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Research article

Global existence and energy decay for a transmission problem under a boundary fractional derivative type

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Abstract: The paper considers the effects of fractional derivative with a high degree of accuracy in the boundary conditions for the transmission problem. It is shown that the existence and uniqueness of the solutions for the transmission problem in a bounded domain with a boundary condition given by a fractional term in the second equation are guaranteed by using the semigroup theory. Under an appropriate assumptions on the transmission conditions and boundary conditions, we also discuss the exponential and strong stability of solution by also introducing the theory of semigroups.

Keywords: fractional derivatives; wave equation; transmission problem; delay term; exponential stability
Mathematics Subject Classification: 35B37, 35L55, 74D05, 93D15, 93D20

1. Introduction and position of problem

A generalization of power series, in which each term has an integer exponent in mathematical analysis, and Laurent series in the theory of functions of a complex variable, was Hadamard series (Hadamard operator) and Frobenius series in mathematical physics with fractional exponents for each term. It is possible to generalize the derivative of an integer order to a derivative of a fractional order (Riemann-Liouville derivatives and Caputo-Gerasimov derivatives). It is natural to write solutions of differential equations of fractional order in terms of Hadamard or Frobenius series. Fractional derivatives appear in new physical, technical and chemical problems arising in research activities. Here, we are interested in taking this phenomenon (fractional derivatives) in the boundary conditions with respect to the time variable, for more detail, please see [5–7, 9, 19].

A new class of initial boundary value problems is a transmission problem given by the equations

$$\begin{cases} \rho_1 \partial_{tt} u - \tau_1 u_{xx} + \varpi_1 \partial_t u(x,t) = 0, & x \in (0, l_0), \\ \rho_2 \partial_{tt} v - \tau_2 v_{xx} + \varpi_2 \partial_t v(x,t) = 0, & x \in (l_0, L), \quad t \in (0, \infty), \end{cases} \tag{1.1}$$

subject to the initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad \partial_t u(x,0) = u_1(x), \quad x \in (0, l_0), \\ v(x,0) &= v_0(x), \quad \partial_t v(x,0) = v_1(x), \quad x \in (l_0, L), \end{aligned} \tag{1.2}$$

transmission conditions

$$u(l_0,t) = v(l_0,t), \rho_2 \tau_1 u_x(l_0,t) = \rho_1 \tau_2 v_x(l_0,t), \quad \forall t > 0, \tag{1.3}$$

boundary conditions

$$u(0,t) = 0, \tau_2 v_x(L,t) + \gamma \rho_2 \tau_2 \partial_t^{\alpha,\zeta} v(L,t) = 0, \quad \forall t > 0, \tag{1.4}$$

and compatibility conditions

$$u_0(l_0) = v_0(l_0), u_1(l_0) = v_1(l_0), \rho_2 \tau_1 u_{0x}(l_0) = \rho_1 \tau_2 v_{0x}(l_0), \tag{1.5}$$

where $0 < l_0 < L, \varpi_1, \varpi_2$ are positive constants, $\rho_1, \rho_2, \tau_1, \tau_2 > 0$ represent the densities and tensions of the strings u and v, respectively, $\gamma > 0$, the initial data (u_0, u_1, v_0, v_1) belong to a suitable function space which will be defined later. We will mention some works related to the stabilization of transmission problems with mechanism of damping (see [8, 20, 21]). In [18], the authors consider a transmission problem in viscoelasticity. The exponential decay of the solutions is obtained and it is proved that the linear model is well posed. In [12], a transmission problem involving two Euler-Bernoulli equations which model the vibrations of a composite beam is considered. By one boundary damping term, the global existence and decay property of the solutions are showed.

Recently in [2], Benaissa and Atoui consider the following transmission problem

$$\begin{cases} \rho_1 \partial_{tt} u - \tau_1 u_{xx} = 0, & x \in (0, l_0), \\ \rho_2 \partial_{tt} v - \tau_2 v_{xx} = 0, & x \in (l_0, L), \\ u(l_0,t) = v(l_0,t), \rho_2 \tau_1 u_x(l_0,t) = \rho_1 \tau_2 v_x(l_0,t), \\ u(0,t) = 0, \tau_2 v_x(L,t) + \gamma \rho_2 \tau_2 \partial_t^{\alpha,\zeta} v(L,t) = 0, & t > 0. \end{cases} \tag{1.6}$$

The lack of exponential decay of the energy is proved and also the polynomial decay rate is showed by using the spectrum method and the Borichev-Tomilov theorem [10].

The paper is organized as follows. In section 1, we introduce our model in (1.1) and the actual state-of-the-art is given. In section 2, the well-posedness of strong/weak solutions of the system is given by using the Hille-Yosida theorem. In section 3, we treat the question of stability where we find that the augmented model is strongly stable in the absence of compactness of the resolvent by using a criteria of Arendt-Batty. In section 4, we show the lack of exponential stability by spectral analysis and the polynomial type decay rate is proved which depends on a parameter α. We finished our work with section 5 by dealing with the polynomial stability for $\zeta \neq 0$.

2. Preliminary and well-posedness of solution

The beginning of this section concerns to write the system (1.1) by another way. For this aim, we will use the following result.

Theorem 2.1. *[14] Let ϱ be a function defined by*

$$\varrho(s) = |s|^{(2\alpha-1)/2}, \quad s \in (-\infty, +\infty), \alpha \in (0,1).$$

Then the relationship between the 'input' U and the 'output' O of the following system

$$\partial_t \Phi(s,t) + \left(s^2 + \zeta\right) \Phi(s,t) - U\varrho(s) = 0, \quad s \in (-\infty, +\infty), 0 \leq \zeta, 0 < t, \tag{2.1}$$

$$\Phi(s,0) = 0,$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \varrho(s) \Phi(s,t)\, ds,$$

is introduced by

$$O = I^{1-\alpha,\zeta} U,$$

where

$$\left[I^{\alpha,\zeta} f\right](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\zeta}(t-\tau) f(\tau)\, d\tau.$$

Lemma 2.1. *[1] Define*

$$D = \{\varpi \in \mathbb{C} / \Re\varpi + \zeta > 0\} \cup \{\Im\varpi \neq 0\}.$$

If $\varpi \in D$, then

$$\begin{aligned} F_1(\varpi) &= \int_{-\infty}^{+\infty} \frac{\varrho^2(s)}{\varpi + \zeta + s^2}\, ds \\ &= \frac{\pi}{\sin\alpha\pi} (\varpi + \zeta)^{\alpha-1}, \end{aligned}$$

and

$$\begin{aligned} F_2(\varpi) &= \int_{-\infty}^{+\infty} \frac{\varrho^2(s)}{\left(\varpi + \zeta + s^2\right)^2}\, ds \\ &= (1-\alpha) \frac{\pi}{\sin\alpha\pi} (\varpi + \zeta)^{\alpha-2}. \end{aligned}$$

We need now to reformulate the system (1.1). For this aim, we take $U = \partial_t v$ in (2.1) and using (1.4), the system (1.1) becomes for $t \in (0, +\infty)$

$$\begin{cases} \rho_1 \partial_{tt} u - \tau_1 u_{xx} + \varpi_1 \partial_t u = 0, & x \in (0, l_0), \\ \rho_2 \partial_{tt} v - \tau_2 v_{xx} + \varpi_2 \partial_t v = 0, & x \in (l_0, L), \\ \partial_t \Phi(\xi, t) + \left(\xi^2 + \zeta\right) \Phi(\xi, t) - \partial_t v(L, t) \varrho(\xi) = 0, & \xi \in \mathbb{R}, \\ u(l_0, t) = v(l_0, t), \rho_2 \tau_1 u_x(l_0, t) = \rho_1 \tau_2 v_x(l_0, t), & \\ u(0, t) = 0, & \\ \tau_2 v_x(L, t) + \varsigma \rho_2 \int_{-\infty}^{+\infty} \varrho(\xi) \Phi(\xi, t) \, d\xi = 0, & \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x), & x \in (0, l_0), \\ v(x, 0) = v_0(x), \quad \partial_t v(x, 0) = v_1(x), & x \in (l_0, L), \end{cases} \tag{2.2}$$

where $\varsigma = (\pi)^{-1} \sin(\alpha\pi) \gamma$. The energy associated with the solutions (u, v, Φ) of (2.2) is defined as follows

$$\mathcal{E}(t) = \frac{1}{2} \int_0^{l_0} \left(|\partial_t u|^2 + \frac{\tau_1}{\rho_1} |u_x|^2 \right) dx + \frac{1}{2} \int_{l_0}^{L} \left(|\partial_t v|^2 + \frac{\tau_2}{\rho_2} |v_x|^2 \right) dx + \frac{\varsigma}{2} \int_{-\infty}^{+\infty} |\Phi(\xi, t)|^2 \, d\xi. \tag{2.3}$$

Lemma 2.2. *Let* (u, v, Φ) *be a regular solution of* (2.2). *Then, the energy* (2.3) *satisfies*

$$\mathcal{E}'(t) = -\left(\varsigma \int_{-\infty}^{+\infty} \left(\xi^2 + \zeta\right) |\Phi(\xi, t)|^2 \, d\xi + \frac{\varpi_1}{\rho_1} \int_0^{l_0} |\partial_t u|^2 \, dx + \frac{\varpi_2}{\rho_2} \int_{l_0}^{L} |\partial_t v|^2 \, dx. \right) \leq 0. \tag{2.4}$$

Proof. By multiplication of $(2.2)_1$ by $\partial_t \overline{u}$ and then integrating by parts over $(0, l_0)$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^{l_0} \left(|\partial_t u|^2 + \frac{\tau_1}{\rho_1} |u_x|^2 \right) dx + \frac{\varpi_1}{\rho_1} \int_0^{l_0} |\partial_t u|^2 \, dx - \frac{\tau_1}{\rho_1} \mathcal{R}e u_x(l_0) \partial_t \overline{u}(l_0) = 0.$$

Now, we multiply $(2.2)_2$ by $\partial_t \overline{v}$ and then we integrate by parts over (l_0, L) and we arrive at

$$\frac{1}{2} \frac{d}{dt} \int_{l_0}^{L} \left(|\partial_t v|^2 + \frac{\tau_1}{\rho_1} |v_x|^2 \right) dx + \varpi_2 \int_{l_0}^{L} |\partial_t v|^2 \, dx - \frac{\tau_2}{\rho_2} v_x(L) \partial_t \overline{v}(L) + \mathcal{R}e \frac{\tau_2}{\rho_2} v_x(l_0) \partial_t \overline{v}(l_0) = 0.$$

