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Research article

A successive midpoint method for nonlinear differential equations with classical and Caputo-Fabrizio derivatives

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Abstract: In this study, we present a numerical scheme for solving nonlinear ordinary differential equations with classical and Caputo–Fabrizio derivatives using consecutive interval division and the midpoint approach. By doing so, we increased the accuracy of the midpoint approach, which is dependent on the number of interval divisions. In the example of the Caputo–Fabrizio differential operator, we established the existence and uniqueness of the solution using the Caratheodory-Tonelli sequence. We solved numerous nonlinear equations and determined the global error to test the accuracy of the proposed scheme. When the differential equation met the circumstances under which it was generated, the results revealed that the procedure was quite accurate.

Keywords: midpoint method; Caratheodory-Tonelli sequence; Caputo–Fabrizio fractional derivative
Mathematics Subject Classification: 26A33, 34A12, 65L05

Nomenclature and Unit

α : The fractional order;
CF : The abbreviation of Caputo–Fabrizio;
${}_0D_t$: The differential operator;
${}_0^{CF}D_t^{\alpha}$: Caputo Fabrizio fractional derivative with fractional order α;
$(y_n)_{n\in\mathbb{N}}$: The sequence where $n \in \mathbb{N}$;
$\widehat{y}_{n+1}$: The predictor formula for the solution of y.
$A-B$: Two-step Adams-Bashforth method; MP : Midpoint method;
SM : Suggested method; PM : Parametrized method;
EOC : Experimental order of convergence; IVP : Initial value problem.

1. Introduction

Nonlinear differential equations appear in different fields of pure and applied mathematics. They are used to model processes having nonlinear behavior that are prevalent in nature, especially in applied mathematics. Even though this class of equations is useful, mathematicians are severely constrained because there are no analytical solutions to these equations. Since there is now a wider variety of numerical schemes available for solving nonlinear equations, researchers increasingly rely on numerical methods [1–10]. Numerical integration, which in analysis refers to a large family of procedures for determining the numerical value of definite integrals, is also occasionally used to refer to the numerical solution of differential equations. The calculation of definite integrals is the main topic of this exercise. The midway technique is a one-step approach for numerically solving nonlinear differential equations with initial values in numerical analysis [11–15]. Because of its simplicity and the fact that the local error at each step of the midpoint technique [2] is of order three and the global error is of order two, this method has been utilized on multiple occasions, outperforming the simple Euler approximation [7,8]. This formula was adapted in the framework of fractional ordinary differential equations with the change that, at the endpoint, the technique becomes implicit and therefore the idea of the predictor-corrector is utilized to get rid of such difficulties [1]. While the precision of the classical midpoint scheme should not be overlooked, there is room for improvement. When the number of divisions is two, the divided interval can be split again, and repeating this process n times results in a new numerical scheme that may be more precise than the classical version. In this paper, we will investigate the aforementioned approach in depth to evaluate if the method's accuracy can be enhanced primarily through consecutive division of intervals. We will also look at the initial value problems with classical and Caputo–Fabrizio differential operators [4].

2. Midpoint pocess

It is well known in geometry that the midpoint of the segment $(t_n, y(t_n))$ to $(t_{n+1}, y(t_{n+1}))$ is

$$\left(\frac{t_n + t_{n+1}}{2}, \frac{y(t_n) + y(t_{n+1})}{2}\right). \tag{2.1}$$

The above is equivalent to

$$\left(t_n + \frac{h}{2}, \frac{y(t_n) + y(t_{n+1})}{2}\right). \tag{2.2}$$

Applying the midpoint two times, we get the following middle points

$$\left(t_n + \frac{h}{4}, \frac{y(t_n) + y\left(t_n + \frac{h}{2}\right)}{2}\right), \left(t_n + \frac{3h}{4}, \frac{y\left(t_n + \frac{h}{2}\right) + y(t_{n+1})}{2}\right). \tag{2.3}$$

For three times, we get the following middle points

$$\left\{\begin{array}{c} \left(t_n + \frac{h}{8}, \frac{y(t_n)+y\left(t_n+\frac{h}{4}\right)}{2}\right), \left(t_n + \frac{3h}{8}, \frac{y\left(t_n+\frac{h}{4}\right)+y\left(t_n+\frac{h}{2}\right)}{2}\right), \\ \left(t_n + \frac{5h}{8}, \frac{y\left(t_n+\frac{h}{2}\right)+y\left(t_n+\frac{3h}{4}\right)}{2}\right), \left(t_n + \frac{7h}{8}, \frac{y\left(t_n+\frac{3h}{4}\right)+y(t_{n+1})}{2}\right) \end{array}\right\}. \tag{2.4}$$

If we apply the midpoint k times, we have the following

$$\left\{t_n + \frac{(2j+1)\,h}{2^k}, y\left(t_n + \frac{(2j+1)\,h}{2^k}\right)\right\}_{j=0,..,2k-1}, k \geq 1, \tag{2.5}$$

noting that we get the the midpoint while $j = 0$ and $k = 1$. The successive midpoint suggested here is a repetitive process where the midpoint of the endpoints is taken k times the midpoint.

3. Derivation of successive midpoint formula

In this section, we derive the successive midpoint idea to solve the Cauchy problem given by

$$\begin{cases} y\prime\,(t) = \psi\,(t, y\,(t))\,, \\ y\,(0) = y_0. \end{cases} \tag{3.1}$$

We shall apply the integral on both sides, thus, we obtain

$$y\,(t) = y\,(0) + \int_0^t \psi\,(\tau, y\,(\tau))\,d\tau. \tag{3.2}$$

If we write the above equation at the points $t = t_{n+1}$ and $t = t_n$ and take their difference, we get

$$y\,(t_{n+1}) = y\,(t_n) + \int_0^{t_{n+1}} \psi\,(\tau, y\,(\tau))\,d\tau. \tag{3.3}$$

Note that the midpoint between t_{n+1} and t_n is $t_n + \frac{h}{2}$. Then, the midpoint method [2] is derived as follows:

$$\begin{aligned} y_{n+1} &= y_n + \psi\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right)\int_{t_n}^{t_{n+1}} d\tau \qquad (3.4) \\ &= y_n + h\psi\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right), \end{aligned}$$

where

$$\psi\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) = \psi\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}\psi\,(t_n, y_n)\right). \tag{3.5}$$

