AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 28210–28218.
DOI: 10.3934/math.20231444
Received: 03 August 2023
Revised: 07 October 2023
Accepted: 09 October 2023
Published: 13 October 2023

Research article

Blow-up in a p-Laplacian mutualistic model based on graphs

Ling Zhou*and Zuhan Liu

School of Mathematical Science, Yangzhou University, Yangzhou 225002, China

* **Correspondence:** Email: zhoul@yzu.edu.cn.

Abstract: In this paper, we study a p-Laplacian ($p > 2$) reaction-diffusion system based on weighted graphs that is used to describe a network mutualistic model of population ecology. After overcoming difficulties caused by the nonlinear p-Laplacian, we develop a new strong mutualistic condition, and the blow-up properties of the solution for any nontrivial initial data are proved under this condition. In this sense, we extend the blow-up results of models with a graph Laplacian ($p = 2$) to a general graph p-Laplacian.

Keywords: p-Laplacian; network; blow-up; strong mutualistic systems; comparison principle
Mathematics Subject Classification: 35K51, 35R35, 92B05, 35B40

1. Introduction

In recent years, evolution problems on complex networks have been studied extensively, for example, in the field of epidemic processes or population ecology [1–5]. A network is mathematically described as a undirected graph $T = (\Omega, E)$, which contains a set Ω of vertices and a set E of edges (x, y) connecting vertex x and vertex y. If vertices x and y are connected by an edge (i.e., they are adjacent), we write $x \sim y$. T is called a finite-dimensional graph if it has a finite number of edges and vertices. A graph is weighted if each adjacent x and y is assigned a weight function $\omega(x, y)$. Here $\omega : \Omega \times \Omega \to [0, +\infty)$ satisfies that $\omega(x, y) = \omega(y, x)$ and $\omega(x, y) > 0$ if and only if $x \sim y$. Throughout this paper, $T = (\Omega, E)$ is assumed to be a weighted finite-dimensional graph with $\Omega = \{1, 2, \ldots, n\}$.

In order to describe our problem more conveniently, we first introduce the following discrete p-Laplacian operators defined on a network.

Definition 1.1. *For a function $v : \Omega \to \mathbb{R}$ and $p \in (2, +\infty)$, the discrete p-Laplacian Δ_ω^p on Ω is defined by*

$$\Delta_\omega^p v(x) := \sum_{y\sim x, y\in\Omega} |v(y) - v(x)|^{p-2}(v(y) - v(x))\omega(x, y). \tag{1.1}$$

When $p = 2$, it is called the discrete Laplacian $\Delta_\omega := \Delta_\omega^2$ on Ω, which is defined by

$$\Delta_\omega v(x) := \sum_{y\sim x, y\in\Omega} (v(y) - v(x))\omega(x, y). \tag{1.2}$$

Recently, the classical Laplacian Δ was substituted by the discrete Laplacian Δ_ω in graph Laplacian problems, and various methods and techniques to study the existence and qualitative properties of solutions have been developed [2, 5–9]. Here we should emphasize that the discrete p-Laplacian operator Δ_ω^p $(p > 2)$ is actually nonlinear, which is different from the classical Laplacian Δ or the discrete Laplacian Δ_ω.

We are mainly interested in studying the blow-up properties of the solution of the following mutualistic model with a p-Laplacian $(p > 2)$ defined on the networks

$$\begin{cases} \frac{\partial v_1}{\partial t} - d_1\Delta_\omega^p v_1 = v_1(a_1 - b_1 v_1 + c_1 v_2), & x \in \Omega,\ t \in (0, +\infty), \\ \frac{\partial v_2}{\partial t} - d_2\Delta_\omega^p v_2 = v_2(a_2 + c_2 v_1 - b_2 v_2), & x \in \Omega,\ t \in (0, +\infty), \\ v_1(x, 0) = v_1^0(x) \geq (\not\equiv)0,\ v_2(x, 0) = v_2^0(x) \geq (\not\equiv)0, & x \in \Omega. \end{cases} \tag{1.3}$$

Here v_i represents the spatial density of the i^{th} species at time t and d_i represents its respective diffusion rate. The nonnegative constant a_i is the birth rate, b_i is its respective intraspecific competition and the parameter c_i denotes the interspecific cooperation of the i^{th} species.

