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Research article

New algorithms for solving nonlinear mixed integral equations

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Abstract: In this article, the existence and unique solution of the nonlinear Volterra-Fredholm integral equation (NVFIE) of the second kind is discussed. We also prove the solvability of the second kind of the NVFIE using the Banach fixed point theorem. Using quadrature method, the NVFIE leads to a system of nonlinear Fredholm integral equations (NFIEs). The existence and unique numerical solution of this system is discussed. Then, the modified Taylor's method was applied to transform the system of NFIEs into nonlinear algebraic systems (NAS). The existence and uniqueness of the nonlinear algebraic system's solution are discussed using Banach's fixed point theorem. Also, the stability of the modified error is presented. Some numerical examples are performed to show the efficiency and simplicity of the presented method, and all results are obtained using Wolfram Mathematica 11.

Keywords: Picard's method; nonlinear Volterra-Fredholm integral equation; Banach fixed point theorem; system of nonlinear Fredholm integral equations; nonlinear algebraic system; modified Taylor's method
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1. Introduction

Our goal in this paper is to use a new algorithm based on a modified Taylor's method to solve the following partial integro-differential equation (PIDE):

$$\frac{\partial}{\partial t}\left(\omega\Psi(u,t) - f(u,t)\right) = \lambda\xi(t)\int_0^1 k(u,v)\vartheta(t,v,\Psi(v,t))\mathrm{d}v, \qquad \Psi(u,0) = \phi(u). \tag{1.1}$$

Here, $f(u,t)$ and $\vartheta(t,v,\Psi(v,t))$ are two given functions, while the function $\Psi(u,t)$ is unknown in the Banach space $L_2[0,1] \times C[0,T]$. The kernel of position, for $x, y \in [0,1]$, $k(u,v)$ is continuous. The kernel of time $\xi(t)$, $t \in [0,T]$, $T < 1$, is continuous in the class $C[0,T]$, the constant ω determines the type of the integral equation and λ is a complex constant with a distinct physical meaning [40,41].

Integrating the previous equation, we get

$$\begin{aligned} \omega\Psi(u,t) &= \gamma(u,t) + \lambda \int_0^t \int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi(v,\tau))\mathrm{d}v\mathrm{d}\tau, \\ \gamma(u,t) &= f(u,t) + \omega\phi(u) - f(u,0). \end{aligned} \tag{1.2}$$

Equation (1.2) is called the NVFIE.

These types of NVFIEs appear in a wide variety of applications in many fields including generalized potential theory [5], electromagnetic and electrodynamics [33, 38], theory of elasticity [31], quantum mechanics [18], contact problems in two layers of elastic materials [3], fluid mechanics [36], radiation [19], nonlinear problems theory of boundary value [4, 9], population genetics [9, 39], mathematical economics [10] and spectral relationships in laser theory [11].

Often, finding exact solutions of these equations is very difficult. Therefore, it is better to develop an effective and accurate numerical method to find a solution of these types of problems. To solve the NVFIE given by Eq (1.2), numerous computational techniques have been proposed, such as the separation of variables method [27], Resolvent method [1], modified iterated projection method [13], degenerate kernel method [8, 28], Lagrange polynomials [32], Legendre polynomials [30], Picard iteration method [20], Chebyshev wavelets polynomials [35], Legendre-Chebyshev collocation method [14], block pulse functions [23], hat functions [15, 16], Tau-collocation method [12], Hybrid Functions method [2], collocation methods [9, 17, 24], Lagrange–collocation method [29], operational matrices [26], Bell polynomials [25], Fibonacci collocation method [22], Taylor polynomial method [37] and modification of hat functions [21]. We have developed an accurate and new method to find the numerical solution to the problem presented by Eq (1.2), and this is the main goal of the study.

The study of the problem in space and time is included in this article, which makes it a rare papers in mathematical physics. This provides the authors with a more comprehensive understanding of how to analyze and solve this problem utilizing a variety of numerical techniques.

In the present study, we consider NVFIE of the second type. Then, a new modification of the Taylor series expansion method is proposed for the NVFIE of the second kind (1.2). The integral equations illustrated in the examples can be approximated using this method, which have very effective and simple steps. Using the presented method to transform the system of NFIEs into NAS, we will explain the specific and practical features of this method in the following sections.

The existence and unique solution of the NVFIE of the second kind are discussed in Section 2. In Section 3, the second type of NVFIE is obviously solvable using the Banach fixed point theorem. In Section 4, using the quadrature method, the NVFIE leads to a system of NFIEs. The existence and unique numerical solution of a system of NFIEs are discussed in Section 5. In Section 6, the modified Taylor's method was applied to transform a system of NFIEs into an NAS. The existence and uniqueness of the nonlinear algebraic system's solution are studied using Banach's fixed point theorem in Section 7. The stability of the modified error is defined in Section 8, while Section 9 solves various illustrative examples by using the program Wolfram Mathematica 11 to confirm the efficiency of the approach. Finally, some remarks and conclusions are shown in Section 10.

2. Existence and uniqueness solution of the NVFIE (1.2)

We provide the following assumptions in order to discuss the existence and uniqueness of the solution of Eq (1.2):

(i) The kernel $k(u, v)$ is continuous in $L_2[0, 1]$ and satisfies $|k(u, v)| \leq \beta,\ \forall u, v \in [0, 1]$ and $\omega \in \mathbb{R} - 0$.

(ii) The function $\xi(\tau)$ is continuous in the space $C[0, T]$ and satisfies

$$\|\xi(\tau)\|_{C[0,T]} = \max_{t\in[0,T]} |\xi(\tau)| \leq \alpha.$$

(iii) The norm of the given function $\gamma(u, t)$ is defined as

$$\|\gamma(u, t)\|_{L_2[0,1]\times C[0,T]} = \max_{0\leq t\leq T} \left| \int_0^t \left[\int_0^1 \gamma^2(u, \tau) du \right]^{\frac{1}{2}} \mathrm{d}\tau \right| = \chi.$$

(iv) The known function $\vartheta(t, u, \Psi(u, t))$, for the constants $\upsilon > \delta$ and $\upsilon > \varepsilon$, satisfies:

$$(a) \quad \max_{0\leq t\leq T} \left| \int_0^t \left[\int_0^1 |\vartheta(\tau, v, \Psi(v, \tau))|^2 dv \right]^{\frac{1}{2}} \mathrm{d}\tau \right| \leq \delta \|\Psi(u, t)\|_{L_2[0,1]\times C[0,T]},$$

$$(b) \quad |\vartheta(t, u, \Psi_1(u, t)) - \vartheta(t, u, \Psi_2(u, t))| \leq \Delta(t, u)|\Psi_1(u, t) - \Psi_2(u, t)|,$$

where

$$\|\Delta(t, u)\| = \max_{0\leq\tau\leq t\leq T} \left| \int_0^t \left[\int_0^1 |\Delta(\tau, u)|^2 du \right]^{\frac{1}{2}} \mathrm{d}\tau \right| = \varepsilon.$$

where $\alpha, \beta, \chi, \delta, \varepsilon$ and υ are positive constants.

Theorem 2.1. *If the conditions (i)–(iv-b) are satisfied, and*

$$T < \frac{|\omega|}{\beta\alpha\upsilon|\lambda|}, \tag{2.1}$$

then Eq (1.2) has a unique solution $\Psi(u, t)$ *in the Banach space* $L_2[0, 1] \times C[0, T]$.

Proof. We apply the successive approximation method (*Picard's method*) to prove this theorem.