Taking the midpoint again, we write

$$\begin{aligned} y\,(t_{n+1}) &= y\,(t_n) + \int_{t_n}^{t_{n+1}} \psi\,(\tau, y\,(\tau))\,d\tau \qquad (3.6) \\ &= y\,(t_n) + \int_{t_n}^{t_n+\frac{h}{2}} \psi\,(\tau, y\,(\tau))\,d\tau + \int_{t_n+\frac{h}{2}}^{t_{n+1}} \psi\,(\tau, y\,(\tau))\,d\tau. \end{aligned}$$

We divide the boundaries according to midpoint as follows:

$$y_{n+1} = y_n + \psi\left(t_n + \frac{h}{4}, y\left(t_n + \frac{h}{4}\right)\right)\int_{t_n}^{t_n+\frac{h}{2}} d\tau + \psi\left(t_n + \frac{3h}{4}, y\left(t_n + \frac{3h}{4}\right)\right)\int_{t_n+\frac{h}{2}}^{t_{n+1}} d\tau \tag{3.7}$$

$$= \quad y_n + \frac{h}{2}\psi\left(t_n + \frac{h}{4}, y\left(t_n + \frac{h}{4}\right)\right) + \frac{h}{2}\psi\left(t_n + \frac{3h}{4}, y\left(t_n + \frac{3h}{4}\right)\right).$$

If we apply the midpoint again, we get

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + \int_{t_n}^{t_{n+1}} \psi(\tau, y(\tau))\, d\tau \qquad (3.8)\\ &= y(t_n) + \int_{t_n}^{t_n+\frac{h}{4}} \psi(\tau, y(\tau))\, d\tau + \int_{t_n+\frac{h}{4}}^{t_n+\frac{h}{2}} \psi(\tau, y(\tau))\, d\tau \\ &\quad + \int_{t_n+\frac{h}{2}}^{t_n+\frac{3h}{4}} \psi(\tau, y(\tau))\, d\tau + \int_{t_n+\frac{3h}{4}}^{t_{n+1}} \psi(\tau, y(\tau))\, d\tau. \end{aligned}$$

Using the midpoint yields

$$\begin{aligned} y_{n+1} &= y_n + \psi\left(t_n + \frac{h}{8}, y\left(t_n + \frac{h}{8}\right)\right) \int_{t_n}^{t_n+\frac{h}{4}} d\tau + \psi\left(t_n + \frac{3h}{8}, y\left(t_n + \frac{3h}{8}\right)\right) \int_{t_n+\frac{h}{4}}^{t_n+\frac{h}{2}} d\tau \qquad (3.9)\\ &\quad + \psi\left(t_n + \frac{5h}{8}, y\left(t_n + \frac{5h}{8}\right)\right) \int_{t_n+\frac{h}{2}}^{t_n+\frac{3h}{4}} d\tau + \psi\left(t_n + \frac{7h}{8}, y\left(t_n + \frac{7h}{8}\right)\right) \int_{t_n+\frac{3h}{4}}^{t_{n+1}} d\tau, \end{aligned}$$

we calculate the above as

$$y_{n+1} = y_n + \frac{h}{4}\left[\begin{array}{c} \psi\left(t_n + \frac{h}{8}, y\left(t_n + \frac{h}{8}\right)\right) + \psi\left(t_n + \frac{3h}{8}, y\left(t_n + \frac{3h}{8}\right)\right) \\ +\psi\left(t_n + \frac{5h}{8}, y\left(t_n + \frac{5h}{8}\right)\right) + \psi\left(t_n + \frac{7h}{8}, y\left(t_n + \frac{7h}{8}\right)\right) \end{array}\right]. \qquad (3.10)$$

Doing the same procedure k times, we have the following successive midpoint formula

$$y_{n+1} = y_n + \frac{h}{2^{k-1}} \sum_{j=0}^{2^{k-1}-1} \psi\left(t_n + \frac{(2j+1)h}{2^k}, y\left(t_n + \frac{(2j+1)h}{2^k}\right)\right) \qquad (3.11)$$

where

$$y\left(t_n + \frac{(2j+1)h}{2^k}\right) = y_n + \frac{(2j+1)h}{2^k}\psi(t_n, y_n). \qquad (3.12)$$

Note that when $j = 0$ and $k = 1$, we obtain the midpoint method. We should, however, attract the attention of readers about the Romberg method [16]. We point out that Romberg's technique analyzes the integrand at evenly spaced places, making it a Newton-Cotes formula. Although only a few derivatives may exist, it is still possible to get acceptable results if the integrand has continuous derivatives. Readers who are interested are urged to read more about this technique in [16, 17].

3.1. Some illustrative examples

In this subsection, we aim to investigate numerical and exact solutions of some Cauchy problems for different k values and calculate the error for the solution of the problem. Moreover, we compare the proposed method with the midpoint (MP), two-step Adams-Bashforth (A-B) and parametrized method (PM) [18] to demonstrate the effectiveness of the method.

Example 1. We consider the following nonlinear equation:

$$\begin{aligned} y'(t) &= t^3 + 2t^2 + t + 1, \\ y(0) &= 1, \end{aligned} \tag{3.13}$$

where the exact solution is as follows:

$$y(t) = \frac{t^4}{4} + \frac{2t^3}{3} + \frac{t^2}{2} + t + 1. \tag{3.14}$$

The phase portrait of the slope-field for (3.13) is presented in Figure 1.

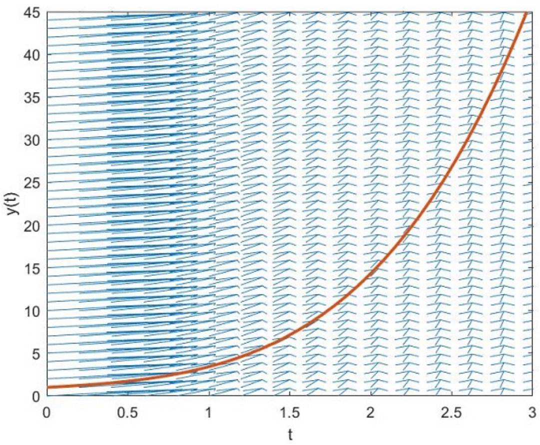

Figure 1. The phase portrait of the slope-field for (3.13) for $h = 0.1, k = 10$.

In Figure 2, we presented the comparison between the exact solution and the numerical solution obtained by the presented method for different k values.

