AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 27513–27527.
DOI: 10.3934/math.20231407
Received: 01 August 2023
Revised: 17 September 2023
Accepted: 20 September 2023
Published: 27 September 2023

Research article

The even vertex magic total labelings of t-fold wheels

Supaporn Saduakdee[1] **and Varanoot Khemmani**[2,*]

[1] Program of Mathematics, Chandrakasem Rajabhat University, Ratchadaphisek Road, Bangkok 10900, Thailand

[2] Department of Mathematics, Srinakharinwirot University, Sukhumvit 23, Bangkok 10110, Thailand

* **Correspondence:** Email: varanoot@g.swu.ac.th.

Abstract: Let G be a graph of order n and size m. A vertex magic total labeling of G is a one-to-one function f: $V(G) \cup E(G) \rightarrow \{1, 2, \cdots, n+m\}$ with the property that for each vertex u of G, the sum of the label of u and the labels of all edges incident to u is the same constant, referred to as the magic constant. Such a labeling is even if $f[V(G)] = \{2, 4, 6, \cdots, 2n\}$. A graph G is called an even vertex magic if there is an even vertex magic total labeling of G. The primary goal of this paper is to study wheel related graphs with the size greater than the order, which have an even vertex magic total labeling. For every integer $n \geq 3$ and $t \geq 1$, the t-fold wheel $W_{n,t}$ is a wheel related graph derived from a wheel W_n by duplicating the t hubs, each adjacent to all rim vertices, and not adjacent to each other. The t-fold wheel $W_{n,t}$ has a size $nt + n$ that exceeds its order $n + t$. In this paper, we determine the magic constant of the t-fold wheel $W_{n,t}$, the bound of an integer t for the even vertex magic total labeling of the t-fold wheel $W_{n,t}$ and the conditions for even vertex magic $W_{n,t}$, focusing on integers n and t are established. Additionally, we investigate the necessary conditions for the even vertex magic total labeling of the n-fold wheel $W_{n,n}$ when n is odd and the n-fold wheel $W_{n,n-2}$ when n is even. Furthermore, our study explores the characterization of an even vertex magic $W_{n,t}$ for integer $3 \leq n \leq 9$.

Keywords: even vertex magic total labeling; even vertex magic; wheel; t-fold wheel; wheel related graph
Mathematics Subject Classification: 05C78

1. Introduction

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Gallian [3] has written a dynamic survey of graph labeling. MacDougall et al. [5] introduced the notion of a vertex magic total labeling of graphs. Let G be a graph of order n and size

m. A *vertex magic total labeling* of G is defined as a one-to-one function

$$f : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, n+m\}$$

with the property that for each vertex u of G,

$$f(u) + \sum_{v \in N(u)} f(uv) = k$$

for some constant k where $N(u)$ is the neighborhood of u. The constant k is called the *magic constant* for f. The vertex-magic total labelings of wheels and related graphs were studied in [6], and later in [11]. The properties of the general graphs such as cycles, paths, complete graphs, wheels, bipartite graphs and trees, which satisfy the vertex magic total labelings, were studied in [10]. MacDougall et al. [4] introduced the concept of a super vertex magic total labeling. They defined a vertex magic total labeling to be *super* if

$$f[V(G)] = \{1, 2, \cdots, n\}.$$

In 2017, Nagaraj et al. [7] introduced the concept of an even vertex magic total labeling. They called a vertex magic total labeling as *even* if

$$f[V(G)] = \{2, 4, \cdots, 2n\}.$$

A graph G is called an *even vertex magic* if there exists an even vertex magic total labeling of G. We note that if G is an even vertex magic, then $n \leq m$. The following results, which appeared in [7], are useful to us.

Theorem 1.1. [7] Let G be a nontrivial graph of order n and size m. If G is an even vertex magic, then magic constant k is given by the following:

$$k = \frac{m^2 + 2mn + m}{n}.$$

A *wheel* W_n, $n \geq 3$, is a graph of order $n + 1$ that contains a cycle C_n, for which every vertex in the cycle C_n is connected to one other vertex known as the *hub*. The edges of the wheel which are incident to the hub are called *spokes*. The vertices and edges of the cycle C_n in W_n are called *rim vertices* and *rim edges*, respectively. It was shown in [7] that a wheel W_n has no even vertex magic total labeling, as we state next.

Theorem 1.2. [7] A wheel W_n is not even vertex magic.

In this paper, the labeling problem is related to the work in [1]. In addition to the aforementioned vertex labeling by even numbers $2, 4, \cdots, 2n$, they studied vertex labelings by using three consecutive numbers $0, 1, 2$ with some specific properties. These labelings were referred to as a *weak Roman dominating function* and a *perfect Roman dominating function*.

From the studies in [8, 9, 12], there exist graphs with the same order and size that are even vertex magics. Moreover, the wheel related graphs, namely fans, cycles and suns, having the even vertex magic total labelings were established in [7]. However, since these graphs have the same order and size, it is interesting and challenging to study wheel related graphs when the size is greater than the order, which have an even vertex magic total labeling.