Summing, we obtain

$$\frac{1}{2} \frac{d}{dt} \left(\int_0^{l_0} \left(|\partial_t u|^2 + \frac{\tau_1}{\rho_1} |u_x|^2 \right) dx + \int_{l_0}^{L} \left(|\partial_t v|^2 + \frac{\tau_2}{\rho_2} |v_x|^2 \right) dx \right)$$
$$+ \frac{\varpi_1}{\rho_1} \int_0^{l_0} |\partial_t u|^2 \, dx + \frac{\varpi_2}{\rho_2} \int_{l_0}^{L} |\partial_t v|^2 \, dx - \mathcal{R}e \frac{\tau_2}{\rho_2} v_x(L) \partial_t \overline{v}(L) = 0.$$

From the boundary condition $(2.2)_6$, we get

$$\frac{1}{2} \frac{d}{dt} \left(\int_0^{l_0} \left(|\partial_t u|^2 + \frac{\tau_1}{\rho_1} |u_x|^2 \right) dx + \int_{l_0}^{L} \left(|\partial_t v|^2 + \frac{\tau_2}{\rho_2} |v_x|^2 \right) dx \right)$$
$$+ \frac{\varpi_1}{\rho_1} \int_0^{l_0} |\partial_t u|^2 \, dx + \frac{\varpi_2}{\rho_2} \int_{l_0}^{L} |\partial_t v|^2 \, dx + \varsigma \partial_t v(L) \int_{-\infty}^{+\infty} \varrho(\xi) \Phi(\xi, t) \, d\xi = 0. \tag{2.5}$$

We multiply $(2.2)_3$ by $\varsigma \Phi$ and then we integrate over $(-\infty, +\infty)$ and we obtain

$$\frac{\varsigma}{2} \frac{d}{dt} \|\Phi\|_2^2 + \varsigma \int_{-\infty}^{+\infty} \left(\xi^2 + \zeta\right) |\Phi(\xi, t)|^2 \, d\xi - \varsigma \mathcal{R}e \partial_t v(L) \int_{-\infty}^{+\infty} \varrho(\xi) \Phi(\xi, t) \, d\xi = 0. \tag{2.6}$$

Then, using (2.5) and (2.6), we arrive at

$$\mathcal{E}'(t) = -\varsigma \int_{-\infty}^{+\infty} \left(\xi^2 - \zeta\right) |\Phi(\xi,t)|^2 \, d\xi - \frac{\varpi_1}{\rho_1} \int_0^{l_0} |\partial_t u|^2 \, dx - \frac{\varpi_2}{\rho_2} \int_{l_0}^{L} |\partial_t v|^2 \, dx \le 0.$$

Now, we will use a semigroup setting for (2.2). For this aim, we introduce the vector $X = (u, \varphi, v, \psi, \Phi)^T$, where $\varphi = \partial_t u$ and $\psi = \partial_t v$. Then we get that the system (2.2) is equivalent to the following system

$$\begin{cases} X' = \mathcal{A}X, \quad 0 < t, \\ X(0) = X_0. \end{cases} \tag{2.7}$$

Here $X_0 := (u_0, u_1, v_0, v_1, \Phi_0)^T$. The operator $\mathcal{A}$ given by

$$\mathcal{A}\begin{pmatrix} u \\ \varphi \\ v \\ \psi \\ \Phi \end{pmatrix} = \begin{pmatrix} \varphi \\ \frac{\tau_1}{\rho_1} u_{xx} - \frac{\varpi_1}{\rho_1}\varphi \\ \psi \\ \frac{\tau_2}{\rho_2} v_{xx} - \frac{\varpi_2}{\rho_2}\psi \\ -\left(\xi^2 + \zeta\right)\Phi(\xi) + \psi(L)\varrho(\xi) \end{pmatrix} \tag{2.8}$$

is a linear operator. We introduce the following Hilbert space (the energy space)

$$H_*^1 = \left\{u \in H^1(0, l_0) : u(0) = 0\right\}.$$

$$\mathcal{H} = \left\{H_*^1(0, l_0) \times L^2(0, l_0) \times H^1(l_0, L) \times L^2(l_0, L) \times L^2(-\infty, +\infty) : u(l_0) = v(l_0)\right\}.$$

For $X = (u, \varphi, v, \psi, \Phi)^T$ and $\overline{X} = (\overline{u}, \overline{\varphi}, \overline{v}, \overline{\psi}, \overline{\Phi})^T$, the inner product in $\mathcal{H}$ is defined as follows

$$\left\langle X, \overline{X} \right\rangle_{\mathcal{H}} = \int_0^{l_0} \left(\varphi\overline{\varphi} + \frac{\tau_1}{\rho_1} u_x \overline{u}_x\right) dx + \int_{l_0}^{L} \left(\psi\overline{\psi} + \frac{\tau_2}{\rho_2} v_x \overline{v}_x\right) dx + \varsigma \int_{-\infty}^{+\infty} \Phi\overline{\Phi} \, dx,$$

where the domain of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, \varphi, v, \psi, \Phi)^T \in \mathcal{H} : u \in H^2(0, l_0) \cap H_0^1(0, l_0), \\ \varphi \in H_*^1(0, l_0), v \in H^2(l_0, L), \psi \in H^1(l_0, L), \\ u(l_0) = v(l_0), \rho_2\tau_1 u_x(l_0) = \rho_1\tau_2 v_x(l_0), \\ \tau_2 v_x(L) + \varsigma\rho_2 \int_{-\infty}^{+\infty} \varrho(\xi)\Phi(\xi)\, d\xi = 0, |\xi|\Phi \in L^2(-\infty, +\infty) \end{array} \right\}. \tag{2.9}$$

We state now a result for existence and uniqueness.

Theorem 2.2. *(1) If $X_0 \in D(\mathcal{A})$, then the system* (2.2) *has a unique strong solution*

$$X \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $X_0 \in \mathcal{H}$, then the system (2.2) *has a unique weak solution*

$$X \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof. Firstly, we will show the dissipativity of the operator $\mathcal{A}$. In fact, we have $\forall X \in D(\mathcal{A})$

$$\begin{aligned}\langle \mathcal{A}X, X\rangle_{\mathcal{H}} = & \int_0^{l_0} \left[\left(\tfrac{\tau_1}{\rho_1}u_{xx} - \tfrac{\varpi_1}{\rho_1}\varphi\right)\varphi + \tfrac{\tau_1}{\rho_1}\varphi_x u_x\right] dx + \int_{l_0}^{L} \left[\left(\tfrac{\tau_2}{\rho_2}v_{xx} - \tfrac{\varpi_2}{\rho_2}\psi\right)\psi + \tfrac{\tau_2}{\rho_2}\psi_x v_x\right] dx \\ & + \varsigma \int_{-\infty}^{+\infty} \left(-\left(\xi^2 + \zeta\right)\Phi(\xi) + \psi(L)\varrho(\xi)\right)\Phi(\xi).\end{aligned}$$

Then, by (2.5) and (2.6), we get

$$\mathcal{R}e\,\langle \mathcal{A}X, X\rangle_{\mathcal{H}} = -\mathcal{R}e\varsigma \int_{-\infty}^{+\infty} \left(\xi^2 + \zeta\right)|\Phi(\xi, t)|^2\, d\xi \le 0. \tag{2.10}$$

Hence, $\mathcal{A}$ is dissipative.

Now, we will prove the surjectivity of the operator $\varpi I - \mathcal{A}$ for $0 < \varpi$. Let $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$. We will prove that there exists $X \in D(\mathcal{A})$ such that

$$(\varpi I - \mathcal{A})\, X = F. \tag{2.11}$$

Here the Eq (2.11) is equivalent to the equation

$$\begin{cases} \varpi u - \varphi = f_1, \\ \left(\varpi + \frac{\varpi_1}{\rho_1}\right)\varphi - \frac{\tau_1}{\rho_1}u_{xx} = f_2, \\ \varpi v - \psi = f_3, \\ \left(\varpi + \frac{\varpi_2}{\rho_2}\right)\psi - \frac{\tau_2}{\rho_2}v_{xx} = f_4, \\ \left(\varpi + \xi^2 + \zeta\right)\Phi - \psi(L)\varrho(\xi) = f_5. \end{cases} \tag{2.12}$$

Suppose that u and v are found with the appropriate regularity. Then, from $(2.12)_1$ and $(2.12)_3$, we find that

$$\begin{aligned} \varphi &= \varpi u - f_1, \\ \psi &= \varpi v - f_3. \end{aligned} \tag{2.13}$$

It is not hard to see that $\varphi \in H_*^1(0, l_0)$ and $\psi \in H^1(l_0, L)$. Furthermore, by $(2.12)_5$, we can find Φ as follows

$$\Phi = \frac{\psi(L)\varrho(\xi) + f_5}{\xi^2 + \zeta + \varpi}. \tag{2.14}$$

By (2.12) and (2.13), we have that u and v satisfy

$$\begin{cases} \varpi^2 u - \frac{\tau_1}{\rho_1}u_{xx} + \frac{\varpi_1}{\rho_1}\varphi = \varpi f_1 + f_2, \\ \varpi^2 v - \frac{\tau_2}{\rho_2}v_{xx} + \frac{\varpi_2}{\rho_2}\psi = \varpi f_3 + f_4. \end{cases} \tag{2.15}$$

The solving of the system (2.15) is equivalent to find $u \in H^2 \cap H_*^1(0, l_0)$ and $v \in H^2(l_0, L)$ so that

$$\begin{cases} \int_0^{l} \left(\varpi^2 u\overline{w} - \frac{\tau_1}{\rho_1}u_{xx}\overline{w} + \frac{\varpi_1}{\rho_1}\varphi\overline{w}\right) dx = \int_0^{l} (\varpi f_1 + f_2)\,\overline{w}\, dx, \\ \int_{l_0}^{L} \left(\varpi^2 v\overline{\chi} - \frac{\tau_2}{\rho_2}v_{xx}\overline{\chi} + \frac{\varpi_2}{\rho_2}\psi\overline{\chi}\right) dx = \int_{l_0}^{L} (\varpi f_3 + f_4)\overline{\chi} dx, \end{cases} \tag{2.16}$$

for all $w \in H_*^1(0, l_0)$ and $\chi \in H^1(l_0, L)$. By (2.14) and (2.16), we get that u and v satisfying

$$\begin{cases} \int_0^{l} \left(\varpi^2 u\overline{w} + \frac{\tau_1}{\rho_1}u_x\overline{w}_x + \frac{\varpi_1}{\rho_1}\varphi\overline{w}\right) dx \\ -\frac{\tau_1}{\rho_1}\left[u_x(l_0)\,\overline{w}(l_0) - u_x(0)\,\overline{w}(0)\right] = \int_0^{l} (\varpi f_1 + f_2)\,\overline{w}\, dx, \\ \int_{l_0}^{L} \left(\varpi^2 v\overline{\chi} + \frac{\tau_2}{\rho_2}v_x\overline{\chi}_x + \frac{\varpi_2}{\rho_2}\psi\overline{\chi}\right) dx \\ -\frac{\tau_2}{\rho_2}\left[v_x(L)\,\overline{\chi}(L) - v_x(l_0)\,\overline{w}(l_0)\right] = \int_{l_0}^{L} (\varpi f_3 + f_4)\overline{\chi} dx. \end{cases} \tag{2.17}$$