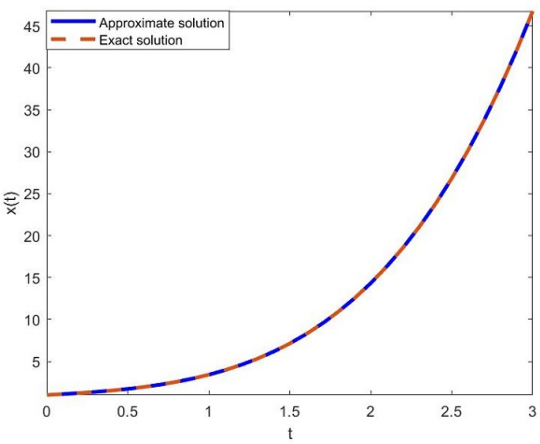

Figure 2. Numerical simulation for Cauchy problem for $h = 0.1, k = 10$.

In Table 1, we presented the error for the function $y(t)$ by employing the suggested method for different k values.

Table 1. Error of the function $y(t)$ for the suggested method.

h	The error for $k = 6$	The error for $k = 10$	The error for $k = 14$	The error for $k = 18$
0.1	$2.8483e-06$	$1.1126e-08$	$4.3461e-11$	$6.7901e-13$
0.05	$7.1208e-07$	$2.7816e-09$	$1.0866e-11$	$1.6875e-13$
0.01	$2.8483e-08$	$1.1126e-10$	$4.3654e-13$	$8.8818e-15$
0.005	$7.1208e-09$	$2.7817e-11$	$1.0969e-13$	$3.9968e-15$
0.001	$2.8483e-10$	$1.1169e-12$	$8.8818e-15$	$1.7764e-15$

In Table 2, we presented the comparison of the error for the function $y(t)$ between the suggested method for $k = 18$, MP, two-step A-B method and PM method.

Table 2. Error of the function $y(t)$ for SM, MP, A-B, PM.

h	Error of SM	Error of MP	Error of A–B	Error of PM
0.1	$6.7901e-13$	0.0163	0.0317	0.0336
0.05	$1.6875e-13$	0.0041	0.0082	0.0179
0.01	$8.8818e-15$	$1.6250e-04$	$3.3917e-04$	0.0038
0.005	$3.9968e-15$	$4.0625e-05$	$8.5104e-05$	0.0019
0.001	$1.7764e-15$	$1.6250e-06$	$3.4142e-06$	$3.7956e-04$

Example 2. We consider the following nonlinear equation:

$$\begin{aligned} y'(t) &= y^3 + \cos y - \sin^6 t + \sin 2t - \cos\left(\sin^2 t\right), \\ y(0) &= 0. \end{aligned} \tag{3.15}$$

The exact solution of the above problem is found as follows:

$$y(t) = \sin^2 t. \tag{3.16}$$

The phase portrait of the slope-field for (3.15) is presented in Figure 3.

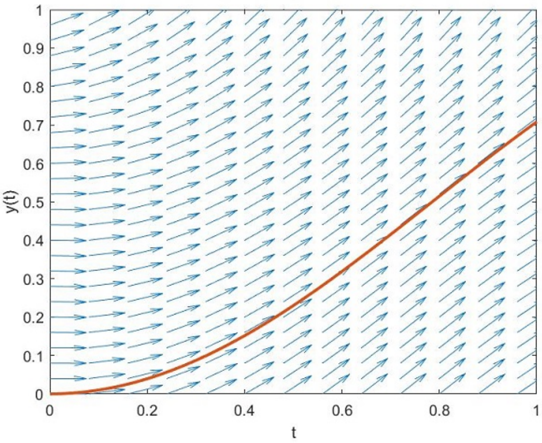

Figure 3. The phase portrait of the slope-field for (3.15).

The numerical simulation for the solution of the above problem is presented in Figure 4.

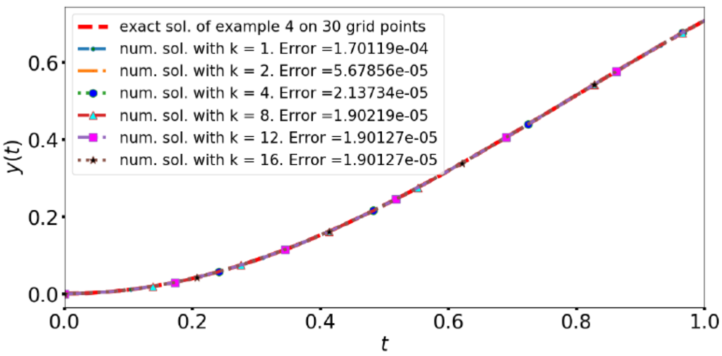

Figure 4. The numerical simulation for the solution of (3.15).

In Table 3, we presented the error for the function $y(t)$ by employing the suggested method for different k values.

Table 3. Error and EOC of the function $y(t)$ for the suggested method for $k = 1, 2, 8$, respectively.

h	Error for $k = 1$	EOC	Error for $k = 8$	EOC	Error for $k = 12$	EOC
0.02564	$9.427e-05$	-1	$1.063e-05$	-1	$1.062e-05$	-1
0.01266	$2.305e-05$	2.032	$2.623e-06$	2.018	$2.622e-06$	2.018
0.00629	$5.698e-06$	2.016	$6.513e-07$	2.01	$6.510e-07$	2.01
0.00313	$1.417e-06$	2.008	$1.622e-07$	2.005	$1.621e-07$	2.005
0.00156	$3.532e-07$	2.004	$4.048e-08$	2.003	$4.046e-08$	2.003

In Table 4, we presented the comparison of the error for the function $y(t)$ between the suggested method for $k = 25$, MP, PM and two-step A-B method.

Table 4. Error of the function $y(t)$ for some methods.

h	Error of SM	Error of MM	Error of A-B	Error of PM
0.02564	$1.062e-05$	$9.427e-05$	$6.5727e-04$	0.0011
0.01266	$2.622e-06$	$2.305e-05$	$1.6027e-04$	$4.2321e-04$
0.00629	$6.510e-07$	$5.698e-06$	$3.9564e-05$	$1.9816e-04$
0.00313	$1.621e-07$	$1.417e-06$	$9.7969e-06$	$9.6604e-05$
0.00156	$4.046e-08$	$3.532e-07$	$2.4336e-06$	$4.7380e-05$

4. Successive midpoint method for a general Cauchy problem with Caputo–Fabrizio derivative

To adapt the suggested scheme for the Caputo–Fabrizio case [4], we consider a general Cauchy problem which is given by

$$\begin{cases} {}_0^{CF}D_t^{\alpha} y(t) = \psi(t, y(t)), \\ y(0) = y_0. \end{cases} \tag{4.1}$$

