Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 28176–28187.
DOI:10.3934/math.20231442
Received: 14 August 2023
Revised: 21 September 2023
Accepted: 25 September 2023
Published: 13 October 2023

Research article

Common best proximity point theorems for proximally weak reciprocal continuous mappings

A. Sreelakshmi Unni[1], V. Pragadeeswarar[1,*] and Manuel De la Sen[2]

[1] Department of Mathematics, Amrita School of Physical Sciences, Coimbatore, Amrita Vishwa Vidyapeetham, India

[2] Institute of Research and Development of Processes IIDP, University of the Basque Country, Campus of Leioa, Leioa 48940, Bizkaia, Spain

* **Correspondence:** Email: v_pragadeeswarar@cb.amrita.edu.

Abstract: The main objective of this paper is to find sufficient conditions for the existence and uniqueness of common best proximity points for discontinuous non-self mappings in the setting of a complete metric space. We introduce and analyze new concepts such as proximally reciprocal continuous, proximally weak reciprocal continuous, R-proximally weak commuting of types M_Λ and M_Γ for non-self mappings. Furthermore, we obtain a common best proximity point theorem for such mappings. In addition, we provide an example to support our main result.

Keywords: compatible mappings; weak reciprocal continuity; R-weakly commuting of type M_Λ and M_Γ
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Fixed point theory has wide applications in various branches of science, engineering and other fields. It deals with the solution of the fixed point equation of the form $\Lambda x = x$, where Λ is a mapping from a metric space (X, D) to itself. Suppose $M, N \subset X$ and the mapping Λ from M to N, where $M \cap N = \emptyset$, then the fixed point equation does not have a solution. In that situation, it is desirable to determine an approximate solution x such that the error $D(x, \Lambda x)$ is minimal. Such an approximate solution is known as the best proximity point. The best proximity point theorems provide sufficient conditions to ensure the optimum solution for this fixed point equation. Hence, the existence of the best proximity points develops the theory of optimization. For more details about the existence of best proximity point, one can go through [1–9] and references therein.

Suppose we have two non-self mappings $\Lambda, \Gamma : M \to N$. The equations $\Lambda x = x$ and $\Gamma x = x$ are likely to have no common solution, known as a common fixed point of the mappings Λ and Γ (For more details about common fixed points, refer to [10–16]). In this situation, one needs to find approximate solution x such that the errors $D(x, \Lambda x)$ and $D(x, \Gamma x)$ are minimal for these two fixed point equations, called a common best proximity point of the mappings Λ and Γ.

Sadiq Basha et al. [17] have investigated a common best proximity theorm for mappings that satisfy a contraction-like condition. A common best proximity point theorem for pairs of contractive non-self mappings and for pairs of contraction non-self mappings has been explored in [18]. Sadiq Basha [19] has investigated a common best proximity point theorem for proximally commuting non-self mappings, an improved approach to which has been discussed in [20]. For detailed analysis on common best proximity point for several types of contractions in different spaces, we direct the reader to see [21–27] and references therein.

On the other hand, in 1999, Pant [28] defined a new class of mappings, called reciprocal continuous mappings which is larger than the class of continuous mappings. He gave an example of a reciprocal continuous mapping which is not continuous. He also obtained sufficient conditions for common fixed point theorem even though the mappings may be discontinuous and some of the mappings may not satisfy the compatibility condition.

After that, in 2011, Pant et al. [29] obtained a common fixed point theorem for discontinuous mappings by introducing a new class of mappings called weak reciprocal continuous mappings which is weaker than the assumption of reciprocal continuous mapping. The theorem is stated below.

Theorem 1.1. *[29] Let Λ and Γ be weakly reciprocally continuous self mappings defined on a complete metric space (X, D) such that,*

1) $\Lambda(X) \subseteq \Gamma(X)$,
2) $D(\Lambda x, \Lambda y) \leq aD(\Gamma x, \Gamma y) + bD(\Lambda x, \Gamma x) + cD(\Lambda y, \Gamma y), 0 \leq a, b, c < 1, 0 \leq a + b + c < 1.$

If Λ and Γ are either compatible or R-weakly commuting of type M_Γ or R-weakly commuting of type M_Λ, then Λ and Γ have a unique common fixed point.

Motivated by the above literature survey on common best proximity point and common fixed point for weak reciprocal continuous mappings, in this paper, we introduce a new concept of weak reciprocal continuity to non-self mappings called proximally weak reciprocal continuous mappings. Furthermore, we provide sufficient conditions to ensure the existence of a common best proximity point for this new class of non-self mappings.

2. Preliminaries

Let us recall some notations which will be used in the sequel.

Let M, N be two non-empty subsets of a metric space (X, D).

$D(M, N) = dist(M, N) = \inf\{D(m, n) : m \in M \text{ and } n \in N\}$;
$D(m, N) = \inf\{D(m, n) : n \in N\}$;
$M_0 = \{m \in M : D(m, n') = D(M, N) \text{ for some } n' \in N\}$;
$N_0 = \{n \in N : D(m', n) = D(M, N) \text{ for some } m' \in M\}$.