A solution for Eq (1.2) can be formed as a sequence of functions $\{\Psi_m(u, t)\}$ as $\{m\}$ tends to ∞; thus,

$$\Psi(u, t) = \lim_{m\to\infty} \Psi_m(u, t),$$

where

$$\Psi_m(u, t) = \sum_{l=0}^{m} S_l(u, t), \quad t \in [0, T], \qquad m = 0, 1, 2, \ldots \tag{2.2}$$

in which the functions $S_l(u, t),\ l = 0, 1, \ldots, m$ are continuous functions and take the following form:

$$\left.\begin{aligned} S_m(u, t) &= \Psi_m(u, t) - \Psi_{m-1}(u, t) \\ S_0(u, t) &= \gamma(u, t). \end{aligned}\right\}. \tag{2.3}$$

We have to consider the following lemmas in order to prove the previous theorem. □

Lemma 2.1. *If the series* $\sum_{l=0}^{m} S_l(u,t)$ *is uniformly convergent, then* $\Psi(u,t)$ *represents a solution of Eq (1.2).*

Proof. We establish a sequence $\Psi_m(u,t)$ that is specified by

$$\begin{aligned}\Psi_m(u,t) &= \frac{1}{\omega}\gamma(u,t) + \frac{\lambda}{\omega}\int_0^t \int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi_{m-1}(v,\tau))\mathrm{d}v\mathrm{d}\tau,\\ \Psi_0(u,t) &= \frac{1}{\omega}f(u,t) + \phi(u) - \frac{1}{\omega}f(u,0).\end{aligned} \tag{2.4}$$

Then, we obtain

$$\Psi_m(u,t) - \Psi_{m-1}(u,t) = \frac{\lambda}{\omega}\int_0^t \int_0^1 \xi(\tau)k(u,v)[\vartheta(\tau,v,\Psi_{m-1}(v,\tau)) - \vartheta(\tau,v,\Psi_{m-2}(v,\tau))]\mathrm{d}v\mathrm{d}\tau.$$

From Eq (2.3) and properties of the norm, we get

$$\|S_m(u,t)\| \leq \frac{|\lambda|}{|\omega|}\left\|\int_0^t \int_0^1 \xi(\tau)k(u,v)[\vartheta(\tau,v,\Psi_{m-1}(v,\tau)) - \vartheta(\tau,v,\Psi_{m-2}(v,\tau))]\mathrm{d}v\mathrm{d}\tau\right\|, \tag{2.5}$$

using (iv-b), we have

$$\begin{aligned}\|S_m(u,t)\| \leq& \frac{|\lambda|}{|\omega|}\left\|\int_0^t \int_0^1 \xi(\tau)k(u,v)\Delta(\tau,v)|\Psi_{m-1}(v,\tau) - \Psi_{m-2}(v,\tau)|\mathrm{d}v\mathrm{d}\tau\right\|\\ \leq& \frac{|\lambda|}{|\omega|}\left\|\int_0^t \int_0^1 \xi(\tau)k(u,v)\Delta(\tau,v)|S_{m-1}(v,\tau)|\mathrm{d}v\mathrm{d}\tau\right\|\\ \leq& \frac{\upsilon|\lambda|}{|\omega|}\left\|\int_0^t \int_0^1 \xi(\tau)k(u,v)|S_{m-1}(v,\tau)|\mathrm{d}v\mathrm{d}\tau\right\|.\end{aligned}$$

Conditions (i)–(ii) have led to

$$\|S_m(u,t)\| \leq \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}\left\|\int_0^t \int_0^1 |S_{m-1}(v,\tau)|\mathrm{d}v\mathrm{d}\tau\right\|, \tag{2.6}$$

for $m = 1$ and using condition (iii), we get from formula (5.2)

$$\begin{aligned}\|S_1(u,t)\| &\leq \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}\left\|\int_0^t \int_0^1 |S_0(v,\tau)|\mathrm{d}v\mathrm{d}\tau\right\|\\ &\leq \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}\|t\|\chi,\end{aligned} \tag{2.7}$$

where

$$T = \max_{0<t\leq T}|t|.$$

Therefore, formula (2.7) becomes

$$\|S_1(u,t)\| \leq \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}T\chi,$$

and by induction, we have

$$\|S_m(u,t)\| \leq \Theta^m \chi, \qquad \Theta = \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}T < 1, \quad m = 0,1,2,\ldots \tag{2.8}$$

Since

$$T < \frac{|\omega|}{\beta\alpha\upsilon|\lambda|},$$

which allows us to conclude that the sequence $\Psi_m(u,t)$ has a convergent solution. Thus, for $m \to \infty$, we get

$$\begin{aligned}\omega\Psi(u,t) &= \lim_{m\to\infty}\left(\gamma(u,t) + \lambda\int_0^t\int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi_m(v,\tau))\mathrm{d}v\mathrm{d}\tau\right)\\ &= \gamma(u,t) + \lambda\int_0^t\int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi(v,\tau))\mathrm{d}v\mathrm{d}\tau.\end{aligned}$$

□

Lemma 2.2. *The function $\Psi(u,t)$ represents a unique solution of NVFIE (1.2).*

Proof. To provide that $\Psi(u,t)$ is a unique solution, assume that there exists another solution $\Phi(u,t)$ of Eq (1.2), then we obtain

$$\omega\Phi(u,t) = \gamma(u,t) + \lambda\int_0^t\int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Phi(v,\tau))\mathrm{d}v\mathrm{d}\tau,$$

and

$$\Psi(u,t) - \Phi(u,t) = \frac{\lambda}{\omega}\int_0^t\int_0^1 \xi(\tau)k(u,v)[\vartheta(\tau,v,\Psi(v,\tau)) - \vartheta(\tau,v,\Phi(v,\tau))]\mathrm{d}v\mathrm{d}\tau.$$

From condition (iv-b), we have

$$\begin{aligned}\|\Psi(u,t) - \Phi(u,t)\| &\leq \frac{|\lambda|}{|\omega|}\left\|\int_0^t\int_0^1 \xi(\tau)k(u,v)\Delta(\tau,v)|\Psi(v,\tau) - \Phi(v,\tau)|\mathrm{d}v\mathrm{d}\tau\right\|\\ &\leq \frac{\upsilon|\lambda|}{|\omega|}\left\|\int_0^t\int_0^1 \xi(\tau)k(u,v)|\Psi(v,\tau) - \Phi(v,\tau)|\mathrm{d}v\mathrm{d}\tau\right\|.\end{aligned}$$

Using conditions (i)–(ii), we have

$$\begin{aligned}\|\Psi(u,t) - \Phi(u,t)\| &\leq \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}T\|\Psi(u,t) - \Phi(u,t)\|\\ &\leq \Theta\|\Psi(u,t) - \Phi(u,t)\|; \quad \Theta < 1.\end{aligned}$$

If $\|\Psi(u,t) - \Phi(u,t)\| \neq 0$, then the last formula yields $\Theta \geq 1$, which is a contradiction. Thus, $\|\Psi(u,t) - \Phi(u,t)\| = 0$ meaning that $\Psi(u,t) = \Phi(u,t)$, implying that the solution is unique. □

3. Normality and continuity of the integral operator

To show the normality and continuity of the NVFIE (1.2), it will be represented in its integral operator form

$$\bar{¥}\Psi = \frac{1}{\omega}\gamma(u,t) + ¥\Psi, \tag{3.1}$$

and

$$¥\Psi = \frac{\lambda}{\omega}\int_0^t \int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi(v,\tau))\mathrm{d}v\mathrm{d}\tau.$$

For the normality of the integral operator

From Eq (3.1), we obtain

$$\|¥\Psi\| \leq \frac{|\lambda|}{|\omega|}\int_0^t \int_0^1 |\xi(\tau)||k(u,v)||\vartheta(\tau,v,\Psi(v,\tau))|\mathrm{d}v\mathrm{d}\tau.$$

Applying conditions (i),(ii) and (iv-a), we get

$$\begin{aligned}\|¥\Psi\| &\leq \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}T\|\Psi(u,t)\| \\ &\leq \Theta\|\Psi(u,t)\|; \quad \Theta = \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}T,\end{aligned}$$

such that,

$$T < \frac{|\omega|}{\beta\alpha\upsilon|\lambda|},$$

Therefore, the integral operator ¥ has a normality, and through the condition (iii), we directly proved that the integral operator $\bar{¥}$ also has a normality.

For the continuity of the integral operator

We consider the two functions $\Psi_1(u,t)$, $\Psi_2(u,t)$ in $L_2[0,1]\times C[0,T]$, satisfies Equation (3.1) then,

$$\bar{¥}\Psi_1 = \frac{1}{\omega}\gamma(u,t) + \frac{\lambda}{\omega}\int_0^t \int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi_1(v,\tau))\mathrm{d}v\mathrm{d}\tau.$$

Subtracting the function $\Psi_2(u,t)$ from $\Psi_1(u,t)$, we get

$$\bar{¥}\Psi_1 - \bar{¥}\Psi_2 = \bar{¥}[\Psi_1 - \Psi_2],$$

Using conditions (i),(ii) and (iv-b), we obtain

$$\|\bar{¥}[\Psi_1 - \Psi_2]\| \leq \frac{\beta\alpha\upsilon|\lambda|}{|\omega|}T\|\Psi_1 - \Phi_2\|,$$

hence, we have

$$\|\bar{¥}[\Psi_1 - \Psi_2]\| \leq \Theta\|\Psi_1 - \Phi_2\|; \quad \Theta < 1. \tag{3.2}$$

Inequality (3.2) shows the continuity of the integral operator $\bar{¥}$. Furthermore, $\bar{¥}$ is a contraction operator in $L_2[0,1]\times C[0,T]$. $\bar{¥}$ has a unique fixed point, as proven by the Banach fixed point theorem. The existence and uniqueness of the NVFIE (1.2) are accepted if the continuity and normality of the integral operator are used.