The *t-fold wheel* $W_{n,t}$, $n \geq 3$, $t \geq 1$, is a wheel related graph derived from a wheel W_n by duplicating the t hubs, each adjacent to all rim vertices, and not adjacent to each other. It is observed that the t-fold wheel $W_{n,t}$ has a size $nt + n$ that exceeds its order $n + t$. The goal of this paper is to study conditions for an even vertex magic $W_{n,t}$ in terms of n and t. Furthermore, we also determine an even vertex magic total labeling of some t-fold wheel $W_{n,t}$.

2. The conditions for an even vertex magic $W_{n,t}$

Since the 1-fold wheel $W_{n,1}$ is isomorphic to the wheel W_n and by Theorem 1.1, W_n is not an even vertex magic. In this section, we consider the t-fold wheel $W_{n,t}$, where n and t are integers with $n \geq 3$ and $t \geq 2$.

In order to present the conditions for an even vertex magic $W_{n,t}$, we initially explore the magic constant of the t-fold wheel $W_{n,t}$ of order $n + t$ and size $nt + n$ by employing Theorem 1.1.

Proposition 2.1. *Let n and t be integers with $n \geq 3$ and $t \geq 2$. If the t-fold wheel $W_{n,t}$ is an even vertex magic, then the magic constant is defined as follows:*

$$k = 2nt + 3n + \frac{n^2t^2 + 2n^2t + n}{n + t}.$$

We are able to show the bound of an integer t for the t-fold wheel having an even vertex magic total labeling as follows.

Proposition 2.2. *Let n and t be integers with $n \geq 3$ and $t \geq 2$. If the t-fold wheel $W_{n,t}$ is an even vertex magic, then $2 \leq t \leq n$.*

Proof. Suppose that the t-fold wheel $W_{n,t}$ is an even vertex magic with magic constant k. By Proposition 2.1, we obtain the following:

$$k = 2nt + 3n + \frac{n^2t^2 + 2n^2t + n}{n + t}.$$

On the contrary, assume that $t > n$. Let $t = n + r$, for some $r \geq 1$. Then,

$$n^2t^2 + 2n^2t + n = n^4 + 2n^3r + n^2r^2 + 2n^3 + 2n^2r + n$$

and

$$n + t = n + (n + r) = 2n + r.$$

Let

$$P(n) = n^4 + 2n^3r + n^2r^2 + 2n^3 + 2n^2r + n.$$

By using the remainder theorem, the remainder when $P(n)$ is divided by $2n + r$ is as follows:

$$P(-\frac{r}{2}) = \frac{r^4 + 4r^3 - 8r}{16}.$$

If

$$P(-\frac{r}{2}) = 0,$$

then $r = -2$, which is a contradiction. Thus,

$$P(-\frac{r}{2}) \neq 0.$$

Specifically, $n^2t^2 + 2n^2t + n$ is not divisible by $n + t$. Thus, k is not an integer, which is a contradiction. Therefore, $2 \leq t \leq n$. □

According to Proposition 2.2, the t-fold wheel $W_{3,t}$ is not an even vertex magic, where $t \geq 4$. Figure 1 shows the even vertex magics $W_{3,2}$ and $W_{3,3}$ with magic constants $k = 36$ and $k = 50$, respectively, where their vertices and edges are labeled by the even vertex magic total labelings.

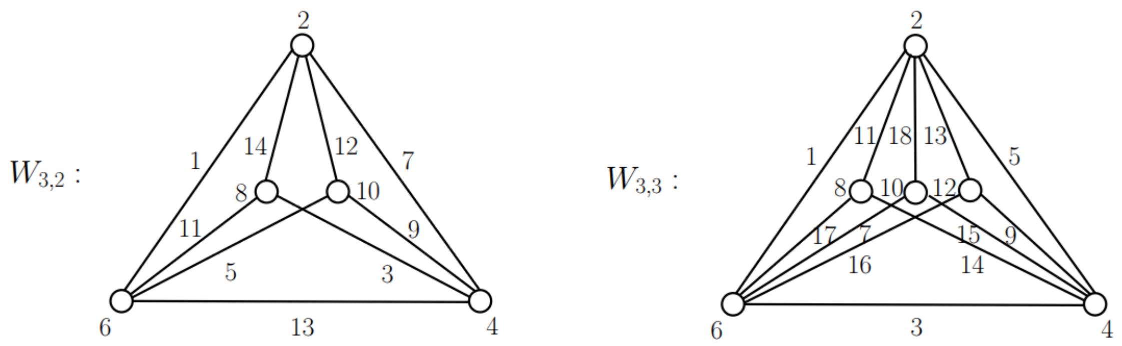

Figure 1. Even vertex magic total labelings of $W_{3,2}$ and $W_{3,3}$.

We present an even vertex magic total labeling of the t-fold wheel $W_{n,t}$ by considering only the integer n as the following results.

Proposition 2.3. *For every integer $n \geq 3$, if the n-fold wheel $W_{n,n}$ is an even vertex magic, then n is odd.*

Proof. Let n be an integer with $n \geq 3$. Suppose that the n-fold wheel $W_{n,n}$ is an even vertex magic with a magic constant k. On the contrary, assume that n is even. There exists an integer q such that $n = 2q$. By Proposition 2.1,

$$k = 2n^2 + 3n + \frac{n^3 + 2n^2 + 1}{2}.$$

Since

$$n^3 + 2n^2 + 1 = 8q^3 + 8q^2 + 1$$

is odd, $n^3 + 2n^2 + 1$ is not divisible by 2. Thus, k is not an integer, which is a contradiction. Therefore, n is odd. □

As we have seen in Figure 1, the 3-fold wheel $W_{3,3}$ is an even vertex magic, as indicated by Proposition 2.3. By an argument similar to the one used in the proof of Proposition 2.3, we obtain the condition for an even vertex magic $W_{n,n-2}$, as we now show.