Definition 2.1. *[23] An element $t \in M$ is said to be a common best proximity point of the non-self mappings $\Lambda_1, \Lambda_2, ..., \Lambda_m : M \to N$ if it satisfies $D(t, \Lambda_1 t) = D(t, \Lambda_2 t) = ... = D(t, \Lambda_m t) = D(M, N)$.*

Definition 2.2. *[23] If M_0 is non-empty, then the pair (M, N) is said to have the P-property if for any $m_1, m_2 \in M_0$ and $n_1, n_2 \in N_0$,*

$$\begin{cases} D(m_1, n_1) = D(M, N) \\ D(m_2, n_2) = D(M, N) \end{cases} \quad \textit{implies} \quad D(m_1, m_2) = D(n_1, n_2).$$

Definition 2.3. *[29] Two self-mappings Λ and Γ of a metric space (X, D) will be called weakly reciprocally continuous if $\lim_{n\to+\infty} \Lambda\Gamma x_n = \Lambda(t)$ or $\lim_{n\to+\infty} \Gamma\Lambda x_n = \Gamma(t)$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} \Lambda x_n = \lim_{n\to+\infty} \Gamma x_n = t$ for some t in X.*

Definition 2.4. *[25] Let M and N be two subsets of a metric space (X, D). Two non-self mappings Λ and Γ, from M to N, are proximally compatible if for any sequences $\{a_n\}, \{u_n\}$ and $\{v_n\} \in M$ with*

$$\begin{cases} D(v_n, \Lambda a_n) = D(M, N) \\ D(u_n, \Gamma a_n) = D(M, N) \end{cases} \quad \text{implies} \quad \lim_{n\to+\infty} D(\Gamma v_n, \Lambda u_n) = 0,$$

whenever $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = t$.

We can note that if $M = N$, then Λ and Γ are proximally compatibile implies they are compatible.

Definition 2.5. *[16] Let (X, D) be a metric space and let Λ and Γ be self-mappings on X.*

- The mappings Λ and Γ are said to be R-weakly commuting of type (M_Λ) if there exists a positive real number R such that $D(\Lambda\Gamma x, \Gamma\Gamma x) \leq RD(\Lambda x, \Gamma x)$, for all $x \in X$.
- The mappings Λ and Γ are said to be R-weakly commuting of type (M_Γ) if there exists a positive real number R such that $D(\Gamma\Lambda x, \Lambda\Lambda x) \leq RD(\Lambda x, \Gamma x)$, for all $x \in X$.

3. Main results

First, we define new notations called proximally reciprocal continuous, proximally weak reciprocal continuous, R-proximally weak commuting of types M_Γ and M_Λ for non-self mappings.

The concept of weak reciprocal continuity in self mappings [29] can be extended to non-self mappings as follows.

Definition 3.1. *Let (X, D) be a metric space and M and N be non-empty subsets of X. Two non-self mappings $\Lambda, \Gamma : M \to N$ are called proximally reciprocal continuous if for all sequences $x_n, u_n, v_n \in M$, such that $D(u_n, \Lambda x_n) = D(M, N) = D(v_n, \Gamma x_n)$, then $\lim_{n\to+\infty} \Gamma(u_n) = \Gamma(t)$ and $\lim_{n\to+\infty} \Lambda(v_n) = \Lambda(t)$ whenever, $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = t$, for some $t \in M$.*

Example 3.2. *Let $X = \mathbb{R}^2, D(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$ where $p = (p_1, p_2)$ and $q = (q_1, q_2)$. Take $M = \{(0, p) : 0 \leq p \leq 2\}$ and $N = \{(1, q) : 0 \leq q \leq 8\}$. Then $D(M, N) = 1$.*
Define $\Lambda, \Gamma : M \to N$ as follows: $\Lambda(0, p) = (1, p^2)$ and $\Gamma(0, p) = (1, 2p^2)$. Now if $x_n = (0, \eta_n)$ such that, $\eta_n \to 0$. Then, we can find u_n and v_n such that,

$$\begin{cases} D((0, \eta_n^2), (1, \eta_n^2)) = 1, \\ D((0, 2\eta_n^2), (1, 2\eta_n^2)) = 1. \end{cases}$$

Here, $u_n = (0, \eta_n^2)$ and $v_n = (0, 2\eta_n^2)$. Since $\eta_n^2 \to 0$, we get, $\lim_{n\to+\infty} \Gamma(0, \eta_n^2) = (1, 0) = \Gamma(t) = \Gamma(0, 0)$ and $\lim_{n\to+\infty} \Lambda(0, 2\eta_n^2) = (1, 0) = \Lambda(t) = \Lambda(0, 0)$. Hence, Λ and Γ are proximally reciprocal continuous mappings.

Definition 3.3. *Let (X, D) be a metric space and M and N be non-empty subsets of X. Two non-self mappings $\Lambda, \Gamma : M \to N$ are called proximally weak reciprocal continuous if for all sequences $x_n, u_n, v_n \in M$, such that $D(u_n, \Lambda x_n) = D(M, N) = D(v_n, \Gamma x_n)$, then $\lim_{n\to+\infty} \Gamma(u_n) = \Gamma(t)$ or $\lim_{n\to+\infty} \Lambda(v_n) = \Lambda(t)$ whenever $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n = t$, for some $t \in M$.*

Example 3.4. *Let $X = \mathbb{R}^2, D(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$ where, $p = (p_1, p_2)$ and $q = (q_1, q_2)$. Take $M = \{(0, p) : 0 \le p \le 2\}$ and $N = \{(1, q) : 0 \le q \le 8\}$. Then, $D(M, N) = 1$.*

Define $\Lambda, \Gamma : M \to N$ as follows, $\Lambda(0, p) = (1, p^2)$ and $\Gamma(0, p) = \begin{cases} (1, 2p^2) & p > 0 \\ (1, 2) & p = 0 \end{cases}$.

Now, let $x_n = (0, \eta_n)$ such that, $\eta_n \to 0$. Then, we can find u_n and v_n such that,

$$\begin{cases} D((0, \eta_n^2), (1, \eta_n^2)) = 1, \\ D((0, 2\eta_n^2), (1, 2\eta_n^2)) = 1. \end{cases}$$

Here $u_n = (0, \eta_n^2)$ and $v_n = (0, 2\eta_n^2)$. Since $\eta_n^2 \to 0$, we get, $\lim_{n\to+\infty} \Gamma(0, \eta_n^2) = (1, 0) = \Gamma(t) = \Gamma(0, 0)$ but $\lim_{n\to+\infty} \Lambda(0, 2\eta_n^2) = (1, 0) \ne \Lambda(t) = \Lambda(0, 0) = (1, 2)$.

