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Research article

A computational method for investigating a quantum integrodifferential inclusion with simulations and heatmaps

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Abstract: We aim to investigate an integro-differential inclusion using a novel computational approach in this research. The use of quantum calculus, and consequently the creation of discrete space, allows the computer and computational algorithms to solve our desired problem. Furthermore, to guarantee the existence of the solution, we use the endpoint property based on fixed point methods, which is one of the most recent techniques in fixed point theory. The above will show the novelty of our work, because most researchers use classical fixed point techniques in continuous space. Moreover, the sensitivity of the parameters involved in controlling the existence of the solution can be recognized from the heatmaps. For a better understanding of the issue and validation of the results, we presented numerical algorithms, tables and some figures in our examples that are presented at the end of the work.

Keywords: fractional calculus; endpoint property; Pompieu-Hausdorff metric; boundary value problem; integro-differential inclusion; quantum calculus; fixed point theory
Mathematics Subject Classification: 34A08, 34A12

1. Introduction

The generalization and fulfillment of fractional operators and their use in differential equations and Boundary Value Problem(BVP) have led to the development of advanced fractional modeling. The non-locality of fractional operators has caused them to be used by researchers in different fields of science to model natural and physical phenomena. For example, we can mention the significant presence of fractional calculus in engineering [1–4], thermodynamics [5–8], physics [9–12], and biomathematics [13–15]. In 2002, Hilfer showed in an important research report that based on laboratory evidence, ordinary calculus is associated with errors in modeling and describing phenomena [16]. Of course, it is worth mentioning that the approach of researchers of different sciences to this property of non-locality has not been the same. For example, physicists used it to model viscosity and heat flow, etc., while mathematicians tried to generalize and present new fractional operators [17]. Today, researchers commonly use famous fractional operators such as Caputo and ψ-Caputo [18–22], Caputo-Fabrizio [23], Hadamard [24], Hilfer [25–27], $\psi-$Hilfer [28, 29], Riemann-Liouville [30], and Atangana-Baleanu [31–34] in their studies. Recently, George et al. showed in new research that one should be careful in using the ψ-Caputo operator to solve the pantograph equation because there is a possibility that there is no solution when using this operator [35]. On the other hand, the prominent role of computer and software packages in the numerical methods of investigating complex equations and modeling cannot be ignored, which requires a discrete space. In this work, we also provide this space with the help of quantum calculus and time scale.

The history of quantum calculus dates back to the works of the British mathematician Frank Hilton Jackson. In 1910, he gave a new definition of the derivative, by which the basic principles of quantum calculus were founded [36, 37]. Jackson introduced two types of operators: q-derivative and h-derivative. Fractional q-derivative has both the advantages of fractional calculus and due to the discreteness of the space, it provides the possibility of using the computer in solving and simulating complex equations. For the same reason, in the last decade, q-derivative has received a lot of attention from researchers and many articles have been published in this field. For example: q-serirs studied in [38], q-Starlike function reviewed in [39], application of q-derivative in differential equations presented in [40], and the existense of positive solutions for boundary value problem by fractional q-derivative investigated in [41–44]. Set-valued mappings, known as multifunctions, have unique features that make them useful in modeling physical phenomena. In 2007, Wlodarczyk et al. studied existence and uniqueness of endpoint of closed set-valued contractions in metric spaces [45]. Wardowski in 2009, investigated the existence of fixed point and endpoint of multifunction in cone metric space [46]. A year later, Amini-Harandi presented an interesting property for multifunction, which plays the main role in this article [47]. Here, we will explore the existence of a solution for a fractional q-integrodifferential inclusion using fractional and quantum calculus and multifunctions.

In 2012, Ahmed and his colleagues investigated the existence and uniqueness of a solution for the following q-difference equations

$$\begin{cases} \mathcal{D}_q^2 \boldsymbol{w}(\kappa) = g(\kappa, \boldsymbol{w}(\kappa)), \quad \kappa \in \mathcal{K}, \\ \boldsymbol{w}(0) = \boldsymbol{w}(K), \quad \mathcal{D}_q \boldsymbol{w}(0) = \mathcal{D}_q \boldsymbol{w}(K), \end{cases}$$

where $q^{\mathbb{N}} := \{q^n : n \in \mathbb{N}\} \cup \{0\}$, $\mathcal{K} = [0, K] \cap q^{\mathbb{N}}$ such that $K \in q^{\mathbb{N}}$ is a fixed constant, and $g \in$

$C(\mathcal{K}\times\mathbb{R},\mathbb{R})$ [48]. In 2012, Agarwal et al. investigated the existence and dimension of the set of mild solutions to following inclusion problem

$$\begin{cases} {}^{C}\mathcal{D}^{\eta}w(\kappa)\in\mathcal{A}w(\kappa)+\mathcal{B}(\kappa,w(\kappa)), \quad \kappa\in[0,K],\ \eta\in(0,1]\\ w(0)+f(0)=w_0, \end{cases}$$

where $\mathcal{A}$ is a sectorial operator (SO), ${}^{C}\mathcal{D}^{\eta}$ is Caputo derivative of fraction order η, and $\mathcal{B}:[0,K]\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$, $f:C([0,K],\mathbb{R}^n)\to\mathbb{R}^n$ [49]. In 2013, Zhao et al. studied BVP of fractional q-derivative equation as follows

$$\begin{cases} \mathcal{D}_q^{\eta}w(\kappa)+\mathcal{B}(\kappa,w(\kappa))=0, \quad \kappa\in(0,1),\ \eta\in(0,1],\\ w(0)=0, \quad w(1)=\int_0^{\alpha}\dfrac{(\alpha-qp)^{\nu-1}}{\Gamma_q(\nu)}w(p)\,\mathrm{d}_qp, \end{cases}$$

such that $\eta\in(1,2]$, $\nu\in(0,2]$, $\alpha\in(0,1)$, $\mathcal{B}:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$, and $\mathcal{D}_q^{\eta}$ is the q-derivative of Riemann-Liouville type of order η [50].

