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Research article

Further properties of Tsallis extropy and some of its related measures

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Abstract: This article introduces the concept of residual and past Tsallis extropy as a continuous information measure within the context of continuous distribution. Moreover, the characteristics and their relationships with other models are evaluated. Several stochastic comparisons are provided, along with outcomes concerning order statistics. Additionally, the models acquired include instances such as uniform and power function distributions. The measure incorporates its monotonic traits, and the outcomes defining its characteristics are presented. On the other hand, a different portrayal of the Tsallis extropy is introduced, expressed in relation to the hazard rate function. The Tsallis extropy of the lifetime for both mixed and coherent systems is explored. In the case of mixed systems, components' lifetimes are considered independent and identically distributed. Additionally, constraints on the Tsallis extropy of these systems are established, along with a clarification of their practical applicability. Non-parametric estimation using an alternative form of Tsallis function extropy for simulated and real data is performed.

Keywords: extropy; Tsallis entropy; Tsallis extropy; stochastic orders; non-parametric estimation
Mathematics Subject Classification: 62B10, 94A15, 94A17

1. Introduction

Shannon [27] illustrated the classical Shannon of entropy measure of uncertainty. Supported with $\mathbb{R}$, the continuous Shannon entropy function for the random variable (RV) X with the probability density function (PDF) $f(x)$ is provided as follows:

$$SN(X) = -E(\ln f(X)) = -\int_R f(x) \ln f(x) dx. \tag{1.1}$$

Lad et al. [12] presented the idea of an extropy measure as a complementary measure to Shannon's entropy. The extropy of the discrete RV X, which is defined over the set $Q = \{x_1, ..., x_N\}$ and has a probability vector denoted as $p = (p_1, ..., p_N)$, can be formulated in the following manner:

$$Ext(X) = -\sum_{i=1}^{N} (1 - p_i) \ln(1 - p_i); \tag{1.2}$$

see additional details in [10] and the references therein. Additionally, the concept of the continuous RV (C-RV) X's extropy, which is defined over the set of real numbers $\mathbb{R}$, has been presented by Raqab and Qiu [23] and Qiu [21], and it is defined as

$$Ext(X) = -\frac{1}{2} \int_R f^2(x) dx. \tag{1.3}$$

Denote the non-negative and absolutely C-RV X as the new system life length with time t and cumulative distribution function (CDF) $F(.)$. The residual lifetime of an individual is denoted by $X_t^{(R)} := [X - t | X \geq t]$ with the PDF $f^{(R)}(x; t) = \frac{f(x)}{\overline{F}(t)}$, $x \geq t$. Moreover, the past lifetime of an item is denoted by $X_t^{(P)} := [X | X \leq t]$ with the PDF $f^{(P)}(x; t) = \frac{f(x)}{F(t)}$, $x \leq t$, $\overline{F}(t) = 1 - F(t)$. Qiu and Jia [22] presented the extropy for the residual life time $X_t^{(R)}$ as

$$RExt(X_t^{(R)}) = -\frac{1}{2} \int_t^{\infty} \left(\frac{f(x)}{1 - F(t)} \right)^2 dx. \tag{1.4}$$

Moreover, Krishnan et al. [11] studied the past extropy as

$$PExt(X_t^{(P)}) = -\frac{1}{2} \int_0^t \left(\frac{f(x)}{F(t)} \right)^2 dx. \tag{1.5}$$

Numerous researchers proposed multiple measures of entropy and their extensions. Among these extensions to account for different forms of uncertainty, Tsallis [29] illustrated the so-called Tsallis entropy. In the context of a C-RV X defined over $\mathbb{R}$, with $\beta \neq 1, \beta > 0$, the continuous Tsallis entropy is defined as

$$Tn_\beta(X) = \frac{1}{\beta - 1} \left(1 - \int_R f^\beta(x) dx \right), \tag{1.6}$$

where $\lim_{\beta \to 1} Tn_\beta(X) = SN(X)$. In addition, the connection between Tsallis and Renyi entropy can be found in Mariz [18], as we can see that $Rn_\beta(X) = \log(1 + (1 - \beta) Tn_\beta(X))/(1 - \beta)$, with $\beta \neq 1$ and $\beta > 0$, where $Rn_\beta(X)$ is the Renyi entropy defined in [24]. Moreover, for cumulative Tsallis entropy see Mohamed et al. [17] and the references therein.

Extropy can be used as an alternate measure to examine uncertainty because it has many matching effects to entropy (Meng et al. [15]; Xie et al. [32]). Under the concept of extropy, numerous dual structures for the entropy have been offered (Zhou and Deng [34]). Jahanshahi et al. [6] suggested a cumulative residual extropy. In addition, a negative cumulative extropy (Tahmasebi and Toomaj [28]) has been presented. Moreover, ordered variables (Noughabi and Jarrahiferiz [20]; Raqab and Qiu [23]; Qiu [21]), lifetime distribution (Kamari and Buono [8]), forecast distribution (Lad et al. [13]) and estimators of RVs (Noughabi and Jarrahiferiz [19]) have been discussed.

Drawing from the context of a discrete distribution lifetime, Xue and Deng [33] suggested the Tsallis extropy model, which serves as the dual counterpart to the Tsallis entropy, and explored its maximization value. Additionally, Balakrishnan et al. [3] investigated the Tsallis extropy, employing it in the realm of pattern recognition. Using the softmax function, Jawa et al. [7] delved into the residual and past aspects of Renyi and Tsalliss extropy.

Recently, Mohamed et al. [16] presented the continuous Tsallis extropy of the RV X backed by **[a,b]**, $-\infty < a < b < \infty$, as follows

$$\begin{aligned} Tx_\beta(X) &= \frac{1}{\beta-1}\left(\int_a^b (1-f(x))dx - \int_a^b (1-f(x))^\beta dx\right) \\ &= \frac{1}{\beta-1}\left(b-a-1-\int_a^b (1-f(x))^\beta dx\right), \end{aligned} \tag{1.7}$$

where the conditions on β can be given as follows:

(1) $\beta \neq 1, \beta > 0$ if $f(x) \leq 1$.

(2) $\beta \in \mathbb{Z}^+\backslash\{1\}$ if $f(x) > 1$.

Moreover, regarding the relation to dynamical information measures and the use of survival functions, we can see Contreras-Reyes et al. [4].

This paper introduces the continuous dynamical version of Tsallis extropy for a continuous distribution lifetime. The residual and past functions for Tsallis extropy and their properties are obtained. Another alternative representation of the Tsallis extropy with additional features is given. The remaining part of the article is therefore structured as follows. In Section 2, the past and residual functions for Tsallis extropy with some bounds, as well as monotone characterization results are introduced. Furthermore, the relation between our models and other measures is obtained. In Section 3, a thorough exploration of multiple properties of the dynamic versions is conducted. Moving on to Section 4, an analysis is presented concerning the Tsallis extropy and its characteristics in terms of both coherent and mixed structures under the conditions of the independent and identically distributed (iid) condition. Additionally, the section provides limits for the Tsallis entropy of system life times. Finally, in Section 4, the Tsallis extropy estimator is presented.

2. Properties of residual and past functions for Tsallis extropy

Inspired by the concepts of Tsallis entropy and extropy functions, this section presents the related measures of residual and past functions for Tsallis extropy as follows.

Likewise, following the approach presented by Lad et al. [12], we can express the residual and past functions for extropy, respectively, as shown below:

$$RExt(X_t^{(R)}) = -\int_t^\infty (1-\frac{f(x)}{\overline{F}(t)})\ln(1-\frac{f(x)}{\overline{F}(t)})dx, \tag{2.1}$$

with $\frac{f(x)}{\overline{F}(t)} < 1$, and

$$PExt(X_t^{(P)}) = -\int_0^{\infty} (1 - \frac{f(x)}{F(t)}) \ln(1 - \frac{f(x)}{F(t)}) dx, \tag{2.2}$$

with $\frac{f(x)}{F(t)} < 1$. There is a significant amount of literature on the use of Eqs (1.4) and (1.5) to discuss extropy. In our research, we will also address Eqs (2.1) and (2.2) as an illustrative representation of extropy. In what follows, we will present the definitions of the residual and past functions for Tsallis extropy.

Definition 2.1. *Let X be a C-RV backed by* $[a, b]$, $-\infty < a < b < \infty$, *with a PDF* $f(.)$. *Then, the residual function for the Tsallis extropy of the residual lifetime* $X_t^{(R)}$ *can be provided as*

$$\begin{aligned} RTx_\beta(X;t) &= \frac{1}{\beta - 1}\left(\int_a^b (1 - f(x))dx - \int_t^b \left(1 - \frac{f(x)}{\overline{F}(t)}\right)^\beta dx\right) \\ &= \frac{1}{\beta - 1}\left(b - a - 1 - \int_t^b \left(1 - \frac{f(x)}{\overline{F}(t)}\right)^\beta dx\right), \end{aligned} \tag{2.3}$$

where the conditions on β *are as follows:*

(1) $\beta \neq 1, \beta > 0$ *if* $\frac{f(x)}{\overline{F}(t)} \leq 1$.

(2) $\beta \in \mathbb{Z}^+ \backslash \{1\}$ *if* $\frac{f(x)}{\overline{F}(t)} > 1$.

Definition 2.2. *Let X be a C-RV backed by* $[a, b]$, $-\infty < a < b < \infty$, *with a PDF* $f(.)$. *Then, the past function for the Tsallis extropy of the past lifetime* $X_t^{(P)}$ *can be provided as*

$$\begin{aligned} PTx_\beta(X;t) &= \frac{1}{\beta - 1}\left(\int_a^b (1 - f(x))dx - \int_a^t \left(1 - \frac{f(x)}{F(t)}\right)^\beta dx\right) \\ &= \frac{1}{\beta - 1}\left(b - a - 1 - \int_a^t \left(1 - \frac{f(x)}{F(t)}\right)^\beta dx\right), \end{aligned} \tag{2.4}$$

where the conditions on β *are as follows:*

(1) $\beta \neq 1, \beta > 0$ *if* $\frac{f(x)}{F(t)} \leq 1$.

(2) $\beta \in \mathbb{Z}^+ \backslash \{1\}$ *if* $\frac{f(x)}{F(t)} > 1$.

Proposition 2.1. *Suppose that X is a non-negative C-RV backed by* $[a, b]$, $0 < a < b < \infty$. *Then, from Eqs (2.1)–(2.4), we have*

$$\lim_{\beta \to 1} RTx_\beta(X;t) = RExt(X_t^{(R)}), \tag{2.5}$$

where $\beta \neq 1, \beta > 0$ *and* $\frac{f(x)}{\overline{F}(t)} \leq 1$.

$$\lim_{\beta \to 1} PTx_\beta(X;t) = PExt(X_t^{(R)}), \tag{2.6}$$

where $\beta \neq 1, \beta > 0$ *and* $\frac{f(x)}{F(t)} \leq 1$.