Hence, Λ and Γ are proximally weak reciprocal continuous but not proximally reciprocal continuous mappings.

One can note that Λ and Γ do not have a common best proximity point.

When Λ and Γ are self-maps, proximally reciprocal continuity becomes reciprocal continuity, and proximally weak reciprocal continuity becomes weak reciprocal continuity.

Now, the concept of R-weakly commuting of type M_Γ and M_Λ for self mappings [29] can be extended to non-self mappings as follows.

Definition 3.5. *Let (X, D) be a metric space and $M, N \subset X$. Let Λ and Γ be two non-self mappings from M to N. The mappings Λ and Γ are said to be R-proximally weak commuting of type (M_Γ) if there exists $R > 0$ such that $D(\Lambda u, \Gamma v) \le RD(u, v)$ whenever $D(u, \Lambda x) = D(M, N) = D(v, \Gamma x)$ for all $u, v, x \in M$.*

Definition 3.6. *Let (X, D) be a metric space. And $M, N \subset X$. Let Λ and Γ be two non-self mappings from M to N. The mappings Λ and Γ are said to be R-proximally weak commuting of type (M_Λ) if there exists $R > 0$ such that $D(\Lambda v, \Gamma v) \le RD(u, v)$ whenever $D(u, \Lambda x) = D(M, N) = D(v, \Gamma x)$ for all $u, v, x \in M$.*

Example 3.7. *Let $X = \mathbb{R}^2, D(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$ where $p = (p_1, p_2)$ and $q = (q_1, q_2)$. Take $M = \{0\} \times [1, +\infty)$ and $N = \{1\} \times [1, +\infty)$. Then, $D(M, N) = 1$. Define $\Lambda, \Gamma : M \to N$ as follows: $\Lambda(0, p) = (1, 2p)$ and $\Gamma(0, p) = (1, 3p)$. It can be verified that Λ and Γ are R-proximally weak commuting of type M_Γ for $R = 2$ and R-proximally weak commuting of type M_Λ for $R = 3$.*

Here, we can see that when $M = N$, R-proximally weak commuting of type M_Γ and M_Λ reduces to R-weakly commuting of type M_Γ and M_Λ.

Now, we state and prove our main result.

Theorem 3.8. *Let Λ and Γ be two proximally weak reciprocal continuous non-self mappings from M to N where M and N are subsets of a complete metric space (X, D) such that*

1) $D(\Lambda x, \Lambda y) \leq aD(\Gamma x, \Gamma y)+bD(\Lambda x, \Gamma x)+cD(\Lambda y, \Gamma y), \forall x, y \in M$, where $0 \leq a, b, c < 1, a+b+c < 1$,
2) $\Lambda(M_0) \subseteq \Gamma(M_0)$ and $\Lambda(M_0) \subseteq N_0$,
3) The pair (M, N) satisfies P-property,
4) $M_0 \neq \emptyset$ and N_0 are closed.

If Λ and Γ are either proximally compatible or R-proximally weak commuting of type M_Γ or R-proximally weak commuting of type M_Λ, then Λ and Γ have a unique common best proximity point.

Proof. Let us fix any point $a_0 \in M_0$. Now, by using condition (2), $\exists a_1 \in M_0$ such that $\Lambda(a_0) = \Gamma(a_1)$. Similarly we can find $a_2 \in M_0$ such that $\Lambda(a_1) = \Gamma(a_2)$. Hence, in general, we can say that there exists a sequence of points $a_0, a_1, a_2, ..., a_n, ..$ where $\Lambda(a_n) = \Gamma(a_{n+1}), n = 0, 1, 2, 3, ...$

Our claim is that $\{\Lambda a_n\}$ is a Cauchy sequence. Using condition (1), we can write

$$\begin{aligned} D(\Lambda a_n, \Lambda a_{n+1}) &\leq aD(\Gamma a_n, \Gamma a_{n+1}) + bD(\Lambda a_n, \Gamma a_n) + cD(\Lambda a_{n+1}, \Gamma a_{n+1}) \\ &\leq aD(\Lambda a_{n-1}, \Lambda a_n) + bD(\Lambda a_n, \Lambda a_{n-1}) + cD(\Lambda a_{n+1}, \Lambda a_n) \\ &\leq (\frac{a+b}{1-c})D(\Lambda a_{n-1}, \Lambda a_n). \end{aligned}$$

Let $p = (\frac{a+b}{1-c})$. Then, we have $D(\Lambda a_n, \Lambda a_{n+1}) \leq pD(\Lambda a_{n-1}, \Lambda a_n)$. Let $n, m \in \mathbb{N}$. Then, we derive

$$\begin{aligned} D(\Lambda a_n, \Lambda a_{n+m}) &\leq D(\Lambda a_n, \Lambda a_{n+1}) + D(\Lambda a_{n+1}, \Lambda a_{n+2}) + ... + D(\Lambda a_{n+(m-1)}, \Lambda a_{n+m}) \\ &\leq (1 + p + p^2 + ... + p^{m-1})D(\Lambda a_n, \Lambda a_{n+1}) \\ &\leq (1 + p + p^2 + ... + p^{m-1})p^n D(\Lambda a_0, \Lambda a_1) \\ &\leq (\frac{p^n}{1-p})D(\Lambda a_0, \Lambda a_1). \end{aligned}$$

Since $p < 1$, we have $p^n \to +\infty$ as $n \to +\infty$. This means that $D(\Lambda a_n, \Lambda a_{n+m}) \to 0$ as $n \to +\infty$. Hence, $\Lambda a_n\}$ is a Cauchy sequence.