We applied the integral on both side yields

$$\begin{cases} y(t) = y(0) + (1-\alpha)\psi(t, y(t)) + \alpha \int_0^t \psi(\tau, y(\tau))\, d\tau, \\ y(0) = y_0. \end{cases} \tag{4.2}$$

When $t = t_{n+1}$, we have

$$y(t_{n+1}) = y(0) + (1-\alpha)\psi(t_{n+1}, y(t_{n+1})) + \alpha \int_0^{t_{n+1}} \psi(\tau, y(\tau))\, d\tau \tag{4.3}$$

and at $t = t_n$, we have

$$y(t_n) = y(0) + (1-\alpha)\psi(t_n, y(t_n)) + \alpha \int_0^{t_n} \psi(\tau, y(\tau))\, d\tau. \tag{4.4}$$

Taking the difference of the last two equation yields,

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + (1-\alpha)(\psi(t_{n+1}, y(t_{n+1})) - \psi(t_n, y(t_n))) \\ &\quad + \alpha\left[\int_0^{t_{n+1}} \psi(\tau, y(\tau))\, d\tau - \int_0^{t_n} \psi(\tau, y(\tau))\, d\tau\right] \\ &= y(t_n) + (1-\alpha)(\psi(t_{n+1}, y(t_{n+1})) - \psi(t_n, y(t_n))) + \alpha \int_{t_n}^{t_{n+1}} \psi(\tau, y(\tau))\, d\tau. \end{aligned} \tag{4.5}$$

By applying the midpoint consecutively, the following numerical algorithm is obtained:

$$\begin{aligned} y_{n+1} &= y_n + (1-\alpha)(\psi(t_{n+1}, \widehat{y}_{n+1}) - \psi(t_n, y_n)) \\ &\quad + \frac{\alpha h}{2^{k-1}} \sum_{j=0}^{2^{k-1}-1} \psi\left(t_n + \frac{(2j+1)h}{2^k}, y\left(t_n + \frac{(2j+1)h}{2^k}\right)\right), \end{aligned} \tag{4.6}$$

where

$$\widehat{y}_{n+1} = y_n + h\psi(t_n, y_n) \tag{4.7}$$

and

$$y\left(t_n + \frac{(2j+1)h}{2^k}\right) = y_n + \frac{(2j+1)h}{2^k}\psi(t_n, y_n). \tag{4.8}$$

5. Theoretical analysis of the suggested method for Caputo–Fabrizio case

5.1. Existence and uniqueness of Caputo–Fabrizio IVP backward steps of Carathéodory-Tonelli

In this subsection, we will present the existence and the uniqueness of the IVP with the Caputo–Fabrizio derivative. To achieve this, we shall present the existence using the Tonelli sequence [19] and the framework of Carathéodory [5] and we will work within the interval $[0, 1]$.

$$\begin{cases} {}_{0}^{CF}D_t^{\alpha} y(t) = \psi(t, y(t),), & \text{if } t \in (0, 1], \\ y(0) = y_0, & \text{if } t = 0. \end{cases} \tag{5.1}$$

$\psi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The approach of Carathéodory and Tonelli to show the existence for the above equation will consist in the defined sequence $\forall n \in \mathbb{N}$, and we define the function $y_n(t)$ by

$$y_n(t) = \begin{cases} y_0, & \text{if } 0 \le t \le \frac{1}{n}, \\ y_0 + (1-\alpha)\psi(t_n, y_n) + \alpha\int_0^{t-\frac{1}{n}} \psi(\tau, y_n(\tau))\, d\tau, & \text{if } \frac{1}{n} \le t \le 1. \end{cases} \tag{5.2}$$

Indeed, all $n \ge 1$, when the computation of $y_n(t)$ is performed within $\left[\frac{j}{n}, \frac{j+1}{n}\right]$ with $(1 \le j \le n-1)$.

5.2. Sufficient conditions for convergence

Theorem 1. We shall adopt the method presented in [6]. The defined iteration (y_n) converges to a global solution y of our problem if

a) $f(t, y_0) = 0, \forall t \in [0, 1]$.

b) $f(t, y)$ increases with respect to y and $f(t, y) \ge 0$, for $\forall (t, y) \in [0, 1] \times \mathbb{R}$ such that $\{y_n\}_{n\in\mathbb{N}}$ converges uniformly to the lower integral of the problem.

c) $f(t, y)$ is increasing with respect to y and $f(t, y) < 0$, for $\forall (t, y) \in [0, 1] \times \mathbb{R}$ such that $\{y_n\}_{n\in\mathbb{N}}$ converges uniformly to the upper integral of the problem.

d) $(y_n)_{n\in\mathbb{N}}$ converges in a point to a value β such that (t_0, β) is not a Peano point for

$$y(t) = y(0) + (1-\alpha)\psi(t, y(t)) + \alpha\int_0^t \psi(\tau, y(\tau))\, d\tau. \tag{5.3}$$

Proof. Within the interval $[0, 1], 0 < 1 - \alpha, \alpha \le 1$, we have that by

a) Every $y_n(t)$ is a constant solution.

b) $\forall n \in \mathbb{N}$, and every $t \in \left[0, \frac{1}{n}\right]$

$$y_{n+1}(t) - y_n(t) = (1-\alpha)\{\psi(t, y_{n+1}) - \psi(t, y_n)\} + \alpha\int_0^{t-\frac{1}{n+1}} \psi(\tau, y_{n+1}(\tau))\, d\tau \tag{5.4}$$

$$
\begin{aligned}
& -\alpha \int_0^{t-\frac{1}{n}} \psi(\tau, y_n(\tau))\, d\tau \\
\geq\ & (1-\alpha)\{\psi(t, y_{n+1}) - \psi(t, y_n)\} + \alpha \int_0^{t-\frac{1}{n}} \left[\begin{array}{c} \psi(\tau, y_{n+1}(\tau)) \\ -\psi(\tau, y_n(\tau)) \end{array} \right] d\tau \\
\geq\ & \{\psi(t, y_{n+1}) - \psi(t, y_n)\} + \alpha \int_0^{t-\frac{1}{n}} \left[\begin{array}{c} \psi(\tau, y_{n+1}(\tau)) \\ -\psi(\tau, y_n(\tau)) \end{array} \right] d\tau \\
\geq\ & 0.
\end{aligned}
$$