Under the condition that Δ_ω^p is replaced by the classical Laplacian in (1.3), the strong mutualistic $(b_1/c_1 < c_2/b_2)$ population-based dynamical system experiences blow-up if the intrinsic growth rates of the population are large or the initial data size is sufficiently large [10]. In the case that $p = 2$ in (1.3), Liu et al. [1] proved that the solution blows up for all $x \in \Omega$, $\min\{v_1^0(x), v_2^0(x)\} \not\equiv 0$, under the strong mutualistic condition $b_1/c_1 < c_2/b_2$ and given $\min\{a_1/d_1,\ a_2/d_2\} \geq 1$.

In this paper, when $p > 2$, we can overcome the difficulties caused by the nonlinear operator p-Laplacian Δ_ω^p and study the blow-up properties for the solution of system (1.3). First, we prove the Green formula for the nonlinear operator Δ_ω^p and consider the eigenvalue problem Δ_ω^p. Second, with the help of the following important inequality (see Lemma 2.4)

$$|b - a|^{p-2}(b - a) \leq 2^{p-2}[|b|^{p-2}b - |a|^{p-2}a] \quad \text{with } b \geq a,$$

the comparison principle of system (1.3) is constructed (see Theorem 2.5). Finally, we propose a new strong mutualistic condition

$$\frac{b_1}{c_1} < \left(\frac{d_1}{d_2}\right)^{\frac{1}{p-2}} < \frac{c_2}{b_2}. \tag{1.4}$$

When condition (1.4) holds, it is proved that the solution of (1.3) blows up for all $x \in \Omega$, $\min\{v_1^0(x), v_2^0(x)\} \not\equiv 0$ (see Theorem 3.2).

2. Preliminaries

Lemma 2.1. *(Green formula for Δ_ω^p) For any functions $u, v : \Omega \to \mathbb{R}$, the p-Laplacian Δ_ω^p satisfies that*

$$2\sum_{x\in\Omega} u(x)(-\Delta_\omega^p)v(x) = \sum_{x,y\in\Omega} |v(y) - v(x)|^{p-2}(v(y) - v(x))(u(y) - u(x))\omega(x, y). \tag{2.1}$$

Moreover, if $u = v$, we have

$$2\sum_{x\in\Omega} v(x)(-\Delta_\omega^p)v(x) = \sum_{x,y\in\Omega} |v(y)-v(x)|^p\omega(x,y). \tag{2.2}$$

Proof. Using (1.1), we get

$$\begin{aligned}\sum_{x\in\Omega} u(x)(-\Delta_\omega^p)v(x) &= -\sum_{x\in\Omega} u(x)\sum_{y\sim x,y\in\Omega} |v(y)-v(x)|^{p-2}(v(y)-v(x))\omega(x,y)\\ &= -\sum_{x,y\in\Omega} u(x)|v(y)-v(x)|^{p-2}(v(y)-v(x))\omega(x,y).\end{aligned} \tag{2.3}$$

Meanwhile, we also deduce that

$$\begin{aligned}\sum_{x\in\Omega} u(x)(-\Delta_\omega^p)v(x) &= -\sum_{x,y\in\Omega} u(y)|v(y)-v(x)|^{p-2}(v(x)-v(y))\omega(x,y)\\ &= \sum_{x,y\in\Omega} u(y)|v(y)-v(x)|^{p-2}(v(y)-v(x))\omega(x,y).\end{aligned} \tag{2.4}$$