Proposition 2.4. *For every integer $n \geq 4$, if the $(n-2)$-fold wheel $W_{n,n-2}$ is an even vertex magic, then n is even.*

The even vertex magic total labeling of the 2-fold wheel $W_{4,2}$ with a magic constant $k = 50$ is shown in Figure 2.

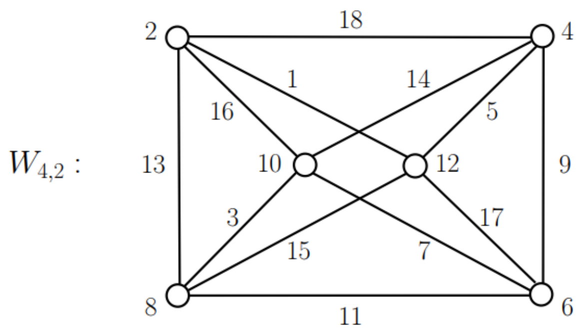

Figure 2. Even vertex magic total labeling of $W_{4,2}$.

In order to deduce an even vertex magic total labeling of the t-fold wheel for achieving the main result, we need some additional notation for the t-fold wheel $W_{n,t}$. For every pair of integers $n \geq 3$ and $t \geq 2$, let

$$V(W_{n,t}) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_t\}$$

and

$$E(W_{n,t}) = \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_i v_j | 1 \leq i \leq n, 1 \leq j \leq t\}.$$

Suppose the t-fold wheel $W_{n,t}$ is an even vertex magic. Then, for any even vertex magic total labeling f of $W_{n,t}$, let

$$S_{rv} = \sum_{i=1}^{n} f(u_i), \quad S_{re} = \sum_{i=1}^{n-1} f(u_i u_{i+1}) + f(u_n u_1)$$

and

$$S_h = \sum_{j=1}^{t} f(v_j), \quad S_s = \sum_{j=1}^{t} \sum_{i=1}^{n} f(u_i v_j).$$

Next, we present the following lemma to show the necessary condition for an even vertex magic $W_{n,t}$ with the following magic constant:

$$k = 2nt + 3n + \frac{n^2t^2 + 2n^2t + n}{n+t}.$$

Note that

$$S_{rv} + 2S_{re} - S_h = (n-t)k.$$

Lemma 2.5. *Let n and t be integers where $n \geq 3$ and $t \geq 2$. If the t-fold wheel $W_{n,t}$ is an even vertex magic, then*

$$S_{rv} + 2S_{re} - S_h = (t^2 + 4t + 3)n^2 + (-2t^3 - 6t^2 - 3t + 1)n + \frac{(t^3 + 2t^2 - 1)(2nt)}{n+t}.$$

With the aid of Lemma 2.5 and Proposition 2.2, the necessary condition for an even vertex magic total labeling of the t-fold wheel $W_{n,t}$ can also be given in terms of n and t.

Proposition 2.6. *Let n and t be integers where $n \geq 3$ and $t \geq 2$. If the t-fold wheel $W_{n,t}$ is an even vertex magic, then*

$$(-t^2 - 2t + 1)n^2 + (2t^3 + 6t^2 + 7t + 1)n - (t^2 + t) - \frac{(t^3 + 2t^2 - 1)(2nt)}{n + t} \geq 0.$$

Proof. Suppose that the t-fold wheel $W_{n,t}$ is an even vertex magic. By Lemma 2.5,

$$S_{rv} + 2S_{re} - S_h = (t^2 + 4t + 3)n^2 + (-2t^3 - 6t^2 - 3t + 1)n + \frac{(t^3 + 2t^2 - 1)(2nt)}{n + t}.$$

Next, we consider the maximum of $(S_{rv} + 2S_{re} - S_h)$.

By Proposition 2.2, $2 \leq t \leq n$, and then $2n + 2t < nt + n + t + 1$. The maximum of

$$\begin{aligned}
(S_{rv} + 2S_{re} - S_h) &= \sum_{\substack{2t+2 \leq i \leq 2n+2t \\ i \text{ is even}}} i + 2\sum_{i=nt+n+t+1}^{nt+2n+t} i - \sum_{\substack{2 \leq i \leq 2t \\ i \text{ is even}}} i \\
&= \left(\sum_{\substack{2 \leq i \leq 2n+2t \\ i \text{ is even}}} i - \sum_{\substack{2 \leq i \leq 2t \\ i \text{ is even}}} i \right) + 2\left(\sum_{i=1}^{nt+2n+t} i - \sum_{i=1}^{nt+n+t} i \right) - \sum_{\substack{2 \leq i \leq 2t \\ i \text{ is even}}} i \\
&= \left(\tfrac{(2n+2t)(2n+2t+2)}{4} - \tfrac{(2t)(2t+2)}{4} \right) \\
&\quad + 2\left(\tfrac{(nt+2n+t)(nt+2n+t+1)}{2} - \tfrac{(nt+n+t)(nt+n+t+1)}{2} \right) - \tfrac{(2t)(2t+2)}{4} \\
&= 2n^2t + 4n^2 + 4nt - t^2 + 2n - t\,.
\end{aligned}$$