Adding the Eqs $(2.2)_6$ and $(2.17)_{1,2}$, we obtain

$$\begin{aligned}&\int_0^l \left(\varpi^2 u\overline{w} + \tfrac{\tau_1}{\rho_1} u_x \overline{w}_x + \tfrac{\varpi_1}{\rho_1}\varphi\overline{w}\right) dx + \int_{l_0}^L \left(\varpi^2 v\overline{\chi} + \tfrac{\tau_2}{\rho_2} v_x\overline{\chi}_x + \tfrac{\varpi_2}{\rho_2}\psi\overline{\chi}\right) dx = \int_0^l (\varpi f_1 + f_2)\,\overline{w}\, dx \\ &+ \int_{l_0}^L (\varpi f_3 + f_4)\overline{\chi} dx - \varsigma\overline{\chi}(L)\int_{-\infty}^{+\infty} \tfrac{\varrho(\xi)}{\xi^2+\zeta+\varpi} f_5\, d\xi - \varsigma\overline{\chi}(L)\int_{-\infty}^{+\infty} \tfrac{\varpi v(L)-F_3(L)}{\xi^2+\zeta+\varpi}\varrho^2(\xi)\, d\xi,\end{aligned} \tag{2.18}$$

where $\overline{\varsigma} = \varsigma(L)\int_{-\infty}^{+\infty} \frac{\varrho^2(\xi)}{\xi^2+\zeta+\varpi} d\xi$. Then, (2.18) is equivalent to

$$a((u,v),(w,\chi)) = L(w,\chi). \tag{2.19}$$

Here the linear form

$$L : H_*^1(0,l_0)\times H^1(l_0,L) \longrightarrow \mathbb{C},$$

and the bilinear form

$$a : \left(H_*^1(0,l_0)\times H^1(l_0,L)\right)^2 \longrightarrow \mathbb{C},$$

are given as follows

$$a((u,v),(w,\chi)) = \int_0^l \left(\varpi^2 u\overline{w} + \frac{\tau_1}{\rho_1}u_x\overline{w}_x + \frac{\varpi_1}{\rho_1}\varphi\overline{w}\right) dx + \int_{l_0}^L \left(\varpi^2 v\overline{\chi} + \frac{\tau_2}{\rho_2}v_x\overline{\chi}_x + \frac{\varpi_2}{\rho_2}\psi\overline{\chi}\right) dx,$$

and

$$\begin{aligned}L(w,\chi) = \;& \int_0^l (\varpi f_1 + f_2)\,\overline{w}\, dx + \int_{l_0}^L (\varpi f_3 + f_4)\overline{\chi} dx - \varsigma\overline{\chi}(L)\int_{-\infty}^{+\infty} \tfrac{\varrho(\xi)}{\xi^2+\zeta+\varpi} f_5\, d\xi \\ & - \varsigma\overline{\chi}(L)\int_{-\infty}^{+\infty} \tfrac{\varpi v(L)-f_3(L)}{\xi^2+\zeta+\varpi}\varrho^2(\xi)\, d\xi,\end{aligned} \tag{2.20}$$

respectively. It is not hard to check that a is coercive and continuous and L is continuous. By using the Lax-Milgram theorem, we find that $\forall (w,\chi) \in H_*^1(0,l_0)\times H^1(l_0,L)$. Then the problem (2.19) has a unique solution

$$(u,v) \in H_*^1(0,l_0)\times H^1(l_0,L).$$

Using the classical elliptic regularity and (2.20), we find that

$$(u,v) \in H^2(0,l_0)\times H^2(l_0,L).$$

Then, $\varpi I - \mathcal{A}$ is surjective $\forall 0 < \varpi$. Owing to the Hille-Yosida theorem, the result in Theorem (2.2) yields.

3. On the strong stability of solution

In this part, as in [4, 11], we use the Arendt-Batty theorem and we see that a C_0-semigroup of contractions $e^{\mathcal{A}t}$ in a Banach space is strongly stable whenever $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of elements and $\mathcal{A}$ has no pure imaginary eigenvalues. The following theorem is our next main result.

Theorem 3.1. *[3] The C_0-semigroup $e^{\mathcal{A}t}$ is strongly stable in $\mathcal{H}$, i.e, $\forall X_0 \in \mathcal{H}$, and the solution of* (2.7) *satisfies*

$$\lim_{t\longrightarrow\infty} \left\|e^{\mathcal{A}t}X_0\right\|_{\mathcal{H}} = 0.$$

To prove this result, we will have a need of the following lemma.

Lemma 3.1. *$\mathcal{A}$ has no eigenvalues on $i\mathbb{R}$.*

Proof. The proof has two stages. The first one is $i\varpi = 0$ and the second one is $i\varpi \neq 0$.
Step 1. It is easy to see, using the boundary conditions in domain (2.9), that the equation $\mathcal{A}X = 0$ leads to $X = 0$. Then, $i\varpi = 0$ can not be an eigenvalue of $\mathcal{A}$.
Step 2. We will use a contradiction argument. Suppose that there exists $\varpi \in \mathbb{R}, \varpi \neq 0$, and $X \neq 0$, such that $\mathcal{A}X = i\varpi X$. Then, we have

$$\begin{cases} i\varpi u - \varphi = 0, \\ \left(i\varpi + \frac{\varpi_1}{\rho_1}\right)\varphi - \frac{\tau_1}{\rho_1}u_{xx} = 0, \\ i\varpi v - \psi = 0, \\ \left(i\varpi + \frac{\varpi_2}{\rho_2}\right)\psi - \frac{\tau_2}{\rho_2}v_{xx} = 0, \\ i\varpi\Phi + \left(\xi^2 + \zeta\right)\Phi(\xi) - \psi(L)\varrho(\xi) = 0. \end{cases} \tag{3.1}$$

Using (2.10), we find

$$\Phi \equiv 0.$$

Using $(3.1)_5$, we get

$$\psi(L) = 0.$$

Hence, applying $(3.1)_3$ and $(2.9)_4$, we obtain

$$v(L) = 0 \text{ and } v_x(L) = 0. \tag{3.2}$$

Inserting $(3.1)_3$ into $(3.1)_4$, we arrive at

$$\begin{cases} -\varpi^2 v - \frac{\tau_2}{\rho_2}v_{xx} = 0, \\ \frac{\varpi\varpi_2}{\rho_2}v = 0. \end{cases} \tag{3.3}$$

The solution of (3.3) is given by

$$\begin{cases} v(x) = c_1 \cos\frac{\varpi}{r_2}x + c_2 \sin\frac{\varpi}{r_2}x, \quad r_2 = \sqrt{\frac{\tau_2}{\rho_2}}. \\ v = 0. \end{cases} \tag{3.4}$$

By (3.2), we get

$$v \equiv 0.$$

Due to the transmission and boundary conditions, we obtain

$$u(l_0) = u_x(l_0).$$

Similarly, we deduce that

$$u \equiv 0.$$

By the Picard Theorem, we get $X = 0$. Then, $\mathcal{A}$ has no purely imaginary eigenvalues.

Lemma 3.2. *For $\varpi \neq 0$, we have the operator $i\varpi I - \mathcal{A}$ is surjective. If $\varpi = 0$ and $\zeta \neq 0$, then the operator $i\varpi I - \mathcal{A}$ is surjective.*

Proof. **Case 1.** Suppose that $\varpi \neq 0$.
Let $F = (f1, f2, f3, f_4, f_5) \in \mathcal{H}$. We seek $X = (u, \varphi, v, \psi, \Phi) \in D(\mathcal{A})$ as a solution of

$$(i\varpi I - \mathcal{A})X = F,$$

which is equivalent to

$$\begin{cases} i\varpi u - \varphi = f_1, \\ \left(i\varpi + \frac{\varpi_1}{\rho_1}\right)\varphi - \frac{\tau_1}{\rho_1}u_{xx} = f_2, \\ i\varpi v - \psi = f_3, \\ \left(i\varpi + \frac{\varpi_2}{\rho_2}\right)\psi - \frac{\tau_2}{\rho_2}v_{xx} = f_4, \\ i\varpi\Phi + \left(\xi^2 + \zeta\right)\Phi(\xi) - \psi(L)\varrho(\xi) = f_5. \end{cases} \tag{3.5}$$

The proof is divided into several steps.

Inserting $(3.5)_1$ into $(3.5)_2$ and inserting $(3.5)_3$ into $(3.5)_4$, we get

$$\begin{cases} \left(-\varpi^2 + i\frac{\varpi\varpi_1}{\rho_1}\right)u - \frac{\tau_1}{\rho_1}u_{xx} = f_2 + \left(i\varpi + \frac{\varpi_1}{\rho_1}\right)f_1, \\ \left(-\varpi^2 + i\frac{\varpi\varpi_2}{\rho_2}\right)v - \frac{\tau_2}{\rho_2}v_{xx} = f_4 + \left(i\varpi + \frac{\varpi_2}{\rho_2}\right)f_3. \end{cases} \tag{3.6}$$

The solving of the system (3.6) is equivalent to find $(u, v) \in H^2 \cap H^1_*(0, l_0) \times H^2(l_0, L)$ such that

$$\begin{cases} \int_0^{l_0}\left(\left(\rho_1^2\varpi^2 + \varpi_1\right)u\overline{w} + \tau_1\rho_1 u_{xx}\overline{w}\right)dx = -\int_0^{l_0}\rho_1 f_2\overline{w}\,dx, \\ \int_{l_0}^{L}\left(\left(\rho_2^2\varpi^2 + \varpi_2\right)v\overline{\chi} + \tau_2\rho_2 v_{xx}\overline{\chi}\right)dx = -\int_{l_0}^{L}\rho_2 f_4\overline{\chi}\,dx, \end{cases}$$