Based on previous research, we want here to examine the existence of a solution for the following fractional quantum integrodifferential inclusion problem

$${}^{C}\mathcal{D}_q^{\eta}w(\kappa)\in\mathcal{T}\Big(\kappa,w(\kappa),w'(\kappa),{}^{C}\mathcal{D}_q^{\sigma}w(\kappa),\int_0^{\kappa}w(\nu)\,\mathrm{d}_q\nu\Big), \quad \kappa\in\mathcal{K}=[0.1] \tag{1.1}$$

under new sum and product boundary conditions

$$\begin{cases} w(0)+\mathcal{S}w'(0)=\int_0^{\theta}w(p)\,\mathrm{d}p,\\ w(1)+\mathcal{P}\,{}^{C}\mathcal{D}_q^{\sigma}w(1)=\int_0^{\lambda}w(p)\,\mathrm{d}p, \end{cases} \tag{1.2}$$

where $\mathcal{S}=\sum_{j=1}^{j=m}\nu_j$, $\mathcal{P}=\prod_{j=1}^{j=m}u_j$, $\nu_j,u_j\in\mathbb{R}$, and $\alpha\in(0,1)$. In our problem ${}^{C}\mathcal{D}_q^{\eta}$ is Caputo quantum operator of fractional order $1\le\eta<2$, and $\sigma,\theta,\lambda\in(0,1)$, such that $\mathcal{T}:\mathcal{K}\times\mathbb{R}^4\to\mathcal{P}(\mathbb{R})$, is multifunction where $\mathcal{P}(\mathbb{R})$ set of all subsets of real numbers. Note that we will continue to do all our calculations on the time scale, namely $TS_{\kappa_0}=\{\kappa_0,\kappa_0q,\kappa_0q^2,...\}\cup\{0\}$, where $\kappa_0\in\mathbb{R}$, and $q\in(0,1)$.

2. Preliminaries

Definition 2.1. *[36] Assume that $v,p\in\mathbb{R}$, $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, then the q-analogue of v and power function $(v-p)^{(n)}$ defined as follows, respectively*

$$[v]_q=\frac{1-q^s}{1-q}=1+q+\cdots+q^{v-1},$$

and

$$\begin{cases} (v-p)_q^{(n)}=\prod_{j=0}^{n-1}(v-pq^i) \quad \textit{for} \quad n\ge 1.\\ (v-p)_q^{(0)}=1. \end{cases}$$

Definition 2.2. *[37] Let $v \in \mathbb{R} - \{0, -1, -2, \dots\}$, then the quantum gamma function formulated as follows*

$$\Gamma_q(v) = \frac{(1-q)^{(v-1)}}{(1-q)^{v-1}},$$

also, it is worth mentioning that $\Gamma_q(v+1) = [v]_q \Gamma_q(v)$ holds true.

In the following, we present an algorithm for calculating the quantum gamma function. Moreover, we computed for some values of q in Tables 1 and 2. Also, the heatmaps of data in Tables 1 and 2 are presented in Figures 1 and 2.

Algorithm 1 The proposed procedure to calculate $\Gamma_q(v)$

function quantum gamma = qG(q,v,r)
$t = 1$;
for $j = 0 : r$
$t = t * (1 - q^{(j+1)})/(1 - q^{(v+j)})$;
end
$qG = t/(1-q)^{(v-1)}$;
end

Table 1. Numerical results for $\Gamma_q(1.25)$ for different value of q.

r	$q = 0.2$	$q = 0.31$	$q = 0.49$	$q = 0.69$	$q = 0.83$	$q = 0.98$
			$v = 1.25$			
1	0.9632	0.9590	0.9726	1.0360	1.1575	1.9005
2	0.9606	0.9516	0.9519	0.9930	1.0912	1.7587
3	0.9601	0.9493	0.9425	0.9678	1.0481	1.6593
4	0.9600	0.9487	0.9380	0.9522	1.0180	1.5843
5	0.9600	0.9484	0.9359	0.9421	0.9960	1.5247
6	0.9600	0.9484	0.9348	0.9354	0.9794	1.4757
7	0.9600	0.9484	0.9343	0.9310	0.9666	1.4345
8	0.9600	0.9483	0.9341	0.9280	0.9566	1.3992
9	0.9600	0.9483	0.9340	0.9259	0.9487	1.3684
...	...	...	...	...	...	...
11	0.9600	0.9483	0.9339	0.9235	0.9297	1.2756
12	0.9600	0.9483	0.9338	0.9229	0.9270	1.2577
...	...	...	...	...	...	...
21	0.9600	0.9483	0.9338	0.9215	0.9149	1.1008
22	0.9600	0.9483	0.9338	0.9214	0.9148	1.0945
..	...	...	...	...	...	...
47	0.9600	0.9483	0.9338	0.9214	0.9142	1.0201
48	0.9600	0.9483	0.9338	0.9208	0.9141	1.0171
...	...	...	...	...	...	...
385	0.9600	0.9483	0.9338	0.9214	0.9141	0.9074
386	0.9600	0.9483	0.9338	0.9214	0.9141	0.9073