Since $S_{rv} + 2S_{re} - S_h$ does not exceed the maximum of $(S_{rv} + 2S_{re} - S_h)$, the maximum of

$$(S_{rv} + 2S_{re} - S_h) - (S_{rv} + 2S_{re} - S_h) \geq 0.$$

Therefore,

$$(-t^2 - 2t + 1)n^2 + (2t^3 + 6t^2 + 7t + 1)n - (t^2 + t) - \frac{(t^3 + 2t^2 - 1)(2nt)}{n + t} \geq 0.$$

□

Now, we investigate the sufficient condition for a labeling f that can be an even vertex magic total labeling of $W_{n,n}$ when n is odd.

Theorem 2.7. *Let n be an odd integer where $n \geq 3$. For every n-fold wheel $W_{n,n}$, let*

$$f : V(W_{n,n}) \cup E(W_{n,n}) \to \{1, 2, \ldots, n^2 + 3n\}$$

be defined by the following:

$$\begin{aligned}
f(u_i) &= 2i, && \text{if } 1 \leq i \leq n, \\
f(v_j) &= 2n + 2j, && \text{if } 1 \leq j \leq n, \\
f(u_i u_{i+1}) &= 2n + 1 - 2i, && \text{if } 1 \leq i \leq n - 1, \\
f(u_n u_1) &= 1, && \\
f(u_{n+1-j} v_j) &= n^2 + 3n + 1 - 2j, && \text{if } 1 \leq j \leq n, \\
f[EH] - f[\{u_{n+1-j} v_j | 1 \leq j \leq n\}] &= \{2n + 1, 2n + 3, \ldots, n^2 + n - 1\} && \\
&\quad \cup \{4n + 2, 4n + 4, \ldots, n^2 + 3n\}, && \text{if } EH = \{u_i v_j | 1 \leq i, j \leq n\}\,.
\end{aligned}$$

If

$$\sum_{j=1}^{n-1} f(u_1 v_j) = \frac{n^3 + 4n^2 - 5}{2},$$

then f can be an even vertex magic total labeling of $W_{n,n}$.

Proof. Assume that

$$\sum_{j=1}^{n-1} f(u_1 v_j) = \frac{n^3 + 4n^2 - 5}{2}.$$

We have that

$$\begin{aligned}
S_s - \sum_{j=1}^{n} f(u_{n+1-j} v_j) &= \sum_{\substack{2n+1 \le i \le n^2+n-1 \\ i \text{ is odd}}} i + \sum_{\substack{4n+2 \le i \le n^2+3n \\ i \text{ is even}}} i \\
&= \left(\sum_{\substack{1 \le i \le n^2+n-1 \\ i \text{ is odd}}} i - \sum_{\substack{1 \le i \le 2n-1 \\ i \text{ is odd}}} i \right) + \left(\sum_{\substack{2 \le i \le n^2+3n \\ i \text{ is even}}} i - \sum_{\substack{2 \le i \le 4n \\ i \text{ is even}}} i \right) \\
&= \left(\frac{(n^2+n)^2}{4} - \frac{(2n)^2}{4} \right) + \left(\frac{(n^2+3n)(n^2+3n+2)}{4} - \frac{4n(4n+2)}{4} \right) \\
&= \frac{n^4+4n^3-4n^2-n}{2},
\end{aligned}$$

and then,

$$\left(S_s - \sum_{j=1}^{n} f(u_{n+1-j} v_j) \right) - \sum_{j=1}^{n-1} f(u_1 v_j) = \frac{n^4 + 3n^3 - 8n^2 - n + 5}{2}.$$

Next, we consider the sum of the label of each vertex and the labels of all edges incident to this vertex. By the assumption, for $1 \le j \le n$,

$$\begin{aligned}
f(v_j) + \sum_{i=1}^{n} f(u_i v_j) &= f(v_j) + f(u_{n+1-j} v_j) + \sum_{\substack{1 \le i \le n \\ i \ne n+1-j}} f(u_i v_j) \\
&= f(v_j) + f(u_{n+1-j} v_j) + \frac{S_s - \sum_{j=1}^{n} f(u_{n+1-j} v_j)}{n} \\
&= (2n + 2j) + (n^2 + 3n + 1 - 2j) + \frac{n^3+4n^2-4n-1}{2} \\
&= \frac{n^3+6n^2+6n+1}{2},
\end{aligned}$$

$$\begin{aligned}
f(u_1) + f(u_1 u_2) &+ f(u_n u_1) + \sum_{j=1}^{n} f(u_1 v_j) \\
&= f(u_1) + f(u_1 u_2) + f(u_n u_1) + f(u_1 v_n) + \sum_{j=1}^{n-1} f(u_1 v_j) \\
&= 2 + (2n + 1 - 2) + 1 + (n^2 + n - 1 + 2) + \frac{n^3+4n^2-5}{2} \\
&= \frac{n^3+6n^2+6n+1}{2}.
\end{aligned}$$