$\forall (w, \chi) \in H^1_*(0, l_0) \times H^1(l_0, L)$. By using $(3.1)_5$, $(2.2)_3$ and $(3.5)_5$, the functions u and v satisfy the following equation

$$\begin{aligned} &\int_0^{l_0}\left(\left(\rho_1^2\varpi^2 + \varpi_1\right)u\overline{w} - \tau_1\rho_1 u_x\overline{w}_x\right)dx + \int_{l_0}^{L}\left(\left(\rho_2^2\varpi^2 + \varpi_2\right)v\overline{\chi} - \tau_2\rho_2 v_x\overline{\chi}_x\right)dx + i\varpi\rho_2^2\overline{\varsigma}\overline{\chi}(L)v(L) \\ &= -\int_{l_0}^{L}\rho_2 f_4\overline{\chi}\,dx - \int_0^{l_0}\rho_1 f_2\overline{w}\,dx + \rho_2^2\overline{\varsigma}\overline{\chi}(L)f_3(L) + \rho_2^2\varsigma\overline{\chi}(L)\int_{-\infty}^{+\infty}\frac{f_5}{i\varpi+\xi^2+\zeta}\varrho(\xi)\,d\xi, \end{aligned} \tag{3.7}$$

where $\overline{\varsigma} = \varsigma\int_{-\infty}^{+\infty}\frac{\varrho^2(\xi)}{i\varpi+\xi^2+\zeta}\,d\xi$. We can rewrite (3.7) as follows

$$-\langle L_\varpi X, Y\rangle_{H^1_R} + \langle X, Y\rangle_{H^1_R} = l(Y), \tag{3.8}$$

where

$$H^1_R = \left\{(u, v) \in H^1_*(0, l_0) \times H^1(l_0, L) / u(l_0) = v(l_0)\right\},$$

with

$$\langle X, Y\rangle_{H^1_R} = \tau_1\rho_1\int_0^{l_0}u_x\overline{w}_x dx + \tau_2\rho_2\int_{l_0}^{L}v_x\overline{\chi}_x dx,$$

and

$$\langle L_\varpi X, Y\rangle_{H^1_R} = \left(\rho_1^2\varpi^2 + \varpi_1\right)\int_0^{l_0}u\overline{w}\,dx + \left(\rho_2^2\varpi^2 + \varpi_2\right)\int_{l_0}^{L}v\overline{\chi}\,dx - i\varpi\rho_2^2\overline{\varsigma}\overline{\chi}(L)v(L).$$

Using the principle of compactness embedding from $\left(L^2(0, l_0) \times L^2(l_0, L)\right)$ into $\left(H^1_R(0, L)\right)'$ and from $H^1_R(0, L)$ into $L^2(0, l_0) \times L^2(l_0, L)$, we find that L_ϖ is compact from $L^2(0, l_0) \times L^2(l_0, L)$

into $L^2(0,l_0) \times L^2(l_0,L)$. Consequently, by using the Fredholm alternative, to prove that X is a solution of (3.8) we will prove that 1 can not be an eigenvalue of L_ϖ. Thus, if 1 is an eigenvalue, then $\exists X \neq 0$ and

$$\langle L_\varpi X, Y\rangle_{H^1_R} = \langle X, Y\rangle_{H^1_R}, \ \forall Y \in H^1_R. \tag{3.9}$$

In particular, if $Y = X$, then we have

$$\begin{aligned}\left(\rho_1^2\varpi^2 + \varpi_1\right)\|u\|^2_{L^2(0,l_0)} + \left(\rho_2^2\varpi^2 + \varpi_2\right)\|v\|^2_{L^2(l_0,L)} - i\varpi\rho_2^2\overline{\varsigma}\,\|v(L)\|^2_{(l_0,L)} \\ = \tau_1\rho_1\|u_x\|^2_{L^2(0,l_0)} + \tau_2\rho_2\|v_x\|^2_{L^2(l_0,L)}.\end{aligned} \tag{3.10}$$

From the definition of a null complex number, we find

$$v(L) = 0.$$

By (3.9), we have

$$v_x(L) = 0$$

and

$$\begin{cases} -n_1^2 u - s_1 u_{xx} = 0, \\ -n_2^2 v - s_2 v_{xx} = 0, \end{cases} \tag{3.11}$$

where $n_1 = \sqrt{\rho_1^2\varpi^2 + \varpi_1}, n_2 = \sqrt{\rho_2^2\varpi^2 + \varpi_2}, s_1 = \tau_1\rho_1, s_2 = \tau_2\rho_2$. We deduce now that the general solutions of (3.11) are given in the form

$$\begin{cases} u(x) = c_1 \cos\frac{n_1}{\sqrt{s_1}}x + c_2 \sin\frac{n_1}{\sqrt{s_1}}x, \\ v(x) = c_3 \cos\frac{n_2}{\sqrt{s_2}}x + c_4 \sin\frac{n_2}{\sqrt{s_2}}x. \end{cases}$$

With the boundary conditions $u(0) = 0$ and $v(L) = v_x(L) = 0$, we have

$$c_1 = c_3 = c_4 = 0.$$

Under the transmission conditions, $u(l_0) = v(l_0)$ and $s_1 u_x(l_0) = s_2 v_x(l_0)$, we have

$$c_2 \sin\frac{n_1}{\sqrt{s_1}}x = 0.$$

Then $c_2 = 0$. So, $X = 0$.

In this case the operator $i\varpi - \mathcal{A}$ is surjective $\forall \varpi \in \mathbb{R}^*$.

Case 2. Assume that $\varpi = 0$ and $\zeta \neq 0$.
Then, the problem (3.5) can be reduced to the problem

$$\begin{cases} -\varphi = f_1, \\ \frac{\varpi_1}{\rho_1}\varphi - \frac{\tau_1}{\rho_1}u_{xx} = f_2, \\ -\psi = f_3, \\ \frac{\varpi_2}{\rho_2}\psi - \frac{\tau_2}{\rho_2}v_{xx} = f_4, \\ \left(\xi^2 + \zeta\right)\Phi(\xi) - \psi(L)\varrho(\xi) = f_5, \end{cases} \tag{3.12}$$

which gives the following system

$$\begin{cases} -\frac{\tau_1}{\rho_1}u_{xx} = f_2 + \frac{\varpi_1}{\rho_1}f_1, \\ -\frac{\tau_2}{\rho_2}v_{xx} = f_4 + \frac{\varpi_2}{\rho_2}f_3, \\ \left(\xi^2+\zeta\right)\Phi(\xi) - \psi(L)\varrho(\xi) = f_5. \end{cases}$$

With $(3.12)_2$ and $(3.12)_4$, using that $u(x) = 0$, we see that

$$\begin{cases} u(x) = -\frac{\rho_1}{\tau_1}\int_0^x\int_0^s\left(f_2 + \frac{\varpi_1}{\rho_1}f_1\right)(r)\ drds + Cx \\ v(x) = -\frac{\rho_2}{\tau_2}\int_{l_0}^x\int_{l_0}^s\left(f_4 + \frac{\varpi_2}{\rho_2}f_3\right)(r)\ drds + C'x + C''. \end{cases}$$

From $(2.2)_6$, $(3.12)_3$, Lemma 2.1 and $(3.12)_5$, we arrive at

$$-\theta f_3(L)\zeta^{\alpha-1} + \frac{\tau_2}{\rho_2}v_x(L,t) + \varsigma\int_{-\infty}^{+\infty}\frac{f_5\varrho(\xi)}{\xi^2+\zeta}d\xi = 0,$$

where $\theta = \varsigma\frac{\pi}{\sin\alpha\pi}$.

We have,

$$v_x(x,t) = -\frac{\rho_2}{\tau_2}\int_{l_0}^x\left(f_4 + \frac{\varpi_2}{\rho_2}f_3\right)(r)\,dr + C'.$$

We substitute into the equation (3.13) and we find

$$C' = \frac{\rho_2}{\tau_2}\left[\theta f_3(L)\zeta^{\alpha-1} + \int_{l_0}^L\left(f_4 + \frac{\varpi_2}{\rho_2}f_3\right)(r)\,dr - \varsigma\int_{-\infty}^{+\infty}\frac{f_5\varrho(\xi)}{\xi^2+\zeta}d\xi\right].$$

Using the boundary transmission conditions, we get

$$u(l_0) = v(l_0) \Rightarrow l_0C' - C'' = -\frac{\rho_1}{\tau_1}\int_0^{l_0}\int_0^s\left(f_2 + \frac{\varpi_1}{\rho_1}f_1\right)(r)\,drds + Cl_0,$$

$$\rho_2\tau_1u_x(l_0) = \rho_1\tau_2v_x(l_0) \Rightarrow C = -\rho_1\rho_2\int_0^x\left(f_2 + \frac{\varpi_1}{\rho_1}f_1\right)(r)\,dr - \rho_1\rho_2\int_{l_0}^x\left(f_4 + \frac{\varpi_2}{\rho_2}f_3\right)(r)\,dr + C'.$$

Finally, we get that $\mathcal{A}$ is surjective and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \Phi$. The proof is now completed.

4. The absence of exponential stability

This section is devoted to the study of the absence of exponential decay of the solutions associated with (2.7). We will need some results and useful lemmas.

Theorem 4.1. *[11–19] Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

$$\overline{\lim_{|\beta|\longrightarrow\infty}}\left\|(i\beta I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

Our main result is given in the following theorem.

Theorem 4.2. *The semigroup generated by the operator $\mathcal{A}$ can not be exponentially stable.*

Proof. We have the following two cases.
Case 1. Let $\zeta = 0$. We will show that $i\varpi = 0$ can not be in the resolvent set of $\mathcal{A}$. Note that $(\sin x, 0, \sin x, 0, 0) \in \mathcal{H}$, and let $(u, \varphi, v, \psi, \Phi)$ be the image of $(\sin x, 0, \sin x, 0, 0)$ with the operator $\mathcal{A}^{-1}$. We see that $\Phi(\xi) = -|\xi|^{\frac{2\alpha-5}{2}} \sin L$. Then $\Phi \notin L^2(-\infty, +\infty)$, since $0 < \alpha < 1$. So $(u, \varphi, v, \psi, \Phi) \notin D(\mathcal{A})$.
Case 2. Assume that $\zeta \neq 0$. We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the wave system (1.1) from being exponentially stable. Indeed, we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let ϖ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $(u, \varphi, v, \psi, \Phi)$. Then $\mathcal{A}X = \varpi X$ is equivalent to

$$\begin{cases} \varpi u - \varphi = 0, \\ \left(\varpi + \frac{\varpi_1}{\rho_1}\right)\varphi - \frac{\tau_1}{\rho_1} u_{xx} = 0, \\ \varpi v - \psi = 0, \\ \left(\varpi + \frac{\varpi_2}{\rho_2}\right)\psi - \frac{\tau_2}{\rho_2} v_{xx} = 0, \\ \left(\varpi + \xi^2 + \zeta\right)\Phi(\xi) - \psi(L)\varrho(\xi) = 0. \end{cases} \tag{4.1}$$

Inserting $(4.1)_1$, $(4.1)_3$ into $(4.1)_2$, $(4.1)_4$ and $(4.1)_5$, respectively, we get

$$\begin{cases} \left(\varpi^2 + \frac{\varpi\varpi_1}{\rho_1}\right) u - \frac{\tau_1}{\rho_1} u_{xx} = 0, \\ \left(\varpi^2 + \frac{\varpi\varpi_2}{\rho_2}\right) v - \frac{\tau_2}{\rho_2} v_{xx} = 0, \\ \left(\varpi + \xi^2 + \zeta\right)\Phi(\xi) - \varpi v(L)\varrho(\xi) = 0. \end{cases} \tag{4.2}$$

By $(4.2)_3$, Lemma2.1, $(2.2)_6$ and the boundary conditions, we have

$$\frac{\tau_2}{\rho_2} v_x(L, t) + \theta\varpi(\varpi + \zeta)^{\alpha-1} v(L) = 0, \tag{4.3}$$

where $\theta = \varsigma \frac{\pi}{\sin\alpha\pi}$.