Table 2. Numerical results for $\Gamma_q(2.25)$ for different value of q.

r	$q = 0.2$	$q = 0.45$	$q = 0.69$	$q = 0.77$	$q = 0.89$	$q = 0.95$
			$v = 2.25$			
1	1.0486	1.1997	1.7704	2.3102	4.9645	12.3195
2	1.0413	1.1283	1.4983	1.8719	3.7499	8.9705
3	1.0399	1.0986	1.3513	1.6262	3.0530	7.0484
4	1.0396	1.0858	1.2643	1.4738	2.6063	5.8134
5	1.0395	1.0800	1.2100	1.3730	2.2984	4.9589
6	1.0395	1.0775	1.1749	1.3036	2.0753	4.3355
..	...	...	...	...	...	...
11	1.0395	1.0754	1.1138	1.1581	1.5240	2.7536
12	1.0395	1.0754	1.1105	1.1470	1.4679	2.5843
...	...	...	...	...	...	...
25	1.0395	1.0754	1.1031	1.1125	1.1855	1.6136
26	1.0395	1.0754	1.1031	1.1122	1.1783	1.5808
...	...	...	...	...	...	...
42	1.0395	1.0754	1.1031	1.1113	1.1312	1.2946
43	1.0395	1.0754	1.1031	1.1113	1.1303	1.2853
..	...	...	...	...	...	...
83	1.0395	1.0754	1.1031	1.1113	1.1231	1.1467
84	1.0395	1.0754	1.1031	1.1113	1.1230	1.1458
...	...	...	...	...	...	...
207	1.0395	1.0754	1.1031	1.1113	1.1230	1.1286
208	1.0395	1.0754	1.1031	1.1113	1.1230	1.1285

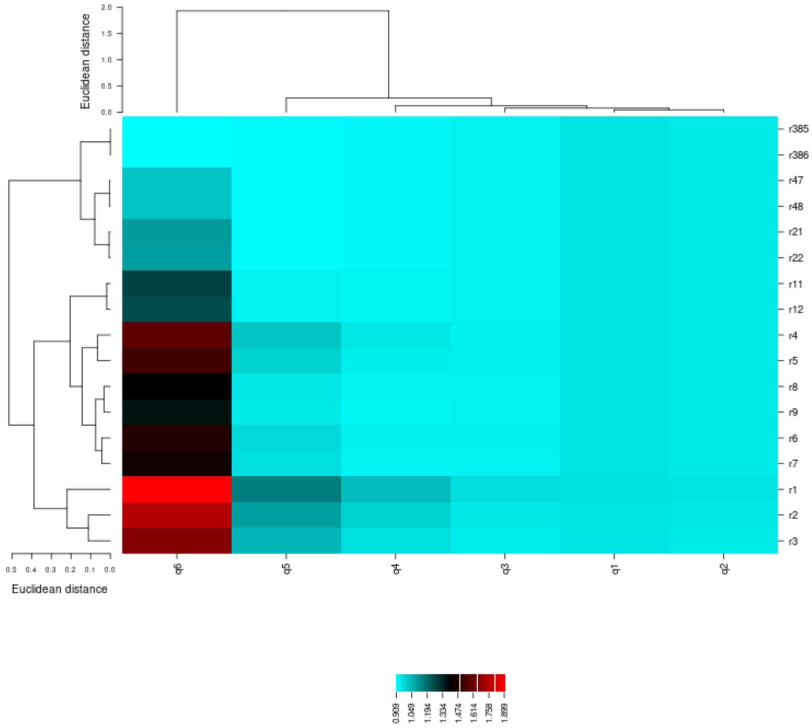

Figure 1. The heatmap of Table 1.

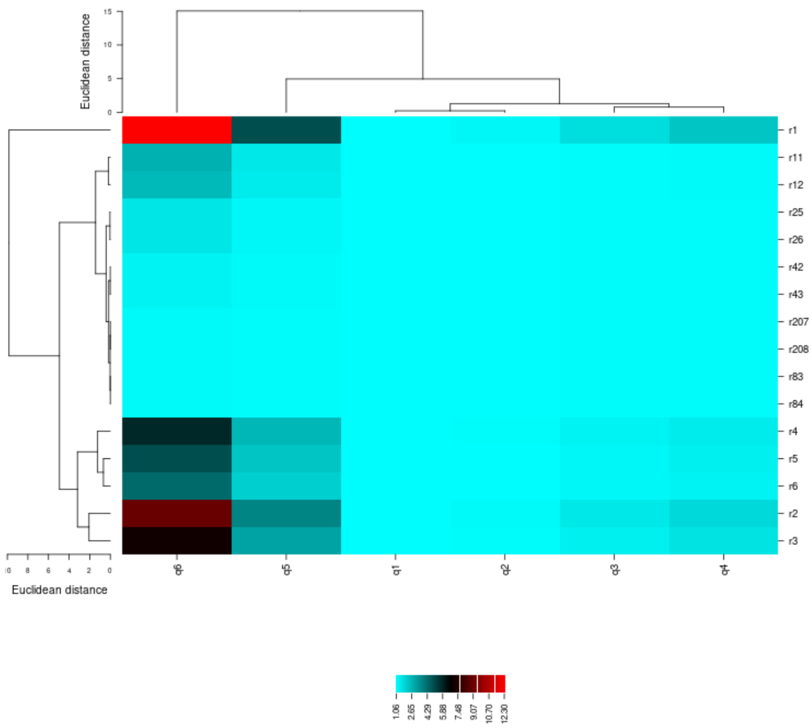

Figure 2. The heatmap of Table 2.

