionistic Fuzzy Set (IFS) context.

The seminal contributions of this manuscript can be delineated into two salient dimensions. First, an evident lacuna remains in the academic realm regarding the nexus between the MADM approach, the GBM operator and TPFNs. Second, a preponderant segment of contemporary literature on information amalgamation operators predicates on the notion that decision attributes are discretely autonomous, neglecting the intricate interrelations that weave them together. The advanced GTPFWBM and GTPFWBGM operators proffered in this treatise adeptly heighten the fidelity of information consolidation in real-world decision-making contexts characterized by intertwined attributes. This furnishes not only a groundbreaking trajectory for navigating the MADM quandary but also bolsters the theoretical edifice of aggregation paradigms within Pythagorean fuzzy numbers (PFNs).

The structure of this paper unfolds as follows. Section 2 undertakes a scholarly exposition on PFS and BM operators, subsequently elucidating the lacunae in extant literature that this manuscript seeks to address. Section 3 commences with a cursory overview of quintessential notions, followed by an in-depth dissection of the conceptual underpinnings and properties inherent to the advanced operators; culminating with a detailed elucidation of the MADM methodology predicated upon these operators. Section 4, through meticulous sensitivity and comparative analyses—epitomized by a venture capital firm selection paradigm—, fortifies the robustness and efficacy of the proffered operators. Section 5 extrapolates managerial sagacity and real-world ramifications of this discourse, both from an academic and pragmatic lens. Conclusively, Section 6 encapsulates the core tenets of this treatise and proffers potential avenues for future scholarly exploration.

2. Literature review

2.1. Review on PFS

Owing to the distinguished attributes of PFS in representing membership, it possesses a formidable aptitude to delineate fuzzy information, adeptly circumventing the attrition of attribute details [19]. Consequently, the refinement of PFS and its application to address the MADM quandaries have emerged as focal points of contemporary research. Within the domain of PFS enhancement, Pan et al. [20] formulated a circumscribed PFS, employing ordered dyads to characterize both fuzziness and stochasticity within an ambivalent milieu, thus eschewing paradoxical outcomes. Meanwhile, Liang et al. [21] pioneered a Bayesian decision-centric Pythagorean fuzzy decision theory rough set paradigm tailored for the archetypal scenarios of information systems bereft of class labels. This model elucidated the selection modus operandi for individual entities, accompanied by pertinent semantic expositions. In a similar vein, Wan et al. [22], anchoring on PF-positive ideal solution (PFPIS) and PF-negative ideal solution (PFNIS), and with an aim to concurrently diminish dual inconsistency indicators, architected a bi-objective Pythagorean fuzzy (PF) mathematical programmatic framework to derive holistic attribute weights. This model, accentuating inconsistency indicators rooted in both the positive ideal solution (PIS) and negative ideal solution (NIS), adeptly redresses the lacunae inherent in the linear programming technique for multidimensional analysis of preference (LINMAP)—a seminal MADM methodology that had hitherto overlooked the NIS in its deliberative schema.

Within the practical domain of PFS, Deb et al. [23] harnessed the Pythagorean fuzzy analytical hierarchical process to calibrate the severity weight pertaining to software defined networks (SDN). This was orchestrated with the objective of discerning associated perils, thereby facilitating preemptive strategic interventions prior to the SDN's deployment. Concurrently, Jana et al. [24] integrated the Dombi operation, thereby culminating in the inception of six Pythagorean fuzzy Dombi aggregation operators, inclusive of the Pythagorean fuzzy Dombi weighted average operator. These operators were subsequently employed to navigate the intricacies of multi-attribute decision-making within a Pythagorean fuzzy milieu. In a similar context, Wan et al. [25] architectured a triphasic strategy for multi-attribute group decision-making (MAGDM) under the auspices of Pythagorean fuzzy numbers (PFNs), a methodology they elegantly applied to the nuances of haze management. Their seminal contribution lay in delineating the normalized projection of PFN and formulating an augmented TOPSIS methodology, premised upon this normalized projection. This paradigm was adept at ascertaining the weights of decision-makers, thereby judiciously obviating the subjective capriciousness endemic to the decision-making process.

2.2. Review on BM operators

The Bonferroni mean (BM) operator [18], esteemed for its adeptness at elucidating the interrelation amongst input variables, has garnered substantial scholarly interest. Yager [26] elegantly expanded the BM operator into a nuanced fuzzy information aggregation mechanism, seamlessly integrating it within the MADM domain to aptly represent the symbiotic interplay amidst evaluative information. Subsequent to this pioneering effort, the BM operator has been ubiquitously incorporated into fuzzy environmental MADM research paradigms. In this context, Nie et al. [27] ingeniously amalgamated the Shapley fuzzy measure and the BM operator in scenarios of indeterminate weights, leading to the conception of a Pythagorean fuzzy partitioned normalized weighted Bonferroni mean operator; a mechanism specifically crafted to navigate the intricacies of the Pythagorean fuzzy MADM conundrum. Chiao [28] masterminded aggregation schemas tailored to disentangle multi-criteria decision-making (MCDM) challenges inherent to ambiguous milieus, specifically harnessing variants of the BM operator such as those juxtaposed with ordered weighted averaging metrics and those combined with OWA weights underscored by individual significance. Taking a slightly divergent trajectory, Fatma et al. [29] ventured to synergize graph fuzzy numbers with the BM operator, embarking on the exploration of an avant-garde graph fuzzy information set operator to untangle the MCDM enigma. Progressing this discourse, Wan et al. [30] sculpted three Bonferroni harmonic mean operators, meticulously crafted to assimilate the attribute value information of MAGDM, underpinned by triangular intuitionistic fuzzy numbers (TIFNs). This architecture adeptly encapsulated the holistic inclinations of decision-makers, especially within the confines of imperative stipulations.

While the BM operator adeptly elucidates the interrelation between paired evaluations, its capacity remains curtailed when confronted with the multifaceted nuances of practical MADM scenarios. Addressing this lacuna, Beloakov et al. [31] introduced the generalized Bonferroni mean (GBM) operator, crafting a more expansive and intricate tapestry of correlations amidst evaluative data. Building upon this foundational work, Xia et al. [32] postulated the generalized weighted Bonferroni mean operator and its geometric counterpart, astutely calibrating them to articulate the varying gravitas of disparate attributes; these innovations found applicability in deciphering MADM conundrums within the intuitionistic fuzzy milieu. Venturing further into this domain, Liu et al. [33] unveiled the dual generalized Bonferroni mean operator, a tool designed to augment the veracity of evaluative data. This was achieved by deftly modulating the embedding parameters, thereby capturing the intricate interplay among varied quantitative attributes. In a synergistic meld, Wang et al. [34] amalgamated the GBM operator with the 2-tuple linguistic neutrosophic numbers, giving rise to the dual generalized 2-tuple linguistic neutrosophic numbers weighted Bonferroni mean operator, accompanied by its cognate multi-objective optimization algorithm.

2.3. Literature summary

To more aptly align with our research trajectory, which centers upon the generalized triangular Pythagorean fuzzy weighted Bonferroni operators, our literature scrutiny bifurcates into two predominant vectors: PFS and BM operators. Pertinent insights harvested from the extant literature are succinctly encapsulated in Table 1.

Table 1. The main information from the relevant literature.