Hence, using (2.3) and (2.4), we obtain

$$2\sum_{x\in\Omega} v(x)(-\Delta_\omega^p)u(x) = \sum_{x,y\in\Omega} |u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x))\omega(x,y),$$

which completes the proof. □

Lemma 2.2. *Consider the following eigenvalue problem:*

$$\begin{cases} -\Delta_\omega^p\varphi(x) = \lambda\varphi(x), & x\in\Omega,\\ \sum_{x\in\Omega}\varphi(x) = 1. \end{cases} \tag{2.5}$$

There exists

$$\lambda_1 := \min_{\phi\not\equiv 0} \frac{\sum_{x,y\in\Omega} |\varphi(y)-\varphi(x)|^p\omega(x,y)}{2\sum_{x\in\Omega}\varphi^2} \quad \textit{for}\ \ \varphi:\Omega\to\mathbb{R} \tag{2.6}$$

and $\Phi_1(x) > 0$ in Ω satisfying the conditions of the above system (2.5), and they are called the first eigenvalue and eigenfunction of (2.5), respectively. Furthermore, we have that $\lambda_1 = 0$.

Proof. Multiplying the first equation of (2.5) by φ and integrating with respect to Ω, we get

$$\sum_{x\in\Omega}\varphi(x)(-\Delta_\omega^p)\varphi(x) = \sum_{x\in\Omega}\lambda\varphi^2.$$

By (2.2), we deduce that

$$\lambda = \frac{\sum_{x,y\in\Omega} |\varphi(y)-\varphi(x)|^p\omega(x,y)}{2\sum_{x\in\Omega}\varphi^2}.$$

Hence we obtain

$$\lambda_1 := \min_{\varphi\not\equiv 0} \frac{\sum_{x,y\in\Omega} |\varphi(y)-\varphi(x)|^p\omega(x,y)}{2\sum_{x\in\Omega}\varphi^2},$$

where the minimum can be attained by taking $\Phi_1 = \frac{1}{n}$, where n is the number of vertices in Ω and Φ_1 satisfies that $\sum_{x\in\Omega}\Phi_1(x) = 1$. Therefore, by taking $\Phi_1 = \frac{1}{n}$, we can get that $\lambda_1 = 0$; the proof is completed. □

Definition 2.3. *For any $T > 0$, assume that for each $x \in \Omega$, $\hat{v}_1(x,\cdot), \hat{v}_2(x,\cdot) \in C([0,T])$ are differentiable in the range of $(0,T]$. If $(\hat{v}_1, \hat{v}_2)$ satisfies the following:*

$$\begin{cases} \frac{\partial \hat{v}_1}{\partial t} - d_1\Delta_\omega^p \hat{v}_1 \le (\ge) \hat{v}_1(a_1 - b_1\hat{v}_1 + c_1\hat{v}_2), & x \in \Omega,\ t \in (0,T], \\ \frac{\partial \hat{v}_2}{\partial t} - d_2\Delta_\omega^p \hat{v}_2 \le (\ge) \hat{v}_2(a_2 + c_2\hat{v}_1 - b_2\hat{v}_2), & x \in \Omega,\ t \in (0,T], \\ \hat{v}_1(x,0) \le (\ge) v_1^0(x),\ \hat{v}_2(x,0) \le (\ge) v_2^0(x), & x \in \Omega, \end{cases} \tag{2.7}$$

$(\hat{v}_1, \hat{v}_2)$ is called a lower solution (an upper solution) of (1.3) on $\Omega \times [0,T]$.

It is worth noting that the existence of the nonlinear operator $\Delta_\omega^p\,(p > 2)$ introduces difficulties when we construct the comparison principle of system (1.3). We introduce the following classical inequalities which will be used in the proof of the comparison principle. For the proofs the readers can refer to [11] (Section 10).

Lemma 2.4. *(Lemma B.4 in [12]) For $p > 2$, $J_p(t) := |t|^{p-2}t$, we have*

$$2^{2-p}|b-a|^p \le (b-a)(J_p(b) - J_p(a)), \qquad a,\ b \in \mathbb{R}.$$

Moreover, if $b \ge a$, we have

$$J_p(b-a) \le 2^{p-2}[J_p(b) - J_p(a)]. \tag{2.8}$$

With the help of inequality (2.8), we propose the following important comparison principle.