For $2 \leq i \leq n$,

$$\begin{aligned}
f(u_i) + f(u_iu_{i+1}) &+ f(u_{i-1}u_i) + \sum_{j=1}^{n} f(u_iv_j) \\
&= f(u_i) + f(u_iu_{i+1}) + f(u_{i-1}u_i) + f(u_iv_{n+1-i}) + \sum_{\substack{1\leq j\leq n \\ j\neq n+1-i}} f(u_iv_j) \\
&= f(u_i) + f(u_iu_{i+1}) + f(u_{i-1}u_i) + f(u_iv_{n+1-i}) + \frac{\left(S_s - \sum_{j=1}^{n} f(u_{n+1-j}v_j)\right) - \sum_{j=1}^{n-1} f(u_1v_j)}{n-1} \\
&= 2i + (2n+1-2i) + (2n+1-2i+2) + (n^2+n-1+2i) + \frac{n^3+4n^2-4n-5}{2} \\
&= \frac{n^3+6n^2+6n+1}{2}.
\end{aligned}$$

Therefore, f can be an even vertex magic total labeling of $W_{n,n}$ with a magic constant

$$k = \frac{n^3+6n^2+6n+1}{2}.$$

□

Now, we investigate the sufficient condition for a labeling f that can be an even vertex magic total labeling of $W_{n,n-2}$ when n is even.

Theorem 2.8. *Let n be an even integer with $n \geq 4$. For every $(n-2)$-fold wheel $W_{n,n-2}$, let*

$$f : V(W_{n,n-2}) \cup E(W_{n,n-2}) \to \{1, 2, \ldots, n^2+n-2\}$$

be defined by the following:

$$\begin{aligned}
f(u_i) &= 2i, && \text{if } 1 \leq i \leq n, \\
f(v_j) &= 2n+2j, && \text{if } 1 \leq j \leq n-2, \\
f[EC] &= \{a_1, a_2, \cdots, a_n\}, && \text{if } EC = \{u_iu_{i+1}, u_nu_1 | 1 \leq i \leq n-1\}, \\
f[EH] &= \{1, 2, \cdots, n^2+n-2\} \\
&\quad -\{2, 4, \cdots, 2n+4, a_1, a_2, \cdots, a_n\}, && \text{if } EH = \{u_iv_j | 1 \leq i \leq n, 1 \leq j \leq n-2\}.
\end{aligned}$$

If

$$S_s = \frac{n^4+n^3-15n^2+20n-4}{2},$$

then f can be an even vertex magic total labeling of $W_{n,n-2}$.

Proof. Assume that

$$S_s = \frac{n^4+n^3-15n^2+20n-4}{2}.$$

It suffices to show that for each vertex u of $W_{n,n-2}$,

$$f(u) + \sum_{v\in N(u)} f(uv) = k,$$

where

$$k = \frac{n^3+3n^2-3n}{2}.$$

To do this, we consider the relevant sums, as follows.

Since the sum of the labels of all rim edges is equal to the sum of the labels of all vertices and the labels of all edges subtracted by the sum of the labels of all vertices and the labels of all spokes, it follows that

$$\begin{aligned} S_{re} &= \sum_{i=1}^{n^2+n-2} i - \sum_{\substack{2\leq i\leq 4n-4 \\ i \text{ is even}}} i - S_s \\ &= \tfrac{(n^2+n-2)(n^2+n-1)}{2} - \tfrac{(4n-4)(4n-2)}{4} - \tfrac{n^4+n^3-15n^2+20n-4}{2} \\ &= \tfrac{n^3+5n^2-11n+2}{2}. \end{aligned}$$

Since the sum of the labels of all hubs is equal to the sum of even integers from $2n+2$ to $4n-4$,

$$\begin{aligned} S_h + S_s &= \sum_{\substack{2n+2\leq i\leq 4n-4 \\ i \text{ is even}}} i + S_s \\ &= \left(\sum_{\substack{2\leq i\leq 4n-4 \\ i \text{ is even}}} i - \sum_{\substack{2\leq i\leq 2n \\ i \text{ is even}}} i \right) + S_s \\ &= \tfrac{(4n-4)(4n-2)}{4} - \tfrac{(2n)(2n+2)}{4} + \tfrac{n^4+n^3-15n^2+20n-4}{2} \\ &= \tfrac{n^4+n^3-9n^2+6n}{2}. \end{aligned}$$

Since the sum of the labels of all rim vertices is equal to the sum of even integers from 2 to $2n$,

$$\begin{aligned} S_{rv} + 2S_{re} + S_s &= \sum_{\substack{2\leq i\leq 2n \\ i \text{ is even}}} i + 2S_{re} + S_s \\ &= \tfrac{(2n)(2n+2)}{4} + 2\left(\tfrac{n^3+5n^2-11n+2}{2}\right) + \tfrac{n^4+n^3-15n^2+20n-4}{2} \\ &= \tfrac{n^4+3n^3-3n^2}{2}. \end{aligned}$$

Next, we consider the sum of the label of each vertex and the labels of all edges incident to this vertex. We have the sum of the label of each hub and the labels of all edges incident to this hub as follows.

For $1 \leq j \leq n-2$,

$$f(v_j) + \sum_{i=1}^{n} f(u_i v_j) = \frac{S_h + S_s}{n-2} = \frac{n^3 + 3n^2 - 3n}{2}.$$

We obtain the sum of the label of each rim vertex and the labels of all edges incident to this rim vertex as follows.