Category	Reference	Main research contents	Main research contributions
PFS	Pan et al. [20]	A constrained PFS	Avoiding counterintuitive results
	Liang et al. [21]	A Bayesian decision-based Pythagorean fuzzy decision theory rough set model	Solving the problem of information systems without class labels
	Wan et al. [22]	A dual-objective PF mathematical programming model	Making up for the shortcomings of LINMAP ignoring NIS in the decision-making process
	Deb et al. [23]	The Pythagorean fuzzy analytical hierarchical process	Identifying the related risks of network and taking appropriate countermeasures
	Jana et al. [24]	Six Pythagorean fuzzy Dombi aggregation operators	Solving the multi-attribute decision making problem in the Pythagorean fuzzy environment
	Wan et al. [25]	A three-phase method for MAGDM with PFNs	Defining the normalized projection of PFN

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Category	Reference	Main research contents	Main research contributions
BM operators	Yager [26]	Generalizing the BM operator as a fuzzy information aggregation operator	Reflecting the mutual influence between evaluation information
	Nie et al. [27]	A Pythagorean fuzzy partitioned normalized weighted Bonferroni mean operator	Solving the Pythagorean fuzzy MADM problem
	Chiao [28]	The BM operator with ordered weighted averaging weights, et al.	Solving MCDM problems in uncertain environments
	Fatma et al. [29]	A new type of graph fuzzy information set operator	Combining graph fuzzy numbers with the BM operator to solve the MCDM problem.
	Wan et al. [30]	Three Bonferroni harmonic mean operators	Modeling the overall preferences of decision makers under mandatory requirements.
	Beloakov et al. [31]	The GBM operator	Describing more correlations between evaluation information
	Xia et al. [32]	The generalized weighted Bonferroni mean operator and the generalized weighted Bonferroni geometric mean operator	Solving the MADM problem in the intuitionistic fuzzy environment
	Liu et al. [33]	The dual generalized Bonferroni mean operator	Enhancing the reliability of the evaluation information
	Wang et al. [34]	A dual generalized 2-tuple linguistic neutrosophic numbers weighted Bonferroni mean operator	Combining the GBM operator and the 2-tuple linguistic neutrosophic numbers

3. Methods

3.1. Question formulation

From the scrutiny of literature pertaining to PFS, it emerges that PFS holds a distinctive edge in navigating intricate attribute values within pragmatic decision-making contexts. Given that the membership degree of PFS is an absolute metric, it often falters in delineating the inherent uncertainty of decisional data. This shortcoming of PFS can be ameliorated by triangular PFS. Presently, research on triangular PFS remains constrained to scenarios where attributes operate in isolation. However, in real-world decision-making matrices, attributes frequently exhibit complementary and redundant interplay. Delving into the literature on BM operators, one discerns that these operators adeptly capture inter-variable relationships, thus rectifying this limitation inherent to triangular PFS. Moreover, juxtaposing the BM operator, which merely captures dyadic evaluative correlations, the GBM operator incorporates a three-parameter input perspective, bestowing it with the capability to holistically represent variable interconnections.

From the aforementioned literature review, the significance of the nexus between information ambiguity and attribute values in the domain of multi-attribute decision-making is patently evident. Concurrently, extant studies offer no insights into the amalgamation of the GBM operator with the triangular PFS operator. Hence, this investigation paves a novel avenue to address the MADM conundrum and augments the theoretical landscape of triangular PFS integration methodologies.

3.1.1. TPFNs definition

Definition 1. [20] Let X be a non-empty set, and any PFS expression in X is as $P = \{\langle x, \mu_p(x), \nu_p(x) | x \in X \rangle\}$. The functions $\mu_P(x)$ and $\nu_P(x)$ are the membership degree and non-membership degree of the element $x \in X$ in the set P, respectively, satisfying the constraint $0 \le (\mu_P(x))^2 + (\nu_P(x))^2 \le 1, \mu_P(x) \in [0,1], \nu_P(x) \in [0,1]$. $\pi_P(x) = \sqrt{1-(\mu_P(x))^2-(\nu_P(x))^2}$ represents the hesitancy degree that the element x belongs to X. The smaller the value of $\pi_P(x)$, indicates that there is more useful information about x and vice versa.

Definition 2. [17] Let X be a non-empty set, and any triangular Pythagorean fuzzy set (TPFS) expression in X is as $\tilde{P} = \{\langle x, \tilde{\mu}_p(x), \tilde{\nu}_p(x)\rangle | x \in X\}$. The functions $\tilde{\mu}_P(x) \subset [0,1]$ and $\tilde{\nu}_P(x) \subset [0,1]$ are two triangular fuzzy numbers $\tilde{\mu}_P(x) = (\mu_P^l(x), \mu_P^m(x), \mu_P^u(x)): X \to [0,1]$ and $\tilde{\nu}_P(x) = (\nu_P^l(x), \nu_P^m(x), \nu_P^u(x)): X \to [0,1]$ and they are also the degree of membership and non-membership of the element x in the set P belonging to X, and $0 \le (\mu_p^u(x))^2 + (\nu_p^u(x))^2 \le 1, x \in X$. The hesitancy degree of TPFS is $\tilde{\pi}_P(x) = \left(\tilde{\pi}_P^l(x), \tilde{\pi}_P^m(x), \tilde{\pi}_P^u(x)\right) = (\sqrt{1-(\mu_P^u(x))^2-(\nu_P^u(x))^2}, \sqrt{1-(\mu_P^m(x))^2-(\nu_P^m(x))^2}, \sqrt{1-(\mu_P^l(x))^2-(\nu_P^l(x))^2})$. When $\mu_P^l(x) = \mu_P^m(x) = \mu_P^u(x), \nu_P^l(x) = \nu_P^m(x) = \nu_P^u(x)$, TPFS degenerates into PFS. Note that the elements of TPFS are TPFNs, $\tilde{\alpha} = P(\mu_p, \nu_p) = P\langle(\mu_P^l, \mu_P^m, \mu_P^u), (\nu_P^l, \nu_P^m, \nu_P^u)\rangle$, $\pi_\alpha = (\pi_P^l, \pi_P^m, \pi_P^u) = (\sqrt{1-(\mu_P^u)^2-(\nu_P^u)^2}, \sqrt{1-(\mu_P^m)^2-(\nu_P^m)^2}, \sqrt{1-(\mu_P^l)^2-(\nu_P^l)^2})$, where $0 \le (\mu_P^u)^2 + (\nu_P^u)^2 \le 1$. Shorthand $\langle(a,b,c),(d,e,f)\rangle$. When $\mu_P^l = \mu_P^m = \mu_P^u$, $\nu_P^l = \nu_P^m = \nu_P^u$, TPFNs degenerate into PFNs.

3.1.2. TPFNs algorithm

Definition 3. [17] Let $\tilde{\alpha}_1 = \langle(a_1,b_1,c_1),(d_1,e_1,f_1)\rangle$ and $\tilde{\alpha}_2 = \langle(a_2,b_2,c_2),(d_2,e_2,f_2)\rangle$ be two arbitrary TPFNs, the real number $\lambda > 0$, the algorithm is as shown in Eqs (1)–(4).

$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left\langle (\sqrt{a_1^2+a_2^2-a_1^2a_2^2}, \sqrt{b_1^2+b_2^2-b_1^2b_2^2}, \sqrt{c_1^2+c_2^2-c_1^2c_2^2}), (d_1d_2, e_1e_2, f_1f_2) \right\rangle \tag{1}$$

$$\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \left\langle (a_1a_2, b_1b_2, c_1c_2), (\sqrt{d_1^2+d_2^2-d_1^2d_2^2}, \sqrt{e_1^2+e_2^2-e_1^2e_2^2}, \sqrt{f_1^2+f_2^2-f_1^2f_2^2}) \right\rangle \tag{2}$$

$$\lambda\tilde{\alpha}_1 = \left\langle (\sqrt{1-(1-a_1^2)^\lambda}, \sqrt{1-(1-b_1^2)^\lambda}, \sqrt{1-(1-c_1^2)^\lambda}), (d_1^\lambda, e_1^\lambda, f_1^\lambda) \right\rangle \tag{3}$$

$$\tilde{\alpha}_1^\lambda = \left\langle (a_1^\lambda, b_1^\lambda, c_1^\lambda), (\sqrt{1-(1-d_1^2)^\lambda}, \sqrt{1-(1-e_1^2)^\lambda}, \sqrt{1-(1-f_1^2)^\lambda}) \right\rangle. \tag{4}$$

3.1.3. TPFN sorting method

Definition 4. [17] Let $\tilde{\alpha} = \langle(a,b,c),(d,e,f)\rangle$ be TPFN, then its score function and exact function are Eqs (5) and (6) respectively.

$$S(\tilde{\alpha}) = \frac{1}{3}(\frac{a^2+2b^2+c^2-(d^2+2e^2+f^2)}{4} + 2) \tag{5}$$

$$H(\tilde{\alpha}) = \frac{1}{3}(\frac{a^2+2b^2+c^2+d^2+2e^2+f^2}{4} + 2). \tag{6}$$

The larger the value of $S(\tilde{\alpha}) \in [0,1]$, the larger the corresponding TPFN $\tilde{\alpha}$. Two TPFNs can be compared according to the calculated score functions, and when the score functions are equal, the size of the two can be compared according to their exact functions.