Theorem 2.5. *(Comparison principle) Suppose that (v_1, v_2) is a solution of system (1.3). If $(\hat{v}_1, \hat{v}_2)$ is a lower solution of (1.3) on $\Omega \times [0,T]$, then $(v_1, v_2) \ge (\hat{v}_1, \hat{v}_2)$ for $\Omega \times [0,T]$.*

Proof. Denote $z_1 := (v_1 - \hat{v}_1)e^{-Kt}$ and $z_2 := (v_2 - \hat{v}_2)e^{-Kt}$, where $K > 0$ will be determined later. Notice that Ω is finite and $z_i(x,t)\,(i = 1,2)$ is continuous in the range of $[0,T]$ for each $x \in \Omega$; there exists $(x_0, t_0) \in \Omega \times [0,T]$ such that

$$z_1(x_0, t_0) = \min_{x \in \Omega, t \in [0,T]} z_1(x,t), \tag{2.9}$$

which immediately implies that

$$z_1(x_0,t_0) \le z_1(y,t_0) \text{ for any } y \in \Omega.$$

This is equivalent to

$$v_1(x_0,t_0) - \hat{v}_1(x_0,t_0) \le v_1(y,t_0) - \hat{v}_1(y,t_0) \text{ for any } y \in \Omega, \tag{2.10}$$

and

$$v_1(y,t_0) - v_1(x_0,t_0) \ge \hat{u}_1(v,t_0) - \hat{u}_1(v_0,t_0) \text{ for any } y \in \Omega. \tag{2.11}$$

Recalling the definition of Δ_ω^p, we have

$$\Delta_\omega^p z_1(x_0,t_0) \ge 0. \tag{2.12}$$

At the same time, due to the differentiability of $z_1(x,t)$ in the range of $(0,T]$, we obtain

$$\frac{\partial z_1}{\partial t}(x_0,t_0) \le 0. \tag{2.13}$$

Note that

$$\begin{aligned}
\Delta_\omega^p z_1(x,t) &= e^{-Kt(p-1)}\Delta_\omega^p(v_1-\hat{v}_1)(x,t)\\
&= e^{-Kt(p-1)}\sum_{y\sim x, y\in\Omega}\Big|(v_1(y,t)-\hat{v}_1(y,t))-(v_1(x,t)-\hat{v}_1(x,t))\Big|^{p-2}\\
&\qquad [(v_1(y,t)-\hat{v}_1(y,t))-(v_1(x,t)-\hat{v}_1(x,t))]\omega(x,y)\\
&= e^{-Kt(p-1)}\sum_{y\sim x, y\in\Omega}\Big|(v_1(y,t)-v_1(x,t))-(\hat{v}_1(y,t)-\hat{v}_1(x,t))\Big|^{p-2}\\
&\qquad [(v_1(y,t)-v_1(x,t))-(\hat{v}_1(y,t)-\hat{v}_1(x,t))]\omega(x,y);
\end{aligned} \tag{2.14}$$

we have

$$\begin{aligned}
\Delta_\omega^p(v_1-\hat{v}_1)(x_0,t_0) = \sum_{y\sim x_0, y\in\Omega}&\Big|(v_1(y,t_0)-v_1(x_0,t_0))-(\hat{v}_1(y,t_0)-\hat{v}_1(x_0,t_0))\Big|^{p-2}\\
&[(v_1(y,t_0)-v_1(x_0,t_0))-(\hat{v}_1(y,t_0)-\hat{v}_1(x_0,t_0))]\omega(x_0,y).
\end{aligned} \tag{2.15}$$

Denote

$$b_y := v_1(y,t_0)-v_1(x_0,t_0),\ \ a_y := \hat{v}_1(y,t_0)-\hat{v}_1(x_0,t_0) \quad \text{and} \quad J_p(t) := |t|^{p-2}t.$$