By the fact that $u(0) = 0, u(l_0) = v(l_0), \tau_1\rho_2 u_x(l_0) = \tau_2\rho_1 v_x(l_0)$ and (4.3), we get

$$\begin{cases} \left(\varpi^2 + \frac{\varpi\varpi_1}{\rho_1}\right) u - \frac{\tau_1}{\rho_1} u_{xx} = 0, \\ \left(\varpi^2 + \frac{\varpi\varpi_2}{\rho_2}\right) v - \frac{\tau_2}{\rho_2} v_{xx} = 0, \\ u(0) = 0, u(l_0) = v(l_0), \tau_1\rho_2 u_x(l_0) = \tau_2\rho_1 v_x(l_0), \\ \frac{\tau_2}{\rho_2} v_x(L, t) + \theta\varpi(\varpi + \zeta)^{\alpha-1} v(L) = 0. \end{cases} \tag{4.4}$$

The general solutions of the equations $(4.4)_1$ and $(4.4)_2$ are given by

$$u(x) = \sum_{i=1}^{i=2} c_i e^{t_i x} \quad \text{and} \quad v(x) = \sum_{i=3}^{i=4} c_i e^{t_i x},$$

where $t_1 = \sqrt{\frac{\rho_1\varpi^2 + \varpi\varpi_1}{\tau_1}}$, $t_2 = -t_1$, $t_3 = \sqrt{\frac{\rho_2\varpi^2 + \varpi\varpi_1}{\tau_2}}$, and $t_4 = -t_3$.

Thus,

$$\begin{cases} c_1 + c_2 = 0, \\ e^{t_1 l_0} c_1 + e^{-t_1 l_0} c_2 - e^{t_3 l_0} c_3 - e^{-t_3 l_0} c_4 = 0, \\ \frac{\tau_1}{\rho_1} t_1 e^{t_1 l_0} c_1 - \frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0} c_2 - \frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} c_3 + \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} c_4 = 0, \\ h(t_3) e^{t_3 L} c_3 + h(-t_3) e^{-t_3 L} c_4 = 0, \end{cases}$$

where $h(r) = \frac{\tau_2}{\rho_2} r + \theta\varpi(\varpi+\zeta)^{\alpha-1}$,
and

$$M(\varpi)C(\varpi) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ e^{t_1 l_0} & e^{-t_1 l_0} & -e^{t_3 l_0} & -e^{-t_3 l_0} \\ \frac{\tau_1}{\rho_1} t_1 e^{t_1 l_0} & -\frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0} & -\frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} & \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} \\ 0 & 0 & h(t_3) e^{t_3 L} & h(-t_3) e^{-t_3 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, a non-trivial solution φ exists if and only if the determinant of $M(\varpi)$ vanishes. Let $f(\varpi) = \det M(\varpi)$. Thus, the characteristic equation is $f(\varpi) = 0$. Our purpose in the sequel is to prove by Rouché's theorem that there is a subsequence of eigenvalues for which their real parts tend to 0. Since $\mathcal{A}$ is dissipative, we treat the asymptotic behavior of the large eigenvalues ϖ of $\mathcal{A}$ in the strip $-\alpha_0 \le \mathcal{R}e(\varpi) \le 0$, for some $0 < \alpha_0$ large enough and for such ϖ, we remark that e^{t_i}, $i = 1, 2$ remains bounded.

The operator $\mathcal{A}$ has no exponential decaying branch of eigenvalues. Thus, the proof is now completed. **Case** $\frac{\tau_1}{\rho_1} = \frac{\tau_2}{\rho_2}$:

Lemma 4.1. *[2] There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbb{Z}^*, |k|\ge N} \subset \sigma(\mathcal{A}), \tag{4.5}$$

where

$$\lambda_k = i\frac{1}{rL}\left(k+\frac{1}{2}\right)\pi + \frac{\widetilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{|k|^{1-\alpha}} + o\left(\frac{1}{k^{3-\alpha}}\right),\ k \ge N, \widetilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,\ r = \sqrt{\frac{\rho_1}{\tau_1}}.$$
$$\lambda_k = \overline{\lambda_{-k}},\ \text{if}\, k \le -N.$$

Moreover for all $|k| \ge N$, the eigenvalues λ_k are simple.

Proof.

$$\begin{aligned} f(\lambda) &= \begin{vmatrix} e^{-t_1 l_0} & -e^{t_3 l_0} & -e^{-t_3 l_0} \\ -\frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0} & -\frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} & \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} \\ 0 & h(t_3) e^{t_3 L} & h(-t_3) e^{-t_3 L} \end{vmatrix} - \begin{vmatrix} e^{t_1 l_0} & -e^{t_3 l_0} & -e^{-t_3 l_0} \\ \frac{\tau_1}{\rho_1} t_1 e^{t_1 l_0} & -\frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} & \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} \\ 0 & h(t_3) e^{t_3 L} & h(-t_3) e^{-t_3 L} \end{vmatrix} \\ &= e^{-t_1 l_0}\left[-\frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} h(-t_3) e^{-t_3 L} - \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} h(t_3) e^{t_3 L}\right] + \frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0}\left[-e^{t_3 l_0} h(-t_3) e^{-t_3 L} + h(t_3) e^{t_3 L} e^{-t_3 l_0}\right] \\ &= e^{-t_1 l_0}\left[-\frac{\tau_2}{\rho_2} t_3 e^{t_3(l_0-L)} h(-t_3) - \frac{\tau_2}{\rho_2} t_3 e^{t_3(L-l_0)} h(t_3)\right] + \frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0}\left[-e^{t_3(l_0-L)} h(-t_3) + h(t_3) e^{t_3(L-l_0)}\right] \\ &= -\frac{\tau_2}{\rho_2} t_3 e^{-t_1 l_0}\left[-e^{t_3(l_0-L)}\frac{\tau_2}{\rho_2} t_3 + \theta\lambda(\lambda+\eta)^{\alpha-1} e^{t_3(l_0-L)} + e^{t_3(L-l_0)}\frac{\tau_2}{\rho_2} t_3 + \theta\lambda(\lambda+\eta)^{\alpha-1} e^{t_3(L-l_0)}\right] \\ &\quad -\frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0}\left[-e^{t_3(l_0-L)}\frac{\tau_2}{\rho_2} t_3 + \theta\lambda(\lambda+\eta)^{\alpha-1} e^{t_3(l_0-L)} - e^{t_3(L-l_0)}\frac{\tau_2}{\rho_2} t_3 + \theta\lambda(\lambda+\eta)^{\alpha-1} e^{t_3(L-l_0)}\right] \\ &= -r_{\substack{2\\1}} t_{\substack{2\\3}}\left[\left(e^{2t_3(l_0-L)} + 1\right)\right] + \theta\lambda\frac{e^{t_3(l_0-L)} + e^{t_3(L-l_0)} + e^{t_3(l_0-L)} + e^{t_3(L-l_0)}}{(\lambda+\eta)^{1-\alpha}}. \end{aligned}$$

We set

$$\begin{aligned} \widetilde{f}(\lambda) &= \left(e^{2t_3(l_0-L)} + 1\right) + \theta\lambda\frac{e^{t_3(l_0-L)} + e^{t_3(L-l_0)} + e^{t_3(l_0-L)} + e^{t_3(L-l_0)}}{(\lambda+\eta)^{1-\alpha}} + \circ\left(\frac{1}{\lambda^{1-\alpha}}\right) \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-\alpha}} + \circ\left(\frac{1}{\lambda^{1-\alpha}}\right), \end{aligned} \tag{4.6}$$

where

$$f_0(\lambda) = e^{2t_3(l_0-L)} + 1, \tag{4.7}$$

and

$$f_1(\lambda) = \theta\lambda\frac{e^{t_3(l_0-L)} + e^{t_3(L-l_0)} + e^{t_3(l_0-L)} + e^{t_3(L-l_0)}}{(\lambda+\eta)^{1-\alpha}}.$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \mathcal{R}e\,(\lambda) \leq 0$.
Setep 2. We look at the roots of f_0. From (4.7), f_0 has one familie of roots that we denote λ_0^k.

$$f_0\,(\lambda) = 0 \Leftrightarrow e^{2t_3(l_0-L)} = -1.$$

Hence

$$2\sqrt{\frac{\rho_2\lambda^2 + \lambda\lambda_1}{\tau_2}}\,(l_0 - L) = i\,(2k+1)\,\pi,$$

i.e.,

$$\lambda_0^k = \frac{i\,(2k+1)\,\pi}{2\sqrt{\frac{\rho_2\lambda^2+\lambda\lambda_1}{\tau_2}}\,(l_0 - L)}, \quad k \in \mathbb{Z}.$$

Now with the help of Rouche's Theorem, we will show that the roots of $\widetilde{f}$ are close to those of f_0. Changing in (4.6) the unknown λ by $u = 2\sqrt{\frac{\rho_1}{\tau_1}}\lambda L$ then (4.6) becomes

$$\widetilde{f}\,(u) = (e^u + 1) + O\left(\frac{1}{u^{(1-\alpha)}}\right) = f_0\,(u) + O\left(\frac{1}{u^{(1-\alpha)}}\right). \tag{4.8}$$