Reference [17] gives the sorting method of TPFN, let $\tilde{\alpha}_1, \tilde{\alpha}_2$ be two TPFNs

(1) If $S(\tilde{\alpha}_1) < S(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$.

(2) If $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$, then when $H(\tilde{\alpha}_1) < H(\tilde{\alpha}_2)$, $\tilde{\alpha}_1 < \tilde{\alpha}_2$, when $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2)$, $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

3.1.4. GBM operator theory

Definition 5. [35] Let $p, q, r \geq 0$, the set of non-negative real numbers is $\{a_1, a_2, \cdots, a_n\}$. If $\{w_1, w_2, \cdots, w_n\}$ is $a_i (i = 1,2, \cdots, n)$ corresponding weights, satisfying $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1, i = 1,2, \cdots, n$, and there is Eq (7).

$$GWBM^{p,q,r}(a_1, a_2, \cdots, a_n) = (\sum_{i,j,k=1}^{n} w_i w_j w_k a_i^p \, a_j^q a_k^r)^{\frac{1}{p+q+r}}. \tag{7}$$

Then the function $GWBM^{p,q,r}$ is called the generalized weighted Bonferroni mean (GWBM) operator.

Definition 6. [35] Let $p, q, r \geq 0$, the set of non-negative real numbers is $\{a_1, a_2, \cdots, a_n\}$. If $\{w_1, w_2, \cdots, w_n\}$ is $a_i (i = 1,2, \cdots, n)$ corresponding weights, satisfying $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1, i = 1,2, \cdots, n$, and there is Eq (8).

$$GWBGM^{p,q,r}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q+r} \sum_{i,j,k=1}^{n} pa_i + qa_j + ra_k)^{w_i w_j w_k}. \tag{8}$$

Then the function $GWBGM^{p,q,r}$ is called the generalized weighted Bonferroni geometric mean (GWBGM) operator.

The GWBM operator and the GWBGM operator not only have excellent properties such as reducibility, idempotency, monotonicity and boundedness, but also expand the input variables to the three-parameter case in the process of information aggregation, which can effectively capture more associated information between the input variables [35]. The GWBM and GWBGM operators, while boasting commendable attributes like reducibility, idempotency, monotonicity and boundedness, also extend the input variables to a tri-parametric context during information aggregation, adeptly encapsulating the intricate interrelations amongst the input variables [35].

3.2. Mathematical model

3.2.1. GTPFWBM operator concept

Definition 7. Let $\tilde{\alpha}_i = \langle (a_i, b_i, c_i), (d_i, e_i, f_i) \rangle$ be a set of TPFNs and $p, q, r \geq 0$. If $\{w_1, w_2, \cdots, w_n\}$ is the corresponding weights of $\tilde{\alpha}_i (i = 1,2, \cdots, n)$, satisfy $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1, i = 1,2, \cdots, n$, then the GTPFWBM operator is Eq (9).

$$GTPFWBM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\oplus_{i,j,k=1}^{n} w_i w_j w_k \left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q \otimes \tilde{\alpha}_k^r\right)\right)^{\frac{1}{p+q+r}}. \tag{9}$$

Theorem 1. Let $\tilde{\alpha}_i = \langle (a_i, b_i, c_i), (d_i, e_i, f_i) \rangle$ be a set of TPFNs and $p, q, r \geq 0$. If $\{w_1, w_2, \cdots, w_n\}$ is the corresponding weights of $\tilde{\alpha}_i(i = 1,2,\cdots,n)$, satisfy $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1, i = 1,2,\cdots,n$, then the result after aggregation by Definition 7 is still TPFN and satisfies Eq (10).

$$GTPFWBM^{p,q,r}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n) = \left\langle\left(\left(\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-a_i^{2p}a_j^{2q}a_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}},\right.\right.\right.$$
$$\left.\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-b_i^{2p}b_j^{2q}b_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}},\ \sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-c_i^{2p}c_j^{2q}c_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\right),$$
$$\left(\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}(1-(1-d_i^2)^p(1-d_j^2)^q(1-d_k^2)^r)\right)^{w_iw_jw_k})^{\frac{1}{p+q+r}}},\right.$$
$$\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}(1-(1-e_i^2)^p(1-e_j^2)^q(1-e_k^2)^r)\right)^{w_iw_jw_k})^{\frac{1}{p+q+r}}},$$
$$\left.\left.\sqrt{1-\left(1-\prod_{i,j,k=1}^{n}(1-(1-f_i^2)^p(1-f_j^2)^q(1-f_k^2)^r)\right)^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\right)\right\rangle. \tag{10}$$

Proof of Theorem 1. Because $\alpha_i = \langle(a_i,b_i,c_i),(d_i,e_i,f_i)\rangle$ and $\alpha_j = \langle(a_j,b_j,c_j),(d_j,e_j,f_j)\rangle$, α_i^p, α_j^q, α_k^r can be get from Eq (4).

$$\alpha_i^p = \left\langle(a_i^p,b_i^p,c_i^p),(\sqrt{1-(1-d_i^2)^p},\sqrt{1-(1-e_i^2)^p},\ \sqrt{1-(1-f_i^2)^p}\right\rangle$$

$$\alpha_j^q = \left\langle(a_j^q,b_j^q,c_j^q),(\sqrt{1-(1-d_j^2)^q},\sqrt{1-(1-e_j^2)^q},\ \sqrt{1-(1-f_j^2)^q})\right\rangle$$

$$\alpha_k^r = \left\langle(a_k^r,b_k^r,c_k^r),(\sqrt{1-(1-d_k^2)^r},\sqrt{1-(1-e_k^2)^r},\ \sqrt{1-(1-f_k^2)^r})\right\rangle.$$

According to Eq (2), Eq (11) can be obtained.

$$\alpha_i^p\otimes\alpha_j^q\otimes\alpha_k^r = \left\langle(a_i^pa_j^qa_k^r, b_i^pb_j^qb_k^r, c_i^pc_j^qc_k^r),(\sqrt{1-(1-d_i^2)^p(1-d_j^2)^q(1-d_k^2)^r},\right.$$
$$\left.\sqrt{1-(1-e_i^2)^p(1-e_j^2)^q(1-e_k^2)^r},\sqrt{1-(1-f_i^2)^p(1-f_j^2)^q(1-f_k^2)^r})\right\rangle. \tag{11}$$

Therefore, Eq (12) can be obtained.

$$\bigoplus_{i,j,k=1}^{n} w_iw_jw_k(\alpha_i^p\otimes\alpha_j^q\otimes\alpha_k^r)$$
$$=\left\langle\left(\left(\sqrt{1-\prod_{i,j,k=1}^{n}(1-a_i^{2p}a_j^{2q}a_k^{2r})^{w_iw_jw_k}},\sqrt{1-\prod_{i,j,k=1}^{n}(1-b_i^{2p}b_j^{2q}b_k^{2r})^{w_iw_jw_k}},\right.\right.\right.$$
$$\left.\sqrt{1-\prod_{i,j,k=1}^{n}(1-c_i^{2p}c_j^{2q}c_k^{2r})^{w_iw_jw_k}}\right),\left(\left(\prod_{i,j,k=1}^{n}(1-(1-d_i^2)^p(1-d_j^2)^q(1-d_k^2)^r)\right)^{\frac{w_iw_jw_k}{2}},\right.$$

$$\left(\prod_{i,j,k=1}^{n}(1-(1-e_i^2)^p(1-e_j^2)^q(1-e_k^2)^r)\right)^{\frac{w_iw_jw_k}{2}}, \left(\prod_{i,j,k=1}^{n}(1-(1-f_i^2)^p(1-f_j^2)^q(1-e_k^2)^r)\right)^{\frac{w_iw_jw_k}{2}}\right)\Bigg\rangle. \tag{12}$$