4. System of nonlinear Fredholm integral equations

The solution of Eq (1.2) is usually reduced to a system of NFIEs by using the quadrature method [7]. We divide the interval $[0,T]$, $0 \leq t \leq T$, as $0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T$, where $t = t_n$, $n = 0,1,\ldots,N$; to get

$$\begin{aligned} \omega\Psi(u,t_n) &= \gamma(u,t_n) + \lambda \int_0^{t_n}\int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi(v,\tau))\mathrm{d}v\mathrm{d}\tau, \\ \gamma(u,t_n) &= f(u,t_n) + \omega\phi(u) - f(u,0), \end{aligned} \tag{4.1}$$

and the term for the Volterra integral are as follows:

$$\int_0^{t_n}\int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi(v,\tau))\mathrm{d}v\mathrm{d}\tau = \sum_{i=0}^{n}\mu_i\xi(t_i)\int_0^1 k(u,v)\vartheta(t_i,v,\Psi(v,t_i))\mathrm{d}v + O(\hbar_n^{p+1}), \tag{4.2}$$

where

$$\hbar_n^{p+1} \longrightarrow 0,\ p > 0,\ \hbar_n = \max_{0\leq i\leq n} h_i \quad \text{and} \quad h_i = t_{i+1} - t_i.$$

The constant p and the values of the weight formula μ_i depend on the number of derivatives $\xi(\tau)$, $\forall\tau \in [0,T]$, with respect to t. Here, $O(\hbar_n^{p+1})$ is the order of sum errors of the numerical approach of splitting the interval $[0,T]$, and the difference between the integration and summation, where the error is defined by:

$$R_n = \int_0^{t_n}\int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi(v,\tau))\mathrm{d}v\mathrm{d}\tau - \sum_{i=0}^{n}\mu_i\xi(t_i)\int_0^1 k(u,v)\vartheta(t_i,v,\Psi(v,t_i))\mathrm{d}v. \tag{4.3}$$

Using Eq (4.2) in Eq (4.1) and neglecting $O(\hbar_n^{p+1})$, we obtain

$$\begin{aligned} \omega\Psi(u,t_n) &= \gamma(u,t_n) + \lambda\sum_{i=0}^{n}\mu_i\xi(t_i)\int_0^1 k(u,v)\vartheta(t_i,v,\Psi(v,t_i))\mathrm{d}v, \\ \gamma(u,t_n) &= f(u,t_n) + \omega\phi(u) - f(u,0). \end{aligned} \tag{4.4}$$

And then using the notations below:

$$\Psi(u,t_n) = \Psi_n(u), \quad \gamma(u,t_n) = \gamma_n(u), \quad \vartheta(t_i,v,\Psi(v,t_i)) = \vartheta_i(v,\Psi_i(v)).$$

Equation (4.4) can be rewritten in the following form:

$$\begin{aligned} \omega\Psi_n(u) &= \gamma_n(u) + \lambda\sum_{i=0}^{n}\mu_i\xi_i\int_0^1 k(u,v)\vartheta_i(v,\Psi_i(v))\mathrm{d}v, \\ \gamma_n(u) &= f_n(u) + \omega\phi(u) - f(u,0). \end{aligned} \tag{4.5}$$

When $\omega = 0$, we get a system of NFIEs of the first-type, whereas Eq (4.5) represents a system of NFIEs of the second-type when $\omega \neq 0$.

5. The Existence of a unique solution of the system of nonlinear Fredholm integral equations

To prove the existence of a unique solution of the system of NFIEs (4.5), we can define the following conditions:

(i*) The kernel of continuous position satisfies $|k(u,v)| \leq \beta$.
(ii*) The function $\mu_i\xi_i$ satisfies $\max_i |\mu_i\xi_i| \leq \alpha^*$.
(iii*) $\max_n |\gamma_n(u)| \leq \chi^*$.
(iv*) The function $\vartheta_i(u,\Psi_i(u))$ satisfies: $|\vartheta_i(u,\Psi_{i,1}(u)) - \vartheta_i(u,\Psi_{i,2}(u))| \leq \varepsilon^*|\Psi_{i,1}(u) - \Psi_{i,2}(u)|$.

Theorem 5.1. *If the series $\sum_{n=0}^{\infty}\{\Xi_{n,l}(u)\}$, $\Xi_{n,m}(u) = \Psi_{n,m}(u)-\Psi_{n,m-1}(u)$ is uniformly convergent, then $\Psi_n(u)$ represents a solution of a system of NFIEs (4.5).*

Proof. We create a sequence $\Psi_{n,m}(u)$ described by

$$\omega\Psi_{n,m}(u) = \gamma_n(u) + \lambda\sum_{i=0}^{n}\mu_i\xi_i\int_0^1 k(u,v)\vartheta_i(v,\Psi_{i,m-1}(v))\mathrm{d}v. \tag{5.1}$$

Introduce the function $\Xi_{n,l}(u)$ such that $\Xi_{n,m}(u) = \Psi_{n,m}(u)-\Psi_{n,m-1}(u)$. In this case, the integral Eq (5.1), becomes

$$\Xi_{n,m}(u) = \frac{\lambda}{\omega}\sum_{i=0}^{n}\mu_i\xi_i\int_0^1 k(u,v)[\vartheta_i(v,\Psi_{i,m-1}(v)) - \vartheta_i(v,\Psi_{i,m-2}(v))]\mathrm{d}v.$$

By utilizing the properties of the norm, we get

$$||\Xi_{n,m}(u)|| \leq \frac{|\lambda|}{|\omega|}\left\|\sum_{i=0}^{n}\mu_i\xi_i\int_0^1 k(u,v)[\vartheta_i(v,\Psi_{i,m-1}(v)) - \vartheta_i(i,\Psi_{i,m-2}(v))]\mathrm{d}v\right\|.$$

Using (iv*), we have

$$||\Xi_{n,m}(u)|| \leq \frac{\varepsilon^*|\lambda|}{|\omega|}\left\|\sum_{i=0}^{n}\mu_i\xi_i\int_0^1 k(u,v)\Xi_{i,m-1}(v)\mathrm{d}v\right\|.$$

For conditions (i*) and (ii*), we have

$$||\Xi_{n,m}(u)|| \leq \frac{\beta\alpha^*\varepsilon^*|\lambda|}{|\omega|}\left\|\int_0^1 \Xi_{i,m-1}(v)\mathrm{d}v\right\|, \tag{5.2}$$

for $m = 1$ and using condition (iii*), we get from the last formula

$$\begin{aligned}||\Xi_{n,1}(u)|| &\leq \frac{\beta\alpha^*\varepsilon^*|\lambda|}{|\omega|}\left\|\int_0^1 \Xi_{i,0}(v)\mathrm{d}v\right\| \\ &\leq \frac{\beta\alpha^*\varepsilon^*|\lambda|}{|\omega|}\chi^*,\end{aligned}$$

and by induction, we get

$$||\Xi_{n,m}(u)|| \leq (\Theta_n)^m\chi, \qquad \Theta_n = \frac{\beta\alpha^*\varepsilon^*|\lambda|}{|\omega|} < 1, \quad n = 0,1,2,\ldots,N. \tag{5.3}$$

The result of inequality (5.3) shows that the sequence of the system of NFIEs (4.5) is uniformly convergent and the system has a unique solution when $m \to \infty$. □

6. Modified Taylor's method

We construct the Taylor expansion approach in this section to arrive at the numerical solution of Eq (4.5) and the method depends on differentiating both sides of (4.5) rth times. Then, we replace the Taylor polynomial for the unknown function in the resulting equation and after convert to NAS. The existence and uniqueness of the solution of the NAS are discussed, and next the solution of the system will be acquired.

Assume the solution of (4.5) takes the form:

$$\Psi_n(u) = \sum_{r=0}^{M} \frac{1}{r!}\Psi_n^{(r)}(a)(u-a)^{(r)}; \quad 0 \le u, a \le 1, \tag{6.1}$$

which is a Taylor polynomial of degree M at $u = a$, where $\Psi_n^{(r)}(a),\ r = 0, 1, \ldots, M$ are coefficients that need to be determined.