Let $\{u_n\}$ be a sequence of elements in M_0 such that $D(u_n, \Lambda a_n) = D(M, N)$ for all $n \geq 0$. By the P-property, we obtain $D(u_n, u_m) = D(\Lambda a_n, \Lambda a_m), \forall n, m \in \mathbb{N}$. Clearly, the sequence $\{u_n\}$ is a Cauchy sequence. Since M_0 is closed, there exists $u \in M_0$ such that $u_n \to u$ as $n \to +\infty$.

The proof can be followed by the following three cases.

Case 1: Suppose that Λ and Γ are proximally compatible mappings.

Then, we have $D(u_n, \Lambda a_n) = D(M, N) = D(u_{n-1}, \Gamma a_n)$ and $u_n \to u$ as $n \to +\infty$. By proximal weak reciprocal continuity of Λ and Γ, as $n \to +\infty$, either $\Gamma u_n \to \Gamma u$ or $\Lambda u_n \to \Lambda u$.

Subcase 1: Let $\Gamma u_n \to \Gamma u$.

By proximal compatibility of the pair (Λ, Γ), $\Lambda u_{n-1} \to \Gamma u$. Using condition (1) we can write $D(\Lambda u, \Lambda u_n) \leq aD(\Gamma u, \Gamma u_n) + bD(\Lambda u, \Gamma u) + cD(\Lambda u_n, \Gamma u_n)$. $D(\Lambda u, \Gamma u) \leq bD(\Lambda u, \Gamma u)$, when $n \to +\infty$. Since $b < 1$, we get $\Lambda u = \Gamma u$. Now, we have to show the existence of the common best proximity point for the given mappings Λ and Γ. Since $\Lambda(M_0) \subseteq N_0$, there exists $v \in M_0$ such that $D(v, \Lambda u) = D(M, N) = D(v, \Gamma u)$. By proximal compatibility of Λ and Γ, we have $\Lambda(v) = \Gamma(v)$. Similarly, since $\Lambda(M_0) \subseteq N_0$, there exists $w \in M_0$ such that $D(w, \Lambda v) = D(M, N) = D(w, \Gamma v)$. Now, we obtain

$$D(v, w) = D(\Lambda u, \Lambda v)$$

$$\begin{aligned} &\leq aD(\Gamma u, \Gamma v) + bD(\Lambda u, \Gamma u) + cD(\Lambda v, \Gamma v) \\ &\leq aD(\Gamma u, \Gamma v) \\ &\leq aD(v, w). \end{aligned}$$

Since $a < 1, v = w$. We have $D(v, \Lambda v) = D(M, N) = D(v, \Gamma v)$. Hence, v is a common best proximity point of Λ and Γ.

Subcase 2: Let $\Lambda u_n \to \Lambda u$ as $n \to +\infty$.

By proximal compatability of the pair (Λ, Γ), $\Gamma u_n \to \Lambda u$. Since $\Lambda(M_0) \subseteq \Gamma(M_0)$, there exists some $t \in M_0$ such that $\Lambda u = \Gamma t$. Now, we have $\Gamma u_n \to \Gamma t$ and $\Lambda u_n \to \Gamma t$. Using condition (1) we can write $D(\Lambda t, \Lambda u_n) \leq aD(\Gamma t, \Gamma u_n) + bD(\Lambda t, \Gamma t) + cD(\Lambda u_n, \Gamma u_n)$. $D(\Lambda t, \Gamma t) \leq bD(\Lambda t, \Gamma t)$, when $n \to +\infty$. Since $b < 1$, then $\Lambda t = \Gamma t$. Existence can be proved in the same method used in Subcase 1.

Case 2: Suppose that Λ and Γ are R-proximally weak commuting of type M_Γ.

By proximal weak reciprocal continuity of Λ and Γ, as $n \to +\infty$, either $\Gamma u_n \to \Gamma u$ or $\Lambda u_n \to \Lambda u$.

Subcase 1: Let $\Gamma u_n \to \Gamma u$ as $n \to +\infty$.

Let Λ and Γ be R- proximally weak commuting of type M_Γ. We have $D(u_{n+1}, \Lambda a_{n+1}) = D(M, N) = D(u_n, \Lambda a_n) = D(u_n, \Gamma a_{n+1})$ for $u_n, u_{n+1} \in M_0$. Then, $D(\Lambda u_{n+1}, \Gamma u_{n+1}) \leq RD(u_n, u_{n+1})$, for some $R > 0$. Then, $D(\Lambda u_{n+1}, \Gamma u_{n+1}) \to 0$ when $n \to +\infty$.

Since $\Gamma u_n \to \Gamma u$ as $n \to +\infty$, $\Lambda u_n \to \Gamma u$ as $n \to +\infty$. Using condition (1) we can write, $D(\Lambda u, \Lambda u_n) \leq aD(\Gamma u, \Gamma u_n) + bD(\Lambda u, \Gamma u) + cD(\Lambda u_n, \Gamma u_n)$. As $n \to +\infty, D(\Lambda u, \Gamma u) \leq aD(\Gamma u, \Gamma u) + bD(\Lambda u, \Gamma u) + cD(\Gamma u, \Gamma u)$. Since $b < 1, \Lambda u = \Gamma u$.

Existence of the common best proximity point can be proved as follows.