Furthermore, from the Eq (4), the Eq (13) can be obtained.

$$GTPFWBM^{p,q,r}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=\left(\bigoplus_{i,j,k=1}^{n}w_iw_jw_k\left(\tilde{\alpha}_i^p\otimes\tilde{\alpha}_j^q\otimes\tilde{\alpha}_j^r\right)\right)^{\frac{1}{p+q+r}}=$$
$$\Bigg\langle\Bigg(\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-a_i^{2p}a_j^{2q}a_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}},\ \sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-b_i^{2p}b_j^{2q}b_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}},$$
$$\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-c_i^{2p}c_j^{2q}c_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\Bigg),$$
$$\Bigg(\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-d_i^2)^p(1-d_j^2)^q(1-d_k^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}},$$
$$\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-e_i^2)^p(1-e_j^2)^q(1-e_k^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}},$$
$$\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-f_i^2)^p(1-f_j^2)^q(1-f_k^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\Bigg)\Bigg\rangle. \tag{13}$$

In Eq (13), $0\le\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-a_i^{2p}a_j^{2q}a_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\le\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-b_i^{2p}b_j^{2q}b_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\le\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-c_i^{2p}c_j^{2q}c_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\le 1$. At the same time, $0\le\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-d_i^2)^p(1-d_j^2)^q(1-d_k^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\le\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-e_i^2)^p(1-e_j^2)^q(1-e_k^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\le\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-f_i^2)^p(1-f_j^2)^q(1-f_k^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\le 1$, and satisfies $0\le\left(1-\prod_{i,j,k=1}^{n}(1-c_i^{2p}c_j^{2q}c_k^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}+\left(1-(1-\prod_{i,j,k=1}^{n}(1-(1-f_i^2)^p(1-f_j^2)^q(1-f_k^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}\right)\le 1$. So, Theorem 1 is proved.

3.2.2. GTPFWBM properties

I. Idempotency

Let TPFN $\tilde{\alpha}_i=\langle(a_i,b_i,c_i),(d_i,e_i,f_i)\rangle=\tilde{\alpha}=\langle(a,b,c),(d,e,f)\rangle$ for all $(i=1,2,\cdots,n)$ satisfy Eq (14).

$$GTPFWBM^{p,q,r}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=GTPFWBM^{p,q,r}(\tilde{\alpha},\tilde{\alpha},\cdots,\tilde{\alpha})=\tilde{\alpha}. \tag{14}$$

Proof. Because $\tilde{\alpha}_i = \langle(a_i, b_i, c_i), (d_i, e_i, f_i)\rangle = \tilde{\alpha} = \langle(a, b, c), (d, e, f)\rangle$, we can get $GTPFWBM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\oplus_{i,j,k=1}^{n} w_i w_j w_k (\tilde{\alpha}^p \otimes \tilde{\alpha}^q \otimes \tilde{\alpha}^r)\right)^{\frac{1}{p+q+r}} = \left(\sum_{i,j,k=1}^{n} w_i w_j w_k \tilde{\alpha}\right)^{\frac{1}{p+q+r}} = \sum_{i=1}^{n} w_i \sum_{j=1}^{n} w_j \sum_{k=1}^{n} w_k \tilde{\alpha} = \tilde{\alpha}$.

II. Permutation invariance

Let $(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$ be a set of TPFNs, and $(\bar{\tilde{\alpha}}_1, \bar{\tilde{\alpha}}_2, \cdots, \bar{\tilde{\alpha}}_n)$ be any permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$, then there is Eq (15).

$$GTPFWBM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = GTPFWBM^{p,q,r}(\bar{\tilde{\alpha}}_1, \bar{\tilde{\alpha}}_2, \cdots, \bar{\tilde{\alpha}}_n). \tag{15}$$

Proof. Because $(\bar{\tilde{\alpha}}_1, \bar{\tilde{\alpha}}_2, \cdots, \bar{\tilde{\alpha}}_n)$ is any permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$

$$GTPFWBM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\bigoplus_{i,j,k=1}^{n} w_i w_j w_k \left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q \otimes \tilde{\alpha}_j^r\right)\right)^{\frac{1}{p+q+r}}$$

$$= \left(\oplus_{i,j,k=1}^{n} w_i w_j w_k \left(\bar{\tilde{\alpha}}_i^p \otimes \bar{\tilde{\alpha}}_j^q \otimes \bar{\tilde{\alpha}}_j^r\right)\right)^{\frac{1}{p+q+r}} = GTPFWBM^{p,q,r}(\bar{\tilde{\alpha}}_1, \bar{\tilde{\alpha}}_2, \cdots, \bar{\tilde{\alpha}}_n).$$

III. Monotonicity

Let $A = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\}$ and $B = \{\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n\}$ be two different TPFN sets, where $\tilde{\alpha}_i = \left\langle(a_{\alpha_i}, b_{\alpha_i}, c_{\alpha_i}), (d_{\alpha_i}, e_{\alpha_i}, f_{\alpha_i})\right\rangle$ and $\tilde{\beta}_i = \left\langle(a_{\beta_i}, b_{\beta_i}, c_{\beta_i}), (d_{\beta_i}, e_{\beta_i}, f_{\beta_i})\right\rangle$. If for any i, there are $a_{\beta_i} \geq a_{\alpha_i}, b_{\beta_i} \geq b_{\alpha_i}, c_{\beta_i} \geq c_{\alpha_i}, d_{\beta_i} \leq d_{\alpha_i}, e_{\beta_i} \leq e_{\alpha_i}, f_{\beta_i} \leq f_{\alpha_i}$, that is, $\tilde{\alpha}_i \leq \tilde{\beta}_i$, then there is Eq (16).

$$GTPFWBM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leq GTPFWBM^{p,q,r}(\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n). \tag{16}$$

Proof. Because for any i, there are $a_{\beta_i} \geq a_{\alpha_i}, b_{\beta_i} \geq b_{\alpha_i}, c_{\beta_i} \geq c_{\alpha_i}, d_{\beta_i} \leq d_{\alpha_i}, e_{\beta_i} \leq e_{\alpha_i}, f_{\beta_i} \leq f_{\alpha_i}$, then there are

$$a_{\alpha_i}^p a_{\alpha_j}^q a_{\alpha_k}^r \leq a_{\beta_i}^p a_{\beta_j}^q a_{\beta_k}^r, b_{\alpha_i}^p b_{\alpha_j}^q b_{\alpha_k}^r \leq b_{\beta_i}^p b_{\beta_j}^q b_{\beta_k}^r, c_{\alpha_i}^p c_{\alpha_j}^q c_{\alpha_k}^r \leq c_{\beta_i}^p c_{\beta_j}^q c_{\beta_k}^r$$
$$\Rightarrow \begin{cases} \prod_{i,j,k=1}^{n} (1 - a_{\alpha_i}^{2p} a_{\alpha_j}^{2q} a_{\alpha_k}^{2r})^{w_i w_j w_k} \geq \prod_{i,j,k=1}^{n} (1 - a_{\beta_i}^{2p} a_{\beta_j}^{2q} a_{\beta_k}^{2r})^{w_i w_j w_k} \\ \prod_{i,j,k=1}^{n} (1 - b_{\alpha_i}^{2p} b_{\alpha_j}^{2q} b_{\alpha_k}^{2r})^{w_i w_j w_k} \geq \prod_{i,j,k=1}^{n} (1 - b_{\beta_i}^{2p} b_{\beta_j}^{2q} b_{\beta_k}^{2r})^{w_i w_j w_k} \\ \prod_{i,j,k=1}^{n} (1 - c_{\alpha_i}^{2p} c_{\alpha_j}^{2q} c_{\alpha_k}^{2r})^{w_i w_j w_k} \geq \prod_{i,j,k=1}^{n} (1 - c_{\beta_i}^{2p} c_{\beta_j}^{2q} c_{\beta_k}^{2r})^{w_i w_j w_k} \end{cases}$$