In view of (2.11), we have that $b_y \geq a_y$ for any $y \sim x_0$ and $y \in \Omega$. Combining this with (2.8) in Lemma 2.4, we deduce that

$$|b_y-a_y|^{p-2}(b_y-a_y) = J_p(b_y-a_y) \leq 2^{p-2}[J_p(b_y)-J_p(a_y)] = 2^{p-2}[|b_y|^{p-2}b_y-|a_y|^{p-2}a_y],$$

which implies that

$$\begin{aligned}
\Delta_\omega^p(v_1-\hat{v}_1)(x_0,t_0) &= \sum_{y\sim x_0, y\in\Omega}|b_y-a_y|^{p-2}(b_y-a_y)\omega(x_0,y)\\
&\leq 2^{p-2}\sum_{y\sim x_0, y\in\Omega}[|b_y|^{p-2}b_y-|a_y|^{p-2}a_y]\omega(x_0,y)\\
&= 2^{p-2}\Big[\sum_{y\sim x_0, y\in\Omega}|b_y|^{p-2}b_y\,\omega(x_0,y)-\sum_{y\sim x_0, y\in\Omega}|a_y|^{p-2}a_y\,\omega(x_0,y)\Big]\\
&= 2^{p-2}\Big[\Delta_\omega^p v_1(x_0,t_0)-\Delta_\omega^p\hat{v}_1(x_0,t_0)\Big].
\end{aligned} \tag{2.16}$$

Combining (2.16) with (2.14), we have

$$\Delta_\omega^p z_1(x_0,t_0) \leq 2^{p-2}e^{-Kt_0(p-1)}\Big[\Delta_\omega^p v_1(x_0,t_0)-\Delta_\omega^p\hat{v}_1(x_0,t_0)\Big]. \tag{2.17}$$

Note that (v_1, v_2) is a solution and $(\hat{v}_1, \hat{v}_2)$ is a lower solution to system (1.3). That is, (v_1, v_2) and $(\hat{v}_1, \hat{v}_2)$ respectively satisfy

$$\frac{\partial v_1}{\partial t} - d_1\Delta_\omega^p v_1 = v_1(a_1-b_1v_1+c_1v_2) \tag{2.18}$$

and

$$\frac{\partial \hat{v}_1}{\partial t} - d_1\Delta_\omega^p \hat{v}_1 \leq \hat{v}_1(a_1-b_1\hat{v}_1+c_1\hat{v}_2). \tag{2.19}$$

Recall that $z_1 := (v_1 - \hat{v}_1)e^{-Kt}$; we have

$$\frac{\partial z_1}{\partial t} = -Kz_1 + e^{-Kt}(\frac{\partial v_1}{\partial t} - \frac{\partial \hat{v}_1}{\partial t}). \tag{2.20}$$

Combining (2.18)–(2.20), we obtain

$$\begin{aligned}\frac{\partial z_1}{\partial t} &\geq -Kz_1 + e^{-Kt}(d_1\Delta_\omega^p v_1 + v_1(a_1 - b_1 v_1 + c_1 v_2) - d_1\Delta_\omega^p \hat{v}_1 + \hat{v}_1(a_1 - b_1\hat{v}_1 - c_1\hat{v}_2)) \\ &= d_1 e^{-Kt}[\Delta_\omega^p v_1 - \Delta_\omega^p \hat{v}_1] + (-K + a_1 - b_1(v_1 + \hat{v}_1) + c_1 v_2)z_1 + c_1\hat{v}_1 z_2.\end{aligned} \tag{2.21}$$

Combining (2.17) with (2.21), we deduce that

$$2^{p-2}e^{-Kt_0(p-2)}\frac{\partial z_1}{\partial t}(x_0,t_0) - d_1\Delta_\omega^p z_1(x_0,t_0) \geq 2^{p-2}e^{-Kt_0(p-2)}\Big[(-K + b_{11})z_1(x_0,t_0) + b_{12}z_2(x_0,t_0)\Big], \tag{2.22}$$

where

$$b_{11} := a_1 - b_1(v_1(x_0,t_0) + \hat{v}_1(x_0,t_0)) + c_1 v_2(x_0,t_0), \quad b_{12} := c_1\hat{v}_1(x_0,t_0). \tag{2.23}$$