Using the condition $\Lambda(M_0) \subseteq N_0$, there exists $u^* \in M_0$ such that $D(u^*, \Lambda u) = D(M, N) = D(u^*, \Gamma u)$. Since Λ and Γ are R-proximally weak commuting of type M_Γ, we have $D(\Lambda u^*, \Gamma u^*) \leq RD(u^*, u^*)$ for some $R > 0$. Therefore, $\Lambda(u^*) = \Gamma(u^*)$. Similarly, using the same condition, as $\Lambda(M_0) \subseteq N_0$, there exists $v^* \in M_0$ such that $D(v^*, \Lambda u^*) = D(M, N) = D(v^*, \Gamma u^*)$. Now, consider

$$\begin{aligned} D(u^*, v^*) &= D(\Lambda u, \Lambda u^*) \\ &\leq aD(\Gamma u, \Gamma u^*) + bD(\Lambda u, \Gamma u) + cD(\Lambda u^*, \Gamma u^*) \\ &\leq aD(\Gamma u, \Gamma u^*) \\ &\leq aD(u^*, v^*). \end{aligned}$$

Since $a < 1$, $u^* = v^*$. We have $D(u^*, \Lambda u^*) = D(M, N) = D(u^*, \Gamma u^*)$. Hence, u^* is a common best proximity point of Λ and Γ.

Subcase 2: Let $\Lambda u_n \to \Lambda u$ as $n \to +\infty$.

Since $\Lambda(M_0) \subseteq \Gamma(M_0)$, there exists some $t \in M_0$ such that $\Lambda u = \Gamma t$. Now, we have $\Lambda u_n \to \Gamma t$ and $\Gamma u_n \to \Gamma t$ since $D(\Lambda u_{n+1}, \Gamma u_{n+1}) \to 0$ as $n \to +\infty$. Using condition (1), we can write $D(\Lambda t, \Lambda u_n) \leq aD(\Gamma t, \Gamma u_n) + bD(\Lambda t, \Gamma t) + cD(\Lambda u_n, \Gamma u_n)$. $D(\Lambda t, \Gamma t) \leq bD(\Lambda t, \Gamma t)$, when $n \to +\infty$. Since $b < 1$, $\Lambda t = \Gamma t$. Existence can be proved in the same method used in Subcase 1.

Case 3: Suppose that Λ and Γ are R-proximally weak commuting of type M_Λ.

By proximal weak reciprocal continuity of Λ and Γ, as $n \to +\infty$, either $\Gamma u_n \to \Gamma u$ or $\Lambda u_n \to \Lambda u$.

Subcase 1: Let $\Gamma u_n \to \Gamma u$ as $n \to +\infty$.

Let Λ and Γ be R- proximally weak commuting of type M_Λ. Already, we know $D(u_{n+1}, \Lambda a_{n+1}) = D(M, N) = D(u_n, \Lambda a_n) = D(u_n, \Gamma a_{n+1})$ for $u_n, u_{n+1} \in M_0$. Then, $D(\Lambda u_n, \Gamma u_n) \leq RD(u_n, u_{n+1})$ for some $R > 0$. Then, $D(\Lambda u_n, \Gamma u_n) \to 0$ when $n \to +\infty$.

Since $\Gamma u_n \to \Gamma u$ as $n \to +\infty$, $\Lambda u_n \to \Gamma u$ as $n \to +\infty$. Using condition (1), we can write $D(\Lambda u, \Lambda u_n) \leq aD(\Gamma u, \Gamma u_n) + bD(\Lambda u, \Gamma u) + cD(\Lambda u_n, \Gamma u_n)$. As $n \to +\infty$, $D(\Lambda u, \Gamma u) \leq aD(\Gamma u, \Gamma u) + bD(\Lambda u, \Gamma u) + cD(\Gamma u, \Gamma u)$. Since $b < 1$, $\Lambda u = \Gamma u$.

Existence of the common best proximity point can be proved as follows.

Using the condition $\Lambda(M_0) \subseteq N_0$, there exists $u^* \in M_0$ such that $D(u^*, \Lambda u) = D(M, N) = D(u^*, \Gamma u)$. Since Λ and Γ are R-proximally weak commuting of type M_Λ, we have $D(\Lambda u^*, \Gamma u^*) \leq RD(u^*, u^*)$, for some $R > 0$. Therefore, $\Lambda(u^*) = \Gamma(u^*)$. Similarly, using the same condition, as $\Lambda(M_0) \subseteq N_0$, there exists $v^* \in M_0$ such that $D(v^*, \Lambda u^*) = D(M, N) = D(v^*, \Gamma u^*)$. Now, consider

$$\begin{aligned} D(u^*, v^*) &= D(\Lambda u, \Lambda u^*) \\ &\leq aD(\Gamma u, \Gamma u^*) + bD(\Lambda u, \Gamma u) + cD(\Lambda u^*, \Gamma u^*) \\ &\leq aD(\Gamma u, \Gamma u^*) \\ &\leq aD(u^*, v^*). \end{aligned}$$

Since $a < 1$, $u^* = v^*$. We have, $D(u^*, \Lambda u^*) = D(M, N) = D(u^*, \Gamma u^*)$. Hence, u^* is a common best proximity point of Λ and Γ.

Subcase 2: Let $\Lambda u_n \to \Lambda u$ as $n \to +\infty$.

Since $\Lambda(M_0) \subseteq \Gamma(M_0)$, there exists some $t \in M_0$ such that $\Lambda u = \Gamma t$. Now, we have $\Lambda u_n \to \Gamma t$ and $\Gamma u_n \to \Gamma t$ since $D(\Lambda u_n, \Gamma u_n) \to 0$ as $n \to +\infty$. Using condition (1) we can write $D(\Lambda t, \Lambda u_n) \leq aD(\Gamma t, \Gamma u_n) + bD(\Lambda t, \Gamma t) + cD(\Lambda u_n, \Gamma u_n)$. $D(\Lambda t, \Gamma t) \leq bD(\Lambda t, \Gamma t)$, when $n \to +\infty$. Since $b < 1$, $\Lambda t = \Gamma t$. Existence can be proved in the same method used in Subcase 1.