$$\Rightarrow\begin{cases}1-\prod_{i,j,k=1}^{n}(1-a_{\alpha_i}^{2p}a_{\alpha_j}^{2q}a_{\alpha_k}^{2r})^{w_iw_jw_k}\leq 1-\prod_{i,j,k=1}^{n}(1-a_{\beta_i}^{2p}a_{\beta_j}^{2q}a_{\beta_k}^{2r})^{w_iw_jw_k}\\ 1-\prod_{i,j,k=1}^{n}(1-b_{\alpha_i}^{2p}b_{\alpha_j}^{2q}b_{\alpha_k}^{2r})^{w_iw_jw_k}\leq 1-\prod_{i,j,k=1}^{n}(1-b_{\beta_i}^{2p}b_{\beta_j}^{2q}b_{\beta_k}^{2r})^{w_iw_jw_k}\\ 1-\prod_{i,j,k=1}^{n}(1-c_{\alpha_i}^{2p}c_{\alpha_j}^{2q}c_{\alpha_k}^{2r})^{w_iw_jw_k}\leq 1-\prod_{i,j,k=1}^{n}(1-c_{\beta_i}^{2p}c_{\beta_j}^{2q}c_{\beta_k}^{2r})^{w_iw_jw_k}\end{cases}$$

$$\Rightarrow\begin{cases}\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-a_{\alpha_i}^{2p}a_{\alpha_j}^{2q}a_{\alpha_k}^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\leq\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-a_{\beta_i}^{2p}a_{\beta_j}^{2q}a_{\beta_k}^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\\ \sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-b_{\alpha_i}^{2p}b_{\alpha_j}^{2q}b_{\alpha_k}^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\leq\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-b_{\beta_i}^{2p}b_{\beta_j}^{2q}b_{\beta_k}^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}.\\ \sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-c_{\alpha_i}^{2p}c_{\alpha_j}^{2q}c_{\alpha_k}^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\leq\sqrt{\left(1-\prod_{i,j,k=1}^{n}(1-c_{\beta_i}^{2p}c_{\beta_j}^{2q}c_{\beta_k}^{2r})^{w_iw_jw_k}\right)^{\frac{1}{p+q+r}}}\end{cases}$$

At the same time,

$$d_{\beta_i}\leq d_{\alpha_i}, e_{\beta_i}\leq e_{\alpha_i}, f_{\beta_i}\leq f_{\alpha_i}$$

$$\Rightarrow\begin{cases}(1-d_{\alpha_i}^2)^p(1-d_{\alpha_j}^2)^q(1-d_{\alpha_k}^2)^r\leq(1-d_{\beta_i}^2)^p(1-d_{\beta_j}^2)^q(1-d_{\beta_k}^2)^r\\ (1-e_{\alpha_i}^2)^p(1-e_{\alpha_j}^2)^q(1-e_{\alpha_k}^2)^r\leq(1-e_{\beta_i}^2)^p(1-e_{\beta_j}^2)^q(1-e_{\beta_k}^2)^r\\ (1-f_{\alpha_i}^2)^p(1-f_{\alpha_j}^2)^q(1-f_{\alpha_k}^2)^r\leq(1-f_{\beta_i}^2)^p(1-f_{\beta_j}^2)^q(1-f_{\beta_k}^2)^r\end{cases}$$

$$\Rightarrow\begin{cases}\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-d_{\alpha_i}^2)^p(1-d_{\alpha_j}^2)^q(1-d_{\alpha_k}^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\\ \geq\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-d_{\beta_i}^2)^p(1-d_{\beta_j}^2)^q(1-d_{\beta_k}^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\\ \sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-e_{\alpha_i}^2)^p(1-e_{\alpha_j}^2)^q(1-e_{\alpha_k}^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\\ \geq\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-e_{\beta_i}^2)^p(1-e_{\beta_j}^2)^q(1-e_{\beta_k}^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\\ \sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-f_{\alpha_i}^2)^p(1-f_{\alpha_j}^2)^q(1-f_{\alpha_k}^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\\ \geq\sqrt{1-(1-\prod_{i,j,k=1}^{n}(1-(1-f_{\beta_i}^2)^p(1-f_{\beta_j}^2)^q(1-f_{\beta_k}^2)^r))^{w_iw_jw_k})^{\frac{1}{p+q+r}}}\end{cases}.$$

Therefore, $GTPFWBM^{p,q,r}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)\leq GTPFWBM^{p,q,r}(\tilde{\beta}_1,\tilde{\beta}_2,\cdots,\tilde{\beta}_n)$ is proved.

IV. Boundedness

Let $\tilde{\alpha}_i=\langle(a_i,b_i,c_i),(d_i,e_i,f_i)\rangle$ be a set of TPFNs, then there is Eq (17).

$$\tilde{\alpha}^-\leq GTPFWBM^{p,q,r}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)\leq\tilde{\alpha}^+. \tag{17}$$

In Eq (17),

$$\tilde{\alpha}^- = \left\langle \left(\min_i a_i, \min_i b_i, \min_i c_i\right), \left(\min_i d_i, \min_i e_i, \min_i f_i\right) \right\rangle$$

$$\tilde{\alpha}^+ = \left\langle \left(\max_i a_i, \max_i b_i, \max_i c_i\right), \left(\max_i d_i, \max_i e_i, \max_i f_i\right) \right\rangle.$$

3.2.3. The GTPFWBGM operator

Definition 8. Let $\tilde{\alpha}_i = \langle (a_i, b_i, c_i), (d_i, e_i, f_i) \rangle$ be a set of TPFNs and $p, q, r \geq 0$. If $\{w_1, w_2, \cdots, w_n\}$ is the corresponding weight of $\tilde{\alpha}_i (i = 1,2, \cdots, n)$, $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1, i = 1,2, \cdots, n$, then the GTPFWBGM operator is Eq (18).

$$GTPFWBGM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \frac{1}{p+q+r}\left(\otimes_{i,j,k=1}^{n}\left(p\tilde{\alpha}_i \oplus q\tilde{\alpha}_j \oplus r\tilde{\alpha}_k\right)^{w_i w_j w_k}\right). \tag{18}$$

Theorem 2. Let $\tilde{\alpha}_i = \langle (a_i, b_i, c_i), (d_i, e_i, f_i) \rangle$ be a set of TPFNs and $p, q, r \geq 0$. If $\{w_1, w_2, \cdots, w_n\}$ is the corresponding weight of $\tilde{\alpha}_i (i = 1,2, \cdots, n)$, $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1, i = 1,2, \cdots, n$, then the result of aggregation by Definition 8 is still TPFN.

$$\begin{aligned}
&GTPFWBGM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \\
&= \left\langle \left(\sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - (1 - a_i^2)^p (1 - a_j^2)^q (1 - a_k^2)^r\right)\right)^{w_i w_j w_k}\right)^{\frac{1}{p+q+r}}}, \right.\right. \\
&\sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - (1 - b_i^2)^p (1 - b_j^2)^q (1 - b_k^2)^r\right)\right)^{w_i w_j w_k}\right)^{\frac{1}{p+q+r}}} \\
&\left. \sqrt{1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - (1 - c_i^2)^p (1 - c_j^2)^q (1 - c_k^2)^r\right)\right)^{w_i w_j w_k}\right)^{\frac{1}{p+q+r}}} \right), \\
&\left(\sqrt{\left(1 - \prod_{i,j,k=1}^{n} (1 - d_i^{2p} d_j^{2q} d_k^{2r})^{w_i w_j w_k}\right)^{\frac{1}{p+q+r}}}, \sqrt{\left(1 - \prod_{i,j,k=1}^{n} (1 - e_i^{2p} e_j^{2q} e_k^{2r})^{w_i w_j w_k}\right)^{\frac{1}{p+q+r}}}, \right. \\
&\left.\left. \sqrt{\left(1 - \prod_{i,j,k=1}^{n} (1 - f_i^{2p} f_j^{2q} f_k^{2r})^{w_i w_j w_k}\right)^{\frac{1}{p+q+r}}} \right) \right\rangle.
\end{aligned} \tag{19}$$