For $2 \leq i \leq n-1$,

$$f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + \sum_{j=1}^{n-2} f(u_i v_j) = \frac{S_{rv} + 2S_{re} + S_s}{n} = \frac{n^3 + 3n^2 - 3n}{2}.$$

Similarly,

$$f(u_n) + f(u_n u_1) + f(u_{n-1} u_n) + \sum_{j=1}^{n-2} f(u_n v_j) = \frac{n^3 + 3n^2 - 3n}{2}$$

and

$$f(u_1) + f(u_1u_2) + f(u_nu_1) + \sum_{j=1}^{n-2} f(u_1v_j) = \frac{n^3 + 3n^2 - 3n}{2}.$$

Therefore, f can be an even vertex magic total labeling of $W_{n,n-2}$ with the following magic constant:

$$k = \frac{n^3 + 3n^2 - 3n}{2}.$$

□

3. An even vertex magic $W_{n,t}$ where $3 \le n \le 9$

In this section, we establish a characterization of an even vertex magic $W_{n,t}$ for an integer $3 \le n \le 9$. First, we present an n-fold wheel $W_{n,n}$ which has an even vertex magic total labeling for every odd integer $3 \le n \le 9$ as follows.

Theorem 3.1. *For every odd integer $3 \le n \le 9$, the n-fold wheel $W_{n,n}$ is an even vertex magic.*

Proof. Let n be an odd integer where $3 \le n \le 9$. We define

$$f : V(W_{n,n}) \cup E(W_{n,n}) \to \{1, 2, \cdots, n^2 + 3n\},$$

as the sufficient condition of Theorem 2.7, by

$$\begin{aligned} f(u_i) &= 2i, && \text{if } 1 \le i \le n, \\ f(v_j) &= 2n + 2j, && \text{if } 1 \le j \le n, \\ f(u_iu_{i+1}) &= 2n + 1 - 2i, && \text{if } 1 \le i \le n-1, \\ f(u_nu_1) &= 1, \end{aligned}$$

and for $1 \le i, j \le n$, $f(u_iv_j)$ are shown in Tables 1–4,

Table 1. Labels of edges u_iv_j of $W_{3,3}$ by f, for $1 \le i, j \le 3$.

$f(u_iv_j)$	v_1	v_2	v_3
u_1	11	18	13
u_2	14	15	9
u_3	17	7	16

Table 2. Labels of edges u_iv_j of $W_{5,5}$ by f, for $1 \le i, j \le 5$.

$f(u_iv_j)$	v_1	v_2	v_3	v_4	v_5
u_1	15	21	36	38	31
u_2	13	30	29	33	28
u_3	40	24	35	19	17
u_4	34	37	11	23	32
u_5	39	27	26	22	25

Table 3. Labels of edges u_iv_j of $W_{7,7}$ by f, for $1 \le i, j \le 7$.

$f(u_iv_j)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1	15	51	64	40	29	68	57
u_2	66	43	17	38	19	59	70
u_3	25	45	56	48	61	32	47
u_4	52	31	33	63	54	53	30
u_5	42	35	65	34	60	36	46
u_6	55	67	23	58	49	27	41
u_7	69	50	62	37	44	39	21

Table 4. Labels of edges u_iv_j of $W_{9,9}$ by f, for $1 \le i, j \le 9$.

$f(u_iv_j)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
u_1	73	33	102	69	19	79	63	86	91
u_2	43	56	35	62	49	89	85	93	87
u_3	75	61	48	53	104	92	95	23	50
u_4	80	58	72	25	108	97	77	47	39
u_5	51	96	59	81	99	37	55	57	70
u_6	60	100	44	101	67	40	41	90	64
u_7	38	83	103	82	74	78	29	76	46
u_8	88	105	94	65	42	66	52	31	68
u_9	107	21	54	71	45	27	106	98	84

For every odd integer $3 \le n \le 9$, the labeling f, as defined above, is an even vertex magic total labeling of the n-fold wheel $W_{n,n}$ with magic constants $k = 50, 153, 340$ and 635, respectively. Therefore, $W_{n,n}$ is an even vertex magic. □

As a consequence of an even vertex magic $W_{3,2}$, Proposition 2.2 and Theorem 3.1, in any t-fold wheel $W_{3,t}$, we are able to show that both $W_{3,t}$ and $W_{3,t}$ are only even vertex magics.

Theorem 3.2. *For every integer $t \ge 2$, the t-fold wheel $W_{3,t}$ is an even vertex magic if and only if $t = 2, 3$.*

The following result gives the necessary and sufficient condition for the t-fold wheel $W_{n,t}$ to be an even vertex magic for every odd integer $5 \le n \le 9$.

Theorem 3.3. *For every odd integer $5 \le n \le 9$ and an integer $t \ge 2$, the t-fold wheel $W_{n,t}$ is an even vertex magic if and only if $t = n$.*

Proof. Let n be an odd integer where $5 \le n \le 9$ and t is an integer where $t \ge 2$. Assume that the t-fold wheel $W_{n,t}$ is an even vertex magic. By Proposition 2.2, $2 \le t \le n$.

Case 1. $n = 5, 7$. If $2 \le t \le n-1$, then $n^2t^2 + 2n^2t + n$ is not divisible by $n + t$, and hence k is not an integer, which is a contradiction. Therefore, $t = n$.