To get the solution of (4.5) in the expression form (6.1), we first differentiate both sides of (4.5), rth times with respect to u, to obtain:

$$\begin{aligned} \omega\Psi_n^{(r)}(u) &= \gamma_n^{(r)}(u) + \lambda \sum_{i=0}^{n} \mu_i \xi_i \int_0^1 \frac{\partial^r k(u,v)}{\partial u^r} G_i(v)\mathrm{d}v, \\ \gamma_n^{(r)}(u) &= f_n^{(r)}(u) + \omega\phi^{(r)}(u) - f^{(r)}(u,0), \quad G_i(v) = \vartheta_i(v, \Psi_i(v)). \end{aligned} \tag{6.2}$$

We put $u = a$ in relation (6.2), and then replace the Taylor expansions of $G_i(v)$ at $v = a$, i.e.,

$$G_i(v) = \sum_{j=0}^{\infty} \frac{1}{j!} G_i^{(j)}(a)(v-a)^{(j)},$$

in the resulting relation. The result is

$$\begin{aligned} \omega\Psi_n^{(r)}(a) &= \gamma_n^{(r)}\ (a) + \lambda \sum_{i=0}^{n} \mu_i \xi_i \int_0^1 \left.\frac{\partial^r k(u,v)}{\partial u^r}\right|_{u=a} \left[\sum_{j=0}^{\infty} \frac{1}{j!} G_i^{(j)}(a)(v-a)^{(j)}\right]\mathrm{d}v, \\ \gamma_n^{(r)}(a) &= f_n^{(r)}(a) + \omega\phi^{(r)}(a) - f^{(r)}(a,0). \end{aligned}$$

Or briefly

$$\omega\Psi_n^{(r)}(a) = \gamma_n^{(r)}(a) + \lambda \sum_{i=0}^{n} \sum_{j=0}^{\infty} \mu_i \xi_i k_{r,j} G_i^{(j)}(a), \tag{6.3}$$

where

$$k_{r,j} = \frac{1}{j!} \int_0^1 \left.\frac{\partial^r k(u,v)}{\partial u^r}\right|_{u=a} (v-a)^{(j)}\mathrm{d}v.$$

The quantities $G_i^{(j)}(a)\ \ (i = 0, 1, \ldots, n;\ j = 0, 1, 2, \ldots)$ in Eq (6.3) can be found from the permutation relation

$$G_i^{(j)}(a) = \sum_{\substack{s_1+2s_2+\ldots+\iota s_\iota = j \\ s_1+s_2+\ldots+s_\iota = \Delta}} \binom{j}{s_1 s_2 \ldots s_\iota} [G_i(a)]^{(\Delta)} \left(\frac{\Psi_i'(a)}{1!}\right)^{s_1} \left(\frac{\Psi_i''(a)}{2!}\right)^{s_2} \cdots \left(\frac{\Psi_i^{(\iota)}(a)}{\iota!}\right)^{s_\iota}, \tag{6.4}$$

where

$$\binom{j}{s_1 s_2 \ldots s_\iota} = \frac{j!}{s_1! s_2! \ldots s_\iota!}$$

and $s_1, s_2, \ldots, s_\iota$ are positive integers and zero.

Note that the generalized Leibniz rule can be used to get the relation (6.4).

$$\begin{aligned} G_i^{(j)}(a) &= [\vartheta_i(a, \Psi_i(a))]^{(j)} \\ &= \sum_{\substack{s_1+2s_2+\ldots+\iota s_\iota = j \\ s_1+s_2+\ldots+s_\iota=\Delta}} \binom{j}{s_1 s_2 \ldots s_\iota} [\vartheta_i(a, \Psi_i(a))]^{(\Delta)} \left(\frac{\Psi_i'(a)}{1!}\right)^{s_1} \left(\frac{\Psi_i''(a)}{2!}\right)^{s_2} \cdots \left(\frac{\Psi_i^{(\iota)}(a)}{\iota!}\right)^{s_\iota}, \end{aligned}$$

If we take $r, j = 0, 1, \ldots, M$, then Eq (6.3) becomes

$$\omega \Psi_n^{(r)}(a) = \gamma_n^{(r)}(a) + \lambda \sum_{i=0}^{n} \sum_{j=0}^{M} \mu_i \xi_i k_{r,j} G_i^{(j)}(a), \tag{6.5}$$

which is an algebraic system of $M + 1$ nonlinear equations for the $M + 1$ unknowns $\Psi_n^{(0)}(a), \Psi_n^{(1)}(a), \ldots, \Psi_n^{(M)}(a)$. Standard techniques can be used to solve these problems numerically.

A system of nonlinear algebraic equations is represented by Eq (6.5), and it has the following form

$$\omega A_r - \lambda K_r A^* = B_r, \tag{6.6}$$

where A_r, B_r and $K_r A^*$ are matrices defined by

$$A_r = \begin{pmatrix} \Psi_n^{(0)}(a) \\ \Psi_n^{(1)}(a) \\ . \\ . \\ . \\ \Psi_n^{(M)}(a) \end{pmatrix}, \quad B_r = \begin{pmatrix} \gamma_n^{(0)}(a) \\ \gamma_n^{(1)}(a) \\ . \\ . \\ . \\ \gamma_n^{(M)}(a) \end{pmatrix},$$

and

$$K_r A^* = \begin{pmatrix} \sum_{i=0}^n \mu_i \xi_i k_{0,0} G_i^{(0)}(a) & \sum_{i=0}^n \mu_i \xi_i k_{0,1} G_i^{(1)}(a) & \cdots & \sum_{i=0}^n \mu_i \xi_i k_{0,M} G_i^{(M)}(a) \\ \sum_{i=0}^n \mu_i \xi_i k_{1,0} G_i^{(0)}(a) & \sum_{i=0}^n \mu_i \xi_i k_{1,1} G_i^{(1)}(a) & \cdots & \sum_{i=0}^n \mu_i \xi_i k_{1,M} G_i^{(M)}(a) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \sum_{i=0}^n \mu_i \xi_i k_{M,0} G_i^{(0)}(a) & \sum_{i=0}^n \mu_i \xi_i k_{M,1} G_i^{(1)}(a) & \cdots & \sum_{i=0}^n \mu_i \xi_i k_{M,M} G_i^{(M)}(a) \end{pmatrix}.$$

On the other hand, we can represent the formula (6.6) as

$$\begin{pmatrix} \omega \Psi_n^{(0)}(a) - \lambda \sum_{i=0}^n \mu_i \xi_i k_{0,0} G_i^{(0)}(a) - \lambda \sum_{i=0}^n \mu_i \xi_i k_{0,1} G_i^{(1)}(a) - \cdots - \lambda \sum_{i=0}^n \mu_i \xi_i k_{0,M} G_i^{(M)}(a) \\ \omega \Psi_n^{(1)}(a) - \lambda \sum_{i=0}^n \mu_i \xi_i k_{1,0} G_i^{(0)}(a) - \lambda \sum_{i=0}^n \mu_i \xi_i k_{1,1} G_i^{(1)}(a) - \cdots - \lambda \sum_{i=0}^n \mu_i \xi_i k_{1,M} G_i^{(M)}(a) \\ . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \\ \omega \Psi_n^{(M)}(a) - \lambda \sum_{i=0}^n \mu_i \xi_i k_{M,0} G_i^{(0)}(a) - \lambda \sum_{i=0}^n \mu_i \xi_i k_{M,1} G_i^{(1)}(a) - \cdots - \lambda \sum_{i=0}^n \mu_i \xi_i k_{M,M} G_i^{(M)}(a) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_n^{(0)}(a) \\ \gamma_n^{(1)}(a) \\ . \\ . \\ . \\ \gamma_n^{(M)}(a) \end{pmatrix}.$$

From this nonlinear system, the unknown Taylor coefficients $\Psi_n^{(r)}(a)$ ($r = 0, 1, \ldots, M$) are determined and replaced in (6.1); thus we find the Taylor polynomial solution

$$\Psi_n(u) \cong \sum_{r=0}^{M} \frac{1}{r!} \Psi_n^{(r)}(a)(u-a)^{(r)}.$$

7. The existence of a unique solution of the system of nonlinear algebraic equations

Here in this section, under some conditions, we will give proof of the existence of the unique solution of the NAS of Eq (6.5) and get the truncation error of the numerical solution. The following theorems will help to achieve these aims:

Theorem 7.1. *Under the following conditions:*

(1) The kernel of position* $\left(\sum_{r=0}^{M} \sum_{j=0}^{M} |k_{r,j}|^2\right)^{\frac{1}{2}} \leq \beta^{**}$.

(2)* $\left(\sum_{i=0}^{n} |\mu_i \xi_i|^2\right)^{\frac{1}{2}} \leq \alpha^{**}$.

(3)* $\left(\sum_{r=0}^{M} |\gamma_n^{(r)}(a)|^2\right)^{\frac{1}{2}} \leq \chi^{**}$.

(4) The known function* $\vartheta_i^{(j)}(a, \Psi_i(a))$*, for the constants* $\upsilon^{**} > \delta^{**}$ *and* $\upsilon^{**} > \varepsilon^{**}$*, satisfies:*

$$(a) \quad \left(\sum_{i=0}^{n} \sum_{j=0}^{M} |\vartheta_i^{(j)}(a, \Psi_i(a))|^2\right)^{\frac{1}{2}} \leq \delta^{**} \left(\sum_{i=0}^{n} \sum_{j=0}^{M} |\Psi_i^{(j)}(a)|^2\right)^{\frac{1}{2}}$$

$$(b) \quad \left(\sum_{i=0}^{n} \sum_{j=0}^{M} |\vartheta_i^{(j)}(a, \Psi_{i,1}(a)) - \vartheta_i^{(j)}(a, \Psi_{i,2}(a))|^2\right)^{\frac{1}{2}} \leq \varepsilon^{**} \left(\sum_{i=0}^{n} \sum_{j=0}^{M} |\Psi_{i,1}^{(j)}(a) - \Psi_{i,2}^{(j)}(a)|^2\right)^{\frac{1}{2}}.$$

The NAS of Eq (6.5) has a unique solution.