3.2.4. GTPFWBGM operator properties

I. Idempotency

Let TPFN $\tilde{\alpha}_i = \langle (a_i, b_i, c_i), (d_i, e_i, f_i) \rangle = \tilde{\alpha} = \langle (a, b, c), (d, e, f) \rangle$ for all $(i = 1,2, \cdots, n)$ satisfy Eq (20).

$$GTPFWBGM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = GTPFWBGM^{p,q,r}(\tilde{\alpha}, \tilde{\alpha}, \cdots, \tilde{\alpha}) = \tilde{\alpha}. \tag{20}$$

II. Permutation invariance

Let $(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$ be a set of TPFNs, and $(\bar{\tilde{\alpha}}_1, \bar{\tilde{\alpha}}_2, \cdots, \bar{\tilde{\alpha}}_n)$ be any permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$, then there is Eq (21).

$$GTPFWBGM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = GTPFWBGM^{p,q,r}(\bar{\tilde{\alpha}}_1, \bar{\tilde{\alpha}}_2, \cdots, \bar{\tilde{\alpha}}_n). \quad (21)$$

III. Monotonicity

Let $A = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n\}$ and $B = \{\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n\}$ be two different TPFN sets, where $\tilde{\alpha}_i = \left\langle (a_{\alpha_i}, b_{\alpha_i}, c_{\alpha_i}), (d_{\alpha_i}, e_{\alpha_i}, f_{\alpha_i}) \right\rangle$ and $\tilde{\beta}_i = \left\langle (a_{\beta_i}, b_{\beta_i}, c_{\beta_i}), (d_{\beta_i}, e_{\beta_i}, f_{\beta_i}) \right\rangle$. If for any i, there are $a_{\beta_i} \geq a_{\alpha_i}, b_{\beta_i} \geq b_{\alpha_i}, c_{\beta_i} \geq c_{\alpha_i}$, $d_{\beta_i} \leq d_{\alpha_i}, e_{\beta_i} \leq e_{\alpha_i}, f_{\beta_i} \leq f_{\alpha_i}$, that is, $\tilde{\alpha}_i \leq \tilde{\beta}_i$, then there is Eq (22).

$$GTPFWBGM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leq GTPFWBGM^{p,q,r}(\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n). \quad (22)$$

IV. Boundedness

Let $\tilde{\alpha}_i = \langle (a_i, b_i, c_i), (d_i, e_i, f_i) \rangle$ be a TPFN set, then there is Eq (23).

$$\tilde{\alpha}^- \leq GTPFWBGM^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leq \tilde{\alpha}^+. \quad (23)$$

In Eq (23),

$$\tilde{\alpha}^- = \left\langle (\min_i a_i, \min_i b_i, \min_i c_i), (\min_i d_i, \min_i e_i, \min_i f_i) \right\rangle$$

$$\tilde{\alpha}^+ = \left\langle (\max_i a_i, \max_i b_i, \max_i c_i), (\max_i d_i, \max_i e_i, \max_i f_i) \right\rangle.$$

3.2.5. Weight calculation

We employ the differential weight methodology for weight computation. This method elucidates the significance of a metric via the variance amidst indicators. An indicator's pronounced divergence from its counterparts amplifies its efficacy in discerning the caliber of the scheme.

The Mean Squared Displacement similarity formula is used to calculate the similarity between the indicators, and the difference between the indicators is calculated by taking its opposite number, as shown in Eq (24).

$$d(y_i, y_j) = \frac{\sum_{h=1}^{card(S_{ij})} (b_{hi} - b_{hj})^2}{card(S_{ij})}. \quad (24)$$

In Eq (24), $d(y_i, y_j)$ is the difference between indicators y_i and y_j. S_{ij} is the set of schemes with indicator values on both y_i and y_j. b_{hi} is the standardized evaluation value of scheme s_h on indicator y_i.

The difference matrix is constructed according to the difference between indicators, as shown in Eq (25).

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}. \quad (25)$$

In Eq (25), d_{ij} is the difference between the ith indicator y_i and the jth indicator y_j.

The average difference $\overline{d_i}$ between the indicator y_i and all the other indicators is calculated as shown in Eq (26).

$$\overline{d_i} = \frac{\sum_{j=1}^{n} d_{ij}}{n}. \tag{26}$$

The difference weight ω_i of indicator y_i is shown in Eq (27).

$$\omega_i = \frac{\overline{d_i}}{\sum_{i=1}^{n} \overline{d_i}}. \tag{27}$$

3.3. Solution approach

Aiming at the MADM problem in which the decision information is given by TPFNs, this paper constructs a new method based on the GTPFWBM operator and the GTPFWBGM operator. For a MADM problem, the decision-making scheme set is $A = \{A_1, A_2, \dots, A_t\}$, the attribute set is $C = \{C_1, C_2, \dots, C_n\}$, the attribute weight is $w = (w_1, w_2, \cdots, w_n)$ and $w_j \in [0,1] (j = 1,2, \dots, n)$. $D = \{d_1, d_2, \dots, d_m\}$ is the decision set. $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ is the decision maker's weight vector $\omega_k \in [0,1]$, where $\sum_{k=1}^{m} \omega_k = 1$, $\sum_{j=1}^{n} w_j = 1$. The specific decision-making steps are as follows:

Step 1: Assume that the decision maker d_k gives the evaluation value of the scheme A_i under the attribute C_j as TPFN, and the decision matrix is obtained as $D^{(k)} = (\tilde{\alpha}_{ij}^{(k)})_{n \times t}$.

Step 2: Standardize the decision matrix $D^{(k)}$ using Eq (28) to obtain $\overline{D^{(k)}}$.

$$\overline{\alpha}_{ij} = (\overline{\mu}_{ij}, \overline{\nu}_{ij}) = \begin{cases} (\overline{\mu}_{ij}, \overline{\nu}_{ij}) & C_j \in I_1 \\ (\overline{\nu}_{ij}, \overline{\mu}_{ij}) & C_j \in I_2 \end{cases}, i = 1,2, \cdots, t; j = 1,2, \cdots, n. \tag{28}$$

Among them, I_1 and I_2 represent the benefit attribute and the cost attribute, respectively.

Step 3: Calculate the weights of decision makers and attributes using Eqs (24)–(27).

Step 4: Use the GTPFWBM operator and the GTPFWBGM operator to integrate information on the decision matrix given by the decision experts, and synthesize the weights to obtain the overall evaluation value of the scheme $A_i (i = 1,2, \cdots, t)$.

Step 5: Calculate the score function value and then rank the schemes according to the TPFN sorting method to obtain the best scheme.

4. Results

4.1. Computational results

During the intricacies of venture capital investment, a thorough analysis of prospective entities' multifaceted factors is imperative. A venture capital firm convened a triumvirate of industry decision-making connoisseurs to appraise five prospective entities across four pivotal metrics: Competitive prowess (C1), expansion potential (C2), societal resonance (C3) and environmental imprint magnitude (C4). To encapsulate the inherent uncertainty of attribute values proffered by the experts more authentically, these values are delineated using TPFNs, as depicted in Tables 2–4.

Table 2. Decision matrix given by the expert d_1.