This is achieved as $t \in \left[\frac{1}{n}, \frac{2}{n}\right]$. The process can be repeated until we cover $[0, 1]$, and therefore, in this case the defined sequence $(y_n)_{n\in\mathbb{N}}$ is increasing. Let's assume that y is any solution of the problem. $\forall n \in \mathbb{N}$ and $t \in \left[0, \frac{1}{n}\right]$

$$
\begin{aligned}
y(t) &= y_0 + (1-\alpha)\psi(t, y(t)) + \alpha \int_0^t \psi(\tau, y(\tau))\, d\tau \\
&\geq y_0 = y_n(t).
\end{aligned} \tag{5.5}
$$

If $t \in \left[\frac{1}{n}, \frac{2}{n}\right]$,

$$
\begin{aligned}
y(t) - y_n(t) &\geq (1-\alpha)\{\psi(t, y_{n+1}) - \psi(t, y_n)\} + \alpha \int_0^{t-\frac{1}{n}} \left[\begin{array}{c} \psi(\tau, y(\tau)) \\ -\psi(\tau, y_n(\tau)) \end{array} \right] d\tau \\
&\geq 0
\end{aligned} \tag{5.6}
$$

which indeed provides

$$
y(t) - y_n(t) \geq 0. \tag{5.7}
$$

We also obtain the same with the three condition

$$
y_n(t) \leq y(t), \forall t \in [0, T]. \tag{5.8}
$$

Let

$$
y(t) = y(0) + (1-\alpha)\psi(t, y(t)) + \alpha \int_0^t \psi(\tau, y(\tau))\, d\tau, y(t_0) = \beta. \tag{5.9}
$$

However, (t_0, β) is not a peano point, each subsequence $(y_{nl})_{l\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ must converge to the unique solution of the above problem. The precompactness of $(y_n)_{n\in\mathbb{N}}$ leads to the convergence of (y_n).

Theorem 2. Let $f = f(t, y)$ such that $\forall y_1, y_2 \in M$

$$
|f(t_1, y_1) - f(t_2, y_2)| \leq L|y_1 - y_2|^\beta, (0 \leq t \leq 1), L > 0, 0 < \beta < 1 \tag{5.10}
$$

fixed. Then, two limit functions $\overline{y}_1$ and $\overline{y}_2$ for the sequence $\{y_n\}_{n\in\mathbb{N}}$ defined by satisfy the inequality

$$
\left|\overline{y}_1(t) - \overline{y}_2(t)\right| \leq L. \tag{5.11}
$$

Proof. The sequence $(y_n)_{n\in\mathbb{N}}$ is equibounded, we found a $M > 0$ such that $\forall n \in \mathbb{N}$ and $t \in [0, 1]$

$$
|\psi(t, y_n(t))| \leq M, \quad \forall p, n \in \mathbb{N}, \tag{5.12}
$$

$$\left|y_{n+p}(t)-y_n(t)\right| \leq \frac{M}{n}, \quad \forall t \in \left[0, \frac{1}{n}\right].$$

If $\frac{1}{n} \leq t \leq \frac{2}{n}$, we have

$$\begin{aligned}
\left|y_{n+p}(t)-y_n(t)\right| &\leq (1-\alpha)\left|\psi\left(t, y_{n+p}\right)-\psi\left(t, y_n\right)\right|+\alpha \int_0^{t-\frac{1}{n}}\left|\psi\left(\tau, y_{n+p}(\tau)\right)-\psi\left(\tau, y_n(\tau)\right)\right| d\tau \qquad (5.13)\\
&+\alpha \int_{t-\frac{1}{n}}^{t-\frac{1}{n+1}}\left|\psi\left(\tau, y_{n+p}(\tau)\right)\right| d\tau \\
&\leq (1-\alpha) L\left|y_{n+p}-y_n\right|^\beta+\alpha L \int_0^{t-\frac{1}{n}}\left|y_{n+p}(\tau)-y_n(\tau)\right|^\beta d\tau+\alpha M\left(\frac{1}{n}-\frac{1}{n+1}\right) \\
&\leq (1-\alpha) L\left(\frac{M}{n}\right)^\beta+\frac{\alpha M}{n}+\frac{\alpha L}{n}\left(\frac{M}{n}\right)^\beta \\
&\leq \left(\frac{M}{n}\right)^\beta\left\{(1-\alpha) L+\frac{\alpha L}{n}\right\}+\frac{M}{n} \\
&\leq \left(\frac{M}{n}\right)^\beta \bar{a}+\frac{M}{n}.
\end{aligned}$$

Additionally, we have that $\forall t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$

$$\left|y_{n+p}(t)-y_n(t)\right| \leq a_k=a_k\left(\beta, \frac{M}{n}, \frac{L}{n}\right). \tag{5.14}$$

Since the sequence increases, we have

$$\left|y_{n+p}(t)-y_n(t)\right| \leq a_n\left(\beta, \frac{M}{n}, \frac{L}{n}\right), 0 \leq t \leq 1. \tag{5.15}$$

For $n \rightarrow \infty$, according to lemma presented in [6], we have $a_n \rightarrow L(1-\beta)^{\frac{1}{1-\beta}}$, which concludes the proof. Therefore,

$$y=y(t)=y(0)+(1-\alpha) \psi(t, y(t))+\alpha \int_0^t \psi(\tau, y(\tau)) d\tau, y\left(t_0\right)=\beta. \tag{5.16}$$

5.3. Some illustrative examples

In this subsection, we compare the proposed method with the MP, two-step A-B and PM to show the accuracy of the method, as well as presenting the numerical and exact solutions of some Cauchy problems with simulation and error for different k values.

Example 3. We consider the following initial value problem:

$$\begin{aligned}
{}_0^{CF}D_t^\alpha y(t) &= t^2 \qquad (5.17)\\
y(0) &= 0,
\end{aligned}$$

where the exact solution is

$$y(t)=(1-\alpha) t^2+\alpha \frac{t^3}{3}. \tag{5.18}$$

The phase portrait of the slope-field for (5.17) is presented in Figure 5.

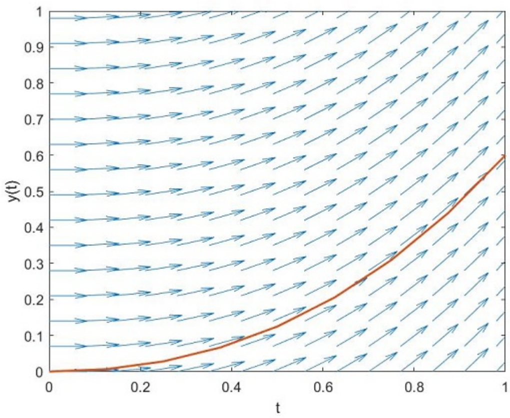

Figure 5. The phase portrait of the slope-field for (5.17).