Proof. We express the NAS (6.5) in the following operator form to prove the theorem:

$$\overline{L}\Psi_n^{(r)}(a) = \frac{1}{\omega}\gamma_n^{(r)}(a) + \frac{\lambda}{\omega} \sum_{i=0}^{n} \sum_{j=0}^{M} \mu_i \xi_i k_{r,j} \vartheta_i^{(j)}(a, \Psi_i(a)). \tag{7.1}$$

Lemma 7.1. *Under the conditions (1*)–(4*-a), the operator* $\overline{L}$ *defined by (7.1) maps the space* ℓ_2 *into itself.*

Proof. From (7.1), we get:

$$|\overline{L}\Psi_n^{(r)}(a)|^2 \leq \left[\frac{1}{|\omega|}|\gamma_n^{(r)}(a)| + \frac{|\lambda|}{|\omega|} \sum_{i=0}^{n} \sum_{j=0}^{M} |\mu_i \xi_i||k_{r,j}||\vartheta_i^{(j)}(a, \Psi_i(a))|\right]^2.$$

Using the Cauchy-Schwarz inequality, then from the conditions (4*-a), and summing from $r = 0$ to $r = M$, we obtain:

$$\left(\sum_{r=0}^{M} |\overline{L}\Psi_n^{(r)}(a)|^2\right)^{\frac{1}{2}} \leq$$
$$\left\{\sum_{r=0}^{M}\left[\frac{1}{|\omega|}|\gamma_n^{(r)}(a)| + \frac{|\lambda|}{|\omega|}\upsilon^{**}\left(\sum_{i=0}^{n}|\mu_i\xi_i|^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{M}|k_{r,j}|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{n}\sum_{j=0}^{M}|\Psi_i^{(j)}(a)|^2\right)^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}}.$$

After applying conditions (1*)–(3*) and allowing $N \to \infty$, the above formula has the following form:

$$\begin{aligned} \|\overline{L}\Psi_n\|_{\ell_2} &\leq \frac{1}{|\omega|}\chi^{**} + \frac{\beta^{**}\alpha^{**}\upsilon^{**}|\lambda|}{|\omega|}\|\Psi_n\|_{\ell_2} \\ &\leq \frac{1}{|\omega|}\chi^{**} + \Theta^{**}\|\Psi_n\|_{\ell_2}; \quad \Theta^{**} = \frac{\beta^{**}\alpha^{**}\upsilon^{**}|\lambda|}{|\omega|}. \end{aligned} \tag{7.2}$$

In view of inequality (7.2), the operator $\overline{L}$ maps into itself. □

Lemma 7.2. *Under the conditions (1*)–(4*-b), $\overline{L}$ defined by (7.1) is a contraction operator in the space ℓ_2.*

Proof. In light of formula (7.1), if $\left\{\Psi_{n,1}^{(r)}(a)\right\}$ and $\left\{\Psi_{n,2}^{(r)}(a)\right\}$ are any functions in the space ℓ_2, we get:

$$|\overline{L}\Psi_{n,1}^{(r)}(a) - \overline{L}\Psi_{n,2}^{(r)}(a)|^2 \leq \left[\frac{|\lambda|}{|\omega|}\sum_{i=0}^{n}\sum_{j=0}^{M}|\mu_i\xi_i||k_{r,j}||\vartheta_i^{(j)}(a,\Psi_{i,1}(a)) - \vartheta_i^{(j)}(a,\Psi_{i,2}(a))|\right]^2.$$

From the Cauchy-Schwarz inequality, then summing from $r = 0$ to $r = M$, and utilizing the conditions (1*), (2*) and (4*-b), the above inequality takes the form:

$$\left(\sum_{r=0}^{M}|\overline{L}\Psi_{n,1}^{(r)}(a) - \overline{L}\Psi_{n,2}^{(r)}(a)|^2\right)^{\frac{1}{2}} \leq \Theta^{**}\left(\sum_{i=0}^{n}\sum_{j=0}^{M}|\vartheta_i^{(j)}(a,\Psi_{i,1}(a)) - \vartheta_i^{(j)}(a,\Psi_{i,2}(a))|^2\right)^{\frac{1}{2}}.$$

The last inequality as $N \to \infty$ becomes

$$\|\overline{L}\Psi_{n,1} - \overline{L}\Psi_{n,2}\|_{\ell_2} \leq \Theta^{**}\|\Psi_{n,1} - \Psi_{n,2}\|_{\ell_2}. \tag{7.3}$$

Under the condition $\Theta^{**} < 1$, if inequality (7.3) shows the continuity of the operator $\overline{L}$ in the space ℓ_2, then $\overline{L}$ is a contraction operator. Hence, by Banach fixed point theorem $\overline{L}$ has a unique fixed point which is the unique solution of the system of NAS (6.5). □

It is obvious that, as $N \to \infty$, the NAS of (6.5) is equivalent to the nonlinear Volterra–Fredholm integral equation (1.2), and consequently the solution is the same.

8. The stability of the modified error

Studying the resulting error is of great importance in developing the programs used as well as the method used, in addition to the degree of approximation required. Therefore, the comparison of one method over another comes by the amount of convergent acceleration between the two methods. Hence, in this section of the paper, we will be interested in studying the error resulting from the approximation.

Assume the approximate solution takes the form

$$\omega\Psi_n(u,t) = \gamma_n(u,t) + \lambda \int_0^t \int_0^1 \xi(\tau)k(u,v)\vartheta(\tau,v,\Psi_n(v,\tau))\mathrm{d}v\mathrm{d}\tau.$$

Hence, we get the error in the form

$$\omega[\Psi(u,t) - \Psi_n(u,t)] = [\gamma(u,t) - \gamma_n(u,t)] + \lambda \int_0^t \int_0^1 \xi(\tau)k(u,v)[\vartheta(\tau,v,\Psi(v,\tau)) - \vartheta(\tau,v,\Psi_n(v,\tau))]\mathrm{d}v\mathrm{d}\tau. \tag{8.1}$$

The above Eq (8.1) takes the form

$$\omega R_n(u,t) = F_n(u,t) + \lambda \int_0^t \int_0^1 \xi(\tau)k(u,v)\vartheta_{error}(\tau,v,\Psi(v,\tau)\mathrm{d}v\mathrm{d}\tau, \tag{8.2}$$

where
$R_n(u,t) = [\Psi(u,t) - \Psi_n(u,t)]$, $F_n(u,t) = \gamma(u,t) - \gamma_n(u,t)$, $\vartheta_{error}(\tau,v,\Psi(v,\tau)) = [\vartheta(\tau,v,\Psi(v,\tau)) - \vartheta(\tau,v,\Psi_n(v,\tau))]$.

From Eq (8.2), we deduce that the modified error represents NVFIE of the second kind.

Theorem 8.1. *Under the same corresponding conditions of Section 2, the modified error (8.2) is stable in the space* $L_2[0,1] \times C[0,T]$.

Proof. Since

$$|\omega| \|R_n(u,t)\| \le \|F_n(u,t)\| + |\lambda| \left\| \int_0^t \int_0^1 |\xi(\tau)||k(u,v)|\vartheta_{error}(\tau,v,\Psi(v,\tau)|\mathrm{d}v\mathrm{d}\tau \right\|,$$

by using the conditions of Section 2, we have

$$\begin{aligned} \|R_n(u,t)\| &\le \|F_n(u,t)\| + \frac{\beta\alpha\upsilon|\lambda|}{|\omega|} T \|\Psi(u,t) - \Psi_n(u,t)\| \\ &\le \|F_n(u,t)\| + \Theta \|\Psi(u,t) - \Psi_n(u,t)\|; \quad \Theta < 1. \end{aligned}$$

As shown by the inequality above, if $n \to \infty$, then $F_n(u,t)$, $R_n \to 0$. □

Theorem 8.2. *The representation of the modified error (8.2) is unique.*

Proof. Assume that there are two different forms to describe the modified error

$$\omega R_n(u,t) - \omega R_m(u,t) = [F_n(u,t) - F_m(u,t)] + \lambda \int_0^t \int_0^1 \xi(\tau)k(u,v)[\vartheta(\tau,v,\Psi(v,\tau)) - \vartheta(\tau,v,\Psi_n(v,\tau))]\mathrm{d}v\mathrm{d}\tau$$

$$-\lambda\int_0^t\int_0^1 \xi(\tau)k(u,v)[\vartheta(\tau,v,\Psi(v,\tau))-\vartheta(\tau,v,\Psi_m(v,\tau))]\mathrm{d}v\mathrm{d}\tau.$$

Then, we have

$$\begin{aligned}\|R_n(u,t)-R_m(u,t)\| &\le \|F_n(u,t)-F_m(u,t)\|+\frac{\beta\alpha\upsilon|\lambda|}{|\omega|}T\|\Psi_n(u,t)-\Psi_m(u,t)\|\\ &\le \|F_n(u,t)-F_m(u,t)\|+\Theta\|\Psi_n(u,t)-\Psi_m(u,t)\|;\quad \Theta<1.\end{aligned}$$

In the above inequality, if $n\to m$, then $\{(F_n(u,t)-F_m(u,t))\}$, $\{(\Psi_n(u,t)-\Psi_m(u,t))\}\to 0 \Leftrightarrow \{(R_n-R_m)\}\to 0$. □

9. Numerical results

The method of this study is useful in finding the solution of the NVFIE in terms of the modified Taylor's method. We provide the following examples to demonstrate it. All computations are performed using Wolfram Mathematica 11.