The roots of f_0 are $u_k = \frac{i\left(k+\frac{1}{2}\right)}{rL}\pi, k \in \mathbb{Z}.$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouche's Theorem. Consequently, there exists a subsequence of roots of $\widetilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f\,(\lambda)$, such that $\lambda_k = \lambda_k^0 + \circ\,(1)$ which tends to the roots $\frac{i\left(k+\frac{1}{2}\right)}{rL}\pi$ of f_0. Finally for $|k| \geq N$, λ_k is simple since λ_k^0 is.
Setep3. From Step 2, we can write

$$\lambda_k = i\frac{1}{rL}\left(k + \frac{1}{2}\right)\pi + \epsilon_k. \tag{4.9}$$

Using (4.9), we get

$$e^{2r\lambda_k L} = -1 - 2rL\epsilon_k - 2rL^2\epsilon_k^2 + \circ\left(\epsilon_k^2\right). \tag{4.10}$$

Substituting (4.10) into (4.6), using the fact that $\widetilde{f}\,(\lambda_k) = 0$,, we get:

$$\widetilde{f}\,(\lambda_k) = -2rL\epsilon_k - \frac{2\gamma}{\sqrt{\tau_1/\rho_1}}\frac{1}{\left(\frac{i(2k+1)\pi}{2rL}\right)^{1-\alpha}} + \circ\,(\epsilon_k) = 0, \tag{4.11}$$

and hence

$$\begin{aligned}\epsilon_k &= -\frac{\gamma r^{1-\alpha}}{L^\alpha\left(\left(K+\frac{1}{2}\right)i\pi\right)^{1-\alpha}} + \circ\,(\epsilon_k) \\ &= -\frac{\gamma r^{1-\alpha}}{L^\alpha\left(\left(K+\frac{1}{2}\right)\pi\right)^{1-\alpha}}\left(\cos\,(1-\alpha)\,\frac{\pi}{2} - i\sin\,(1-\alpha)\,\frac{\pi}{2}\right) + \circ\,(\epsilon_k) \quad \mathit{for}\, k \geq 0.\end{aligned} \tag{4.12}$$

From (4.12) we have in that case $|k|^{1-\alpha}\,\mathcal{R}e\lambda_k \approx \beta$, with

$$\beta = -\frac{\gamma r^{1-\alpha}}{L^\alpha\pi^{1-\alpha}}\cos\,(1-\alpha)\,\frac{\pi}{2}.$$

Case $\frac{\tau_1}{\rho_1} \neq \frac{\tau_2}{\rho_2}$:

Lemma 4.2. *[2] There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbb{Z}^*, |k|\geq N} \subset \sigma(\mathcal{A}), \tag{4.13}$$

where

$$\lambda_k = i\mu_k + \frac{\widetilde{\alpha}}{|k|^{1-\alpha}} + \frac{\beta}{|k|^{1-\alpha}} + \circ\left(\frac{1}{k^{3-\alpha}}\right), k \geq N, \widetilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof.

$$\begin{aligned}
f(\lambda) &= \begin{vmatrix} e^{-t_1 l_0} & -e^{t_3 l_0} & -e^{-t_3 l_0} \\ -\frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0} & -\frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} & \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} \\ 0 & h(t_3) e^{t_3 L} & h(-t_3) e^{-t_3 L} \end{vmatrix} - \begin{vmatrix} e^{t_1 l_0} & -e^{t_3 l_0} & -e^{-t_3 l_0} \\ \frac{\tau_1}{\rho_1} t_1 e^{t_1 l_0} & -\frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} & \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} \\ 0 & h(t_3) e^{t_3 L} & h(-t_3) e^{-t_3 L} \end{vmatrix} \\
&= e^{-t_1 l_0}\left[-\frac{\tau_2}{\rho_2} t_3 e^{t_3 l_0} h(-t_3) e^{-t_3 L} - \frac{\tau_2}{\rho_2} t_3 e^{-t_3 l_0} h(t_3) e^{t_3 L}\right] + \frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0}\left[-e^{t_3 l_0} h(-t_3) e^{-t_3 L} + h(t_3) e^{t_3 L} e^{-t_3 l_0}\right] \\
&= e^{-t_1 l_0}\left[-\frac{\tau_2}{\rho_2} t_3 e^{t_3(l_0-L)} h(-t_3) - \frac{\tau_2}{\rho_2} t_3 e^{t_3(L-l_0)} h(t_3)\right] + \frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0}\left[-e^{t_3(l_0-L)} h(-t_3) + h(t_3) e^{t_3(L-l_0)}\right] \\
&= -\frac{\tau_2}{\rho_2} t_3 e^{-t_1 l_0}\left[-e^{t_3(l_0-L)} \frac{\tau_2}{\rho_2} t_3 + \theta\lambda(\lambda+\eta)^{\alpha-1} e^{t_3(l_0-L)} + e^{t_3(L-l_0)} \frac{\tau_2}{\rho_2} t_3 + \theta\lambda(\lambda+\eta)^{\alpha-1} e^{t_3(L-l_0)}\right] \\
&\quad -\frac{\tau_1}{\rho_1} t_1 e^{-t_1 l_0}\left[-e^{t_3(l_0-L)} \frac{\tau_2}{\rho_2} t_3 + \theta\lambda(\lambda+\eta)^{\alpha-1} e^{t_3(l_0-L)} - e^{t_3(L-l_0)} \frac{\tau_2}{\rho_2} t_3 + \theta\lambda(\lambda+\eta)^{\alpha-1} e^{t_3(L-l_0)}\right] \\
&= -r_1 r_2 t_3^2\left[\left(e^{2t_3(l_0-L)} + 1\right)\right] + \theta\lambda \frac{e^{t_3(l_0-L)} + e^{t_3(L-l_0)} + e^{t_3(l_0-L)} + e^{t_3(L-l_0)}}{(\lambda+\eta)^{1-\alpha}} + \circ\left(\frac{1}{\lambda^{1-\alpha}}\right).
\end{aligned}$$

We set

$$\begin{aligned}
\widetilde{f}(\lambda) &= -r_1 r_2 \frac{\rho_2\lambda^2 + \lambda\lambda_1}{\tau_2}\left[\left(e^{2t_3(l_0-L)} + 1\right)\right] + \theta\lambda \frac{e^{t_3(l_0-L)} + e^{t_3(L-l_0)} + e^{t_3(l_0-L)} + e^{t_3(L-l_0)}}{(\lambda+\eta)^{1-\alpha}} + \circ\left(\frac{1}{\lambda^{1-\alpha}}\right) \\
&= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-\alpha}} + \circ\left(\frac{1}{\lambda^{1-\alpha}}\right),
\end{aligned}$$

where

$$f_0(\lambda) = -r_1 r_2 \frac{\rho_2\lambda^2 + \lambda\lambda_1}{\tau_2}\left[\left(e^{2t_3(l_0-L)} + 1\right)\right], \tag{4.14}$$

and

$$f_1(\lambda) = \theta\lambda \frac{e^{t_3(l_0-L)} + e^{t_3(L-l_0)} + e^{t_3(l_0-L)} + e^{t_3(L-l_0)}}{(\lambda+\eta)^{1-\alpha}} + \circ\left(\frac{1}{\lambda^{1-\alpha}}\right).$$

We look at the roots of f_0. From (4.14), f_0 has one familie of roots that we denote λ_0^k. Indeed, $f_0(\lambda) = 0$ corresponds to the eigenvalues problem to the conservative problem associated with $\left(P'\right)$:

$$\begin{cases}
\rho_1 u_{tt}(x,t) - \tau_1 u_{xx}(x,t) + \lambda_1 u_t(x,t) = 0 & in\ (0,l_0)\times(0,+\infty), \\
\rho_2 v_{tt}(x,t) - \tau_2 v_{xx}(x,t) + \lambda_2 v_t(x,t) = 0 & in\ (l_0,L)\times(0,+\infty), \\
\partial_t\phi(\xi,t) + \left(\xi^2+\eta\right)\phi(\xi,t) - v_t(L,t)\mu(\xi) = 0 & in\ \mathbb{R}\times(0,+\infty), \\
u(l_0,t) = v(l_0,t), \rho_2\tau_1 u_x(l_0,t) = \rho_1\tau_2 v_x(l_0,t) & on\ (0,+\infty) \\
u(0,t) = 0 & on\ (0,+\infty) \\
\tau_2 v_x(L,t) + \varsigma\rho_2\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi = 0 & on\ (0,+\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & on\ (0,l_0), \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x) & on\ (l_0,L),
\end{cases} \tag{4.15}$$

where $\varsigma = (\pi)^{-1}\sin(\alpha\pi)\gamma$. For a solution (u,v,ϕ) of (4.15). The abstract formulation of (P') is

$$\mathcal{A}_0\begin{pmatrix} u \\ \varphi \\ v \\ \psi \\ \phi \end{pmatrix} = \begin{pmatrix} \varphi \\ \frac{\tau_1}{\rho_1}u_{xx} - \frac{\lambda_1}{\rho_1}\varphi \\ \psi \\ \frac{\tau_2}{\rho_2}v_{xx} - \frac{\lambda_2}{\rho_2}\psi \\ -\left(\xi^2+\eta\right)\phi(\xi) + \psi(L)\mu(\xi) \end{pmatrix}. \tag{4.16}$$

The domain of $\mathcal{A}_0$

$$D(\mathcal{A}_0) = \left\{ \begin{array}{l} (u,\varphi,v,\psi,\phi)^T \in \mathcal{H} : u \in H^2(0,l_0) \cap H_0^1(0,l_0), \\ \varphi \in H_*^1(0,l_0), v \in H^2(l_0,L), \psi \in H^1(l_0,L), \\ u(l_0) = v(l_0), \rho_2\tau_1 u_x(l_0) = \rho_1\tau_2 v_x(l_0), \\ \tau_2 v_x(L) + \varsigma\rho_2 \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi = 0, |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{4.17}$$

We introduce the following Hilbert space (the energy space):

$$H_*^1 = \left\{u \in H^1(0,l_0) : u(0) = 0\right\}. \tag{4.18}$$

$$\mathcal{H} = \left\{H_*^1(0,l_0) \times L^2(0,l_0) \times H^1(l_0,L) \times L^2(l_0,L) \times L^2(-\infty,+\infty) : u(l_0) = v(l_0)\right\}. \tag{4.19}$$

$\mathcal{A}_0$ is clearly a skew adjoint operator with a compact resolvent, then there is an orthonormal system of eigenvectors of $\mathcal{A}_0$ which is complete in $\mathcal{H}_0$. All eigenvalues of $\mathcal{A}_0$ are of the form. Now $i\mu_k, \mu_k \in \mathbb{R}$

$$f_0(i\mu_k) = 0 \Leftrightarrow \sqrt{\frac{\rho_2 (i\mu_k)^2 + i\mu_k\lambda_1}{\tau_2}}\,(l_0 - L) = i(2k+1)\pi. \tag{4.20}$$

By representation of graph of the functions tan and cot, we easily have $\mu_k \approx ck$ for large k and a constant c depending on parameters $\rho_1, \rho_1, \tau_1, \tau_2$, l_0 and L. Moreover, the algebraic multiplicity of μ_k is one. Then, we follow exactly as the case $\frac{\tau_1}{\rho_1} = \frac{\tau_1}{\rho_1}$. The operator $\mathcal{A}$ has a non exponential decaying branche of eigenvalues. Thus the proof is completed.