	C_1	C_2	C_3	C_4
A_1	<(0.3,0.4,0.5), (0.4,0.5,0.5)>	<(0.6,0.7,0.8), (0.1,0.1,0.2)>	<(0.6,0.6,0.7), (0.2,0.2,0.3)>	<(0.5,0.6,0.7), (0.1,0.2,0.2)>
A_2	<(0.4,0.5,0.6), (0.2,0.3,0.4)>	<(0.5,0.6,0.6), (0.1,0.2,0.3)>	<(0.4,0.5,0.6), (0.2,0.3,0.4)>	<(0.2,0.3,0.4), (0.4,0.5,0.6)>
A_3	<(0.2,0.3,0.4), (0.4,0.5,0.6)>	<(0.4,0.5 0.6), (0.3,0.3,0.4)>	<(0.7,0.8,0.9), (0.1,0.1,0.1)>	<(0.1,0.2,0.3), (0.5,0.6,0.7)>
A_4	<(0.5,0.6,0.7), (0.1,0.2,0.2)>	<(0.8,0.8,0.8), (0.2,0.2,0.2)>	<(0.5,0.6,0.6), (0.2,0.3,0.4)>	<(0.4,0.5,0.6), (0.3,0.4,0.4)>
A_5	<(0.7,0.7,0.8), (0.1,0.1,0.2)>	<(0.5,0.5,0.5), (0.2,0.3,0.4)>	<(0.7,0.7,0.7), (0.1,0.1,0.1)>	<(0.3,0.4,0.4), (0.4,0.5,0.6)>

Table 3. Decision matrix given by the expert d_2.

	C_1	C_2	C_3	C_4
A_1	<(0.2,0.3,0.4), (0.3,0.4,0.4)>	<(0.5,0.6,0.7), (0.1,0.1,0.1)>	<(0.5,0.5,0.6), (0.1,0.1,0.2)>	<(0.4,0.5,0.6), (0.1,0.1,0.1)>
A_2	<(0.3,0.4,0.5), (0.1,0.2,0.3)>	<(0.4,0.5,0.5), (0.1,0.2,0.2)>	<(0.3,0.4,0.5), (0.1,0.2,0.3)>	<(0.1,0.2,0.3), (0.3,0.4,0.5)>
A_3	<(0.1,0.2,0.3), (0.3,0.4,0.5)>	<(0.3,0.4,0.5), (0.2,0.2,0.3)>	<(0.6,0.7,0.8), (0.1,0.1,0.1)>	<(0.1,0.1,0.2), (0.4,0.5,0.6)>
A_4	<(0.4,0.5,0.6), (0.1,0.1,0.1)>	<(0.7,0.7,0.7), (0.1,0.1,0.1)>	<(0.4,0.5,0.5), (0.1,0.2,0.3)>	<(0.3,0.4,0.5), (0.2,0.3,0.3)>
A_5	<(0.6,0.6,0.7), (0.1,0.1,0.1)>	<(0.4,0.4,0.4), (0.1,0.2,0.3)>	<(0.6,0.6,0.6), (0.1,0.1,0.1)>	<(0.2,0.3,0.3), (0.3,0.4,0.5)>

Table 4. Decision matrix given by the expert d_3.

	C_1	C_2	C_3	C_4
A_1	<(0.1,0.2,0.3), (0.6,0.7,0.7)>	<(04,0.5,0.6), (0.3,0.3,0.4)>	<(0.4,0.4,0.5), (0.4,0.4,0.5)>	<(0.3,0.4,0.5), (0.3,0.4,0.4)>
A_2	<(0.2,0.3,0.4), (0.4,0.5,0.6)>	<(0.3,0.4,0.4), (0.3,0.4,0.5)>	<(0.2,0.3,0.4), (0.4,0.5,0.6)>	<(0.1,0.1,0.2), (0.6,0.7,0.8)>
A_3	<(0.1,0.2,0.2), (0.6,0.7,0.8)>	<(0.2,0.3,0.4), (0.5,0.5,0.6)>	<(0.5,0.6,0.7), (0.3,0.3,0.3)>	<(0.1,0.1,0.1), (0.7,0.8,0.9)>
A_4	<(0.3,0.4,0.5), (0.3,0.4,0.4)>	<(0.6,0.6,0.6), (0.4,0.4,0.4)>	<(0.3,0.4,0.4), (0.4,0.5,0.6)>	<(0.2,0.3,0.4), (0.5,0.6,0.6)>
A_5	<(0.5,0.5,0.6), (0.3,0.3,0.4)>	<(0.3,0.3,0.3), (0.4,0.5,0.6)>	<(0.5,0.5,0.5), (0.3,0.3,0.3)>	<(0.1,0.2,0.2), (0.6,0.7,0.8)>

Step 1: Establish the triangular Pythagorean fuzzy matrix, as shown in Tables 2–4.

Step 2: Because each decision attribute is a benefit attribute, there is no need to standardize the decision matrix $D^{(k)}$.

Step 3: Calculate the weight vector of the three decision makers, adopting the Pythagorean fuzzy weights ω_k = (0.312748098, 0.307266963, 0.37998494), and the weight vectors of the four indicators are w_j = (0.24159086, 0.242205478, 0.262497977, 0.253705685).

Step 4: This paper studies the case of $p = q = r = 1$, and calculates the comprehensive evaluation value of the three experts for the five candidate companies by Eqs (10) and (19). The weighted evaluation results of each candidate company are shown in Table 5.

Step 5: According to the TPFN sorting method, the candidate companies are ranked as $A_2>A_4>A_5>A_1>A_3$ and $A_2>A_1>A_3>A_4>A_5$. Therefore, the optimal company is A_2.

Table 5. Comprehensive TPFNs and scoring functions.

	TPFNs after aggregation of GTPFWBM operator	**Score function**	**TPFNs after aggregation of GTPFWBGM operator**	**Score function**
A_1	<(0.4981,0.5023,0.5964), (0.6103,0.6218,0.7526>	0.6612	<(0.3956,0.5213,0.5969), (0.6623,0.6659,0.7743>	0.3641
A_2	<(0.5416,0.5632,0.5961), (0.4518,0.47220.5103>	0.8851	<(0.3642,0.4518,0.4321), (0.3342,0.3611 0.6481>	0.5542
A_3	<(0.4491,0.4681,0.4964), (0.3651,0.3342,0.4803>	0.6112	<(0.6621,0.6596,0.6802), (0.2351,0.2832,0.3942>	0.3241
A_4	<(0.4862,0.5109,0.6237), (0.3412,0.3699,0.3781>	0.8469	<(0.6723,0.6869,0.7427), (0.4581,0.5152,0.5199>	0.3012
A_5	<(0.2632,0.3214,0.4201), (0.5427,0.5581,0.5742>	0.7812	<(0.1211,0.2453,0.5104), (0.6821,0.6942,0.752>	0.1624

4.2. Sensitivity analysis

A group test experiment is conducted for the p, q, r parameters of the GTPFWBM operator and the GTPFWBGM operator in order to prove the stability and effectiveness of the two operators. The design parameters p, q, r are different numerical combinations to conduct numerical experiments, and the experimental results of group testing are shown in Table 6.

Table 6. Sorting results of candidate companies corresponding to different parameters p, q, r.