Case 2. $n = 9$. If either $t = 2$ or $4 \le t \le n-1$, then $n^2t^2 + 2n^2t + n$ is not divisible by $n + t$, and hence k is not an integer, which is a contradiction. If $t = 3$, then,

$$2nt^3 - n^2t^2 - 2n^2t + 6nt^2 + 7nt + n^2 - t^2 + n - t - \frac{2nt^4 + 4nt^3 - 2nt}{n+t} = -174 < 0,$$

which is a contradiction with Proposition 2.6. Therefore, $t = n$.

Conversely, assume $t = n$. By Theorem 3.1, $W_{n,t}$ is an even vertex magic. □

We show an even vertex magic total labeling of $W_{n,n-2}$ for every even integer $4 \leq n \leq 8$ as follows.

Theorem 3.4. *For every even integer $4 \leq n \leq 8$, the $(n-2)$-fold wheel $W_{n,n-2}$ is an even vertex magic.*

Proof. Let n be an even integer with $4 \leq n \leq 8$. We define

$$f : V(W_{n,n-2}) \cup E(W_{n,n-2}) \to \{1, 2, \ldots, n^2 + n - 2\}$$

as the sufficient condition of Theorem 2.8, by

$$\begin{aligned} f(u_i) &= 2i, && \text{if } 1 \leq i \leq n, \\ f(v_j) &= 2n + 2j, && \text{if } 1 \leq j \leq n-2, \end{aligned}$$

for $1 \leq i \leq n-1$, $f(u_iu_{i+1})$ and $f(u_nu_1)$ are shown in Tables 5–7.

Table 5. Labels of edges u_iu_{i+1} and u_4u_1 of $W_{4,2}$ by f, for $1 \leq i \leq 3$.

$f(u_1u_2)$	$f(u_2u_3)$	$f(u_3u_4)$	$f(u_4u_1)$
18	9	11	13

Table 6. Labels of edges u_iu_{i+1} and u_6u_1 of $W_{6,4}$ by f, for $1 \leq i \leq 5$.

$f(u_1u_2)$	$f(u_2u_3)$	$f(u_3u_4)$	$f(u_4u_5)$	$f(u_5u_6)$	$f(u_6u_1)$
40	39	38	13	17	19

Table 7. Labels of edges u_iu_{i+1} and u_8u_1 of $W_{8,6}$ by f, for $1 \leq i \leq 7$.

$f(u_1u_2)$	$f(u_2u_3)$	$f(u_3u_4)$	$f(u_4u_5)$	$f(u_5u_6)$	$f(u_6u_7)$	$f(u_7u_8)$	$f(u_8u_1)$
70	68	66	64	62	19	13	11

And for $1 \leq i \leq n$ and $1 \leq j \leq n-2$, $f(u_iv_j)$ are shown in Tables 8–10.

Table 8. Labels of edges u_iv_j of $W_{4,2}$ by f, for $1 \leq i \leq 4$ and $1 \leq j \leq 2$.

$f(u_iv_j)$	v_1	v_2
u_1	16	1
u_2	14	5
u_3	7	17
u_4	3	15

Table 9. Labels of edges u_iv_j of $W_{6,4}$ by f, for $1 \le i \le 6$ and $1 \le j \le 4$.

$f(u_iv_j)$	v_1	v_2	v_3	v_4
u_1	36	24	3	29
u_2	11	25	33	1
u_3	26	9	5	30
u_4	22	23	34	15
u_5	37	21	28	27
u_6	7	35	32	31

Table 10. Labels of edges u_iv_j of $W_{8,6}$ by f, for $1 \le i \le 8$ and $1 \le j \le 6$.

$f(u_iv_j)$	v_1	v_2	v_3	v_4	v_5	v_6
u_1	69	3	40	56	39	50
u_2	1	67	5	59	31	35
u_3	21	17	65	7	44	46
u_4	42	38	23	33	9	57
u_5	61	27	37	34	30	15
u_6	41	53	49	32	47	25
u_7	29	60	54	52	63	36
u_8	58	55	45	43	51	48

For every even integer $4 \le n \le 8$, the labeling f, as defined above, is an even vertex magic total labeling of the $(n-2)$-fold wheel $W_{n,n-2}$ with magic constants $k = 50, 153$ and 340, respectively. Therefore, $W_{n,n-2}$ is an even vertex magic. □

There is a similar methodology of the proof of Theorem 3.4, which is also used in the study of graph operations (see [2]). Next, we determine a characterization of the t-fold wheel $W_{n,t}$ to be an even vertex magic for every even integer $4 \le n \le 8$. In order to we need to present the following lemma involving a 3-fold wheel $W_{8,3}$.

Lemma 3.5. *The* 3*-fold wheel* $W_{8,3}$ *is not an even vertex magic.*

Proof. On the contrary, assume that the 3-fold wheel $W_{8,3}$ is an even vertex magic with a magic constant k. Since $W_{8,3}$ has an order 11 and a size 32 and by Proposition 2.1, $k = 160$. We have that

$$S_{rv} = 132, \;\; 2S_{re} = 1,628 - 2S_s$$

and

$$S_{rv} + 2S_{re} + S_s = 8k = 1,280.$$

Thus, $S_s = 480$. However, $S_h + S_s = 3k = 480$. This is a contradiction because $S_h > 0$. Therefore, $W_{8,3}$ is not an even vertex magic. □

We are able to show that the necessary and sufficient condition for the t-fold wheel $W_{n,t}$ is an even vertex magic for every even integer $4 \le n \le 8$.