Uniqueness of the common best proximity point can be proved as follows.
Suppose that v and v^* are two distinct common best proximity points of mappings Λ and Γ. Then, we can write

$$D(v, \Lambda v) = D(M, N) = D(v, \Gamma v)$$

and

$$D(v^*, \Lambda v^*) = D(M, N) = D(v^*, \Gamma v^*).$$

Consider

$$\begin{aligned} D(v, v^*) &= D(\Lambda v, \Lambda v^*) \\ &\leq aD(\Gamma v, \Gamma v^*) + bD(\Lambda v, \Gamma v) + cD(\Lambda v^*, \Gamma v^*) \\ &\leq aD(v, v^*). \end{aligned}$$

Since $a < 1$, $v = v^*$. Hence, uniqueness is proved.

Example 3.9. *Let* $X = \mathbb{R}^n, n \in \mathbb{N}$ *and* $D(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + ... + (p_n - q_n)^2}$ *where* $p = (p_1, p_2, ..., p_n)$ *and* $q = (q_1, q_2, ..., q_n)$. *Take* $M = \underbrace{\{0\} \times \{0\} \times ...\{0\}}_{\text{n-1 times}} \times [2, 20]$ *and* $N = \underbrace{\{0\} \times \{0\} \times ...\{0\}}_{\text{n-2 times}} \times \{1\} \times [2, 20]$. *Then,* $D(M, N) = 1$. *Here,* $M_0 = M$ *and* $N_0 = N$. *Moreover,* M_0 *and* N_0 *are closed and the pair* (M, N) *satisfies P-property.*

Define $\Lambda, \Gamma : M \to N$ as follows.

$$\Lambda(\underbrace{0,0,...,0}_{\text{n-1 times}}, p) = \begin{cases} (\underbrace{0,0,...,0}_{\text{n-2 times}}, 1, 2) & p = 2, p > 5, \\ (\underbrace{0,0,...,0}_{\text{n-2 times}}, 1, 6) & 2 < p \leq 5, \end{cases} \quad \text{and}$$

$$\Gamma(\underbrace{0,0,...,0}_{\text{n-1 times}}, p) = \begin{cases} (\underbrace{0,0,...,0}_{\text{n-2 times}}, 1, 2) & p = 2, \\ (\underbrace{0,0,...,0}_{\text{n-2 times}}, 1, 12) & 2 < p \leq 5, \\ (\underbrace{0,0,...,0}_{\text{n-2 times}}, 1, \dfrac{p+1}{3}) & p > 5. \end{cases}$$

It can be verified that $\Lambda(M_0) \subseteq \Gamma(M_0)$ and $\Lambda(M_0) \subseteq N_0$.

Also, Λ and Γ satisfies the contraction condition for $a = \frac{4}{5}, b = \frac{1}{10}, c = \frac{1}{20}$.

It can also be noted that Λ and Γ are proximally weak reciprocal continuous. To see this, let $(\underbrace{0,0,...,0}_{\text{n-1 times}}, x_n)$, $(\underbrace{0,0,...,0}_{\text{n-1 times}}, u_n)$ and $(\underbrace{0,0,...,0}_{\text{n-1 times}}, v_n)$ be three sequences with

$$\begin{cases} D((\underbrace{0,0,...,0}_{\text{n-1 times}}, u_n), \Lambda(\underbrace{0,0,...,0}_{\text{n-1 times}}, x_n)) = 1; \\ D((\underbrace{0,0,...,0}_{\text{n-1 times}}, v_n), \Gamma(\underbrace{0,0,...,0}_{\text{n-1 times}}, x_n)) = 1, \end{cases}$$

such that $\lim_{n\to+\infty}(\underbrace{0,0,...,0}_{\text{n-1 times}}, u_n) = \lim_{n\to+\infty}(\underbrace{0,0,...,0}_{\text{n-1 times}}, v_n) = (\underbrace{0,0,...,0}_{\text{n-1 times}}, t)$. Then, $t = 2$ and either $x_n = 2$ for each n or $x_n = 5 + \epsilon_n$ where $\epsilon_n \to 0$ as $n \to +\infty$.

If $x_n = 2$ for each n, then $u_n = v_n = 2$. Hence, $\lim_{n\to+\infty} \Lambda(\underbrace{0,0,...,0}_{\text{n-1 times}}, v_n) = \lim_{n\to+\infty} \Gamma(\underbrace{0,0,...,0}_{\text{n-1 times}}, u_n) = (\underbrace{0,0,...,0}_{\text{n-2 times}}, 1, 2) = \Lambda(\underbrace{0,0,...,0}_{\text{n-1 times}}, 2) = \Gamma(\underbrace{0,0,...,0}_{\text{n-1 times}}, 2)$.

If $x_n = 5 + \epsilon_n$, then $u_n = 2$ and $v_n = 2 + \frac{\epsilon_n}{3}$. Hence, $\lim_{n\to+\infty} \Gamma(\underbrace{0,0,...,0}_{\text{n-1 times}}, u_n) = (\underbrace{0,0,...,0}_{\text{n-2 times}}, 1, 2) = \Gamma(\underbrace{0,0,...,0}_{\text{n-1 times}}, 2)$. But, $\lim_{n\to+\infty} \Lambda(\underbrace{0,0,...,0}_{\text{n-1 times}}, v_n) = (\underbrace{0,0,...,0}_{\text{n-2 times}}, 1, 12) \neq \Lambda(\underbrace{0,0,...,0}_{\text{n-1 times}}, 2)$. Hence, Λ and Γ are proximally weak reciprocal continuous.