Proof. By directly applying L'Hôpital's rule, the results are obtained. □

Now, to discuss some further properties, it is useful to discuss the sign of our models. The following proposition discusses the conditions that guarantee the non-negativity of the residual and past functions for Tsallis extropy.

Proposition 2.2. *Assume that X is a non-negative C-RV backed by $[a, b]$, $0 < a < b < \infty$, with a PDF $f(.)$ and CDF $F(.)$. From Eqs (2.3) and (2.4), if $\beta > 1$ $(\beta < 1)$ and $\frac{f(x)}{\overline{F}(t)} \leq 1$, $\forall a < t < b < \infty$, then the residual function for Tsallis extropy is non-negative (negative). Moreover, if $\beta > 1$ $(\beta < 1)$ and $\frac{f(x)}{F(t)} \leq 1$, $\forall a < t < b < \infty$, then the past function for Tsallis extropy is non-negative (negative).*

Proof. Since $\frac{f(x)}{\overline{F}(t)} \leq 1$, then we have

$$0 \leq \int_t^b \left(1 - \frac{f(x)}{\overline{F}(t)}\right)^\beta dx \leq \int_t^b \left(1 - \frac{f(x)}{\overline{F}(t)}\right) dx = b - t - 1.$$

Therefore, from Eq (2.3), when $\beta > 1$ $(\beta < 1)$, we obtain

$$\begin{aligned} RTx_\beta(X;t) &= \frac{1}{\beta - 1}\left(b - a - 1 - \int_t^b \left(1 - \frac{f(x)}{\overline{F}(t)}\right)^\beta dx\right) \\ &\geq (\leq) \frac{1}{\beta - 1}(t - a) \geq (\leq) 0. \end{aligned}$$

Similarly, from Eq (2.4), the result follows. □

Example 2.1. *Suppose that the C-RV X has a continuous uniform distribution over $[a, b]$, $-\infty < a < b < \infty$, denoted by $U(a, b)$, with a CDF $F(x) = \frac{x-a}{b-a}$ and PDF $f(x) = \frac{1}{b-a}$. Then, from (2.3) and (2.4), the residual and past functions for Tsallis extropy are given, respectively, by*

$$RTx_\beta(X;t) = \frac{1}{\beta - 1}\left(b - a - 1 - (b - t)\left(1 - \frac{1}{(b - t)}\right)^\beta\right),$$

$$PTx_\beta(X;t) = \frac{1}{\beta - 1}\left(b - a - 1 - (-a + t)\left(1 - \frac{1}{(-a + t)}\right)^\beta\right).$$

Example 2.2. *Suppose that the C-RV X has a power function distribution with a CDF and PDF shown, respectively, by*

$$F(x) = \left(\frac{x}{\lambda}\right)^\theta,$$

$$f(x) = \frac{\theta x^{(\theta - 1)}}{\lambda^\theta}, 0 \leq x \leq \lambda \text{ and } \theta, \lambda > 0.$$

Then, from Eqs (2.3) and (2.4), the residual and past functions for Tsallis extropy are given, respectively, by

$$RTx_\beta(X;t) = \frac{1}{\beta - 1}\left(\lambda - 1 - \int_t^\lambda \left(1 + \frac{\theta x^{(\theta - 1)}}{(t^\theta - \lambda^\theta)}\right)^\theta dx\right),$$

$$PTx_\beta(X;t) = \frac{1}{\beta - 1}\left(\lambda - 1 - \int_a^t (1 - \theta t^{-\theta} x^{(\theta - 1)})^\theta dx\right).$$

Using different values of θ and λ, Figure 1 gives the residual and past functions for Tsallis extropy of the power function distribution.

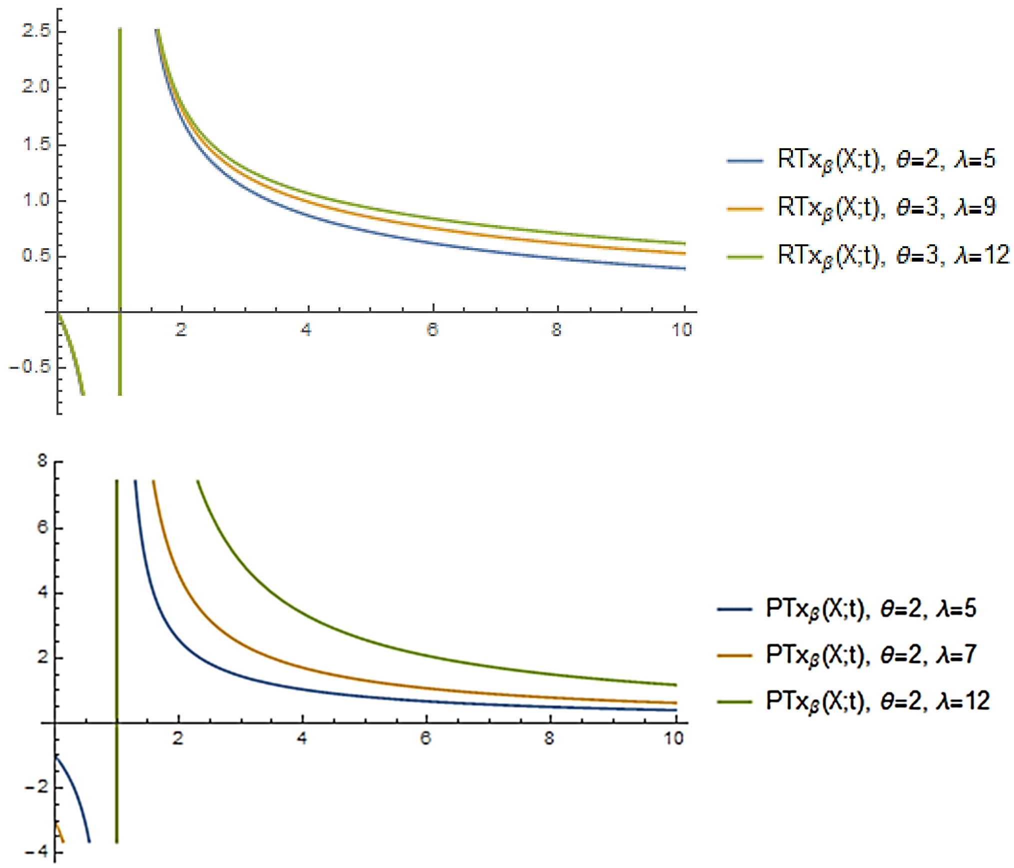

Figure 1. Power function distributions for residual function for Tsallis extropy with $t = 1$ (upper panel) and past function for Tsallis extropy with $t = 3$ (lower panel).

Proposition 2.3. *Suppose that X is a non-negative C-RV backed by* $[a, b]$, $0 < a < b < \infty$, *with the PDF* $f(.)$ *and CDF* $F(.)$. *From Eqs (2.3) and (2.4), we have the following properties.*

i) From Eq (2.3) and under the conditions that $0 < \frac{f(x)}{\overline{F}(t)} < 1$, $\beta \neq 1$ *and* $\beta > 0$, *we have*

$$RTx_\beta(X;t) \leq \frac{1}{\beta - 1}\left(t - a - 1 + \beta\right).$$

ii) From Eq (2.4) and under the conditions that $0 < \frac{f(x)}{F(t)} < 1$, $\beta \neq 1$ *and* $\beta > 0$, *we have*

$$PTx_\beta(X;t) \leq \frac{1}{\beta - 1}\left(b - t - 1 + \beta\right).$$

Proof. *i*) Property (*i*) can be obtained from Eq (2.3) and under the conditions $0 < \frac{f(x)}{\overline{F}(t)} < 1, \beta \neq 1$ and $\beta > 0$. Furthermore, employing Bernoulli's inequality, we have

$$\begin{aligned} RTx_\beta(X;t) &= \frac{1}{\beta-1}\left(b-a-1-\int_t^b\left(1-\frac{f(x)}{\overline{F}(t)}\right)^\beta dx\right) \\ &\leq \frac{1}{\beta-1}\left(b-a-1-\int_t^b\left(1-\beta\frac{f(x)}{\overline{F}(t)}\right)dx\right) \\ &= \frac{1}{\beta-1}\left(t-a-1+\beta\int_t^b\frac{f(x)}{\overline{F}(t)}dx\right) \\ &= \frac{1}{\beta-1}\left(t-a-1+\beta\right). \end{aligned}$$

Similarly, (*ii*) can be obtained. □

Definition 2.3. *Suppose that X is a C-RV backed by* $[a,b]$, $-\infty < a < b < \infty$. *Then,*

(1) X is a decreasing (increasing) residual function for Tsallis extropy of order β *(*$DRTEX_\beta$ *(*$IRTEX_\beta$*)) if* $RTx_\beta(X;t)$ *is decreasing (increasing) in terms of t, where* β *is defined in Eq (2.3).*

(2) X is a decreasing (increasing) past function for Tsallis extropy of order β *(*$DPTEX_\beta$ *(*$IPTEX_\beta$*)) if* $PTx_\beta(X;t)$ *is decreasing (increasing) in terms of t, where* β *is defined in Eq (2.4).*

Proposition 2.4. *Let X be a C-RV backed by* $[a,b]$, $-\infty < a < b < \infty$, *with a PDF* $f(.)$. *Therefore, an alternative representation of the residual function for Tsallis extropy with respect to the hazard rate function* $\psi(x) = \frac{f(x)}{\overline{F}(x)}$ *is given by*

$$RTx_\beta(X;t) = \frac{1}{\beta-1}\left(b-a-1+\sum_{i=0}^{A_\beta}\binom{\beta}{i}\frac{1}{i}E[(-\psi(X_{i,t}))^{i-1}]\right), \tag{2.7}$$

where

$$A_\beta = \begin{cases} \beta, & \beta \in \mathbb{Z}^+\backslash\{1\}; \\ \infty, & \beta \neq 1, \beta > 0 \text{ when } \frac{f(x)}{\overline{F}(t)} < 1, \end{cases}$$

and the RV $X_{i,t}$ *has the PDF*

$$f_{X_{i,t}}(x,t) = \frac{if(x)}{\overline{F}(t)}\left(\frac{\overline{F}(x)}{\overline{F}(t)}\right)^{i-1}, x \geq t > 0.$$

Proposition 2.5. *Let X be a C-RV backed by* $[a,b]$, $-\infty < a < b < \infty$, *with a PDF* $f(.)$. *Therefore, an alternative representation of the past function for Tsallis extropy with respect to the reversed hazard rate function* $\Omega(x) = \frac{f(x)}{F(x)}$ *is given by*

$$PTx_\beta(X;t) = \frac{1}{\beta-1}\left(b-a-1+\sum_{i=0}^{B_\beta}\binom{\beta}{i}\frac{1}{i}E[(-\Omega(X_{i,t}))^{i-1}]\right), \tag{2.8}$$

where

$$B_\beta = \begin{cases} \beta, & \beta \in \mathbb{Z}^+ \backslash \{1\}; \\ \infty, & \beta \neq 1, \beta > 0 \text{ when } \frac{f(x)}{F(t)} < 1, \end{cases}$$

and the RV $X_{i,t}$ has the PDF

$$g_{X_{i,t}}(x,t) = \frac{if(x)}{F(t)} \left(\frac{F(x)}{F(t)} \right)^{i-1}, a < t < x < b.$$

Lemma 2.1. *According to Eqs (2.7) and (2.8), we have*

$$\frac{d}{dt} RTx_\beta(X;t) = \frac{\psi(t)}{\beta - 1} \left(-\sum_{i=0}^{A_\beta} \binom{\beta}{i} (-\psi(t))^{i-1} + \sum_{i=0}^{A_\beta} \binom{\beta}{i} E[(-\psi(X_{i,t}))^{i-1}] \right), \tag{2.9}$$

$$\frac{d}{dt} PTx_\beta(X;t) = \frac{\Omega(t)}{\beta - 1} \left(\sum_{i=0}^{B_\beta} \binom{\beta}{i} (-\Omega(t))^{i-1} - \sum_{i=0}^{B_\beta} \binom{\beta}{i} E[(-\Omega(X_{i,t}))^{i-1}] \right) \tag{2.10}$$

for all $t \geq 0$.