Theorem 3.6. *For every even integer $4 \leq n \leq 8$ and integer $t \geq 2$, the t-fold wheel $W_{n,t}$ is an even vertex magic if and only if $t = n - 2$.*

Proof. Let n be an even integer where $4 \leq n \leq 8$ and t is an integer where $t \geq 2$. Assume that the t-fold wheel $W_{n,t}$ is an even vertex magic. By Proposition 2.2, $2 \leq t \leq n$.

Case 1. $n = 4, 6$. If either $2 \leq t \leq n-3$ or $n-1 \leq t \leq n$, then $n^2t^2 + 2n^2t + n$ is not divisible by $n + t$, and hence k is not an integer, which is a contradiction. Therefore, $t = n - 2$.

Case 2. $n = 8$. If either $4 \leq t \leq n - 3$ or $n - 1 \leq t \leq n$, then $n^2t^2 + 2n^2t + n$ is not divisible by $n + t$, and hence k is not an integer, which is a contradiction. If $t = 3$, then, by Lemma 3.5, $W_{n,t}$ is not an even vertex magic, which is a contradiction. If $t = 2$, then,

$$2nt^3 - n^2t^2 - 2n^2t + 6nt^2 + 7nt + n^2 - t^2 + n - t - \frac{2nt^4 + 4nt^3 - 2nt}{n+t} = -62 < 0,$$

which is a contradiction with Proposition 2.6. Therefore, $t = n - 2$.

Conversely, assume $t = n - 2$. By Theorem 3.4, $W_{n,t}$ is an even vertex magic. □

4. Conclusions

In this paper, we have not only established the bound of an integer t for the even vertex magic total labeling of the t-fold wheel, but have also presented the necessary condition for such labeling in terms of n and t. Furthermore, we have conducted an investigation into the sufficient conditions for labelings that can serve as even vertex magic total labelings for $W_{n,n}$ when n is odd, and $W_{n,n-2}$ when n is even.

Our research has led us to the following significant conclusions:

- For every integer $t \geq 2$, the t-fold wheel $W_{3,t}$ is an even vertex magic total labeling if and only if $t = 2, 3$.
- For every odd integer $5 \leq n \leq 9$ and an integer $t \geq 2$, the t-fold wheel $W_{n,t}$ is an even vertex magic total labeling if and only if $t = n$.
- For every even integer $4 \leq n \leq 8$ and an integer $t \geq 2$, the t-fold wheel $W_{n,t}$ is an even vertex magic total labeling if and only if $t = n - 2$.

In essence, our work has discussed the characterizations of t-fold wheel $W_{n,t}$ to possess an even vertex magic total labeling for an integer $3 \leq n \leq 9$. It would be interesting to apply the results of this paper to further study under what conditions for $W_{n,t}$ will be an even vertex magic, especially for a larger n.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. A. Alhevaz, M. Darkooti, H. Rahbani, Y. Shang, Strong equality of perfect Roman and weak Roman domination in trees, *Mathematics*, **7** (2019), 997. http://doi.org/10.3390/math7100997
2. A. Alhevaz, M. Baghipur, H. A. Ganie, Y. Shang, The generalized distance spectrum of the join of graphs, *Symmetry*, **12** (2020), 169. http://doi.org/10.3390/sym12010169
3. J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Comb.*, 2009.
4. J. A. MacDougall, M. Miller, K. A. Sugeng, Super vertex-magic total labeling of graphs, *Proceedings of the 15th Australasian Workshop on Combinatorial Algorithms*, 2004.
5. J. A. MacDougall, M. Miller, Slamin, W. D. Wallis, Vertex-magic total labeling of graphs, *Utilitas Math.*, **61** (2002), 3–21.
6. J. A. MacDougall, M. Miller, W. D. Wallis, Vertex-magic total labelings of wheels and related graphs, *Utilitas Math.*, **62** (2002), 175–183.
7. C. T. Nagaraj, C. Y. Ponnappan, G. Prabakaran, Even vertex magic total labeling, *Int. J. Pure Appl. Math.*, **115** (2017), 363–374.
8. C. T. Nagaraj, C. Y. Ponnappan, G. Prabakaran, Even vertex magic total labeling of isomorphic and non isomorphic suns, *Int. J. Math. Trends Technol.*, **52** (2017), 458–467.
9. C. T. Nagaraj, C. Y. Ponnappan, G. Prabakaran, Even vertex magic total labeling of some 2-regular graphs, *Int. J. Math. Trends Technol.*, **54** (2018), 52–59.
10. V. A. Nageswari, V. Maheswari, Vertex magic total labeling of some general graphs and its properties, *Int. J. Pure Appl. Math.*, **118** (2018), 4637–4643.
11. M. T. Rahim, I. Tomescu, Slamin, On vertex-magic total labeling of some wheel related graphs, *Utilitas Math.*, **73** (2007), 97–104.
12. M. Sindhu, S. C. Kumar, Even vertex in-magic total labeling of some 2-regular digraphs, *J. Electron. Comput. Networking Appl. Math.*, **1** (2021), 1–11. http://doi.org/10.55529/jecnam.12.1.11

AIMS Press