Now, take sequences $a_n = (\underbrace{0,0,...,0}_{\text{n-1 times}}, 5 + \frac{1}{n})$, $u_n = (\underbrace{0,0,...,0}_{\text{n-1 times}}, 2)$ and $v_n = \underbrace{0,0,...,0}_{\text{n-1 times}}, 2 + \frac{1}{3n})$. For which $D(u_n, \Lambda a_n) = D(v_n, \Gamma a_n) = 1$. But, $\lim_n D(\Lambda v_n, \Gamma u_n) \neq 0$. Therefore, Λ and Γ are not proximally compatible mappings.

For $R = 1$, Λ and Γ are R-proximally weak commuting of type M_Γ.

Hence, given Λ and Γ satisfies all the conditions of the Theorem 3.8 and hence they have a unique common best proximity point $(\underbrace{0,0,...,0}_{\text{n-1 times}}, 2) \in M_0$.

Corollary 3.1. *Let Λ and Γ be two proximally weak reciprocal continuous non-self mappings from M to N, where M and N are subsets of a metric space (X, D) such that*

1) $D(\Lambda x, \Lambda y) \leq kD(\Gamma x, \Gamma y), \forall x, y \in M, 0 \leq k < 1,$

2) $\Lambda(M_0) \subseteq \Gamma(M_0)$ and $\Lambda(M_0) \subseteq N_0$,
3) The pair (M, N) satisfies P-property,
4) $M_0 \neq \emptyset$ and N_0 are closed.

If Λ and Γ are either proximally compatible or R-proximally weak commuting of type M_Γ or R-proximally weak commuting of type M_Λ, then Λ and Γ have a unique common best proximity point.

Corollary 3.2. *Let Λ and Γ be two proximally weak reciprocal continuous non-self mappings from M to N, where M and N are subsets of a complete metric space (X, D) such that*

1) $D(\Lambda x, \Lambda y) \leq kD(\Lambda x, \Gamma x), \forall x, y \in M$, where $0 \leq k < 1$,
2) $\Lambda(M_0) \subseteq \Gamma(M_0)$ and $\Lambda(M_0) \subseteq N_0$,
3) The pair (M, N) satisfies P-property,
4) $M_0 \neq \emptyset$ and N_0 are closed.

If Λ and Γ are either proximally compatible or R-proximally weak commuting of type M_Γ or R-proximally weak commuting of type M_Λ, then Λ and Γ have a unique common best proximity point.

Corollary 3.3. *Let Λ and Γ be two proximally weak reciprocal continuous non-self mappings from M to N, where M and N are subsets of a complete metric space (X, D) such that*

1) $D(\Lambda x, \Lambda y) \leq kD(\Lambda y, \Gamma y), \forall x, y \in M$, where $0 \leq k < 1$,
2) $\Lambda(M_0) \subseteq \Gamma(M_0)$ and $\Lambda(M_0) \subseteq N_0$,
3) The pair (M, N) satisfies P-property,
4) $M_0 \neq \emptyset$ and N_0 are closed.

If Λ and Γ are either proximally compatible or R-proximally weak commuting of type M_Γ or R-proximally weak commuting of type M_Λ, then Λ and Γ have a unique common best proximity point.

Remark 3.10. *In Theorem 3.8, when $\Lambda = \Gamma$, we cannot conclude anything as the contraction condition fails. Regarding best proximity point theory, we can observe that when $M = N$, the best proximity point reduces to a fixed point and a common best proximity point reduces to a common fixed point. Here, the Theorem 3.8 subsumes the following common fixed point theorem due to Pant [29], as a particular case when $M = N$.*

Corollary 3.4. *Let Λ and Γ be weakly reciprocally continuous self mappings of a complete metric space (X, D) such that*

1) $\Lambda(X) \subseteq \Gamma(X)$,
2) $D(\Lambda x, \Lambda y) \leq aD(\Gamma x, \Gamma y) + bd(\Lambda x, \Gamma x) + cD(\Lambda y, \Gamma y), 0 \leq a, b, c < 1, 0 \leq a + b + c < 1.$

If Λ and Γ are either compatible or R-weakly commuting of type M_Γ or R-weakly commuting of type M_Λ, then Λ and Γ have a unique common fixed point.

4. Conclusions

The fixed point and best proximity point results guarantee the existence of solutions for many problems in non-linear analysis. Pant et al. [29] introduced the concept of weak reciprocal continuity and obtained fixed point theorems by employing this new concept. In our paper, the above concept of reciprocal continuity of self mappings is extended to non-self mappings and derived the sufficient conditions which ensure the existence of a common best proximity point for a given pair of non-self mappings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work has been partially funded by the Basque Government through Grant IT1207-19 and Grant IT1155-22.

Conflict of interest

The authors declare that they have no competing interest.