Definition 2.3. *[51] The quantum derivative of a continuous function as $\boldsymbol{w}(\kappa)$ is as follows*

$$(\mathcal{D}_q\boldsymbol{w})(\kappa) = \frac{\boldsymbol{w}(\kappa) - \boldsymbol{w}(q\kappa)}{(1-q)\kappa},$$

in addition, $(\mathcal{D}_q\boldsymbol{w})(0) = \lim_{\kappa\to 0}(\mathcal{D}_q\boldsymbol{w})(\kappa)$. Furthermore, for all $n \in \mathbb{N}$, the relation $(\mathcal{D}_q^n\boldsymbol{w})(\kappa) = \mathcal{D}_q(\mathcal{D}_q^{n-1}\boldsymbol{w})(\kappa)$ holds true.

Definition 2.4. *[52] Suppose that $\boldsymbol{w}(\kappa) : [0,\infty] \to \mathbb{R}$, be a continuous function, then its fractional Riemann-Liouville quantum integral and its fractional Caputo quantum derivative are expressed respectively by*

$$\mathcal{I}_q^{\eta}\boldsymbol{w}(\kappa) = \frac{1}{\Gamma_q(\eta)}\int_0^{\kappa}(\kappa - qp)^{\eta-1}\boldsymbol{w}(p)\,\mathrm{d}_q p,$$

and

$${}^{c}\mathcal{D}^{\eta}\boldsymbol{w}(\kappa) = \frac{1}{\Gamma_q(n-\eta)}\int_0^{\kappa}(\kappa - qp)^{n-\eta-1}\mathcal{D}_q^{n}\boldsymbol{w}(p)\,\mathrm{d}_q p, \quad n = [\eta] + 1.$$

Lemma 2.5. *[53] Assume that $n = [\eta] + 1$, then the following relation holds true*

$$({}^{C}\mathcal{I}_q^{\eta}\,{}^{C}\mathcal{D}_q^{\eta}\boldsymbol{w})(\kappa) = \boldsymbol{w}(\kappa) - \sum_{j=0}^{n-1}\frac{w^j}{\Gamma_q(j+1)}(\mathcal{D}_q^{j}\boldsymbol{w})(0),$$

which is deduced from it, the general solution for ${}^{C}\mathcal{D}_q^{\eta}\boldsymbol{w}(\kappa) = 0$, expressed by

$$\boldsymbol{w}(\kappa) = \ell_0 + \ell_1\kappa + \ell_2\kappa^2 + \cdots + \ell_{n-1}\kappa^{n-1},$$

where $\ell_0, \ldots, \ell_{n-1} \in \mathbb{R}$.

Notation 2.6. *Here, we introduce symbols used in the topology of the space. Consider $(\mathcal{G}, d_{\mathcal{G}})$ be a metric space, also suppose that $\mathcal{P}(\mathcal{G})$ and $2^{\mathcal{G}}$ represent the set of all subsets of $\mathcal{G}$ and the set of all non-empty subsets of $\mathcal{G}$, respectively. In the sequel, we mean the symbols $\mathcal{P}_{cl}(\mathcal{G})$, $\mathcal{P}_{bd}(\mathcal{G})$, $\mathcal{P}_{cx}(\mathcal{G})$ and $\mathcal{P}_{ct}(\mathcal{G})$ respectively as the class of all closed, bounded, convex and compact subsets of $\mathcal{G}$, respectively.*

Definition 2.7. *[47] A fixed point of a multifunction $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$ is an element $\kappa \in \mathcal{K}$, such that $\kappa \in \mathcal{E}(\kappa)$. As well as, if we have $\mathcal{E}(\kappa) = \{\kappa\}$, then this element, namely κ, is called an end point of $\mathcal{E}$.*

Definition 2.8. *[47] Let $(\mathcal{G}, d_{\mathcal{G}})$ be a metric space and $\mathcal{E} : \mathcal{G} \to 2^{\mathcal{G}}$ is a multifunction, then $\mathcal{E}$, has an approximate property if $\inf_{\kappa\in\mathcal{G}}\sup_{r\in\mathcal{E}(\kappa)} d_{\mathcal{G}}(\kappa, r) = 0$.*

Definition 2.9. *[54] If $(\mathcal{G}, d_{\mathcal{G}})$ is a metric space, then the Pompeiu-Hausdorff meter, namely $\mathcal{HM} : 2^{\mathcal{G}} \times 2^{\mathcal{G}} \to [0.\infty]$, is defined as follows*

$$\mathcal{HM}(\mathcal{W}, \mathcal{Z}) = \left\{\sup_{w\in\mathcal{W}} d_{\mathcal{G}}(w, \mathcal{Z}), \sup_{z\in\mathcal{Z}} d_{\mathcal{G}}(\mathcal{W}, z)\right\},$$

where $\mathcal{HM}(\mathcal{W}, z) = \inf_{w\in\mathcal{W}} d_{\mathcal{G}}(w.z)$. Then $(\mathcal{P}_{bd,cl}(\mathcal{G}), \mathcal{HM})$, and $(\mathcal{P}_{cl}(\mathcal{G}), \mathcal{HM})$ represent a metric space and a generalized metric space, respectively.