Substituting (2.12) and (2.13) into (2.22), we deduce that

$$((-K + b_{11})z_1 + b_{12}z_2)(x_0,t_0) \leq 0. \tag{2.24}$$

Next, we will prove that $z_1(x_0,t_0) \geq 0$ by contradiction. Alternatively, suppose that $z_1(x_0,t_0) = -\delta < 0$. Choosing

$$K := \frac{|z_2(x_0,t_0)|}{\delta}|b_{12}(x_0,t_0)| + |b_{11}(x_0,t_0)| + 1,$$

we obtain that $((-K + b_{11})z_1 + b_{12}z_2)(x_0,t_0) > 0$, which contradicts (2.24). Hence we have that $z_1(x_0,t_0) \geq 0$. In view of (2.9), it follows that $z_1(x,t) \geq 0$ for $x \in \Omega$, $t \in [0,T]$. By a similar argument to that for z_2, we can also obtain that $z_2(x,t) \geq 0$ for $x \in \Omega$, $t \in [0,T]$. Thus, we obtain that $v_i \geq \hat{v}_i$ $(i = 1,2)$ for $x \in \Omega$, $t \in [0,T]$. □

3. Blow up result

Theorem 3.1. *Let $U(x,t)$ be a solution of the following problem:*

$$\begin{cases} \frac{\partial U}{\partial t} - d\Delta_\omega^p U = U(\alpha + \beta U), & x \in \Omega,\ t \in (0,+\infty), \\ U(x,0) \geq (\not\equiv) 0, & x \in \Omega, \end{cases} \tag{3.1}$$

where d, α and β are constants satisfying that $d > 0$ and $\beta > 0$. We have the following blow-up properties:

(i) *When $\alpha \geq 0$, $U(x,t)$ blows up for all nontrivial initial data.*

(ii) *When $\alpha < 0$, $U(x,t)$ blows up for sufficiently large initial data.*

Proof. Denote $M(t) := \sum_{x\in\Omega}\Phi_1(x)U(x,t)$, where $\Phi_1(x)$ is defined in Lemma 2.2. Deriving $M(t)$ with respect to t and using (3.1), we have

$$dM'(t) = \sum_{x\in\Omega}\Phi_1(x)[d\Delta_\omega^p U + U(\alpha+\beta U)].$$

Note that $\Phi_1(x)$ is a constant; then, due to (2.1) in Lemma 2.1, we have

$$\sum_{x\in\Omega}\Phi_1(x)\Delta_\omega^p U = -\frac{1}{2}\sum_{x,y\in\Omega}|U(y)-U(x)|^{p-2}(U(y,t)-U(x,t))(\Phi_1(y)-\Phi_1(x))\omega(x,y) = 0.$$

Hence, combining the above equation with $\sum_{x\in\Omega}\Phi_1(x) = 1$ in Lemma 2.2, we deduce that

$$\begin{aligned} dM'(t) &= \sum_{x\in\Omega}\Phi_1(x)U(\alpha+\beta U) = \alpha M(t) + \beta\sum_{x\in\Omega}\Phi_1(x)U^2 \\ &= \alpha M(t) + \beta\sum_{x\in\Omega}\Phi_1(x)U^2\sum_{x\in\Omega}\Phi_1(x) \geq \alpha M(t) + \beta(\sum_{x\in\Omega}\Phi_1(x)U)^2 \\ &= \alpha M(t) + \beta M^2(t), \end{aligned} \tag{3.2}$$

where Hölder's inequality is used.

When $\alpha \geq 0$, using (3.2), we immediately obtain the blow-up result.