Parameter value	**GTPFWBM operator sorting results**	**GTPFWBGM operator sorting results**
$p=1,q=1,r=1$	$A_2>A_4>A_3>A_1>A_5$	$A_2>A_3>A_5>A_1>A_4$
$p=2,q=2,r=2$	$A_2>A_3>A_5>A_4>A_1$	$A_2>A_5>A_4>A_1>A_3$
$p=3,q=3,r=3$	$A_2>A_5>A_4>A_1>A_3$	$A_2>A_4>A_5>A_1>A_3$
$p=4,q=4,r=4$	$A_2>A_4>A_5>A_1>A_3$	$A_2>A_4>A_5>A_1>A_3$
$p=5,q=5,r=5$	$A_2>A_5>A_1>A_4>A_3$	$A_2>A_4>A_5>A_1>A_3$
$p=6,q=6,r=6$	$A_2>A_4>A_5>A_1>A_3$	$A_2>A_4>A_5>A_1>A_3$
$p=7,q=7,r=7$	$A_2>A_4>A_5>A_1>A_3$	$A_2>A_4>A_5>A_1>A_3$
$p=8,q=8,r=8$	$A_2>A_4>A_5>A_1>A_3$	$A_2>A_4>A_5>A_1>A_3$
$p=9,q=9,r=9$	$A_2>A_4>A_5>A_1>A_3$	$A_2>A_4>A_5>A_1>A_3$
$p=10,q=10,r=10$	$A_2>A_4>A_5>A_1>A_3$	$A_2>A_4>A_5>A_1>A_3$

It can be seen from Table 6 that, on the one hand, although the change of the fixed parameters p, q, r affects the ranking of the candidate companies, the optimal candidate company is still A_2. On the other hand, when the parameters p, q, r become larger, although the score function value and exact function value of each candidate company change, their ranking remains unchanged. This proves that the GTPFWBM operator and the GTPFWBGM operator studied in this paper tend to be stable.

4.3. Discussion

In this paper, the duo of operators introduced are juxtaposed against the TPFWA operator [17], GTPFWA operator [17], TPFWG operator [17], GTPFWG operator [17], Pythagorean fuzzy three-way decisions-based (PFTWDB) operator [36] and Pythagorean fuzzy Einstein weighted averaging (PFEWA) operator [37]. The hierarchical results for each potential entity are elucidated in Table 7. The quartet of operators - TPFWA, GTPFWA, TPFWG and GTPFWG - are elected as reference benchmarks given their kinship with the proposed operators, both sets being enhanced variants predicated on TPFNs. The inclusion of PFTWDB and PFEWA operators as referential entities stems from their affiliation with the Pythagorean Fuzzy Aggregation operator family.

Table 7. Sorting results of each candidate company.

Decision operator	**Ranking of candidate companies**
GTPFWBM operator in this paper	$A_2>A_4>A_5>A_1>A_3$
GTPFWBGM operator in this paper	$A_2>A_1>A_3>A_4>A_5$
GTPFWA operator in [17] ($\lambda_{=2}$)	$A_2>A_3>A_4>A_5>A_1$
TPFWG operator in [17]	$A_2>A_3>A_4>A_5>A_1$
GTPFWG operator in [17] (λ=2)	$A_2>A_3>A_4>A_5>A_1$
TPFWA operator in [17]	$A_2>A_3>A_4>A_5>A_1$
PFTWDB operator in [36]	$A_2>A_4>A_1>A_5>A_3$
PFEWA operator in [37]	$A_2>A_3>A_4>A_1>A_5$

From the aforementioned comparative scrutiny, it becomes evident that, despite the variances in company rankings among the sextet of operators and the two delineated in this treatise, the quintessential candidate consistently emerges as A2. This solidifies the efficacy of the proposed adjudicative methodology. The nuances in ranking owe their existence to the divergent decision-making paradigms inherent to this study and those rooted in varied Pythagorean fuzzy aggregation operators. The adjudicative strategy expounded in the references [17,36,37] seeks to derive a holistic attribute value for each firm via distinctive Pythagorean fuzzy aggregation mechanisms, subsequently hierarchizing each entity based on a scoring function. This approach adeptly captures the inherent ambiguity of data amidst the realm of autonomous attributes.

In real-world company selection scenarios, firms boasting pronounced competitive edges invariably manifest concomitant strengths in their growth potential. This inevitably leads to an overlap of evaluative information provided by experts. By meticulously eliminating such redundancies and fostering complementarity within attribute data, one can safeguard the veracity and cogency of decision outcomes. This manuscript adeptly amalgamates the GBM operator with TPFNs, ensuring that the introduced operators are not merely apt for decision-making in ambiguous contexts but also intricately consider inter-attribute correlations courtesy of the tri-parametric characteristic. In conclusion, when confronting interrelationships among evaluative attributes in tangible decision-making scenarios, only by holistically acknowledging these correlations can decisions achieve optimal soundness. Consequently, the GTPFWBM and GTPFWBGM operators, as expounded in this treatise, resonate with real-world dynamics and adeptly discern the merits and demerits of the proposition in question.

5. Conclusions and outlook

The inherent uncertainty in decision-making attributes amplifies the complexity of multi-attribute decision-making (MADM) endeavors. Oftentimes, decision-makers, constrained by their experiential knowledge, introduce attributes replete with interdependencies. Consider, for instance, the decision attributes integral to the holistic appraisal of potential companies: competitive prowess (C1), growth potential (C2), societal influence (C3) and environmental impact magnitude (C4). Typically, an enterprise exhibiting superior competitive advantage (C1) tends to highlight commensurately elevated growth potential (C2). Hence, an information aggregation operator that accounts for inter-attribute relationships demonstrably aligns with practical decision-making exigencies. This compensates for the limitations of existing TPFN information aggregation operators, which are solely efficacious when attributes are mutually exclusive. The selection outcomes for prospective companies validate the accuracy of the novel decision-making algorithm introduced herein. This avant-garde algorithm adeptly mitigates the influence of attribute interdependencies on decision outcomes, rendering the conclusions more authentic and credible.

Confronted with MADM challenges characterized by escalating intricacy due to ambiguous data, decision-makers, bounded by their cumulative wisdom, often induce notable interrelations amidst evaluative metrics. To illustrate, attributes delineating the overarching assessment of corporations encompass facets like competitive edge and growth trajectory. These attributes bear substantial overlaps. Many, in addressing such decision conundrums, endeavor to obviate commonalities amidst indicators, inadvertently sidelining their intrinsically synergistic decision-making essence. Ergo, an information aggregation methodology that duly acknowledges attribute correlations resonates profoundly with tangible decision-making paradigms.

To address the shortcomings of prevailing TPFNs methodologies, which function optimally solely under mutually exclusive attributes, this manuscript introduces the GTPFWBM and GTPFWBGM operators in tandem with the GBM operator, delving into their inherent characteristics. A decision-making paradigm, predicated upon the GTPFWBM and GTPFWBGM operators, is conceptualized and subsequently applied to the enterprise selection conundrum inherent in venture capital endeavors. Sensitivity analysis underscores that variances in parameters leave the optimal outcome unaltered, attesting to the robustness of the delineated operators. Comparative assessments with six alternative methodologies elucidate a consistent identification of optimal candidate corporations, reinforcing the efficacy of the operators posited herein. Relative to alternate strategies, the considerations integral to this paper's comprehensive ranking appear the most cogent, underscoring the precision of the introduced operators. In summation, the articulated method adeptly obviates the deleterious implications of attribute interdependencies on decision outcomes, yielding results of heightened authenticity and credibility, thus proffering an innovative solution to the MADM quandary. This investigation bridges the extant scholarly lacuna pertaining to MADM approaches premised upon GBM operators and TPFNs, enhancing the theoretical corpus on PFNs aggregation methodologies.

Future refinements of this research will pivot around two salient vectors: First, acknowledging that decision-makers, swayed by external contingencies, may exhibit hesitation in providing evaluative data, forthcoming endeavors will extrapolate generalized triangular Pythagorean fuzzy weighted Bonferroni operators into the domain of Pythagorean hesitant fuzzy sets, crafting a congruent MADM model for indeterminate attribute weights. Second, the decision framework anchored on the GTPFWBM and GTPFWBGM operators caters to scenarios with limited scheme samples. However, as societal evolution mandates optimal verdicts amidst a plethora of candidates, future research, cognizant of the nuance of decision-maker weight sensitivities, will amalgamate the BM operator to probe large-scale collective Pythagorean fuzzy decision-making challenges.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by Soft Science Research Project of STCSM (No. 23692106100), Technical Standards Project of STCSM (No. 23DZ2204500) and the Shanghai Data Exchange under the Research Project Titled "Case Study on Enterprise Data Product—Trading and Data Assetization Practices".

Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

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