Example 9.1. *Consider the following partial integro-differential equation with symmetric kernel:*

$$\begin{aligned}5\frac{\partial}{\partial t}\Psi(u,t) &= \frac{\partial}{\partial t}f(u,t)+0.3t^2\int_0^1(u-v)^2[\Psi(v,t)]^2\mathrm{d}v,\\ \Psi(u,0) &= u^2.\end{aligned}\tag{9.1}$$

where the function $f(u,t)$ is specified by laying $\Psi(u,t)=u^2e^{-t}$ as an exact solution.

$$\begin{aligned}f(u,t)=&5u^2e^{-t}+e^{-2t}(0.0107143-0.025u+0.015u^2+t(0.0214286-0.05u+0.03u^2)\\ &+t^2(0.0214286-0.05u+0.03u^2)).\end{aligned}$$

Integrating Equation (9.1), we obtained NVFIE of the second kind,

$$\begin{aligned}5\Psi(u,t) &= \gamma(u,t)+0.3\int_0^t\int_0^1\tau^2(u-v)^2[\Psi(v,\tau))]^2\mathrm{d}v\mathrm{d}\tau,\\ \gamma(u,t) &= f(u,t)+5u^2-f(u,0),\end{aligned}\tag{9.2}$$

and approximate the solution $\Psi_n(u)$ by the Taylor polynomial at $a=0$

$$\Psi_n(u)=\sum_{r=0}^{5}\frac{1}{r!}\Psi_n^{(r)}(0)(u)^{(r)};\quad 0\le u\le 1.$$

In order to apply the modified Taylor technique of integral problem (9.2), we do the following steps. First, we find the coefficients $k_{r,j}$ $(r,j=0,1,\dots,5)$, and after that we obtain the derived values of the function $\gamma(u,t)$ at $a=0$.

In Table 1, for $u\in[0,1]$, $t\in[0,0.6]$, the numerical computational results of the approximate and exact solution of (9.2) are computed for $M=5$. The maximum absolute errors of the proposed technique are presented in Table 2.

Table 1. Comparison between the exact and the approximate solution for Example 9.1 at $M = 5$.

u	$t = 0, M = 5$	$t = 0.2, M = 5$	$t = 0.4, M = 5$	$t = 0.6, M = 5$
0.0	3.58741×10^{-7}	3.65284×10^{-7}	1.85235×10^{-6}	5.36214×10^{-6}
0.1	6.51784×10^{-7}	7.74528×10^{-7}	3.25874×10^{-6}	9.45698×10^{-6}
0.2	8.02587×10^{-7}	8.52587×10^{-7}	5.74136×10^{-6}	3.14721×10^{-5}
0.3	8.69857×10^{-7}	9.36521×10^{-7}	8.02974×10^{-6}	3.99999×10^{-5}
0.4	9.41875×10^{-7}	9.55647×10^{-6}	1.89789×10^{-5}	4.10257×10^{-5}
0.5	5.13254×10^{-6}	9.79854×10^{-6}	5.74102×10^{-5}	7.96321×10^{-5}
0.6	5.36951×10^{-6}	9.97412×10^{-6}	6.32054×10^{-5}	3.69852×10^{-4}
0.7	6.85272×10^{-5}	6.99998×10^{-5}	8.01111×10^{-5}	5.36214×10^{-4}
0.8	7.69852×10^{-5}	7.85796×10^{-5}	8.69852×10^{-5}	6.74123×10^{-4}
0.9	9.02588×10^{-5}	9.25841×10^{-5}	9.56874×10^{-5}	7.36985×10^{-4}
1	9.65812×10^{-5}	9.85204×10^{-5}	7.85668×10^{-4}	9.10258×10^{-4}

Table 2. Maximum errors for different values of $t = 0, 0.2, 0.4, 0.6$ and $M = 5$ for Eq (9.2).

	$t = 0, M = 5$	$t = 0.2, M = 5$	$t = 0.4, M = 5$	$t = 0.6, M = 5$
Maximum errors	9.65812×10^{-5}	9.85204×10^{-5}	7.85668×10^{-4}	9.10258×10^{-4}

In Figures 1–4, with various values of t, u and $M = 5$, we calculated the absolute error function.

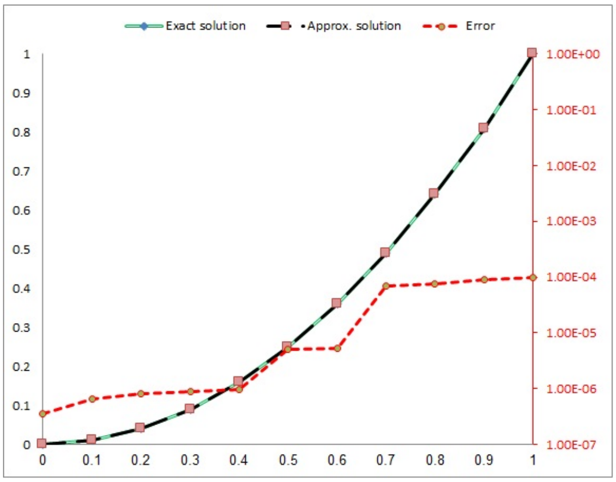

Figure 1. Approximate, exact solution and absolute error for $M = 5,\ t = 0$.

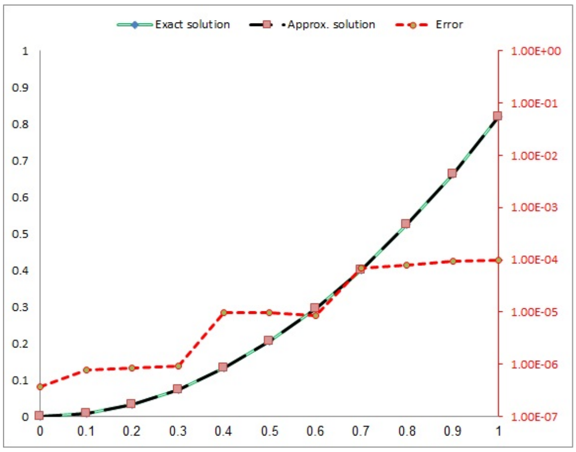

Figure 2. Approximate, exact solution and absolute error for $M = 5,\ t = 0.2$.

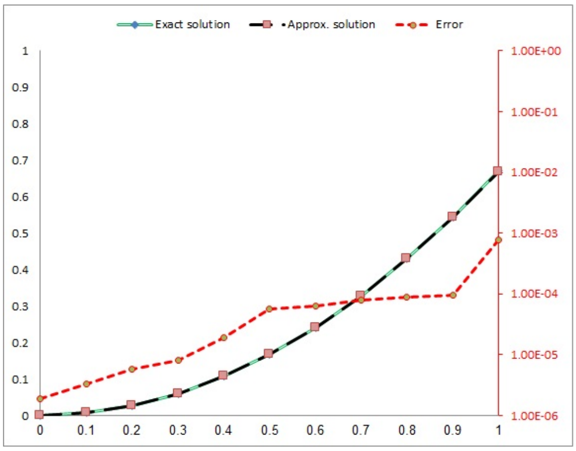

Figure 3. Approximate, exact solution and absolute error for $M = 5,\ t = 0.4$.