When $\alpha < 0$, we choose a sufficiently large initial function $U(x,0)$ which satisfies

$$M(0) = \sum_{x\in\Omega}\Phi_1(x)U(x,0) > -\frac{\alpha}{\beta}.$$

It follows from (3.2) that $U(x,t)$ blows up. □

Theorem 3.2. *If the strong mutualistic condition*

$$\frac{b_1}{c_1} < (\frac{d_1}{d_2})^{\frac{1}{p-2}} < \frac{c_2}{b_2}$$

is satisfied , the solution (v_1, v_2) *of (1.3) blows up for all* $x\in\Omega$, $\min\{v_1^0(x), v_2^0(x)\} \not\equiv 0$.

Proof. We define $(\hat{v}_1(x,t),\hat{v}_2(x,t)) := (\delta_1 U(x,t), \delta_2 U(x,t))$, where the constants δ_1, δ_2 and function $U(x,t)$ will be determined later. In order to ensure that $(\hat{v}_1(x,t),\hat{v}_2(x,t))$ is a lower solution of (1.3), we need to prove that $(\hat{v}_1(x,t),\hat{v}_2(x,t)) \leq (v_1^0(x), v_2^0(x))$ and

$$\begin{cases} \frac{\partial U}{\partial t} - d_1\delta_1^{p-2}\Delta_\omega^p U \leq U(a_1 - b_1\delta_1 U + c_1\delta_2 U), & (x,t)\in\Omega\times(0,\infty), \\ \frac{\partial U}{\partial t} - d_2\delta_2^{p-2}\Delta_\omega^p U \leq U(a_2 + c_2\delta_1 U - b_2\delta_2 U), & (x,t)\in\Omega\times(0,\infty). \end{cases} \tag{3.3}$$

Since the parameters satisfy that $\frac{b_1}{c_1} < (\frac{d_1}{d_2})^{\frac{1}{p-2}} < \frac{c_2}{b_2}$, by calculation, we get

$$c_1(\frac{d_1}{d_2})^{\frac{1}{p-2}} - b_1 > 0 \text{ and } c_2 - b_2(\frac{d_1}{d_2})^{\frac{1}{p-2}} > 0. \tag{3.4}$$

Thus, for a sufficiently small positive constant ε, we choose $\delta_1 := \varepsilon$, $\delta_2 := (\frac{d_1}{d_2})^{\frac{1}{p-2}}\varepsilon$ such that $d_1\delta_1^{p-2} = d_2\delta_2^{p-2}$,

$$-b_1\delta_1 + c_1\delta_2 = \varepsilon[c_1(\frac{d_1}{d_2})^{\frac{1}{p-2}} - b_1] > 0 \text{ and } c_2\delta_1 - b_2\delta_2 = \varepsilon[c_2 - b_2(\frac{d_1}{d_2})^{\frac{1}{p-2}}] > 0. \tag{3.5}$$

Denote

$$d := d_1\delta_1^{p-2} = d_2\delta_2^{p-2},\ \alpha := \min\{a_1, a_2\} \geq 0,\ \beta := \min\{-b_1\delta_1 + c_1\delta_2,\ c_2\delta_1 - b_2\delta_2\}. \tag{3.6}$$

To prove (3.3), it suffices to show the following:

$$\frac{\partial U}{\partial t} - d\Delta_\omega^p U \leq U(\alpha + \beta U), \qquad x \in \Omega,\ t \in (0, T]. \tag{3.7}$$

Hence $(\hat{v}_1(x,t), \hat{v}_2(x,t))$ is a lower solution of (1.3) provided that $(\hat{v}_1(x,0), \hat{v}_2(x,0)) \leq (v_1^0(x), v_2^0(x))$. Let $w(x,0) := \min\{v_1^0(x),\ v_2^0(x)\}$ and ε be small enough such that $\delta_1, \delta_2 < 1$. Thus, $(\hat{v}_1(x,0), \hat{v}_2(x,0)) \leq (v_1^0(x), v_2^0(x))$ holds.