Definition 2.10. *[54] Assume that $\mathcal{V} = C(\mathcal{K}, \mathbb{R})$, then define the space*

$$\mathcal{G} = \{\boldsymbol{w}(\kappa) : \boldsymbol{w}(\kappa), \boldsymbol{w}'(\kappa), {}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa), \int_0^{\kappa} \boldsymbol{w}(v)\,\mathrm{d}v \in \mathcal{V}\}$$

equipped with the norm

$$\|\boldsymbol{w}\| = \sup_{\kappa\in\mathcal{K}} |\boldsymbol{w}(\kappa)| + \sup_{\kappa\in\mathcal{K}} |\boldsymbol{w}'(\kappa)| + \sup_{\kappa\in\mathcal{K}} |{}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa)| + \sup_{\kappa\in\mathcal{K}} \Big|\int_0^{\kappa} \boldsymbol{w}(v)\,\mathrm{d}v\Big|.$$

Now $(\mathcal{G}, \|.\|)$ is a Banach space.

Definition 2.11. *Let $\boldsymbol{w} \in \mathcal{G}$, then for all $\kappa \in \mathcal{K}$, define the set of selection of $\mathcal{S}^*$ as follows*

$$\mathcal{S}^*_{\mathcal{T},\boldsymbol{w}} = \left\{ \mathfrak{g} \in \mathcal{L}^1(\mathcal{K}) : \boldsymbol{w}(\kappa) \in \mathcal{T}\big(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}'(\kappa), {}^{C}\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa), \int_0^{\kappa} \boldsymbol{w}(v)\,\mathrm{d}v\big) \right\},$$

If $\dim(\mathcal{G}) < \infty$*, then the above selection is nonempty which is proved in [54].*

In 2010, Amini-Harandi introduced the end-point technique, which is crucial in proving Theorem 3.2. Now we will express it here.

Lemma 2.12. *[47] Suppose that $(\mathcal{G}, d_{\mathcal{G}})$ is a complete metric space, also consider two map Ψ and $\mathcal{E}$ with the following properties*

- $\Psi : [0, \infty) \to [0, \infty)$ *is upper semi continuous (USC), which $\forall \kappa > 0$ we have $\Psi(\kappa) < \kappa$, and* $\liminf_{\kappa\to\infty}(\kappa - \Psi(\kappa)) > 0$.
- $\forall w, z \in \mathcal{G}$*, for the set-valued map $\mathcal{E} : \mathcal{G} \to \mathcal{P}_{cl,bd}(\mathcal{G})$, the inequality $\mathcal{HM}(\mathcal{E}(w), \mathcal{E}(z)) \leq \Psi(d_{\mathcal{G}}(w, z))$ holds true.*

Then the set-valued map $\mathcal{E}$, has a unique endpoint iff $\mathcal{E}$ has an approximate end-point property.

3. Main results

Now we have provided the prerequisites necessary to express our main results, and only one lemma remains, which we prove here.

Lemma 3.1. *The unique solution for the fractional q-differential problem ${}^{c}\mathcal{D}_q^{\eta}\boldsymbol{w}(\kappa) = g(\kappa)$ under boundary condition* (1.2) *expressed by*

$$\begin{aligned}
\boldsymbol{w}(\kappa) = {} & \frac{1}{\Gamma_q(\eta)} \int_0^{\kappa} (\kappa - qp)^{\eta-1} g(p)\,\mathrm{d}_q p + \frac{2\mathbf{N} + (\theta^2 - 2\mathcal{S})(\lambda - 1)}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)} \int_0^{\theta}\int_0^{p} (p - qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p \\
& + \frac{(\theta^2 - 2\mathcal{S})}{2\mathbf{N}\Gamma_q(\eta)} \int_0^{\lambda}\int_0^{p} (p - qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p + \frac{(2\mathcal{S} - \theta^2)}{2\mathbf{N}\Gamma_q(\eta)} \int_0^{1} (1 - qp)^{\eta-1} g(p)\,\mathrm{d}_q p \\
& + \frac{\mathcal{P}(2\mathcal{S} - \theta^2)}{2\mathbf{N}\Gamma_q(\eta - \sigma)} \int_0^{1} (1 - qp)^{\eta-\sigma-1} g(p)\,\mathrm{d}_q p + \frac{(1-\theta)\kappa}{\mathbf{N}\Gamma_q(\eta)} \int_0^{\lambda}\int_0^{p} (p - qm)^{\eta-1} g(m)\,\mathrm{d}_q m\,\mathrm{d}p
\end{aligned}$$

$$+\frac{(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}g(p)\,\mathrm{d}_qp+\frac{\mathcal{P}(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}g(p)\,\mathrm{d}_qp$$

$$+\frac{(\lambda-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^{\theta}\int_0^{p}(p-qm)^{\eta-1}g(m)\,\mathrm{d}_qm\,\mathrm{d}p.$$

Such that $\eta\in[1,2)$*,* $g(\kappa)\in\mathcal{V}$*, and* $\mathbf{M}=(1+\frac{\mathcal{P}}{\Gamma_q(2-\sigma)}-\frac{\lambda^2}{2})$, $\mathbf{N}=(1-\theta)\mathbf{M}+(\lambda-1)(\mathcal{S}-\frac{\theta^2}{2})$.