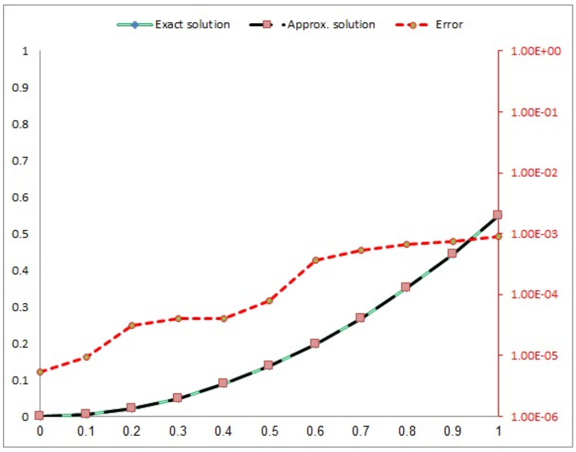

Figure 4. Approximate, exact solution and absolute error for $M = 5,\ t = 0.6$.

Example 9.2. *[6] In Eq (1.2), take* $\omega = 1,\ \lambda = 1,\ \xi(\tau) = \tau^2,\ k(u, v) = v$ *and* $\vartheta(t, v, \Psi(v, t)) = \Psi^2(v, t)$. *When* $\Psi(u, t) = u^2t^2$, *then the given function is* $\gamma(u, t) = -(t^7/42) + t^2u^2$.

In order to determine the approximate solutions, we apply the technique provided in this study for the cases where $t = 0.8,\ M = 8$ *and* $t = 0.8,\ M = 15$. *In Table 3, for* $u = [0, 1],\ t = 0.8$, *the numerical computational results of the approximate and exact solution of our method and the method in [6] are calculated. The maximum absolute errors of the approach used in our paper and [6] are displayed in Table 4.*

Table 3. Comparison between the approximate and the exact solution at $t = 0.8;\ M = 8, 15$.

	$t = 0.8, M = 8$		$t = 0.8, M = 15$	
u	Error of our method	Error of [6]	Error of our method	Error of [6]
0.0	1.05471×10^{-8}	0.114×10^{-7}	0.23584×10^{-10}	0.254×10^{-9}
0.2	4.36215×10^{-8}	0.532×10^{-7}	1.32548×10^{-9}	0.369×10^{-8}
0.4	7.36204×10^{-8}	0.542×10^{-6}	3.02512×10^{-9}	0.521×10^{-8}
0.6	2.32014×10^{-7}	0.856×10^{-6}	6.74124×10^{-9}	0.852×10^{-8}
0.8	5.32147×10^{-7}	0.999×10^{-6}	5.20147×10^{-8}	0.741×10^{-7}
1.0	1.21478×10^{-6}	0.216×10^{-5}	8.95647×10^{-8}	0.902×10^{-7}

Table 4. Maximum errors for different values of M at $t = 0.8$.

	$t = 0.8, M = 8$	$t = 0.8, M = 15$
Maximum errors of our method	6.21478×10^{-6}	8.95647×10^{-8}
Maximum errors of [6]	0.216×10^{-5}	0.902×10^{-7}

In Figures 5–8, with different values of t, u *and* $M = 8,\ 15$, *we calculated the absolute error function.*

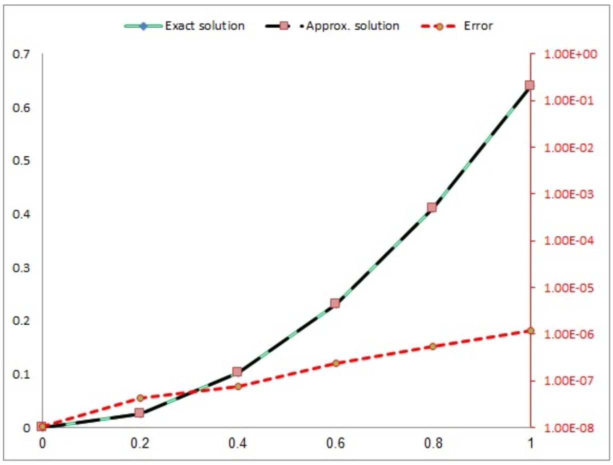

Figure 5. Absolute error, approximate and exact solution for $M = 8$ of our method.

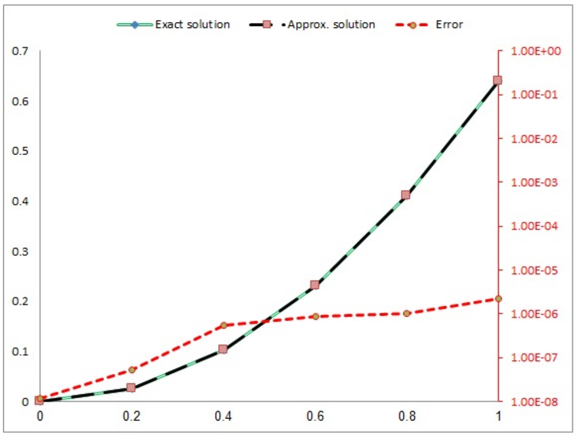

Figure 6. Absolute error, approximate and exact solution for $M = 8$ of [6].

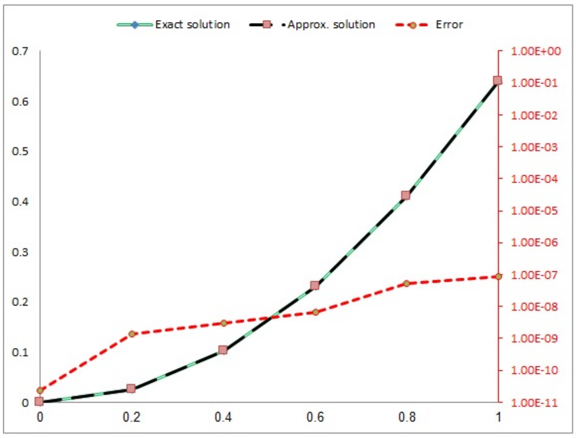

Figure 7. Absolute error, approximate and exact solution for $M = 15$ of our method.

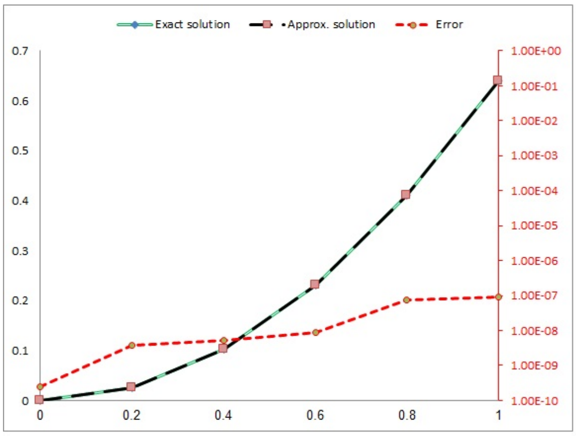

Figure 8. Absolute error, approximate and exact solution for $M = 15$ of [6].

Example 9.3. *Consider the following partial integro-differential equation with continuous kernel of the second type:*

$$\begin{aligned} 9\frac{\partial}{\partial t}\Psi(u,t) &= \frac{\partial}{\partial t}f(u,t) + 4\frac{t^3}{5}\int_0^1 u^3 v[\Psi(v,t)]^3 \mathrm{d}v, \\ \Psi(u,0) &= u, \end{aligned} \tag{9.3}$$

where the function $f(u,t)$ is specified by laying $\Psi(u,t) = u^2 + t$ as an exact solution.

$$f(u,t) = 9t(1+u^2) - \frac{t^3u^3}{30} - \frac{t^4u^3}{10} - \frac{3t^5u^3}{25} - \frac{t^6u^3}{15}.$$

Integrating Eq (9.3), we obtained NVFIE with continuous kernel of the second kind,

$$\begin{aligned} 9\Psi(u,t) &= \gamma(u,t) + 4\int_0^t\int_0^1 \frac{\tau^2}{5}u^3v[\Psi(v,\tau))]^3\mathrm{d}v\mathrm{d}\tau, \\ \gamma(u,t) &= f(u,t) + 9u - f(u,0), \end{aligned} \tag{9.4}$$

and approximated the solution $\Psi_n(u)$ by the Taylor polynomial at $a = 1$

$$\Psi_n(u) = \sum_{r=0}^{10}\frac{1}{r!}\Psi_n^{(r)}(1)(u-1)^{(r)}; \quad 0 \le u \le 1.$$

In order to apply the modified Taylor technique of integral problem (9.4), we do the following steps. First, we find the coefficients $k_{r,j}$ $(r, j = 0, 1, \ldots, 10)$, and after that we obtain the derived values of the function $\gamma(u,t)$ at $a = 1$.

In Table 5, for $u \in [0,1],\ t \in [0,0.9]$, the numerical computational results of the approximate and exact solution of (9.4) are computed for $M = 10$. Table 6 shows the maximum absolute errors of the given method.