Remark 2.1. *(1) If X is $DRTEX_\beta$ ($IRTEX_\beta$), then $\frac{d}{dt}RTx_\beta(X;t) = 0$ and we have* $\sum_{i=0}^{A_\beta} \binom{\beta}{i} E[(-\psi(X_{i,t}))^{i-1}] = \sum_{i=0}^{A_\beta} \binom{\beta}{i} (-\psi(t))^{i-1}$.

(2) If X is $DPTEX_\beta$ ($IPTEX_\beta$), then $\frac{d}{dt}PTx_\beta(X;t) = 0$ and we have $\sum_{i=0}^{B_\beta} \binom{\beta}{i} E[(-\Omega(X_{i,t}))^{i-1}] = \sum_{i=0}^{B_\beta} \binom{\beta}{i} (-\Omega(t))^{i-1}$.

In the upcoming theorem, we examine the connection between $IRTEX_\beta$ with increasing failure rate (IFR) and $DPTEX_\beta$ with decreasing reversed failure rate (DRFR).

Theorem 2.1. *Suppose that X is a non-negative C-RV backed by $[a,b]$, $0 < a < b < \infty$, with a PDF $f(.)$ and CDF $F(.)$.*

(1) From Eqs (2.3) and (2.7), when $\beta > 1$ and $\frac{f(x)}{\overline{F}(t)} \leq 1$, if X is IFR, X is $IRTEX_\beta$.

(2) From Eqs (2.4) and (2.8), when $\beta > 1$ and $\frac{f(x)}{F(t)} \leq 1$, if X is DRFR, X is $DPTEX_\beta$.

Proof. (1) Let X be the IFR; then, $\psi(x)$ is increasing in terms of x. From Eqs (2.3) and (2.7), when $\beta > 1$ and $\frac{f(x)}{\overline{F}(t)} \leq 1$, we have

$$\begin{aligned} \sum_{i=0}^{A_\beta} \binom{\beta}{i} E[(-\psi(X_{i,t}))^{i-1}] &= \sum_{i=0}^{A_\beta} \binom{\beta}{i} \int_t^b (-\psi(x))^{i-1} f_{X_{i,t}}(x,t) dx \\ &\geq \sum_{i=0}^{A_\beta} \binom{\beta}{i} (-\psi(t))^{i-1} \end{aligned}$$

for $t \geq 0$ and from Eq (2.9) we get the result.

(2) Let X be the DRFR; then $\Omega(x)$ is decreasing in terms of x. From Eqs (2.4) and (2.8), when $\beta > 1$ and $\frac{f(x)}{F(t)} \leq 1$, we have

$$\sum_{i=0}^{B_\beta} \binom{\beta}{i} E[(-\Omega(X_{i,t}))^{i-1}] = \sum_{i=0}^{B_\beta} \binom{\beta}{i} \int_t^b (-\Omega(x))^{i-1} g_{X_{i,t}}(x,t)dx$$
$$\leq \sum_{i=0}^{B_\beta} \binom{\beta}{i} (-\Omega(t))^{i-1}$$

for $t \geq 0$ and from Eq (2.10) we get the result.

□

The plots of the residual and past functions for Tsallis extropy in Figure 2 show that X is not $IRTEX_\beta$ or $DPTEX_\beta$.

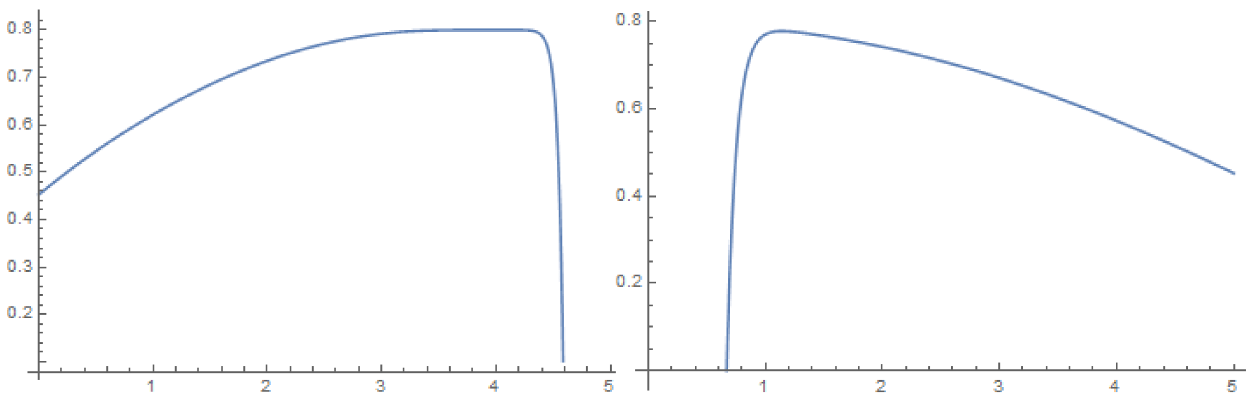

Figure 2. Power function distribution ($\beta = 6, \theta = 2, \lambda = 5$) for the residual function for Tsallis extropy (left panel) and past function for Tsallis extropy (right panel) with respect to t.

In the next part, we will obtain some interesting residual and past functions for Tsallis extropy when the order $\beta = 2$ is selected.

Remark 2.2. *According to Definitions 2.1 and 2.2, the residual and past functions for Tsallis extropy of order $\beta = 2$ is selected; then, they are accurate for both $f(x) \leq 1$ or $f(x) > 1$.*

The following example gives the residual and past functions for Tsallis extropy of order $\beta = 2$ for the finite range.

Example 2.3. *Suppose that the C-RV X has a continuous finite range with a beta distribution function $F(x) = 1 - (1-x)^\theta$ and PDF $f(x) = \theta(1-x)^{\theta-1}$, $x \in (0,1)$, $\theta > 1$. Then, from Eqs (2.3) and (2.4), the residual and past functions for Tsallis extropy of order $\beta = 2$ are given, respectively, by*

$$RTx_2(X;t) = 1 + t + \frac{\theta^2}{((-1+t)(-1+2\theta))},$$
$$PTx_2(X;t) = 2 - t - \frac{((-1+(1-t)^{2\theta}+t)\theta^2)}{((-1+(1-t)^\theta)^2(-1+t)(-1+2\theta))}.$$

Proposition 2.6. *Suppose that X is a C-RV backed by $[a, b]$, $-\infty < a < b < \infty$. Then, from Eqs (2.3), (2.4), (1.4) and (1.5), we have*

(1) $RTx_2(X; t) = t - a + RTn_2(X; t) = t - a + 1 + 2RExt(X_t^{(R)})$,

(2) $PTx_2(X; t) = b - t + PTn_2(X; t) = b - t + 1 + 2PExt(X_t^{(R)})$,

where $RTn_\beta(X; t) = \frac{1}{\beta-1}\left(1 - \int_t^b (1 - \frac{f(x)}{\overline{F}(t)})^\beta dx\right)$ *and* $PTn_\beta(X; t) = \frac{1}{\beta-1}\left(1 - \int_a^t (1 - \frac{f(x)}{F(t)})^\beta dx\right)$ *are the residual and past functions for Tsallis entropy of order β, respectively; for more details about those measures, see [1].*

Proof. From Eq (2.3), when $\beta = 2$, we have

$$\begin{aligned} RTx_2(X; t) &= \frac{1}{2-1}\left(b - a - 1 - \int_t^b \left(1 - \frac{f(x)}{\overline{F}(t)}\right)^2 dx\right) \\ &= b - a - 1 - \left(b - t - 2 + \int_t^b \left(1 - \frac{f(x)}{\overline{F}(t)}\right)^2 dx\right) \\ &= t - a + RTn_2(X; t) = t - a + 1 + 2RExt(X_t^{(R)}). \end{aligned}$$

It is similar for $PTx_2(X; t)$. □

Theorem 2.2. *The residual and past functions for Tsallis entropies of order* 2 *are uniquely determined by the hazard rate function $\psi(t)$ and reversed hazard rate function $\Omega(t)$, $t \geq 0$.*

Proof. From Eq (2.3), when $\beta = 2$, we have

$$\frac{d}{dt}RTx_2(X; t) = 1 + 2(a - t - 1)\psi(t) + \psi^2(t) + 2\psi(t)RTx_2(X; t).$$

Therefore, we get

$$\frac{d}{dt}RTx_2(X; t) - 2\psi(t)RTx_2(X; t) = 1 + 2(a - t - 1)\psi(t) + \psi^2(t). \tag{2.11}$$

We can solve the previous first-order linear ordinary differential equation with a varying coefficient $\psi(t)$ by using the integrating factor method (IFM). Thus

$$RTx_2(X; t) = e^{2\int_t \psi(t_1)dt_1}\left[\int_t (1 + 2(a - t_2 - 1)\psi(t_2) + \psi^2(t_2))e^{-2\int_{t_2}\psi(t_1)dt_1} dt_2 + \mathbb{C}\right], \tag{2.12}$$

where $\mathbb{C}$ is a constant and $RTx_2(X; t)|_{t=0} = Tx_2(X)$. Similarly, for $PTx_2(X; t)$ and from Eq (2.4), when $\beta = 2$, we have

$$\frac{d}{dt}PTx_2(X; t) = -1 - 2(t - b - 1)\Omega(t) - \Omega^2(t) - 2\Omega(t)PTx_2(X; t).$$

Therefore, we get

$$\frac{d}{dt}PTx_2(X; t) + 2\Omega(t)PTx_2(X; t) = -1 - 2(t - b - 1)\Omega(t) - \Omega^2(t). \tag{2.13}$$

We can solve the previous first-order linear ordinary differential equation with a varying coefficient $\Omega(t)$ by using the IFM. Thus

$$PTx_2(X;t) = e^{-2\int_t \psi(t_1)dt_1}\left[\int_t (-1-2(t-b-1)\Omega(t)-\Omega^2(t))e^{2\int_{t_2}\psi(t_1)dt_1}dt_2 + \mathbb{G}\right], \tag{2.14}$$

where $\mathbb{G}$ is a constant and $PTx_2(X;t)|_{t=0} = Tx_2(X)$. This completes the proof. □

Remark 2.3. *According to Eqs (2.11) and (2.13), we can state that*

(1) $RTx_2(X;t)$ is decreasing (increasing) in terms of t if and only if $RTx_2(X;t) \leq (\geq) t-a+1-\frac{1+\psi^2(t)}{2\psi(t)}$;

(2) $PTx_2(X;t)$ is decreasing (increasing) in terms of t if and only if $PTx_2(X;t) \leq (\geq) b-t+1-\frac{1+\Omega^2(t)}{2\Omega(t)}$.