Proof. In view of Lemma 2.5, the problem ${}^{C}\mathcal{D}_q^{\eta}\boldsymbol{w}(\kappa)=g(\kappa)$, has a unique solution which acquired by

$$\boldsymbol{w}(\kappa)=\mathcal{I}_q^{\eta}g(\kappa)+\ell_0+\ell_1\kappa=\frac{1}{\Gamma_q(\eta)}\int_0^{\kappa}(\kappa-qp)^{\eta-1}g(p)\,\mathrm{d}_qp+\ell_0+\ell_1\kappa, \tag{3.1}$$

which $\ell_0,\ell_1\in\mathbb{R}$. To apply the boundary conditions, it is necessary to calculate the first order derivative, namely $\boldsymbol{w}'(\kappa)=\ell_1+\mathcal{I}_q^{\eta-1}g(\kappa)$. Now with regard to boundary condition (1.2), we get

$$\ell_0(1-\theta)+\ell_1(\mathcal{S}-\frac{\theta^2}{2})=\frac{1}{\Gamma_q(\eta)}\int_0^{\theta}\int_0^{p}(p-qm)^{\eta-1}g(m)\,\mathrm{d}_qm\,\mathrm{d}p,$$

and

$$\begin{aligned}\ell_0(1-\lambda)+\ell_1(1+\frac{\mathcal{P}}{\Gamma_q(2-\sigma)}-\frac{\lambda^2}{2})=&\int_0^{\lambda}\int_0^{p}(p-qm)^{\eta-1}g(m)\,\mathrm{d}_qm\,\mathrm{d}p\\&-\frac{1}{\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}g(p)\,\mathrm{d}_qp\\&-\frac{\mathcal{P}}{\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}g(p)\,\mathrm{d}_qp.\end{aligned}$$

If for simplicity in computation we set

$$\mathbf{M}=(1+\frac{\mathcal{P}}{\Gamma_q(2-\sigma)}-\frac{\lambda^2}{2})\quad\text{and}\quad\mathbf{N}=(1-\theta)\mathbf{M}+(\lambda-1)(\mathcal{S}-\frac{\theta^2}{2}).$$

Then, the values of ℓ_0 and ℓ_1 will be as follows:

$$\begin{aligned}\ell_0=&\frac{2\mathbf{N}+(\theta^2-2\mathcal{S})(\lambda-1)}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)}\int_0^{\theta}\int_0^{p}(p-qm)^{\eta-1}g(m)\,\mathrm{d}_qm\,\mathrm{d}p\\&+\frac{(\theta^2-2\mathcal{S})}{2\mathbf{N}\Gamma_q(\eta)}\int_0^{\lambda}\int_0^{p}(p-qm)^{\eta-1}g(m)\,\mathrm{d}_qm\,\mathrm{d}p\\&+\frac{(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}g(p)\,\mathrm{d}_qp\\&+\frac{\mathcal{P}(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}g(p)\,\mathrm{d}_qp,\end{aligned}$$

and

$$\ell_1 = \frac{(1-\theta)}{\mathbf{N}\Gamma_q(\eta)} \int_0^{\lambda} \int_0^{p} (p-qm)^{\eta-1} g(m)\, \mathrm{d}_q m\, \mathrm{d}p$$

$$+ \frac{\theta-1}{\mathbf{N}\Gamma_q(\eta)} \int_0^{1} (1-qp)^{\eta-1} g(p)\, \mathrm{d}_q p$$

$$+ \frac{\mathcal{P}(\theta-1)}{\mathbf{N}\Gamma_q(\eta-\sigma)} \int_0^{1} (1-qp)^{\eta-\sigma-1} g(p)\, \mathrm{d}_q p$$

$$+ \frac{\lambda-1}{\mathbf{N}\Gamma_q(\eta)} \int_0^{\theta} \int_0^{p} (p-qm)^{\eta-1} g(m)\, \mathrm{d}_q m\, \mathrm{d}p.$$

Placing coefficients ℓ_0 and ℓ_1 in Eq (3.1) provides the desired result. □

In order to obtain the result in our inclusion problem, it is necessary to apply the following hypotheses.

$\mathcal{A}_1$) Since $\mathcal{T} : \mathcal{K} \times \mathbb{R}^4 \to P_{ct}(\mathbb{R})$ is integrable and bounded, therefore $\mathcal{T}(., a, b, c, d) : [0,1] \to \mathcal{P}_{ct}(\mathbb{R})$ is measurable.

$\mathcal{A}_2$) For $\Psi : [0,\infty) \to [0,\infty)$, which is nondecreasing and (USC), $\forall p > 0$ we have $\liminf_{p\to\infty}(p - \Psi(p)) > 0$ and $\Psi(p) < p$.