5. Polynomial stability for $\zeta \neq 0$

In this part, we prove that (2.2) is polynomially stable when $\zeta > 0$.

Theorem 5.1. *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$\|S_{\mathcal{A}}(t)X_0\|_{\mathcal{H}} \leq \frac{1}{t^{\frac{1}{4-2\alpha}}}\|X_0\|_{D(\mathcal{A})}.$$

Proof. We have a need to study the resolvent equation $(i\varpi I - \mathcal{A})X = F$, for $\varpi \in \mathbb{R}$, namely,

$$\left\{ \begin{array}{l} i\varpi u - \varphi = f_1, \\ i\varpi\varphi - \frac{\tau_1}{\rho_1}u_{xx} + \frac{\varpi_1}{\rho_1}\varphi = f_2, \\ i\varpi v - \psi = f_3, \\ i\varpi\psi - \frac{\tau_2}{\rho_2}v_{xx} + \frac{\varpi_2}{\rho_2}\psi = f_4, \\ i\varpi\Phi + \left(\xi^2+\zeta\right)\Phi(\xi) - \psi(L)\varrho(\xi) = f_5, \end{array} \right. \tag{5.1}$$

where $F=(f_1,f_2,f_3,f_4,f_5)^T$.
The proof is divided into several steps
Step 1. Inserting $(5.1)_1$ into $(5.1)_2$ and $(5.1)_3$ into $(5.1)_4$, we get

$$\begin{cases} u_{xx}+m_1u=-\left[\left(\frac{\varpi_1}{\rho_1}f_1+f_2\right)+i\varpi f_1\right], \\ v_{xx}+m_2v=-\left[\left(\frac{\varpi_2}{\rho_2}f_3+f_4\right)+i\varpi f_3\right], \end{cases}$$

where $m_j=\frac{\varpi^2\varpi_j}{\tau_j}-i\frac{\varpi\varpi_j}{\tau_j}$ with $j\in\{1,2\}$. We have

$$\begin{cases} u(x) &= C\left(e^{\widetilde{m_1}x}-e^{-\widetilde{m_1}x}\right)-\frac{1}{2\widetilde{m_1}}\int_0^x\left[\left(\frac{\varpi_1}{\rho_1}f_1(\sigma)+f_2(\sigma)\right)+i\varpi f_1(\sigma)\right]\left[e^{\widetilde{m_1}(x-\sigma)}-e^{-\widetilde{m_1}(x-\sigma)}\right]d\sigma, \\ v(x) &= \frac{1}{2}\left(e^{\widetilde{m_2}(x-l_0)}+e^{-\widetilde{m_2}(x-l_0)}\right)v(l_0)+\frac{1}{2\widetilde{m_2}}\left(e^{-\widetilde{m_2}(x-l_0)}-e^{\widetilde{m_2}(x-l_0)}\right)v_x(l_0), \\ & -\frac{1}{2\widetilde{m_2}}\int_{l_0}^x\left[\left(\frac{\varpi_2}{\rho_2}f_3(\sigma)+f_4(\sigma)\right)+i\varpi f_3(\sigma)\right]\left[e^{\widetilde{m_2}(x-\sigma)}-e^{-\widetilde{m_2}(x-\sigma)}\right]d\sigma, \end{cases} \tag{5.2}$$

and hence,

$$\begin{cases} u_x(x) &= -\frac{1}{2}\int_0^x\left[\left(\frac{\varpi_1}{\rho_1}f_1(\sigma)+f_2(\sigma)\right)+i\varpi f_1(\sigma)\right]\left[e^{\widetilde{m_1}(x-\sigma)}-e^{-\widetilde{m_1}(x-\sigma)}\right]d\sigma+C\widetilde{m_1}\left(e^{\widetilde{m_1}x}+e^{-\widetilde{m_1}x}\right), \\ v_x(x) &= \frac{\widetilde{m_2}}{2}\left(e^{\widetilde{m_2}(x-l_0)}-e^{-\widetilde{m_2}(x-l_0)}\right)v(l_0)-\frac{1}{2}\left(e^{-\widetilde{m_2}(x-l_0)}+e^{\widetilde{m_2}(x-l_0)}\right)v_x(l_0), \\ & -\frac{1}{2}\int_{l_0}^x\left[\left(\frac{\varpi_2}{\rho_2}f_3(\sigma)+f_4(\sigma)\right)+i\varpi f_3(\sigma)\right]\left[e^{\widetilde{m_2}(x-\sigma)}-e^{-\widetilde{m_2}(x-\sigma)}\right]d\sigma. \end{cases} \tag{5.3}$$

Step2. With the fifth equation of (5.1), we get

$$\Phi(\xi)=\frac{\psi(L)\varrho(\xi)+f_5}{i\varpi+\xi^2+\zeta}. \tag{5.4}$$

Inserting (5.4) into the boundary condition $(2.2)_6$ and using Lemma 2.1, we deduce that

$$r_2v_x(L,t)+\gamma\varpi(i\varpi+\zeta)^{\alpha-1}v(L)=\gamma(i\varpi+\zeta)^{\alpha-1}f_3(L)-\zeta\int_{-\infty}^{+\infty}\frac{\varrho(\xi)f_5(\xi)}{i\varpi+\xi^2+\zeta}d\xi.$$

Using the Eq (5.2) and the Eq (5.3), we arrive at

$$\begin{aligned} &\tfrac{1}{2}v(l_0)\left[r_2\widetilde{m_2}\left(e^{\widetilde{m_2}(L-l_0)}-e^{-\widetilde{m_2}(L-l_0)}\right)+\beta\left(e^{\widetilde{m_2}(L-l_0)}+e^{-\widetilde{m_2}(L-l_0)}\right)\right] \\ &-\tfrac{1}{2}v_x(l_0)\left[r_2\left(e^{-\widetilde{m_2}(L-l_0)}+e^{\widetilde{m_2}(L-l_0)}\right)-\frac{\beta}{\widetilde{-m_2}}\left(e^{\widetilde{m_2}(L-l_0)}-e^{\widetilde{m_2}(L-l_0)}\right)\right] \\ &=\gamma(i\varpi+\zeta)^{\alpha-1}f_3(L)-\zeta\int_{-\infty}^{+\infty}\frac{\varrho(\xi)f_5(\xi)}{i\varpi+\xi^2+\zeta}d\xi \\ &+\frac{r_2}{2}\int_{l_0}^L\left[\left(\frac{\varpi_2}{\rho_2}f_3(\sigma)+f_4(\sigma)\right)+i\varpi f_3(\sigma)\right]\left(e^{\widetilde{m_2}(L-\sigma)}+e^{\widetilde{m_2}(L-\sigma)}\right)d\sigma \\ &+\frac{\beta}{2\widetilde{m_2}}\int_{l_0}^L\left[\left(\frac{\varpi_2}{\rho_2}f_3(\sigma)+f_4(\sigma)\right)+i\varpi f_3(\sigma)\right]\left(e^{\widetilde{m_2}(L-\sigma)}-e^{\widetilde{m_2}(L-\sigma)}\right)d\sigma, \end{aligned} \tag{5.5}$$

where $\beta=\gamma\varpi(i\varpi+\zeta)^{\alpha-1}$. By $u(l_0)=v(l_0)$ and $r_1u_x(l_0)=r_2v_x(l_0)$, we get

$$\begin{cases} v(l_0) &= -\frac{1}{2\widetilde{m_1}}\int_0^{l_0}\left[\left(\frac{\varpi_1}{\rho_1}f_1(\sigma)+f_2(\sigma)\right)+i\varpi f_1(\sigma)\right]e^{\widetilde{m_1}(l_0-\sigma)}d\sigma \\ & +\frac{1}{2\widetilde{m_1}}\int_0^{l_0}\left[\left(\frac{\varpi_1}{\rho_1}f_1(\sigma)+f_2(\sigma)\right)+i\varpi f_1(\sigma)\right]e^{\widetilde{m_1}(\sigma-l_0)}d\sigma+C\left(e^{\widetilde{m_1}l_0}-e^{-\widetilde{m_1}l_0}\right) \\ v_x(l_0) &= -\frac{r_1}{2r_2}\int_0^{l_0}\left[\left(\frac{\varpi_1}{\rho_1}f_1(\sigma)+f_2(\sigma)\right)+i\varpi f_1(\sigma)\right]e^{\widetilde{m_1}(l_0-\sigma)}d\sigma \\ & -\frac{r_1}{2r_2}\int_0^{l_0}\left[\left(\frac{\varpi_1}{\rho_1}f_1(\sigma)+f_2(\sigma)\right)+i\varpi f_1(\sigma)\right]e^{\widetilde{m_1}(\sigma-l_0)}d\sigma+\frac{C\widetilde{m_1}r_1}{r_2}\left(e^{\widetilde{m_1}l_0}+e^{-\widetilde{m_1}l_0}\right). \end{cases} \tag{5.6}$$