In what follows, we characterize the distribution of the finite range from the perspective of the residual function for Tsallis extropy.

Theorem 2.3. *Suppose that X is a C-RV with failure rate $\psi(.)$. If $RTx_2(X;t) = t+1-2k\psi(t)$, where $t \geq 0$ and the non-negative constant $k \geq 0$; thus, X follows a distribution of finite range if and only if $k > \frac{1}{4}$.*

Proof. According to Example 2.3, the necessary condition is obtained. In what follows, we will discuss the sufficient part, assuming that $RTx_2(X;t) = t+1-2k\psi(t)$, $t \geq 0$. From (2.11), we can see that

$$\frac{\psi'(t)}{\psi^2(t)} = \frac{4k-1}{2k}, t \geq 0.$$

By resolving the equation provided above, we get that $\psi(t) = \frac{1}{qt+s}$, $t \geq 0$; $q = \frac{1-4k}{2k}$ and $s = \frac{1}{\psi(0)}$. Therefore, if $k > \frac{1}{4}$, then $p > 0$ and $\psi(t)$ is the failure rate of the distribution of the finite range, which is uniquely determined by its failure rate. □

Residual and past functions for Tsallis extropy of order statistics

Suppose that $X_1, X_2, ..., X_n$ are n independent random samples from a population with a PDF $f(.)$ and CDF $F(.)$. Then, $X_{1:n}, X_{2:n}, ..., X_{n:n}$ are the order statistics (O.S.) of the random samples, and the *ith* O.S., $1 \leq i \leq n$, is given by

$$f_{i:n}(x) = \frac{F^{i-1}(x)\overline{F}^{n-i}(x)f(x)}{\mathfrak{B}(i, n-i+1)}, \tag{2.15}$$

where $\mathfrak{B}(i, n-i+1)$ is the beta function.

Proposition 2.7. *From Eqs (2.3) and (2.4), suppose $RTx_\beta(X_{i:n};t)$ and $PTx_\beta(X_{i:n};t)$ are the residual and past functions for Tsallis extropy of the* ith *O.S. $X_{i:n}$, $1 \leq i \leq n$, respectively. Then, we can conclude the following:*

(1) From Eq (2.3), we have

$$RTx_\beta(X_{i:n};t) \leq \frac{1}{\beta-1}(t-a-1+\beta),$$

where $0 \leq \frac{f(x)}{\overline{F}(t)} \leq 1$.

(2) From Eq (2.3), we have

$$\begin{aligned} RTx_2(X_{i:n};t) &= t - a + 1 + 2RExt(X_{i:n};t) \\ &= t - a + RTn_2(X_{i:n};t). \end{aligned}$$

(3) From Eq (2.4), we have

$$PTx_\beta(X_{i:n};t) \le \frac{1}{\beta-1}(b - t - 1 + \beta),$$

where $0 \le \frac{f(x)}{F(t)} \le 1$.

(4) From Eq (2.4), we have

$$\begin{aligned} PTx_2(X_{i:n};t) &= b - t + 1 + 2PExt(X_{i:n};t) \\ &= b - t + PTn_2(X_{i:n};t). \end{aligned}$$

Proof. From Eqs (2.3) and (2.4), the residual and past functions for Tsallis extropy of the *ith* O.S. $X_{i:n}$, $1 \le i \le n$, respectively, are given by

$$\begin{aligned} RTx_\beta(X_{i:n};t) &= \frac{1}{\beta-1}\left(b - a - 1 - \int_t^b \left(1 - \frac{f_{i:n}(x)}{\overline{F}_{i:n}(t)}\right)^\beta dx\right), \\ PTx_\beta(X_{i:n};t) &= \frac{1}{\beta-1}\left(b - a - 1 - \int_a^t \left(1 - \frac{f_{i:n}(x)}{F_{i:n}(t)}\right)^\beta dx\right), \end{aligned}$$

where $F_{i:n}(t)$ is the CDF of the *ith* O.S. $X_{i:n}$, $1 \le i \le n$. Then, the results follow. □

Theorem 2.4. *For* $t \ge 0$*, if* $RTx_2(X_{i:n};t) = t + 1 - 2k\psi(t)$*, where* $t \ge 0$ *and the non-negative constant* $k \ge 0$*, then X follows a distribution of finite range if and only if* $k > \frac{n}{4}$.

3. Further properties of Tsallis extropy

From the continuous Tsallis extropy definition presented in Eq (1.7), we can represent the continuous Tsallis extropy of the RV X backed by $[a,b]$, $-\infty < a < b < \infty$, as follows

Proposition 3.1. *Let X be a C-RV backed by* $[a,b]$*,* $-\infty < a < b < \infty$*, with a PDF* $f(.)$*. Therefore, an alternative representation of the Tsallis extropy in terms of the hazard rate function* $\psi(x) = \frac{f(x)}{\overline{F}(x)}$ *is given by*

$$\begin{aligned} Tx_\beta(X) &= \frac{1}{\beta-1}\left(b - a - 1 - \sum_{i=0}^{T_\beta}\binom{\beta}{i}\int_a^b (-f(x))^i dx\right) \\ &= \frac{1}{\beta-1}\left(b - a - 1 + \sum_{i=0}^{T_\beta}\binom{\beta}{i}\frac{1}{i}E[(-\psi(X_i))^{i-1}]\right), \end{aligned} \tag{3.1}$$

where

$$T_\beta = \begin{cases} \beta, & \beta \in \mathbb{Z}^+\backslash\{1\}; \\ \infty, & \beta \ne 1, \beta > 0 \text{ when } f(x) < 1, \end{cases}$$

and the RV X_i *has the PDF*

$$f_{X_i}(x,t) = i\overline{F}^{i-1}(t)f(x).$$

According to Shaked and Shanthikumar [26], we will utilize some stochastic orders known as stochastic order ($\leq_{ST}$), hazard rate ($\leq_{HR}$) order and dispersive order ($\leq_{DIS}$) (the order of variability distribution). Moreover, the previous orders indicate the following:
1- $\leq_{HR} \Longrightarrow \leq_{ST}$;
2- $\leq_{DIS} \Longrightarrow \leq_{ST}$.

Definition 3.1. *Suppose that X_1 and X_2 are non-negative C-RVs backed by $[a, b]$, $0 < a < b < \infty$. Then, X_1 is smaller than X_2 for the case of Tsallis extropy of order β, ($X_1 \leq_{Tx} X_2$) if $Tx_\beta(X_1) \leq Tx_\beta(X_2)$, where β is defined in Eq (1.7).*

Theorem 3.1. *Suppose that X_1 and X_2 are non-negative C-RVs backed by $[a, b]$, $0 < a < b < \infty$, with PDFs f_1, f_2 and CDFs F_1, F_2, respectively. From (3.1), if $X_1 \leq_{DIS} X_2$ then $X_1 \leq_{Tx} X_2$.*

Proof. From (3.1) with $\beta > 1$ (< 1). If $X_1 \leq_{DIS} X_2$, then

$$\begin{aligned}(\beta-1)Tx_\beta(X_2) &= b-a-1-\sum_{i=0}^{T_\beta}\binom{\beta}{i}(-1)^i\int_0^1 f_2^{i-1}(F_2^{-1}(u))du\\ &\leq (\geq) b-a-1-\sum_{i=0}^{T_\beta}\binom{\beta}{i}(-1)^i\int_0^1 f_1^{i-1}(F_1^{-1}(u))du = (\beta-1)Tx_\beta(X_1).\end{aligned}$$

Then the result follows for all values of β defined in Eq (1.7). □

The next theorem presents the effect of a transformation on the Tsallis extropy of an RV.

Theorem 3.2. *Suppose that X_1 is a non-negative C-RV backed by $[a, b]$, $0 < a < b < \infty$, with the PDF f_1, and that $X_2 = \varphi(X_1)$ where φ is a continuous function with the derivative $\varphi'(x)$ such that $E(X_2^2) < \infty$. If $|\varphi'(x)| \geq 1$, $\forall\, x$ supported with X_1, then $Tx_\beta(X_1) \leq Tx_\beta(X_2)$, $\forall\, \beta$ is defined in Eq (1.7).*

Proof. Let $X_2 = \varphi(X_1)$ since the Jacobian transformation $J_\varphi(X_2) = |\frac{d\varphi^{-1}(X_2)}{dX_2}|$. Therefore, $f_{X_2}(x) = f_{X_1}(\varphi^{-1}(x))|\frac{1}{\varphi'(\varphi^{-1}(x))}|$. Then,

$$\begin{aligned}Tx_\beta(X_2) &= \frac{1}{\beta-1}\left[b-a-1-\sum_{i=0}^{T_\beta}\binom{\beta}{i}(-1)^i\int_a^b f_{X_2}^i(x)dx\right]\\ &= \frac{1}{\beta-1}\left[b-a-1-\sum_{i=0}^{T_\beta}\binom{\beta}{i}(-1)^i\int_a^b f_{X_1}^i(\varphi^{-1}(x))\left(\frac{1}{\varphi'(\varphi^{-1}(x))}\right)^i dx\right]\\ &= \frac{1}{\beta-1}\left[b-a-1-\sum_{i=0}^{T_\beta}\binom{\beta}{i}(-1)^i\int_0^1 f_{X_1}^i(u)\left(\frac{1}{\varphi'(u)}\right)^{i-1} du\right],\end{aligned}$$

and the rest of the proof is analogous to Theorem 1 in Ebrahimi et al. [5]. □

We consider some aging restrictions of the associated RVs and the order β. The following theorem shows the importance of the stochastic order.