$\mathcal{A}_3$) For all $\kappa \in \mathcal{K}$, and $w_j, z_j \in \mathbb{R}$, $j = 1,2,3,4$, there exist $\mathbf{\Omega} \in C(\mathcal{K},[0,\infty))$, where

$$\mathcal{HM}\Big(\mathcal{T}(\kappa, w_1, w_2, w_3, w_4), \mathcal{T}(\kappa, z_1, z_2, z_3, z_4)\Big) \le \frac{1}{\mathfrak{X}_1 + \mathfrak{X}_2 + \mathfrak{X}_3 + \mathfrak{X}_4} \mathbf{\Omega}(\kappa) \Psi\Big(\sum_{j=1}^{4} |w_j - z_j|\Big),$$

such that

$$\mathfrak{X}_1 = \|\mathbf{\Omega}\| \Big[\frac{1}{\Gamma_q(\eta+1)} + \Big| \frac{(2\mathbf{N} + (\theta^2 - 2\mathcal{S})(\lambda-1))\theta^{\eta+1}}{2\mathbf{N}(1-\theta)\Gamma_q(\eta+2)} \Big| + \Big| \frac{(\theta^2 - 2\mathcal{S})\lambda^{\eta+1}}{2\mathbf{N}\Gamma_q(\eta+2)} \Big|$$

$$+ \Big| \frac{2\mathcal{S} - \theta^2}{2\mathbf{N}\Gamma_q(\eta+1)} \Big| + \Big| \frac{\mathcal{P}(2\mathcal{S} - \theta^2)}{2\mathbf{N}\Gamma_q(\eta-\sigma+1)} \Big| + \Big| \frac{(1-\theta)\lambda^{\eta+1}}{\mathbf{N}\Gamma_q(\eta+2)} \Big| + \Big| \frac{\theta-1}{\mathbf{N}\Gamma_q(\eta+1)} \Big|$$

$$+ \Big| \frac{\mathcal{P}(\theta-1)}{\mathbf{N}\Gamma_q(\eta-\sigma+1)} \Big| + \Big| \frac{(\lambda-1)\theta^{\eta+1}}{\mathbf{N}\Gamma_q(\eta+2)} \Big| \Big],$$

$$\mathfrak{X}_2 = \|\mathbf{\Omega}\| \Big[\frac{1}{\Gamma_q(\eta)} + \Big| \frac{(1-\theta)\lambda^{\eta+1}}{\mathbf{N}\Gamma_q(\eta+2)} \Big| + \Big| \frac{\theta-1}{\mathbf{N}\Gamma_q(\eta+1)} \Big| + \Big| \frac{\mathcal{P}(\theta-1)}{\mathbf{N}\Gamma_q(\eta-\sigma+1)} \Big| + \Big| \frac{(\lambda-1)\theta^{\eta+1}}{\mathbf{N}\Gamma_q(\eta+2)} \Big],$$

$$\mathfrak{X}_3 = \|\mathbf{\Omega}\| \Big[\frac{1}{\Gamma_q(\eta-\sigma+1)} + \Big| \frac{(1-\theta)\lambda^{\eta+1}}{\mathbf{N}\Gamma_q(\eta+2)\Gamma_q(2-\sigma)} \Big| + \Big| \frac{\theta-1}{\mathbf{N}\Gamma_q(\eta+1)\Gamma_q(2-\sigma)} \Big|$$

$$+\left|\frac{(\lambda-1)\theta^{\eta+1}}{\mathbf{N}\Gamma_q(\eta+2)\Gamma_q(2-\sigma)}\right|+\left|\frac{\mathcal{P}(\theta-1)}{\mathbf{N}\Gamma_q(\eta-\sigma+1)\Gamma_q(2-\sigma)}\right|\Big]$$

$$\begin{aligned}\mathfrak{X}_4 = \|\mathbf{\Omega}\|\Big[&\frac{1}{\Gamma_q(\eta+2)}+\left|\frac{(2\mathbf{N}+(\theta^2-2\mathcal{S})(\lambda-1))\theta^{\eta+1}}{2\mathbf{N}(1-\theta)\Gamma_q(\eta+2)}\right|+\left|\frac{(\theta^2-2\mathcal{S})\lambda^{\eta+1}}{2\mathbf{N}\Gamma_q(\eta+2)}\right|\\ &+\left|\frac{2\mathcal{S}-\theta^2}{2\mathbf{N}\Gamma_q(\eta+1)}\right|+\left|\frac{\mathcal{P}(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta-\sigma+1)}\right|+\left|\frac{(1-\theta)\lambda^{\eta+1}}{2\mathbf{N}\Gamma_q(\eta+2)}\right|+\left|\frac{\theta-1}{2\mathbf{N}\Gamma_q(\eta+1)}\right|\\ &+\left|\frac{\mathcal{P}(\theta-1)}{2\mathbf{N}\Gamma_q(\eta-\sigma+1)}\right|+\left|\frac{(\lambda-1)\theta^{\eta+1}}{2\mathbf{N}\Gamma_q(\eta+2)}\right|\Big].\end{aligned}$$

Theorem 3.2. *Let hypotheses $\mathcal{A}_1-\mathcal{A}_4$ are holds true. If the set-valued map $\mathcal{E}:\mathcal{G}\to 2^{\mathcal{G}}$, has the approximate endpoint property, then the inclusion q-integro-differential problem mentioned in* (1.1) *and* (1.2) *has a solution.*