References

1. M. A. Al-Thagafi, N. Shahzad, Convergence and existence results for best proximity points, *Nonlinear Anal. Theor.*, **70** (2009), 3665–3671. https://doi.org/10.1016/j.na.2008.07.022
2. A. A. Eldred, P. Veeramani, Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, **323** (2006), 1001–1006. https://doi.org/10.1016/j.jmaa.2005.10.081
3. C. Di Bari, T. Suzuki, C. Vetro, Best proximity points for cyclic Meir-Keeler contractions, *Nonlinear Anal. Theor.*, **69** (2008), 3790–3794. https://doi.org/10.1016/j.na.2007.10.014
4. V. Pragadeeswarar, M. Marudai, Best proximity points: approximation and optimization in partially ordered metric spaces, *Optim. Lett.*, **7** (2012), 1883–1892. https://doi.org/10.1007/s11590-012-0529-x
5. S. S. Basha, P. Veeramani, Best approximations and best proximity pairs, *Acta. Sci. Math.*, **63** (1997), 289–300.
6. S. S. Basha, P. Veeramani, D. V. Pai, Best proximity pair theorems, *Indian J. Pure Appl. Math.*, **32** (2001), 1237–1246. https://doi.org/10.1093/imamat/66.4.411
7. S. S. Basha, Best proximity points: optimal solutions, *J. Optim. Theory Appl.*, **151** (2011), 210–216. https://doi.org/10.1007/s10957-011-9869-4
8. V. S. Raj, A best proximity point theorem for weakly contractive non-self-mappings, *Nonlinear Anal. Theor.*, **74** (2011), 4804–4808. https://doi.org/10.1016/j.na.2011.04.052
9. M. R. Haddadi, V. Parvaneh, M. Mursaleen, Global optimal approximate solutions of best proximity points, *Filomat*, **35** (2021), 1555–1564. https://doi.org/10.2298/FIL2105555H
10. M. Jovanović, Z. Kadelburg, S. Radenović, Common fixed point results in metric-type spaces, *Fixed Point Theory Appl.*, **2010** (2010), 978121. https://doi.org/10.1155/2010/978121

11. R. Chugh, S. Kumar, Common fixed points for weakly compatible maps, *Proc. Indian Acad. Sci. (Math. Sci.)*, **111** (2001), 241–247. https://doi.org/10.1007/BF02829594

12. P. Debnath, N. Konwar, S. Radenović, *Metric fixed point theory*, Singapore: Springer, 2021. https://doi.org/10.1007/978-981-16-4896-0

13. Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, Some common fixed point results in ordered partial b-metric spaces, *J. Inequal. Appl.*, **2013** (2013), 562. https://doi.org/10.1186/1029-242X-2013-562

14. J. R. Roshan, N. Shobkolaei, S. Sedghi, V. Parvaneh, S. Radenović, Common fixed point theorems for three maps in discontinuous Gb metric spaces, *Acta Math. Sci.*, **34** (2014), 1643–1654. https://doi.org/10.1016/S0252-9602(14)60110-7

15. H. Isik, V. Parvaneh, B. Mohammadi, I. Altun, Common fixed point results for generalized Wardowski type contractive multi-valued mappings, *Mathematics*, **7** (2019), 1130. https://doi.org/10.3390/math7111130

16. H. K. Pathak, Y. J. Cho, S. M. Kang, Remarks of R-weakly commuting mappings and common fixed point theorems, *Bull. Korean Math. Soc.*, **34** (1997), 247–257.

17. S. S. Basha, N. Shahzad, R. Jeyaraj, Common best proximity points: global optimization of multi-objective functions, *Appl. Math. Lett.*, **24** (2011), 883–886. https://doi.org/10.1016/j.aml.2010.12.043

18. N. Shahzad, S. S. Basha, R. Jeyaraj, Common best proximity points: global optimal solutions, *J. Optim. Theory Appl.*, **148** (2011), 69–78. https://doi.org/10.1007/s10957-010-9745-7

19. S. S. Basha, Common best proximity points: global minimization of multi-objective functions, *J. Glob. Optim.*, **54** (2012), 367–373. https://doi.org/10.1007/s10898-011-9760-8

20. C. Mongkolkeha, P. Kumam, Some common best proximity points for proximity commuting mappings, *Optim. Lett.*, **7** (2013), 1825–1836. https://doi.org/10.1007/s11590-012-0525-1

21. M. Gabeleh, J. Markin, Common best proximity pairs via the concept of complete proximal normal structure, *Ann. Funct. Anal.*, **11** (2020), 831–847. https://doi.org/10.1007/s43034-020-00057-x

22. S. Mondal, L. K. Dey, Some common best proximity point theorems in a complete metric space, *Afr. Mat.*, **28** (2017), 85–97. https://doi.org/10.1007/s13370-016-0432-1

23. P. Lo'lo', S. M. Vaezpour, J. Esmaily, Common best proximity points theorem for four mappings in metric-type spaces, *Fixed Point Theory Appl.*, **2015** (2015), 47. https://doi.org/10.1186/s13663-015-0298-1

24. P. Lo'lo', S. M. Vaezpour, R. Saadati, Common best proximity points results for new proximal C-contraction mappings, *Fixed Point Theory Appl.*, **2016** (2016), 56. https://doi.org/10.1186/s13663-016-0545-0

25. V. Pragadeeswarar, R. Gopi, M. De la Sen, S. Radenovi'c, Proximally compatible mappings and common best proximity points, *Symmetry*, **12** (2020), 353. https://doi.org/10.3390/sym12030353

26. V. Pragadeeswarar, G. Poonguzali, M. Marudai, S. Radenović, Common best proximity point theorem for multivalued mappings in partially ordered metric spaces, *Fixed Point Theory Appl.*, **2017** (2017), 22. https://doi.org/10.1186/s13663-017-0615-y

27. J. Puangpee, S. Suantai, New hybrid algorithms for global minimization of common best proximity points of some generalized nonexpansive mappings, *Filomat*, **33** (2019), 2381–2391. https://doi.org/10.2298/FIL1908381P

28. R. P. Pant, A common fixed point theorm under a new condition, *Indian J. Pure Appl. Math.*, **30** (1999), 147–152.

29. R. P. Pant, R. K. Bisht, D. Arora, Weak reciprocal continuity and fixed point theorems, *Ann. Univ. Ferrara*, **57** (2011), 181–190. https://doi.org/10.1007/s11565-011-0119-3