Proposition 3.2. *Suppose that X_1 and X_2 are non-negative C-RVs backed by $[a, b]$, $0 < a < b < \infty$, with PDFs f_1, f_2 and CDFs F_1, F_2, respectively. If $X_1 \leq_{ST} X_2$, then $Tx_\beta(X_1) \geq (\leq) Tx_\beta(X_2)$ for $\beta > 1$ ($\beta < 1$) defined in Eq (1.7).*

Proof. Since $X_1 \leq_{ST} X_2$, $\overline{F}_1(x) \leq \overline{F}_2(x)$. From (1.7), the result follows. $\square$

Tsallis extropy of a mixture of coherent systems

The particular case of a coherent structure is the k-out-of-n system. Moreover, a mixture of coherent schemes is deemed a mixed system; see Samaniego [25]. Under the iid case, the PDF of the mixed system lifetime M is given by

$$f_M(m) = \sum_{j=1}^{n} q_j f_{j:n}(x), \tag{3.2}$$

where $f_{j:n}(x)$ is defined in Eq (2.15), $1 \leq j \leq n$. The system signature is the vector $\mathbf{q} = (q_1, ..., q_n)$, and $q_j = P(M = X_{j:n})$, $\sum_j^n q_j = 1$, $1 \leq j \leq n$. The O.S. $U_{j:n} = F(X_{j:n})$, $1 \leq j \leq n$, has the PDF

$$h_j(u) = \frac{u^{j-1}(1-u)^{n-j}}{\mathfrak{B}(j, n-j+1)}.$$

Therefore, the PDF of $W = F(M)$ is

$$h_W(w) = \sum_{j=1}^{n} q_j h_j(w). \tag{3.3}$$

By using the previous transformations, the following formula discusses the Tsallis extropy of$M$.

Theorem 3.3. *The Tsallis extropy of the mixed system lifetime M is*

$$Tx_\beta(M) = \frac{1}{1-\beta}\left(\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_0^1 h_W^i(w) f^{i-1}(F^{-1}(w)) dw - b + a + 1\right), \tag{3.4}$$

where $h_W(w)$ is defined in Eq (3.3).

Proof. From (3.1), and using the transformation $w = F(m)$, we have

$$\begin{aligned} Tx_\beta(M) &= \frac{1}{1-\beta}\left(\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_a^b \left(\sum_{j=1}^{n} q_j f_{j:n}(m)\right)^i dm - b + a + 1\right) \\ &= \frac{1}{1-\beta}\left(\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_0^1 \left(\sum_{j=1}^{n} q_j \frac{w^{j-1}(1-w)^{n-j}}{\mathfrak{B}(j, n-j+1)}\right)^i f^{i-1}(F^{-1}(w)) dw - b + a + 1\right) \\ &= \frac{1}{1-\beta}\left(\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_0^1 h_W^i(w) f^{i-1}(F^{-1}(w)) dw - b + a + 1\right). \end{aligned} \tag{3.5}$$

$\square$

Theorem 3.4. *Under the same signature, suppose that the lifetime of two mixed systems are M_{X_1} and M_{X_2} with n iid component lifetimes. Then, we have the following:*

(1) If $X_1 \leq_{DIS} X_2$, then $M_{X_1} \leq_{Tx} M_{X_2}$.

(2) *Suppose that* $R_1 = \{0 < w < 1 \mid \frac{f_2(F_2^{-1}(w))}{f_1(F_1^{-1}(w))} < 1\}$ *and* $R_2 = \{0 < w < 1 \mid \frac{f_2(F_2^{-1}(w))}{f_1(F_1^{-1}(w))} \geq 1\}$. *If* $X_1 \leq_{Tx} X_2$, *then* $M_{X_1} \leq_{Tx} M_{X_2}$, *under the condition that* $R_1 = R_2 = \phi$ *or* $\inf_{w\in R_1} h_W(w) \geq \sup_{w\in R_2} h_W(w)$.

Proof. (1) Since $X_1 \leq_{DIS} X_2$, from Eq (3.1), we have

$$\begin{aligned}(1-\beta)\Big(Tx_\beta(M_{X_1}) - Tx_\beta(M_{X_2})\Big) = &\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_0^1 h_W^i(w) \\ &\times \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \geq 0 (\leq 0),\end{aligned}$$

where $\beta > 1 (0 < \beta < 1)$, and the result follows.

(2) Since $X_1 \leq_{Tx} X_2$, from Eq (3.1) when $\beta > 1$, we have

$$\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_0^1 \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \geq 0. \tag{3.6}$$

In the sequel, we get

$$\begin{aligned}(1-\beta)\Big(Tx_\beta(M_{X_1}) - Tx_\beta(M_{X_2})\Big) = &\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_0^1 h_W^i(w) \\ &\times \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw.\end{aligned}$$

Thus, using (3.6) and the given $\inf_{w\in R_1} h_W(w) \geq \sup_{w\in R_2} h_W(w)$ for $\beta > 1$, we obtain

$$\begin{aligned}&\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_{R_1} h_W^i(w) \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \\ &+ \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_{R_2} h_W^i(w) \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \\ &\geq \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i (\inf_{w\in R_1} h_W(w))^i \int_{R_1} \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \\ &+ \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i (\sup_{w\in R_2} h_W(w))^i \int_{R_2} \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \\ &\geq \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i (\sup_{w\in R_2} h_W(w))^i \int_{R_1} \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \\ &+ \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i (\sup_{w\in R_2} h_W(w))^i \int_{R_2} \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \\ &= \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i (\sup_{w\in R_2} h_W(w))^i \int_0^1 \Big(f_1^{i-1}(F_1^{-1}(w)) - f_2^{i-1}(F_2^{-1}(w))\Big)\, dw \geq 0.\end{aligned}$$

Similarly, the result follows for $0 < \beta < 1$.

□

When the components within the system cannot be quantified, or if the system involves a complex function structure, obtaining the Tsallis extropy often becomes challenging. Consequently, establishing the limits of this measure becomes crucial. The subsequent theorem provides the boundaries for Tsallis extropy of the mixed system.

Theorem 3.5. *Suppose that $Tx_\beta(X) < \infty$, from Eq (3.4) with $\beta > 1$ $(0 < \beta < 1)$, we have*

$$Tx_\beta(M) \geq (\leq) \frac{\left(\sup_{w\in(0,1)} h_W(w)\right)^\beta (b-a-1) - b + a + 1}{1-\beta} + \left(\sup_{w\in(0,1)} h_W(w)\right)^\beta Tx_\beta(X).$$

Proof. From (3.4), we have

$$\begin{aligned}
b - a - 1 + (1-\beta)Tx_\beta(M) &= \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_0^1 h_W^i(w) \left(f^{i-1}(F^{-1}(w))\right) dw \\
&\leq \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \left(\sup_{w\in(0,1)} h_W(w)\right)^i \int_0^1 f^{i-1}(F^{-1}(w)) dw \\
&\leq \left(\sup_{w\in(0,1)} h_W(w)\right)^\beta \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_0^1 f^{i-1}(F^{-1}(w)) dw \\
&= \left(\sup_{w\in(0,1)} h_W(w)\right)^\beta [b - a - 1 + (1-\beta)Tx_\beta(X)],
\end{aligned}$$

which proves the theorem. □

In the case of the decreasing failure rate (DFR) of the lifetimes component, the following theorem indicates that the minimum lifetime has a lower or equal Tsallis extropy order in the iid case than for all of the mixed systems.

Theorem 3.6. *Consider the iid case and the lifetime component to be DFR. Then, $X_{1:n} \leq_{Tx} M$, where M is the mixed lifetime system.*

Proof. According to Bagai and Kochar [2], under the condition of the DFR lifetime, we have that $X_{1:n} \leq_{HR} M \Longrightarrow X_{1:n} \leq_{DIS} M$. From Theorem 3.1, we get that $X_{1:n} \leq_{Tx} M$. □

Theorem 3.7. *Suppose that $Tx_\beta(X_{j:n}) < \infty$, from Eq (3.4), we have*

$$Tx_\beta(M) \geq \sum_{j=1}^{n} q_j Tx_\beta(X_{j:n}),$$

where $Tx_\beta(X_{j:n})$ is the Tsallis extropy of the jth O.S.

Proof. Recall Eq (3.5), we have

$$Tx_\beta(M) = \frac{1}{1-\beta}\left(\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_a^b \left(\sum_{j=1}^{n} q_j f_{j:n}(m)\right)^i dm - b + a + 1\right).$$

Using Jensen's inequality, we obtain

$$\left(\sum_{j=1}^{n} q_j f_{j:n}(m)\right)^{\beta} \geq (\leq) \sum_{j=1}^{n} q_j f_{j:n}^{\beta}(m),$$

where $f_{j:n}^{\beta}$ is concave (convex) when $0 < \beta < 1$ ($\beta > 1$) and $m > 0$. Thus,

$$\int_a^b f_M^{\beta}(m)dm = \int_a^b \left(\sum_{j=1}^{n} q_j f_{j:n}(m)\right)^{\beta} dm \geq (\leq) \sum_{j=1}^{n} q_j \int_a^b f_{j:n}^{\beta}(m)dm,$$

$$\Longrightarrow \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_a^b \left(\sum_{j=1}^{n} q_j f_{j:n}(m)\right)^{i} dm \geq (\leq) \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \sum_{j=1}^{n} q_j \int_a^b f_{j:n}^{i}(m)dm; \tag{3.7}$$

multiplying (3.7) by $\frac{1}{1-\beta}$, and noting that $1-\beta > 0$ ($1-\beta < 0$), it holds that

$$\begin{aligned}
Tx_\beta(M) &\geq \frac{1}{1-\beta}\left[\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \sum_{j=1}^{n} q_j \int_a^b f_{j:n}^{i}(m)dm - b + a + 1\right] \\
&= \frac{1}{1-\beta}\left[\sum_{j=1}^{n} q_j \sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_a^b f_{j:n}^{i}(m)dm - \sum_{j=1}^{n} q_j (b-a-1)\right] \\
&= \sum_{j=1}^{n} q_j \left[\frac{1}{1-\beta}\left(\sum_{i=0}^{T_\beta} \binom{\beta}{i} (-1)^i \int_a^b f_{j:n}^{i}(m)dm - (b-a-1)\right)\right] \\
&= \sum_{j=1}^{n} q_j \left[\frac{1}{1-\beta}\left(\int_a^b (1-f_{j:n}(m))^{\beta} dm - \int_a^b (1-f_{j:n}(m))dm\right)\right] \\
&= \sum_{j=1}^{n} q_j Tx_\beta(X_{j:n}).
\end{aligned}$$

□

4. Tsallis extropy estimator

The process of measuring the information of C-RVs has gained the interest of numerous researchers; see Qiu and Jia [22], Qiu [21], Noughabi and Jarrahiferiz [19], Jahanshahi et al. [6], and Contreras-Reyes et al. [4]. In this section, we show a non-parametric approach for estimating the extropy of the Tsallis.

4.1. The proposed estimator

Using the operator for Vasicek's difference (see Vasicek [30] and Kayal and Balakrishnan [9]), the estimate is produced by utilizing the empirical CDF F_n in place of the CDF F and substituting a difference operator for a differential operator. Then, a function for the the O.S. is applied to estimate the derivative of $F^{-1}(q)$. Therefore, from (3.1), the Tsallis extropy estimator can be provided as follows

$$\begin{aligned} Tx_{\beta,nm}(X) &= \frac{1}{\beta-1}\left(b-a-1-\sum_{i=0}^{T_\beta}\binom{\beta}{i}\int_a^b(-f(x))^i dx\right)\\ &= \frac{1}{\beta-1}\left(b-a-1-\sum_{i=0}^{T_\beta}\binom{\beta}{i}(-1)^i\int_0^1\left[\frac{d}{dq}F^{-1}(q)\right]^{-i+1}dq\right)\\ &= \frac{1}{\beta-1}\left(b-a-1-\sum_{i=0}^{T_\beta}\binom{\beta}{i}\frac{(-1)^i}{n}\sum_{j=1}^{n}\left[\frac{n}{G_j m}(X_{j+m}-X_{j-m})\right]^{-i+1}\right), \end{aligned} \tag{4.1}$$

where

$$G_j = \begin{cases} 1+\frac{j-1}{m}, & 1 \le j \le m \\ 2, & m+1 \le j \le n-m \\ 1+\frac{n-j}{m}, & n-m+1 \le j \le n, \end{cases} \tag{4.2}$$

$a = X_{1:n} \le X_{2:n} \le ... \le X_{n:n} = b$, the window size positive integer $m < \frac{n}{2}$ and $X_i = X_1$ if $i < 1$ and $X_i = X_n$ if $i > n$.