In Figure 6, we presented the comparison between the exact solution and the numerical solution obtained by the presented method for different k values.

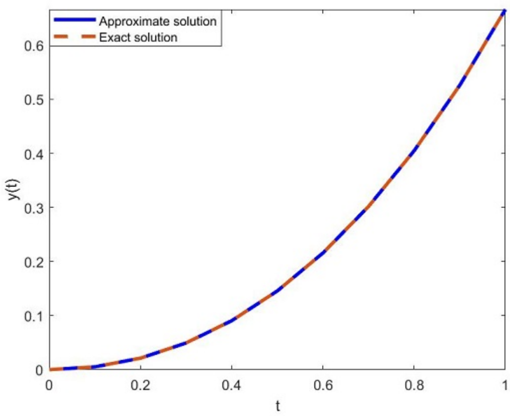

Figure 6. Numerical simulation for Cauchy problem for $\alpha = 0.6, k = 8$.

In Table 5, we presented the error of the function $y(t)$ by employing the suggested method for different k values and fractional orders.

Table 5. Error of the function $y(t)$ for suggested method.

α	Error for $k = 10$	Error for $k = 15$	Error for $k = 20$	Error for $k = 25$
1	$3.1789e-09$	$3.1042e-12$	$2.8866e-15$	$1.1102e-15$
0.9	$2.8610e-09$	$2.7938e-12$	$2.4980e-15$	$1.1102e-15$
0.8	$2.5431e-09$	$2.4833e-12$	$2.2204e-15$	$9.9920e-16$
0.7	$2.2252e-09$	$2.1730e-12$	$2.1094e-15$	$6.6613e-16$
0.6	$1.9073e-09$	$1.8625e-12$	$1.6653e-15$	$6.6613e-16$

In Table 6, we presented the comparison the error for the function $y(t)$ between the suggested method for $k = 25$, MP, PM and two-step A-B method.

Table 6. Error of the function $y(t)$ for some methods for $h = 0.1$.

α	Error of SM	Error of MP	Error of A-B	Error of PM
1	$1.1102e-15$	$8.3333e-04$	0.0078	0.0082
0.9	$1.1102e-15$	$7.5000e-04$	0.0080	0.0074
0.8	$9.9920e-16$	$6.6667e-04$	0.0083	0.0066
0.7	$6.6613e-16$	$5.8333e-04$	0.0085	0.0058
0.6	$6.6613e-16$	$5.0000e-04$	0.0087	0.0049

Example 4. We consider the following nonlinear equation:

$$\begin{aligned} {}_{0}^{CF}D_{t}^{\alpha}y(t) &= -t^{3}+t, \\ y(0) &= 0, \end{aligned} \tag{5.19}$$

where the exact solution is

$$y(t) = (1-\alpha)\left(-t^{3}+t\right) + \alpha\left(-\frac{t^{4}}{4} + \frac{t^{2}}{2}\right). \tag{5.20}$$

The phase portrait of the slope-field for (5.19) is presented in Figure 7.

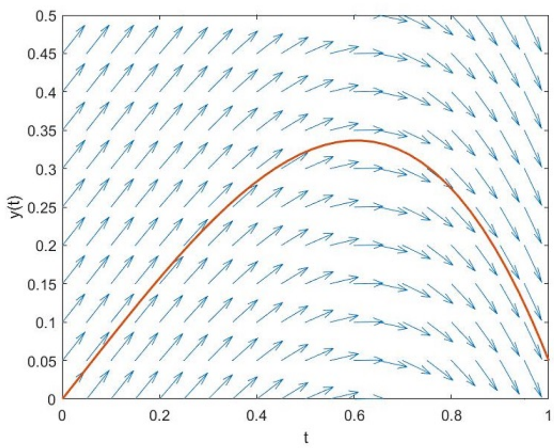

Figure 7. The phase portrait of the slope-field for (5.19).

In Figure 8, we presented the comparison between the exact solution and the numerical solution obtained by the presented method for different k values.

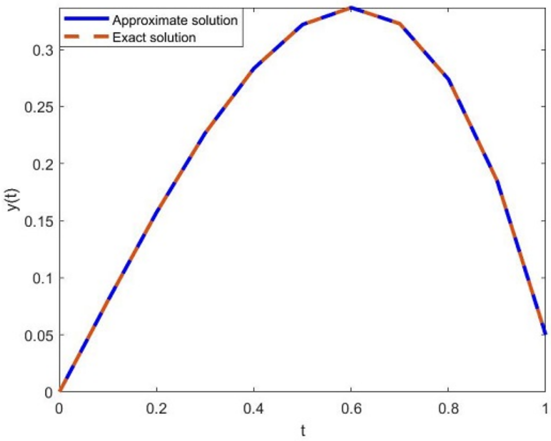

Figure 8. Numerical simulation for Cauchy problem for $\alpha = 0.6, k = 30$.

In Table 7, we presented the error for the function $y(t)$ by employing the suggested method for different k values.

Table 7. Error of the function $y(t)$ for some methods for $h = 0.1$.

α	Error for $k = 7$	Error for $k = 11$	Error for $k = 13$	Error for $k = 17$
1	$3.0518e - 07$	$1.1921e - 09$	$7.4506e - 11$	$2.9088e - 13$
0.9	$2.7466e - 07$	$1.0729e - 09$	$6.7055e - 11$	$2.6179e - 13$
0.8	$2.4414e - 07$	$9.5367e - 10$	$5.9605e - 11$	$2.3273e - 13$
0.7	$2.1362e - 07$	$8.3447e - 10$	$5.2154e - 11$	$2.0359e - 13$
0.6	$1.8311e - 07$	$7.1526e - 10$	$4.4704e - 11$	$1.7450e - 13$

In Table 8, we presented the comparison of the error for the function $y(t)$ between the suggested method for $k = 17$, MP and two step A-B method.

Table 8. Error of the function $y(t)$ for some methods for $h = 0.1$.