By (5.6), we note that we can rewrite (5.5) as an equation in the unknown C

$$\begin{aligned}
& \frac{1}{2}\left(C\left(e^{\widetilde{m_1}l_0}-e^{-\widetilde{m_1}l_0}\right)\right)\left[r_2\widetilde{m_2}\left(e^{\widetilde{m_2}(L-l_0)}-e^{-\widetilde{m_2}(L-l_0)}\right)+\beta\left(e^{\widetilde{m_2}(L-l_0)}+e^{-\widetilde{m_2}(L-l_0)}\right)\right] \\
- & \frac{1}{2}\left(\frac{C\widetilde{m_1}r_1}{r_2}\left(e^{\widetilde{m_1}l_0}+e^{-\widetilde{m_1}l_0}\right)\right)\left[r_2\left(e^{-\widetilde{m_2}(L-l_0)}+e^{\widetilde{m_2}(L-l_0)}\right)-\frac{\beta}{\widetilde{m_2}}\left(e^{-\widetilde{m_2}(L-l_0)}-e^{\widetilde{m_2}(L-l_0)}\right)\right] \\
= & \gamma\left(i\varpi+\zeta\right)^{\alpha-1}f_3\left(L\right)-\zeta\int_{-\infty}^{+\infty}\frac{\varrho\left(\xi\right)F_5\left(\xi\right)}{I\varpi+\xi^2+\zeta}d\xi \\
- & \frac{r_2}{2}\int_{l_0}^{L}\left[\left(\frac{\varpi_2}{\rho_2}f_3\left(\sigma\right)+f_4\left(\sigma\right)\right)+i\varpi f_3\left(\sigma\right)\right]\left(e^{\widetilde{m_2}(L-\sigma)}+e^{-\widetilde{m_2}(L-\sigma)}\right)d\sigma \\
+ & \frac{\beta}{2\widetilde{m_2}}\int_{l_0}^{L}\left[\left(\frac{\varpi_2}{\rho_2}f_3\left(\sigma\right)+f_4\left(\sigma\right)\right)+i\varpi f_3\left(\sigma\right)\right]\left(e^{\widetilde{m_2}(L-\sigma)}-e^{-\widetilde{m_2}(L-\sigma)}\right)d\sigma \\
+ & \frac{1}{4\widetilde{m_1}}\int_{0}^{l_0}\left[\left(\frac{\varpi_1}{\rho_1}f_1\left(\sigma\right)+f_2\left(\sigma\right)\right)+i\varpi f_1\left(\sigma\right)\right]\left(e^{\widetilde{m_1}(l_0-\sigma)}-e^{-\widetilde{m_1}(l_0-\sigma)}\right)d\sigma \\
& \left[r_2\widetilde{m_2}\left(e^{\widetilde{m_2}(L-l_0)}-e^{-\widetilde{m_2}(L-l_0)}\right)+\beta\left(e^{\widetilde{m_2}(L-l_0)}+e^{-\widetilde{m_2}(L-l_0)}\right)\right] \\
+ & \frac{r_1}{2r_2}\int_{0}^{l_0}\left[\left(\frac{\varpi_1}{\rho_1}f_1\left(\sigma\right)+f_2\left(\sigma\right)\right)+i\varpi f_1\left(\sigma\right)\right]\left(e^{\widetilde{m_1}(l_0-\sigma)}+e^{-\widetilde{m_1}(l_0-\sigma)}\right)d\sigma \\
& \left[r_2\left(e^{-\widetilde{m_2}(L-l_0)}+e^{\widetilde{m_2}(L-l_0)}\right)-\frac{\beta}{\widetilde{m_2}}\left(e^{-\widetilde{m_2}(L-l_0)}-e^{\widetilde{m_2}(L-l_0)}\right)\right].
\end{aligned}$$

Step 3. We set

$$\begin{aligned}
g\left(\varpi\right)= & \frac{1}{2}\left(C\left(e^{\widetilde{m_1}l_0}-e^{-\widetilde{m_1}l_0}\right)\right)\left[r_2\widetilde{m_2}\left(e^{\widetilde{m_2}(L-l_0)}-e^{-\widetilde{m_2}(L-l_0)}\right)+\beta\left(e^{\widetilde{m_2}(L-l_0)}+e^{-\widetilde{m_2}(L-l_0)}\right)\right] \\
& -\frac{1}{2}\left(\frac{C\widetilde{m_1}r_1}{r_2}\left(e^{\widetilde{m_1}l_0}+e^{-\widetilde{m_1}l_0}\right)\right)\left[r_2\left(e^{-\widetilde{m_2}(L-l_0)}+e^{\widetilde{m_2}(L-l_0)}\right)-\frac{\beta}{\widetilde{m_2}}\left(e^{-\widetilde{m_2}(L-l_0)}-e^{\widetilde{m_2}(L-l_0)}\right)\right],
\end{aligned}$$

and we have

$$\begin{aligned}
g\left(\varpi\right)= & \frac{1}{2}C\left(e^{\widetilde{m_1}l_0}-e^{-\widetilde{m_1}l_0}\right)\left[r_2\widetilde{m_2}\left(e^{\widetilde{m_2}(L-l_0)}-e^{-\widetilde{m_2}(L-l_0)}\right)+\beta\left(e^{\widetilde{m_2}(L-l_0)}+e^{-\widetilde{m_2}(L-l_0)}\right)\right] \\
& -\frac{C\widetilde{m_1}r_1}{2\widetilde{m_2}r_2}\left(e^{\widetilde{m_1}l_0}+e^{-\widetilde{m_1}l_0}\right)\left[r_2\widetilde{m_2}\left(e^{-\widetilde{m_2}(L-l_0)}+e^{\widetilde{m_2}(L-l_0)}\right)-\beta\left(e^{-\widetilde{m_2}(L-l_0)}-e^{\widetilde{m_2}(L-l_0)}\right)\right].
\end{aligned}$$

As $(f_1, f_2) \in \left(H_*^1\right)^2$ and $(f_3, f_4) \in \left(H^1\left(l_0, L\right)\right)^2$, we have

$$\left|\begin{array}{ll}
\left(\frac{\beta}{2\widetilde{m_2}}-\frac{r_2}{2}\right)\left[\int_{l_0}^{L}\left[\left(\frac{\varpi_2}{\rho_2}f_3\left(\sigma\right)+f_4\left(\sigma\right)\right)+i\varpi f_3\left(\sigma\right)\right]e^{\widetilde{m_2}(L-\sigma)}d\sigma\right] & \leq c_1\left(\|f_4\|_{L^2(l_0,L)}+\|f_3\|_{H^1(l_0,L)}\right) \\
-\left(\frac{\beta}{2\widetilde{m_2}}+\frac{r_2}{2}\right)\left[\int_{l_0}^{L}\left[\left(\frac{\varpi_2}{\rho_2}f_3\left(\sigma\right)+f_4\left(\sigma\right)\right)+i\varpi f_3\left(\sigma\right)\right]e^{\widetilde{m_2}(\sigma-L)}d\sigma\right] & \leq c_2\left(\|f_4\|_{L^2(l_0,L)}+\|f_3\|_{H^1(l_0,L)}\right) \\
\left(\frac{1}{4\widetilde{m_1}}+r_2\widetilde{m_2}\right)\left[\int_{0}^{l_0}\left[\left(\frac{\varpi_1}{\rho_1}f_1\left(\sigma\right)+f_2\left(\sigma\right)\right)+i\varpi f_1\left(\sigma\right)\right]e^{\widetilde{m_1}(l_0-\sigma)}d\sigma\right] & \leq c_3\left(\|f_2\|_{L^2(0,l_0)}+\|f_1\|_{H^1(0,l_0)}\right) \\
\left(\frac{r_1}{2r_2}-\frac{1}{4\widetilde{m_1}}\right)\left[\int_{0}^{l_0}\left[\left(\frac{\varpi_1}{\rho_1}f_1\left(\sigma\right)+f_2\left(\sigma\right)\right)+i\varpi f_1\left(\sigma\right)\right]e^{\widetilde{m_1}(\sigma-l_0)}d\sigma\right] & \leq c_4\left(\|f_2\|_{L^2(0,l_0)}+\|f_1\|_{H^1(0,l_0)}\right).
\end{array}\right.$$

We can easily prove that

$$|g(\varpi)| \geq c|\varpi|^{\alpha} \text{ for } \varpi \text{ large.}$$

Then, we deduce that

$$\|u_x\|_{L^2(0,l_0)} \leq c|\varpi|^{1-\alpha} \text{ for } \varpi \text{ large.}$$

Moreover, the transmission conditions are as follows

$$u(l_0,t) = v(l_0,t), \rho_2\tau_1 u_x(l_0,t) = \rho_1\tau_2 v_x(l_0,t), \quad \forall t \in (0,+\infty).$$

We obtain

$$|v(l_0)| \leq c|\varpi|^{-\alpha} \quad \text{and} \quad |v_x(l_0)| \leq c|\varpi|^{1-\alpha} \text{ as } |\varpi| \longrightarrow \infty.$$

Hence,

$$\|v_x\|_{L^2(l_0,L)} \leq c|\varpi|^{1-\alpha} \text{ as } |\varpi| \longrightarrow \infty.$$

From $(5.1)_1$, $(5.1)_3$ and (5.2), we get

$$\|\varphi\|_{L^2(0,l_0)}, \|\psi\|_{L^2(l_0,L)} \leq c|\varpi|^{1-\alpha} \text{ as } |\varpi| \longrightarrow \infty.$$

From (5.4), we have

$$\begin{aligned} \|\Phi\|_{L^2(-\infty,+\infty)} &\leq |\psi(L)| \left\| \frac{\varrho(\xi)}{i\varpi+\xi^2+\zeta} \right\|_{L^2(-\infty,+\infty)} + \left\| \frac{f_5(\xi)}{i\varpi+\xi^2+\zeta} \right\|_{L^2(-\infty,+\infty)} \\ &\leq c|\varpi|^{\frac{-\alpha}{2}} \left(\|f_2\|_{L^2(l_0,L)} + \|f_1\|_{H^1(0,l_0)} \right) + \frac{c}{|\varpi|} \|f_5\|_{L^2(-\infty,+\infty)}, \end{aligned}$$

for $\varpi \neq 0$. If $|\varpi| > 1$, we get

$$\|X\|_{\mathcal{H}} \leq c|\varpi|^{1-\alpha} \|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\left\|(i\varpi I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leq c|\varpi|^{1-\alpha} \text{ as } |\varpi| \longrightarrow \infty. \tag{5.7}$$

6. Conclusions

In this work, the existence and uniqueness result for the transmission problem is proved in a functional framework by means of the semigroup theory, after a reformulation of the system above into an augmented system according to the transformation introduced in reference [14]. Besides this, in a series of results concerning the asymptotic behavior the following are proved: (i) the strong stability of the semigroup, by using a criteria of Arendt-Batty [4], (ii) the impossibility of exponential decay, and (iii) a polynomial decay by means of the Borichev-Tomilov theorem [10].

Some of previous recent works prove exponential decay but without a fractional derivative in the boundary condition. The recently published article [2] has a very strong relationship with our paper. Indeed, the results here are the same than those proved in [2] with $\varpi_j = 0, j = 1, 2$; that is without inner damping. Moreover, the same techniques are employed.

The authors in [2] proved that the fractional derivative in time can not ensure the exponential stability of the total system, however they shown polynomial stability. In this paper even with the inclusion of linear damping terms in the equations for u, v, the exponential stability of the total system is not achieved under an appropriate conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

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