The proposed Tsallis extropy estimators are demonstrated to be consistent by the following theorem. Vasicek [30] has stated that its proof is apparent, so it is ignored.

Theorem 4.1. *Suppose that the random sample $X_1, X_2, ..., X_n$ has a CDF F, a PDF f and finite variance. Then,*

$$Tx_{\beta,nm}(X) \xrightarrow{P} Tx_\beta(X),$$

as $n \longrightarrow \infty$, $m \longrightarrow \infty$ and $\frac{m}{n} \longrightarrow \infty$.

We have generated the data from $U(a, b)$ distribution and calculated the Tsallis extropy estimation. Table 1 contains the root mean squared error (RMSE) and standard deviation (SD) of the Tsallis extropy estimates after repetition 1000 times for each sample size. If $\beta = 2$, then the Tsallis extropies of $U(0, 1)$, $U(0, 2)$, $U(0, 3)$ are 0, 0.5, $\frac{2}{3}$, respectively. If $\beta = 3$, then the Tsallis extropies of $U(0, 1)$, $U(0, 2)$, $U(0, 3)$ are 0, $\frac{3}{8}$, $\frac{5}{9}$, respectively. Figures 3 and 4 show the behavior of the estimated value to the theoretical value. We can conclude the following from Table 1 and Figures 3 and 4:

(1) Under a fixed n, the RMSE increases by increasing m.

(2) Under a large and fixed n, the RMSE increases by increasing the range of a and b in the $U(a, b)$ distribution.

(3) The SD decreases by increasing n and m.

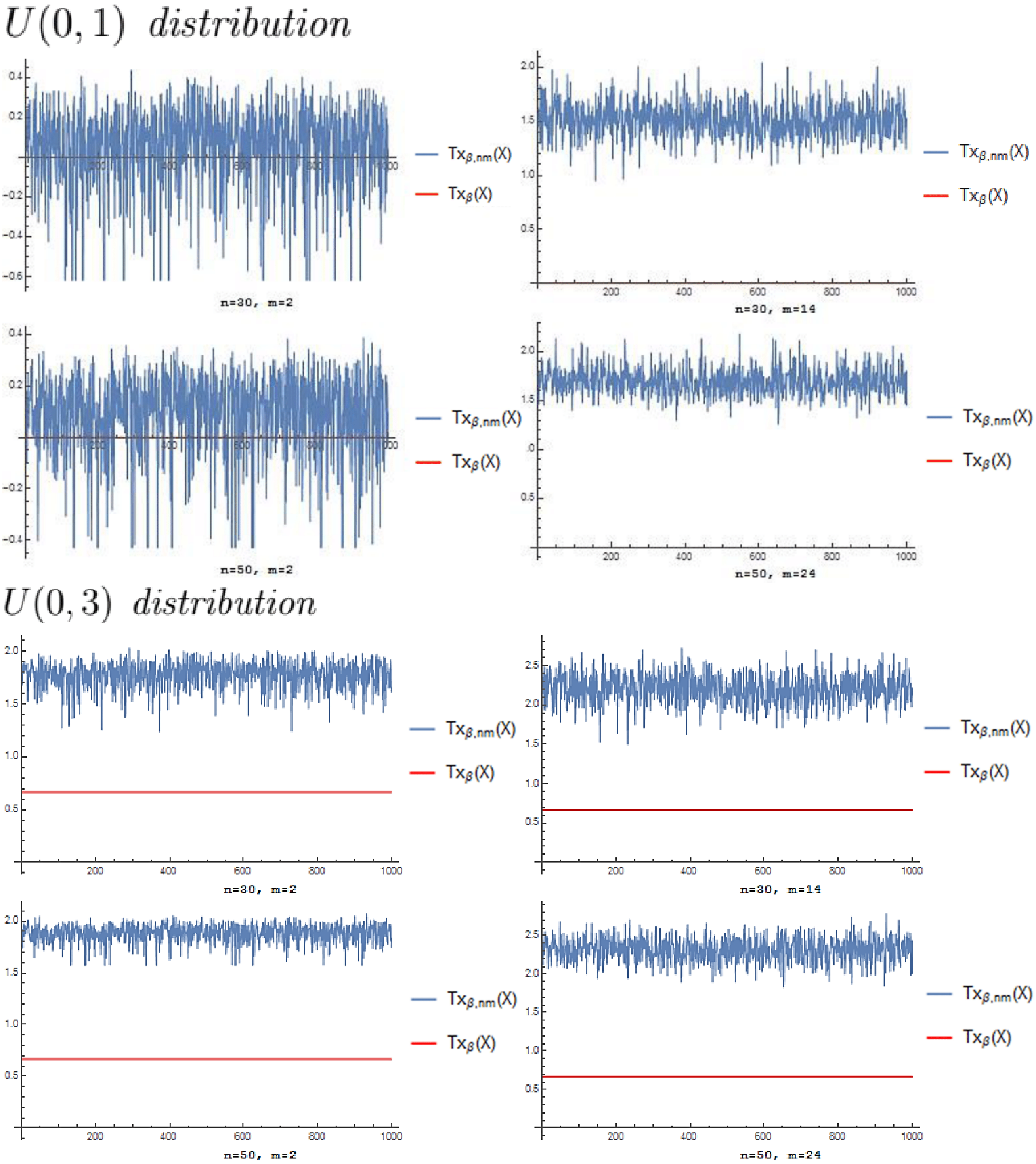

Figure 3. Tsallis extropy estimator for simulated $U(0, 1)$ (upper panel) and $U(0, 3)$ (lower panel) when $\beta = 2$.

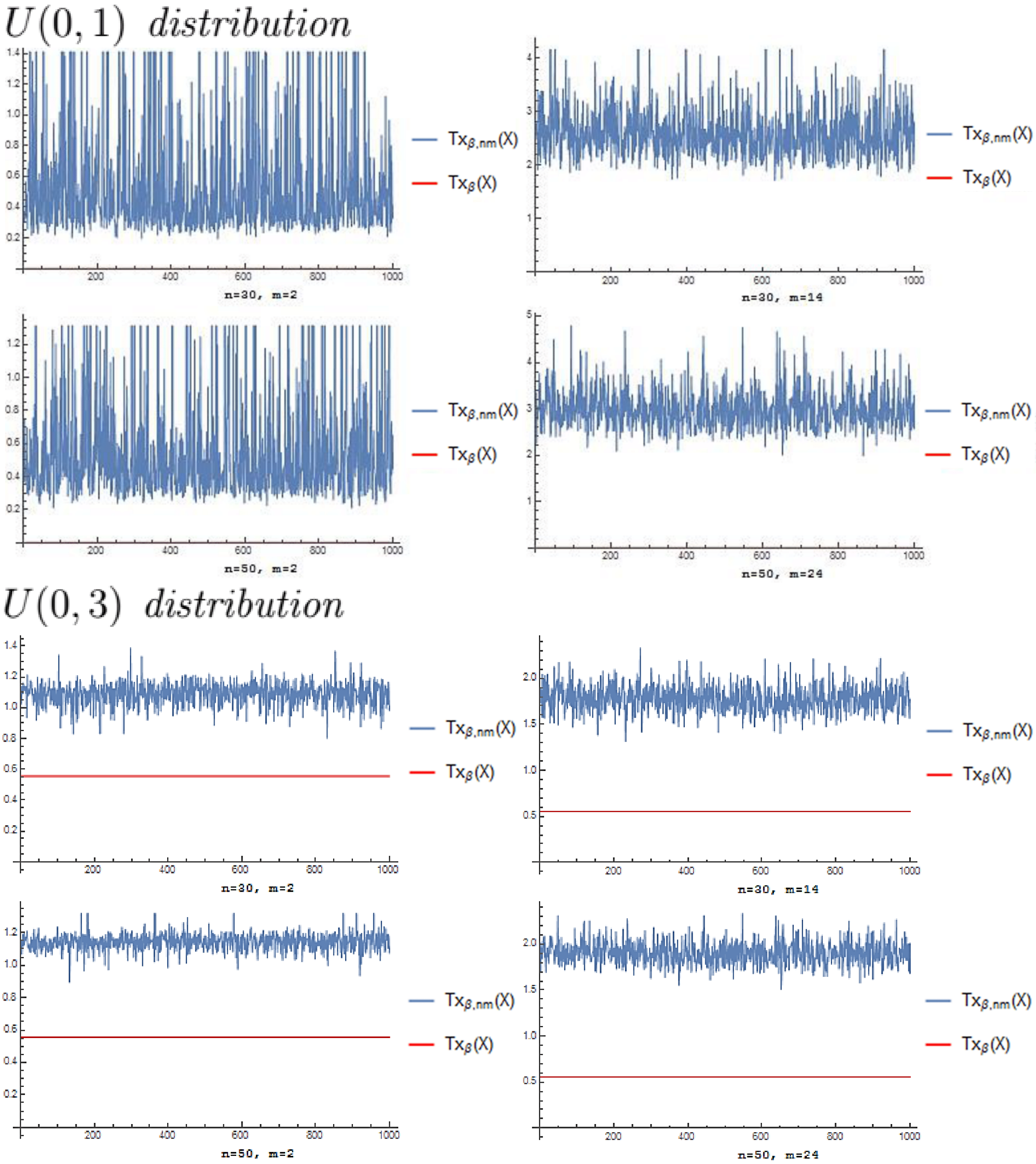

Figure 4. Tsallis extropy estimator for simulated $U(0, 1)$ (upper panel) and $U(0, 3)$ (lower panel) when $\beta = 3$.