α	Error of SM	Error of MP	Error of A-B	Error of PM
1	$2.9088e - 13$	0.0013	0.0061	0.0043
0.9	$2.6179e - 13$	0.0011	0.0144	0.0039
0.8	$2.3273e - 13$	0.0010	0.0238	0.0035
0.7	$2.0359e - 13$	$8.7500e - 04$	0.0332	0.0030
0.6	$1.7450e - 13$	$7.5000e - 04$	0.0426	0.0026

6. Application to chaos

In this section, we will adapt the proposed method for the solution of chaotic systems with classical and Caputo–Fabrizio derivatives. We first considered a chaotic system introduced by Lai et al. [14]

$$\begin{cases} x'(t) = ax - 2yz, \\ y'(t) = -by + 2xz, \\ z'(t) = -cz + xyz + d/2, \\ x(0) = x_0, y(0) = y_0, z(0) = z_0. \end{cases} \tag{6.1}$$

For simplification, we write

$$\begin{cases} x'(t) = f_1(x, y, z, t), \\ y'(t) = f_2(x, y, z, t), \\ z'(t) = f_3(x, y, z, t). \end{cases} \tag{6.2}$$

The numerical solution of the considered chaotic model is presented as:

$$\begin{cases} \left[x_{n+1} = x_n + \frac{h}{2^{k-1}} \sum_{j=0}^{2^{k-1}-1} f_1\left(t_n + \frac{(2j+1)h}{2^k}, x\left(t_n + \frac{(2j+1)h}{2^k}\right), y\left(t_n + \frac{(2j+1)h}{2^k}\right), z\left(t_n + \frac{(2j+1)h}{2^k}\right)\right)\right], \\ \left[y_{n+1} = y_n + \frac{h}{2^{k-1}} \sum_{j=0}^{2^{k-1}-1} f_2\left(t_n + \frac{(2j+1)h}{2^k}, x\left(t_n + \frac{(2j+1)h}{2^k}\right), y\left(t_n + \frac{(2j+1)h}{2^k}\right), z\left(t_n + \frac{(2j+1)h}{2^k}\right)\right)\right], \\ \left[z_{n+1} = z_n + \frac{h}{2^{k-1}} \sum_{j=0}^{2^{k-1}-1} f_3\left(t_n + \frac{(2j+1)h}{2^k}, x\left(t_n + \frac{(2j+1)h}{2^k}\right), y\left(t_n + \frac{(2j+1)h}{2^k}\right), z\left(t_n + \frac{(2j+1)h}{2^k}\right)\right)\right], \end{cases} \tag{6.3}$$

where

$$\begin{cases} x\left(t_n + \frac{(2j+1)h}{2^k}\right) = x_n + \frac{(2j+1)h}{2^k} f_1(t_n, x_n, y_n, z_n), \\ y\left(t_n + \frac{(2j+1)h}{2^k}\right) = y_n + \frac{(2j+1)h}{2^k} f_2(t_n, x_n, y_n, z_n), \\ z\left(t_n + \frac{(2j+1)h}{2^k}\right) = z_n + \frac{(2j+1)h}{2^k} f_3(t_n, x_n, y_n, z_n). \end{cases} \tag{6.4}$$

The presented chaotic system has chaotic properties with the following parameters and initial conditions:

$$a = 2, b = 8, c = 2.9, d = 4, \tag{6.5}$$

and

$$x(0) = \pm 1, y(0) = \pm, z(0) = 1. \tag{6.6}$$

The numerical simulations for the considered chaotic model are presented in Figure 9. For the Caputo–Fabrizio case, we considered the Thomas attractor [9] which is given by

$$\begin{cases} x'(t) = \sin(\exp(y)) - 0.2x, \\ y'(t) = \cos(\sin(z)) - 0.2y, \\ z'(t) = \exp(\cos(x)) - 0.2z, \end{cases} \tag{6.7}$$

where initial conditions are taken as:

$$x(0) = 0, y(0) = 3, z(0) = 11. \tag{6.8}$$

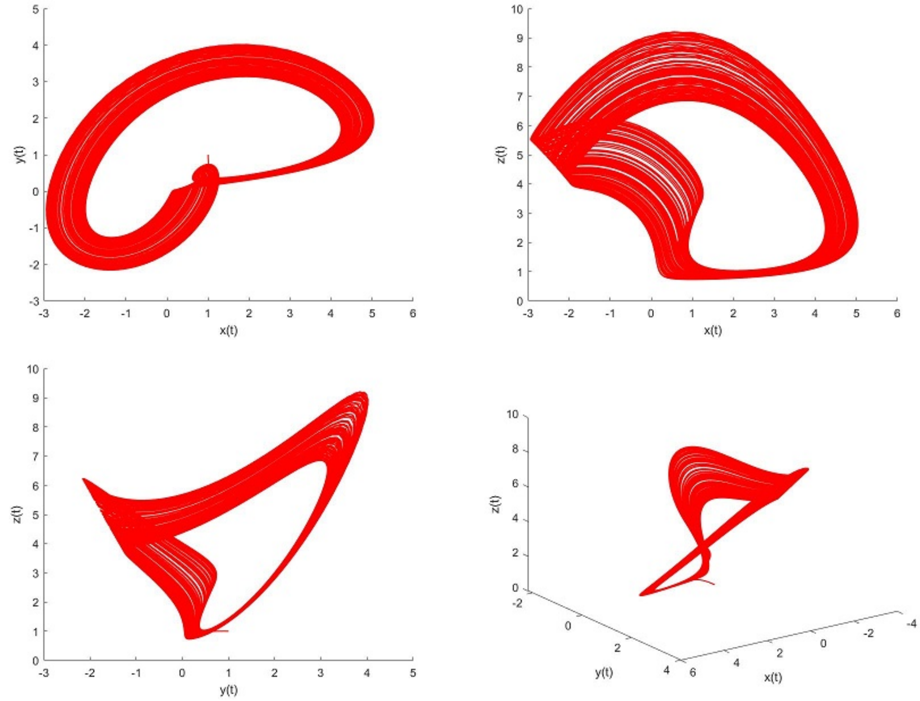

Figure 9. Numerical simulation of chaotic system.