Let U be a solution of the following problem:

$$\begin{cases} \frac{\partial U}{\partial t} - d\Delta_\omega^p U = U(\alpha + \beta U), & x \in \Omega,\ t \in (0, \infty), \\ U(x,0) \geq (\not\equiv) 0, & x \in \Omega. \end{cases} \tag{3.8}$$

By applying Theorem 3.1 to U, we have that $U(x,t)$ blows up, which implies that $(\hat{v}_1(x,t), \hat{v}_2(x,t))$ blows up. With the help of Lemma 2.2, the blow-up properties of the solution of system (1.3) are then obtained. □

4. Conclusions

This research contributes to the blow-up properties of a p-Laplacian ($p > 2$) reaction-diffusion system based on weighted graphs. The discrete p-Laplacian operator Δ_ω^p ($p > 2$) is actually nonlinear, which is different from the classical Laplacian Δ or the discrete Laplacian Δ_ω. To overcome the difficulties caused by the nonlinearity, we establish Green Formula and comparison principle for the p-Laplacian operator. Hence, we develop a new strong mutualistic condition and prove the blow-up properties of the solution for any nontrivial initial data. In this sense, we extend the blow-up results of models with a graph Laplacian ($p = 2$) in [1] to a general graph p-Laplacian ($p > 2$).

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous reviewers for their helpful comments. The work was partially supported by the National Natural Science Foundation of China (11771380) and Natural Science Foundation of Jiangsu Province (BK20191436).

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. Z. Liu, J. Chen, C. Tian, Blow-up in a net work mutualistic model, *Appl. Math. Lett.*, **106** (2020), 106402. https://doi.org/10.1016/j.aml.2020.106402
2. Z. Liu, C. Tian, S. Ruan, On a network model of two competitors with applications to the invasion and competition of Aedes albopictus and Aedes aegypti mosquitoes in the United States, *SIAM J. Appl. Math.*, **80** (2020), 929–950. https://doi.org/10.1137/19M1257950
3. R. Pastor-Satorras, C. Castellano, P. Van Mieghem, A. Vespignani, Epidemic processes in complex networks, *Rev. Modern Phys.*, **87** (2015), 925–986. https://doi.org/10.1103/RevModPhys.87.925
4. R. V. Solé, J. Bascompte, *Self-Organization in Complex Ecosystems*, Princeton: Princeton University Press, 2006.
5. C. Tian, Q. Zhang, L. Zhang, Global stability in a networked SIR epidemic model, *Appl. Math. Lett.*, **107** (2020), 106444. https://doi.org/10.1016/j.aml.2020.106444
6. F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi, S. T. Yau, Li-Yau inequality on graphs, *J. Differential Geom.*, **99** (2015), 359–405. https://doi.org/10.4310/jdg/1424880980
7. A. Grigoryan, Y. Lin, Y. Yang, Yamabe type equations on graphs, *J. Differential Equ.*, **261** (2016), 4924–4943. https://doi.org/10.1016/j.jde.2016.07.011
8. C. Tian, Global stability of a networked predator-prey model, *Appl. Math. Lett.*, **143** (2023), 108685. https://doi.org/10.1016/j.aml.2023.108685
9. C. Tian, S. Ruan, Pattern formation and synchronism in an allelopathic plankton model with delay in a network, *SIAM J. Appl. Dyn. Syst.*, **18** (2019), 531–557. https://doi.org/10.1137/18M1204966
10. K. I. Kim, Z. Lin, Blow-up in a three-species cooperating model, *Appl. Math. Lett.*, **17** (2004), 89–94. https://doi.org/10.1016/S0893-9659(04)90017-1
11. P. Lindqvist, *Notes on the p-Laplace Equation*, Jyvaskyla: University of Jyvaskyla, Department of Mathematics and Statistics, 2017.
12. L. Brasco, E. Parini, The second eigenvalue of the fractional p-Laplacian, *Adv. Calc. Var.*, **9** (2016), 323–355. https://doi.org/10.1515/acv-2015-0007