Table 1. RMSE and SD results for Tsallis extropy estimator for $U(0,1)$, $U(0,2)$, $U(0,3)$, and $\beta = 2, 3$.

n	m	RMSE (SD) with $\beta = 2$			RMSE (SD) with $\beta = 3$		
		$U(0,1)$	$U(0,2)$	$U(0,3)$	$U(0,1)$	$U(0,2)$	$U(0,3)$
5	2	0.7401 (0.7353)	0.4357 (0.4123)	0.4535 (0.3401)	4.21 (3.98)	0.9013 (0.8204)	0.4233 (0.333)
10	2	0.4975 (0.4821)	0.4552 (0.3015)	0.731 (0.2799)	1.963 (1.796)	0.4461 (0.3418)	0.3868 (0.1735)
	3	0.4367 (0.3326)	0.6363 (0.2466)	0.9241 (0.2646)	1.003 (0.7084)	0.4592 (0.1553)	0.5248 (0.1497)
	4	0.7425 (0.2488)	0.8184 (0.2196)	1.0469 (0.2641)	1.3207 (0.4249)	0.7386 (0.1596)	0.7151 (0.1596)
20	2	0.2948 (0.2949)	0.584 (0.188)	1.0032 (0.1811)	1.051 (0.8495)	0.3727 (0.1575)	0.4897 (0.1034)
	3	0.3308 (0.2139)	0.7706 (0.1645)	1.2119 (0.1817)	0.4719 (0.196)	0.4192 (0.0833)	0.6055 (0.1066)
	4	0.4375 (0.19701)	0.8469 (0.1648)	1.2742 (0.19102)	0.4934 (0.1147)	0.4861 (0.0872)	0.6637 (0.1117)
	5	0.5387 (0.1786)	0.8936 (0.1623)	1.2949 (0.1976)	0.6106 (0.0992)	0.5563 (0.0874)	0.7081 (0.1131)
	6	0.6698 (0.1687)	0.9495 (0.1646)	1.3186 (0.20614)	0.8034 (0.0892)	0.6525 (0.0894)	0.7663 (0.1163)
	7	0.8267 (0.1477)	1.0087 (0.1539)	1.3352 (0.1986)	1.0962 (0.1396)	0.7793 (0.0947)	0.8368 (0.1137)
	8	1.034 (0.1527)	1.0915 (0.1605)	1.3668 (0.2085)	1.5439 (0.2671)	0.9617 (0.1274)	0.9395 (0.1274)
	9	1.3099 (0.1788)	1.2096 (0.1686)	1.4243 (0.21206)	2.226 (0.4766)	1.228 (0.1902)	1.0915 (0.1525)
30	2	0.2245 (0.2195)	0.6526 (0.1354)	1.1158 (0.1281)	0.801 (0.5608)	0.3793 (0.1004)	0.5396 (0.0728)
	3	0.32309 (0.1707)	0.8451 (0.1266)	1.3402 (0.1376)	0.4691 (0.2379)	0.442 (0.0677)	0.6586 (0.08)
	4	0.4102 (0.1509)	0.9155 (0.1268)	1.4077 (0.1478)	0.4096 (0.0951)	0.4863 (0.0697)	0.7071 (0.0873)
	5	0.4735 (0.1517)	0.9486 (0.1335)	1.4294 (0.1601)	0.4515 (0.079)	0.5229 (0.0741)	0.7344 (0.0934)
	6	0.5347 (0.1425)	0.97108 (0.134)	1.43307 (0.1649)	0.5066 (0.0454)	0.5592 (0.0754)	0.7554 (0.0954)
	7	0.6008 (0.1368)	0.9926 (0.1354)	1.4334 (0.1707)	0.5929 (0.0413)	0.6023 (0.0756)	0.7785 (0.0971)
	8	0.6714 (0.1279)	1.0129 (0.1308)	1.4294 (0.1677)	0.7097 (0.0477)	0.6532 (0.0726)	0.8038 (0.0942)
	9	0.7569 (0.1253)	1.0389 (0.1318)	1.4275 (0.1707)	0.8596 (0.0625)	0.717 (0.0727)	0.8363 (0.095)
	10	0.8616 (0.1188)	1.0777 (0.1297)	1.4381 (0.1703)	1.0532 (0.0909)	0.8003 (0.0758)	0.8827 (0.0953)
	11	0.9838 (0.1175)	1.1243 (0.1326)	1.4529 (0.1763)	1.3073 (0.1419)	0.9047 (0.0871)	0.9406 (0.1007)
	12	1.1292 (0.1187)	1.1785 (0.1278)	1.4685 (0.16801)	1.6314 (0.2051)	1.034 (0.1006)	1.0111 (0.1016)
	13	1.3103 (0.1326)	1.2578 (0.1373)	1.5091 (0.1789)	2.0697 (0.3103)	1.2073 (0.1329)	1.1114 (0.1175)
	14	1.5193 (0.15704)	1.3459 (0.1485)	1.5501 (0.1877)	2.6384 (0.4579)	1.4218 (0.1805)	1.2317 (0.1405)
50	2	0.1903 (0.1668)	0.719 (0.0997)	1.2183 (0.0912)	1.1642 (0.9907)	0.4633 (0.2167)	0.5997 (0.0953)
	3	0.3328 (0.1201)	0.9231 (0.0871)	1.4644 (0.0939)	0.4082 (0.1315)	0.4697 (0.0449)	0.7143 (0.0555)
	4	0.4139 (0.1083)	0.9957 (0.08608)	1.5447 (0.0982)	0.3755 (0.0708)	0.5082 (0.0468)	0.7632 (0.0584)
	5	0.4658 (0.1031)	1.0309 (0.0896)	1.57703 (0.1069)	0.3774 (0.0449)	0.5331 (0.0511)	0.7881 (0.0632)
	6	0.5021 (0.1021)	1.0471 (0.0937)	1.5851 (0.1146)	0.3953 (0.0353)	0.5512 (0.0544)	0.8009 (0.0672)
	7	0.5312 (0.0988)	1.0549 (0.09269)	1.5824 (0.1147)	0.4214 (0.0326)	0.566 (0.054)	0.8085 (0.0668)
	8	0.5613 (0.09803)	1.0617 (0.0953)	1.5773 (0.1195)	0.4532 (0.0286)	0.583 (0.0554)	0.8159 (0.069)
	9	0.5913 (0.0988)	1.0667 (0.0978)	1.5694 (0.1238)	0.49201 (0.0276)	0.601 (0.0567)	0.8229 (0.071)
	10	0.6213 (0.0986)	1.0716 (0.0995)	1.5612 (0.1268)	0.53902 (0.0262)	0.6207 (0.0571)	0.8306 (0.0723)
	11	0.6562 (0.0979)	1.0785 (0.10019)	1.5539 (0.1284)	0.5931 (0.02604)	0.6438 (0.057)	0.8404 (0.0727)
	12	0.6933 (0.098)	1.0842 (0.10109)	1.5432 (0.13007)	0.6563 (0.0281)	0.6691 (0.0568)	0.8502 (0.0732)
	13	0.7358 (0.09798)	1.0964 (0.1034)	1.5411 (0.1344)	0.7313 (0.0325)	0.7008 (0.0575)	0.8659 (0.075)
	14	0.782 (0.091301)	1.107 (0.1008)	1.534 (0.1332)	0.8161 (0.0382)	0.7353 (0.0563)	0.8815 (0.0736)
	15	0.8353 (0.09132)	1.1226 (0.10102)	1.5322 (0.1332)	0.9158 (0.0461)	0.7764 (0.0569)	0.9021 (0.0739)
	16	0.8937 (0.09125)	1.1394 (0.1029)	1.5296 (0.1366)	1.0314 (0.0616)	0.823 (0.0593)	0.9252 (0.0758)
	17	0.9607 (0.08766)	1.1631 (0.1026)	1.5344 (0.1378)	1.1669 (0.08305)	0.8787 (0.0627)	0.9552 (0.0771)
	18	1.0361 (0.08854)	1.1906 (0.103)	1.5415 (0.1381)	1.3228 (0.1009)	0.9425 (0.0658)	0.99 (0.0779)
	19	1.118 (0.09106)	1.2202 (0.1063)	1.5485 (0.1429)	1.5074 (0.1331)	1.0155 (0.0749)	1.0293 (0.0827)
	20	1.2109 (0.09113)	1.2561 (0.1034)	1.5609 (0.1387)	1.7206 (0.1677)	1.099 (0.0809)	1.076 (0.0823)
	21	1.3155 (0.1008)	1.2992 (0.1084)	1.5794 (0.1431)	1.9763 (0.2201)	1.199 (0.0991)	1.1323 (0.0913)
	22	1.4295 (0.1075)	1.346 (0.1115)	1.5994 (0.1459)	2.2662 (0.2696)	1.3116 (0.1145)	1.1949 (0.0984)
	23	1.559 (0.11906)	1.4014 (0.1157)	1.6258 (0.14801)	2.6181 (0.3391)	1.4449 (0.1366)	1.27 (0.1087)
	24	1.7022 (0.13414)	1.4636 (0.1228)	1.6571 (0.1533)	3.0253 (0.4178)	1.5971 (0.1629)	1.3559 (0.1224)

4.2. *Real data application*

In this subsection, we utilized the breast cancer Wisconsin (diagnostic) dataset [31], which comprises 569 diagnoses, focusing on presenting real-valued attributes calculated for individual cell nuclei. These features encompass the following: 1) smoothness (reflecting local variations in radius length); 2) compactness (calculated as perimeter^2/area -1); 3) concavity (expressing the degree of concavity in contour segments); 4) concave points (tallying the quantity of concave segments within the contour); 5) symmetry, and 6) fractal dimension (measured via "coastline approximation" -1). Furthermore, the "worst" or most considerable value (mean of the three most significant values) of these attributes was computed for each image. Figures 5 and 6 display the correlation between each variable of the Wisconsin worst breast cancer dataset and their respective histograms. Tables 2 and 3 show the Tsallis extropy estimator results for 569 diagnoses of breast cancer Wisconsin data when $\beta = 2, 3$, neglecting any zero or missing values. Furthermore, Figure 7 shows the Tsallis extropy estimator results for 569 diagnoses of breast cancer Wisconsin data when $\beta = 2, 3$ and $m = 2, 3, ..., 200$. Moreover, we can conclude that the Tsallis extropy estimator increases by increasing m and β.