By using the suggested method, the above system is solved numerically as follows

$$\left\{\begin{array}{l} \left\{\begin{array}{l} x_{n+1}=x_n+(1-\alpha)\left(f_1\left(t_{n+1},,\widehat{x}_{n+1},\widehat{y}_{n+1},\widehat{z}_{n+1}\right)-f_1\left(t_n,x_n,y_n,z_n\right)\right) \\ \quad +\frac{\alpha h}{2^{k-1}}\sum\limits_{j=0}^{2^{k-1}-1} f_1\left(\begin{array}{c} t_n+\frac{(2j+1)h}{2^k}, x\left(t_n+\frac{(2j+1)h}{2^k}\right), \\ y\left(t_n+\frac{(2j+1)h}{2^k}\right), z\left(t_n+\frac{(2j+1)h}{2^k}\right) \end{array}\right) \end{array}\right\} \\ \left\{\begin{array}{l} y_{n+1}=y_n+(1-\alpha)\left(f_2\left(t_{n+1},,\widehat{x}_{n+1},\widehat{y}_{n+1},\widehat{z}_{n+1}\right)-f_2\left(t_n,x_n,y_n,z_n\right)\right) \\ \quad +\frac{\alpha h}{2^{k-1}}\sum\limits_{j=0}^{2^{k-1}-1} f_2\left(\begin{array}{c} t_n+\frac{(2j+1)h}{2^k}, x\left(t_n+\frac{(2j+1)h}{2^k}\right), \\ y\left(t_n+\frac{(2j+1)h}{2^k}\right), z\left(t_n+\frac{(2j+1)h}{2^k}\right) \end{array}\right) \end{array}\right\} \\ \left\{\begin{array}{l} z_{n+1}=z_n+(1-\alpha)\left(f_3\left(t_{n+1},,\widehat{x}_{n+1},\widehat{y}_{n+1},\widehat{z}_{n+1}\right)-f_3\left(t_n,x_n,y_n,z_n\right)\right) \\ \quad +\frac{\alpha h}{2^{k-1}}\sum\limits_{j=0}^{2^{k-1}-1} f_3\left(\begin{array}{c} t_n+\frac{(2j+1)h}{2^k}, x\left(t_n+\frac{(2j+1)h}{2^k}\right), \\ y\left(t_n+\frac{(2j+1)h}{2^k}\right), z\left(t_n+\frac{(2j+1)h}{2^k}\right) \end{array}\right) \end{array}\right\} \end{array}\right. \tag{6.9}$$

where

$$\begin{aligned} \widehat{x}_{n+1} &= x_n+hf_1\left(t_n,x_n,y_n,z_n\right), \\ \widehat{y}_{n+1} &= y_n+hf_2\left(t_n,x_n,y_n,z_n\right), \\ \widehat{z}_{n+1} &= z_n+hf_3\left(t_n,x_n,y_n,z_n\right), \end{aligned} \tag{6.10}$$

and

$$x\left(t_n+\frac{(2j+1)h}{2^k}\right) = x_n+\frac{(2j+1)h}{2^k}f_1\left(t_n,x_n,y_n,z_n\right),$$

$$
\begin{aligned}
y\left(t_n+\frac{(2j+1)\,h}{2^k}\right) &= y_n+\frac{(2j+1)\,h}{2^k}f_2\left(t_n,x_n,y_n,z_n\right), \qquad (6.11)\\
z\left(t_n+\frac{(2j+1)\,h}{2^k}\right) &= z_n+\frac{(2j+1)\,h}{2^k}f_3\left(t_n,x_n,y_n,z_n\right).
\end{aligned}
$$

The numerical simulations for the considered chaotic model are presented in Figure 10.

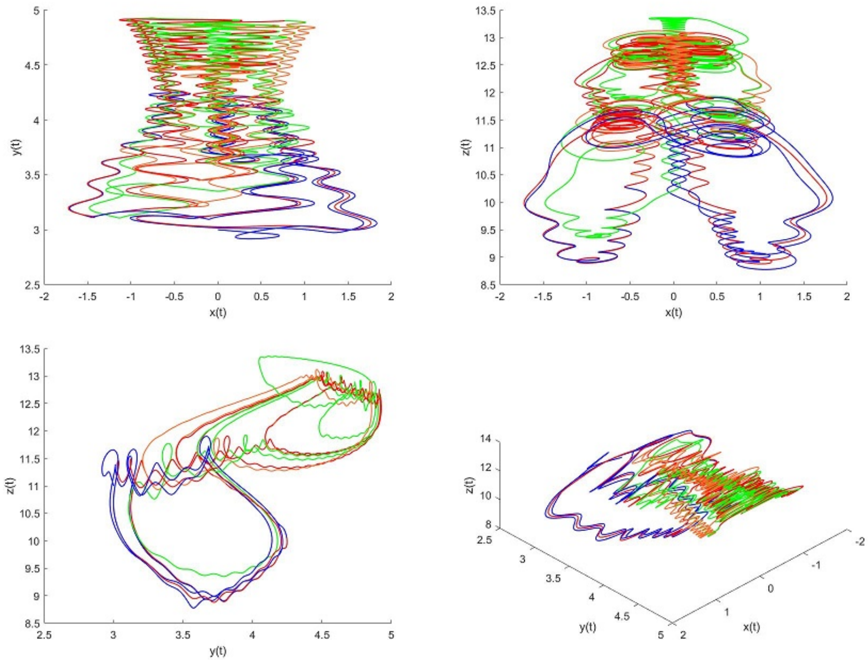

Figure 10. Numerical simulation of chaotic system for $\alpha = 0.93$.

7. Discussion

The MP technique is a one-step approach for solving the differential equation numerically. We will recall that the local error at each stage of the MP approach is of the order $O\left(h^3\right)$, resulting in a global error of the size $O\left(h^2\right)$. As a result, while the MP method is more computationally costly than Euler's method, its error drops faster as h approaches zero. The methods are examples of Runge-Kutta methods, a type of higher-order method. Because this method is based on the MP approach, it benefits from all of the advantages that the midpoint provides, especially when the split of the interval k is decreased to one. We will see that the value k denotes the number of times the MP strategy is applied inside a certain interval $[t_n, t_{n+1}]$. It is observed in both cases, classical nonlinear ordinary differential equations and fractional nonlinear differential equations, that increasing the value k increases the accuracy of the MP, implying that in some cases, the suggested extension provides us with more accuracy when k is greater than one.

8. Conclusions

Numerical integration methods are often defined as combining integrand evaluations to obtain an approximation to the integral. The integrand is assessed at a finite set of places known as integration points, and the integral is approximated using a weighted sum of these values. The integration points and weights are determined by the method used and the level of accuracy desired from the approximation. We have extended the accuracy of the well-known MP method in this study by introducing a sequential approach to interval's division. This derivation was carried out for general Cauchy problems with classical and Caputo–Fabrizio derivatives. In the case of Caputo–Fabrizio, however, we gave an existence and uniqueness solution. Several examples were provided to test the accuracy of the proposed systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Abdon Atangana is an editorial board member for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article. The authors declare that there is no conflict of interests regarding the publication of this paper.

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