Table 5. Comparison between the exact and the approximate solution for Example 9.3 at $M = 10$.

u	$t = 0.3, M = 10$	$t = 0.5, M = 10$	$t = 0.7, M = 10$	$t = 0.9, M = 10$
0.0	9.25487×10^{-11}	2.36521×10^{-10}	2.36587×10^{-9}	4.62587×10^{-9}
0.1	1.36987×10^{-10}	5.36214×10^{-10}	4.32587×10^{-9}	6.47184×10^{-9}
0.2	3.98745×10^{-10}	3.25417×10^{-9}	7.32548×10^{-9}	9.02587×10^{-9}
0.3	4.39201×10^{-10}	5.32014×10^{-9}	8.21471×10^{-9}	5.36214×10^{-8}
0.4	1.87456×10^{-9}	6.21478×10^{-9}	9.36985×10^{-9}	7.00147×10^{-8}
0.5	2.85214×10^{-9}	7.25874×10^{-9}	9.96521×10^{-9}	8.36952×10^{-8}
0.6	5.74120×10^{-9}	1.25417×10^{-8}	9.02587×10^{-8}	9.32658×10^{-8}
0.7	5.99852×10^{-9}	3.65278×10^{-8}	9.32548×10^{-8}	4.21477×10^{-7}
0.8	8.96325×10^{-9}	6.32587×10^{-8}	9.63258×10^{-8}	7.36985×10^{-7}
0.9	9.89652×10^{-9}	7.00024×10^{-8}	9.78521×10^{-8}	8.69854×10^{-7}
1	2.83217×10^{-8}	9.32587×10^{-8}	4.62541×10^{-7}	9.00258×10^{-7}

Table 6. Maximum errors for different values of $t = 0.3, 0.5, 0.7, 0.9$ and $M = 10$ for Eq (9.4).

	$t = 0.3, M = 10$	$t = 0.5, M = 10$	$t = 0.7, M = 10$	$t = 0.9, M = 10$
Maximum errors	2.83217×10^{-8}	9.32587×10^{-8}	4.62541×10^{-7}	9.00258×10^{-7}

In Figures 9–12, we computed the absolute error function with different values of t, u *and* $M = 10$.

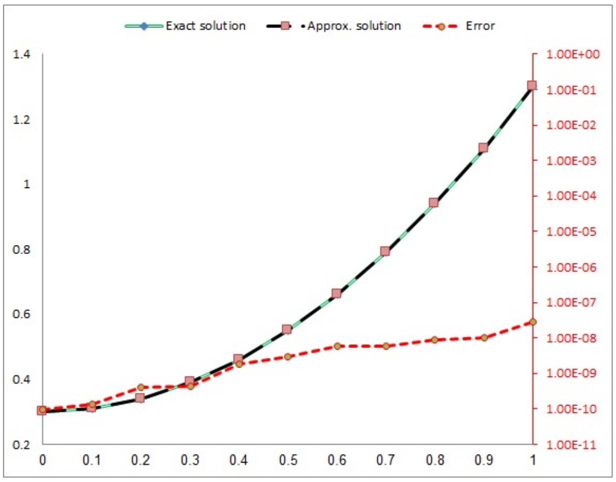

Figure 9. Approximate, exact solution and absolute error for $t = 0.3$, $M = 10$.

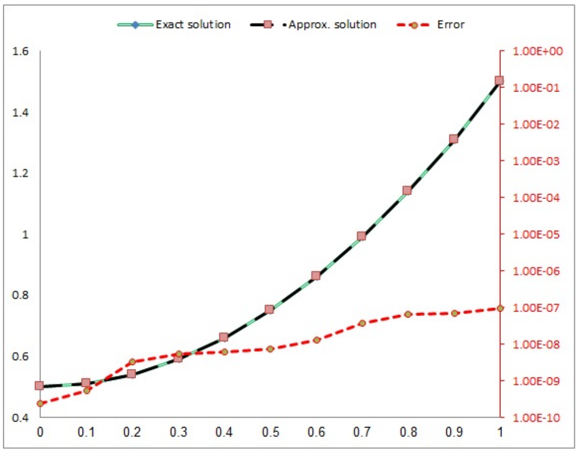

Figure 10. Approximate, exact solution and absolute error for $t = 0.5$, $M = 10$.

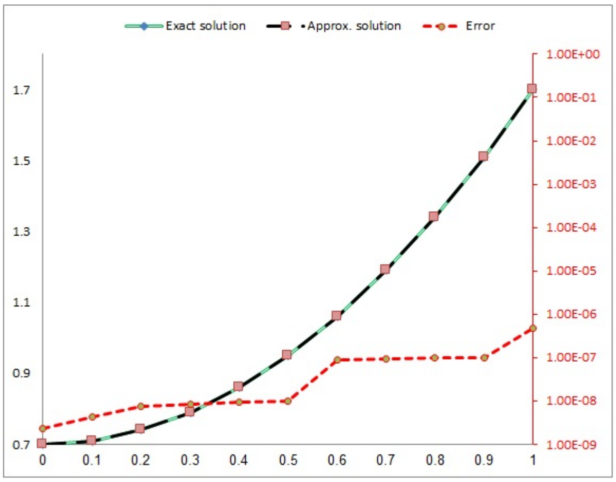

Figure 11. Approximate, exact solution and absolute error for $t = 0.7,\ M = 10$.

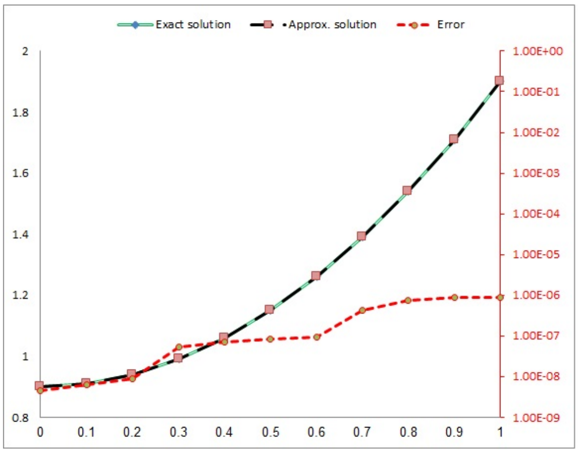

Figure 12. Approximate, exact solution and absolute error for $t = 0.9,\ M = 10$.

10. Conclusions

The tables above and our numerical results lead us to conclude the following:

1) The Nonlinear Volterra–Fredholm integral equation (1.2) has a unique solution $\Psi(u, t)$ in the Banach space $L_2[0, 1] \times C[0, T]$, under some conditions.
2) Since NVFIEs are usually difficult to solve analytically, it is required to obtain the approximate solutions.
3) The modified Taylor's method is considered as one of the best methods to obtain the solution of the NVFIE with continuous kernel, numerically. This is evident by comparison with other methods, see Example 9.2.

3.i) Our achieved results in this paper show that this method is effective and easy to implement.

3.ii) One of the advantages of this method is that the solution is expressed as a truncated Taylor series at $u = a$, then $\Psi_n(u)$ can be easily evaluated for arbitrary values of u at low-computation effort.

3.iii) The method proposed in this paper can be applied to a wide class of NVFIEs of the second kind with smooth and weakly singular kernels.
3.iv) There is a solution that is closer to the exact solution when the Taylor polynomial solution for the conditions that are given is searched for about the points (Example 9.3).
4) From Examples 9.1–9.3, we notice that we obtain an analytical solution in many cases and this is one of the interesting features of this method.
5) From Table 1 in Example 9.1, it is noticeable that the error is 3.58741×10^{-7} at the point $u = 0,\ t = 0$, but at the same point the error increase for $t = 0.6$ and becomes 9.45698×10^{-6}. Also, at the point $u = 1,\ t = 0$ the error is 9.65812×10^{-5} while for $t = 0.6$ at the same point the error becomes 9.10258×10^{-4}. This means that if the time increases, then the error is also increases. This has also been noted in the rest of the tables.
6) By comparing Figure 9 with Figure 10 in Example 9.3, at different times $t = 0.3,\ t = 0.5$ we find that the error is less when the time is smaller. See also Figures 11 and 12.
7) In Example 9.2, we consider an NVFIE with continuous kernel of the second kind, and a comparison was made between the modified Taylor's method and projection-iterated method which is used in [6] in Table 3, at the time $t = 0.8$, the following was noted at the point $u = 0,\ M = 8$: the error of our method is 1.05471×10^{-8} while the error of method used in [6], at the same value of $u,\ M$ is 0.114×10^{-7}. We notice a large difference in the error, and this difference is observed for all values of u in $[0, 1]$. This difference was also noticed in the error when $M = 15$. This shows that the modified Taylor's method is more accurate than the projection-iterated method. Also, the numerical results of Example 9.2 are presented in Figures 5–8.
8) All computations were carried out using the program Wolfram Mathematica 11.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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