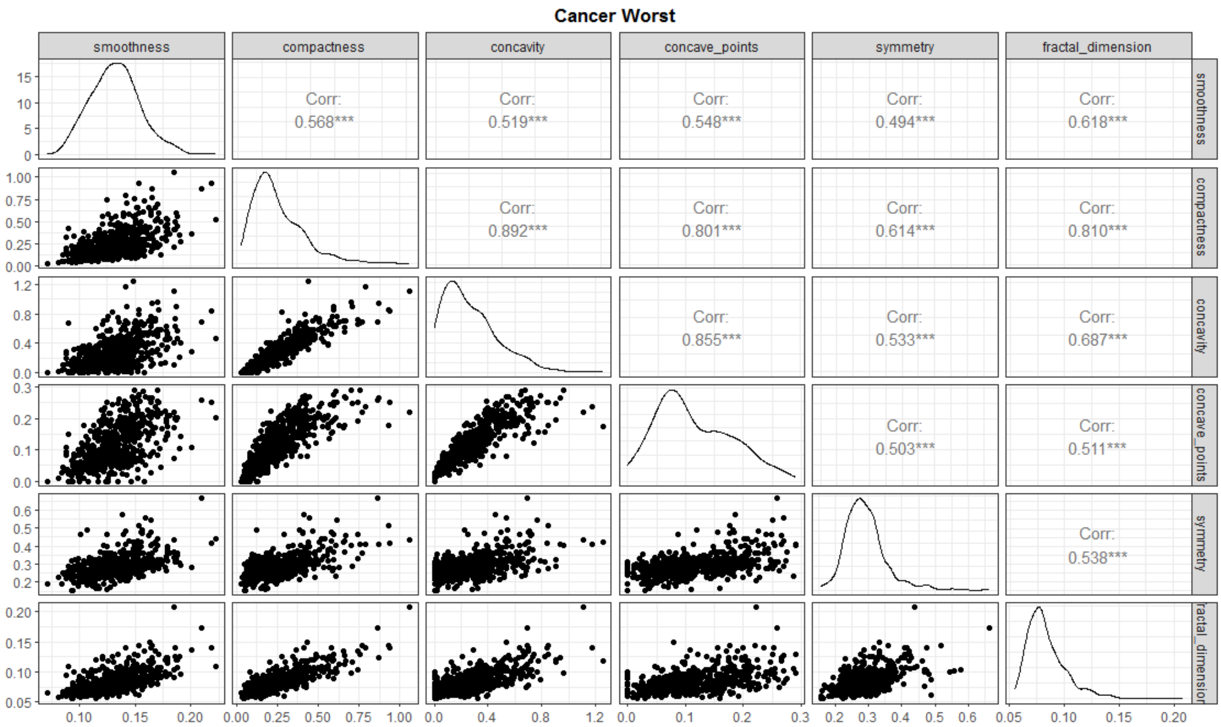

Figure 5. Correlation between each variable for data on the worst breast cancer Wisconsin.

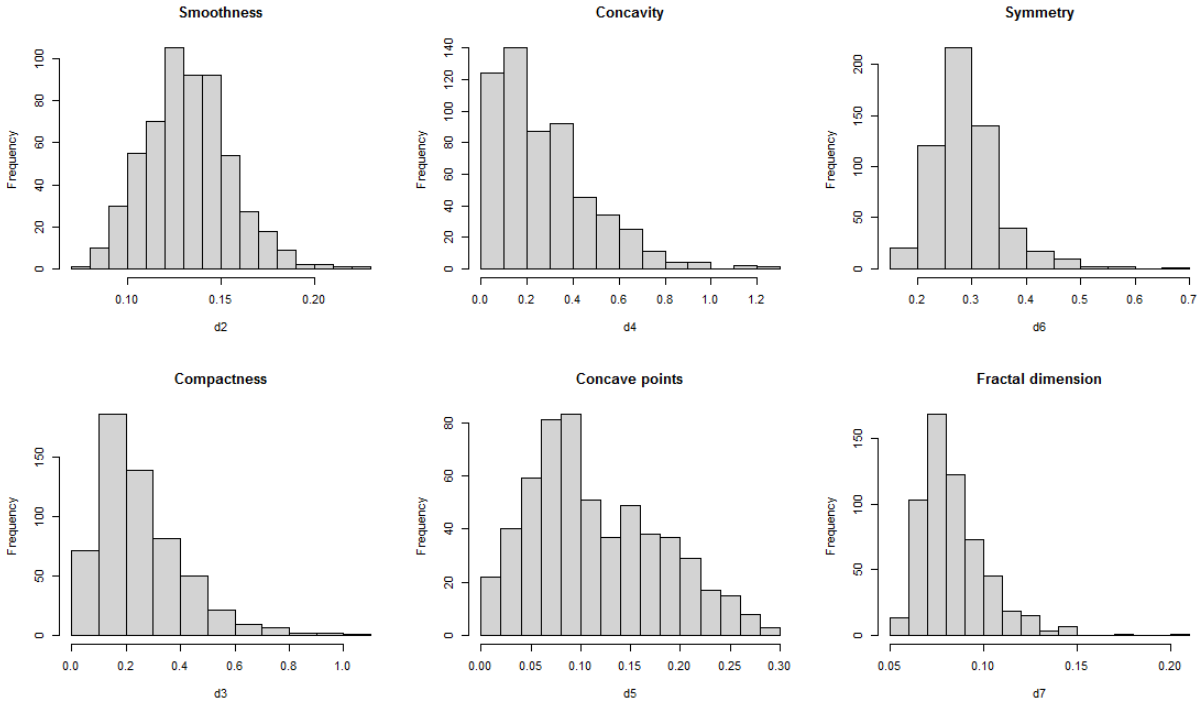

Figure 6. Histograms of worst breast cancer Wisconsin functions.

Table 2. Tsallis extropy estimator results for the 569 diagnosis breast cancer Wisconsin data, with $\beta = 2$.

m	Smoothness	Compactness	Concavity	Concave points	Symmetry	Fractal dimension
2	-16.072	-1.38291	-0.606017	-5.59625	-5.92697	-26.3413
5	-12.9753	-0.817777	-0.0124867	-4.23306	-4.77741	-21.4673
10	-12.1106	-0.634671	0.171713	-3.86458	-4.39443	-19.6003
30	-11.2559	-0.422343	0.389655	-3.40821	-4.03067	-18.1719
50	-10.5923	-0.253445	0.51764	-3.10719	-3.70565	-16.9701
100	-8.49788	0.192588	0.866023	-2.22443	-2.70257	-13.4282
200	-2.84619	1.22606	1.60576	0.0199367	-0.264414	-4.68175

Table 3. Tsallis extropy estimator results for the 569 diagnosis breast cancer Wisconsin data, with $\beta = 3$.

m	Smoothness	Compactness	Concavity	Concave points	Symmetry	Fractal dimension
2	228.075	3.39238	2.19059	34.6507	31.7404	659.563
5	110.648	1.61416	0.794512	14.4676	16.4353	336.679
10	89.3667	1.30072	0.664333	10.7478	13.0594	252.118
30	77.0099	1.25187	0.681	7.7352	11.5776	216.794
50	72.328	1.25625	0.737789	6.93683	10.9402	199.884
100	60.125	1.25529	0.884281	5.50338	9.03925	157.538
200	35.7649	1.3679	1.34623	5.8625	4.57925	58.4933

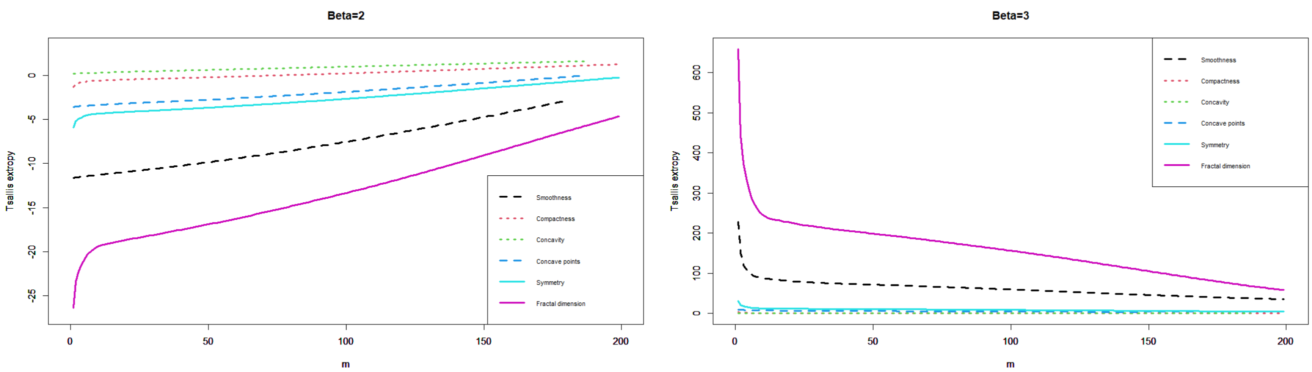

Figure 7. Tsallis extropy estimator results for the 569 diagnosis breast cancer Wisconsin data with $\beta = 2, 3$ and $m = 2, 3, ..., 200$.

Comparative analysis with extropy

In this part, using the breast cancer Wisconsin (diagnostic) dataset, we will compare the Tsallis extropy estimator given by Eq (4.1) with the original extropy estimator proposed in [21] as follows

$$Ex_{\beta,nm}(X) = \frac{-1}{2n}\sum_{j=1}^{n}\frac{G_j m}{n(X_{j+m} - X_{j-m})}, \tag{4.3}$$

where G_j is defined in Eq (4.2). Figure 8 shows the extropy estimator, and in comparison with Figure 7, we can conclude that the Tsallis extropy estimator gives negative and positive values, unlike the extropy estimator, which is known for negative values. Thus, a comparison of Figures 7 and 8 shows that the Tsallis extropy, a complementary dual of the Tsallis entropy, as a new measure of uncertainty, takes more versatile values. This opens the door to a more effective analysis of many disciplines whereby knowledge is evaluated by utilizing probabilistic notions.

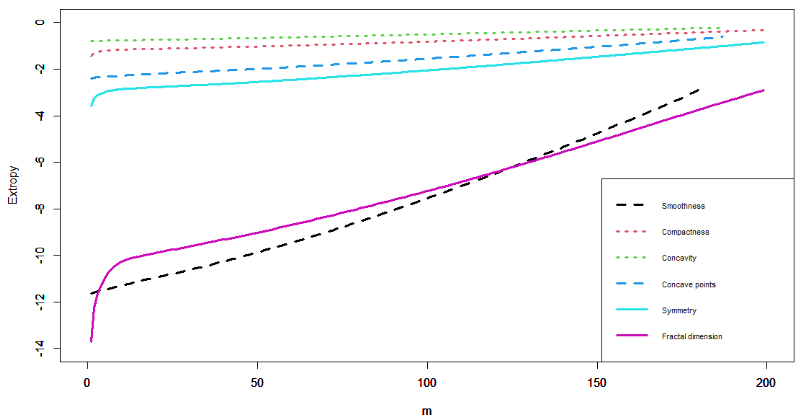

Figure 8. Extropy estimator results for the breast cancer Wisconsin (diagnostic) dataset.

5. Conclusions

We have examined further properties of Tsallis extropy and its related measures under the condition of continuity. The residual and past Tsallis extropy functions were presented and the conditions of negativity and non-negativity were discussed for those models. Examples of different distributions applied to our measures were given. Moreover, bounded and monotonically increasing and decreasing measures were obtained. Besides, the characterization results for those measures were studied. Furthermore, the properties of the corresponding O.S. were discussed. On the other hand, an alternative representation of the continuous Tsallis extropy with connection to stochastic orders was revealed. These discoveries prompted our investigation into Tsalli's extropy for mixed systems and coherent structures within the context of the iid scenario. Besides, we formulated certain limitations on the systems' Tsallis extropy and demonstrated the practicality of the provided constraints. Finally, the Tsallis extropy estimator, as determined by using the Vasicek's difference operator, was applied to simulated data and real data for breast cancer in Wisconsin. The estimators exhibited increases and decreases according to the n, m and β values.

Use of AI tools declaration

The authors affirm that they did not employ Artificial Intelligence (AI) tools in the development of this article.

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Conflict of interest

The authors declare no conflict of interest.

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