Proof. To show that our problems (1.1) and (1.2) has a solution, we go to find the end point of the operator $\mathcal{E}:\mathcal{G}\to 2^{\mathcal{G}}$ which for $\mathrm{g}\in\mathcal{S}^*_{\mathcal{T},w}$ read as follows:

$$\begin{aligned}\mathcal{E}(\hbar)=\Big\{\hbar\in\mathcal{G}:\hbar(\kappa)=&\frac{1}{\Gamma_q(\eta)}\int_0^\kappa(\kappa-qp)^{\eta-1}\mathrm{g}(p)\,\mathrm{d}_qp\\ &+\frac{2\mathbf{N}+(\theta^2-2\mathcal{S})(\lambda-1)}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)}\int_0^\theta\int_0^p(p-qm)^{\eta-1}\mathrm{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p\\ &+\frac{(\theta^2-2\mathcal{S})}{2\mathbf{N}\Gamma_q(\eta)}\int_0^\lambda\int_0^p(p-qm)^{\eta-1}\mathrm{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p\\ &+\frac{(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}\mathrm{g}(p)\,\mathrm{d}_qp\\ &+\frac{\mathcal{P}(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}\mathrm{g}(p)\,\mathrm{d}_qp\\ &+\frac{(1-\theta)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^\lambda\int_0^p(p-qm)^{\eta-1}\mathrm{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p\\ &+\frac{(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}\mathrm{g}(p)\,\mathrm{d}_qp\\ &+\frac{\mathcal{P}(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}\mathrm{g}(p)\,\mathrm{d}_qp\\ &+\frac{(\lambda-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^\theta\int_0^p(p-qm)^{\eta-1}\mathrm{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p\Big\}.\end{aligned}$$

This end point is the solution of our inclusion. We do this in two steps.

Step I. we shall show for all $\hbar \in \mathcal{G}$, $\mathcal{E}(\hbar) \subset \mathcal{G}$ which $\mathcal{E}(\hbar)$ is closed. According to, for all $\hbar \in \mathcal{G}$ the map $\kappa \mapsto \mathcal{T}(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}'(\kappa), {}^C\mathcal{D}_q^\sigma \boldsymbol{w}(\kappa), \int_0^\kappa \boldsymbol{w}(p)dp)$, is measurable and closed value map. Therefore, such a map has a non-empty measurable selection, namely $\mathcal{S}^*_{\mathcal{T},\boldsymbol{w}} \neq \phi$. Now assume that $\{t_n\}_{n\geq 1}$ be a sequence in $\mathcal{E}(\hbar)$, which $t_n \to t$. Choose $\mathfrak{g}_n \in \mathcal{S}^*_{\mathcal{T},\boldsymbol{w}}$, which for all $\kappa \in \mathcal{K}$ and $n \geq 1$

$$\begin{aligned}
t_n &= \frac{1}{\Gamma_q(\eta)} \int_0^\kappa (\kappa - qp)^{\eta-1} \mathfrak{g}_n(p)\, \mathrm{d}_q p \\
&+ \frac{2\mathbf{N} + (\theta^2 - 2\mathcal{S})(\lambda - 1)}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)} \int_0^\theta \int_0^p (p - qm)^{\eta-1} \mathfrak{g}_n(m)\, \mathrm{d}_q m\, \mathrm{d}p \\
&+ \frac{(\theta^2 - 2\mathcal{S})}{2\mathbf{N}\Gamma_q(\eta)} \int_0^\lambda \int_0^p (p - qm)^{\eta-1} \mathfrak{g}_n(m)\, \mathrm{d}_q m\, \mathrm{d}p \\
&+ \frac{(2\mathcal{S} - \theta^2)}{2\mathbf{N}\Gamma_q(\eta)} \int_0^1 (1 - qp)^{\eta-1} \mathfrak{g}_n(p)\, \mathrm{d}_q p \\
&+ \frac{\mathcal{P}(2\mathcal{S} - \theta^2)}{2\mathbf{N}\Gamma_q(\eta - \sigma)} \int_0^1 (1 - qp)^{\eta-\sigma-1} \mathfrak{g}_n(p)\, \mathrm{d}_q p \\
&+ \frac{(1-\theta)\kappa}{\mathbf{N}\Gamma_q(\eta)} \int_0^\lambda \int_0^p (p - qm)^{\eta-1} \mathfrak{g}_n(m)\, \mathrm{d}_q m\, \mathrm{d}p \\
&+ \frac{(\theta - 1)\kappa}{\mathbf{N}\Gamma_q(\eta)} \int_0^1 (1 - qp)^{\eta-1} \mathfrak{g}_n(p)\, \mathrm{d}_q p \\
&+ \frac{\mathcal{P}(\theta - 1)\kappa}{\mathbf{N}\Gamma_q(\eta - \sigma)} \int_0^1 (1 - qp)^{\eta-\sigma-1} \mathfrak{g}_n(p)\, \mathrm{d}_q p \\
&+ \frac{(\lambda - 1)\kappa}{\mathbf{N}\Gamma_q(\eta)} \int_0^\theta \int_0^p (p - qm)^{\eta-1} \mathfrak{g}_n(m)\, \mathrm{d}_q m\, \mathrm{d}p.
\end{aligned}$$

Compactness of $\mathcal{T}$, implies that $\mathfrak{g}_n$ has a subsequence(show this again with $\mathfrak{g}_n$), which converges to some $\mathfrak{g} \in \mathcal{L}^1[0,1]$. It is easy to check that $\mathfrak{g} \in \mathcal{S}^*_{\mathcal{T},\boldsymbol{w}}$, and for all $\kappa \in \mathcal{K}$

$$\begin{aligned}
t_n(\kappa) \to t(\kappa) &= \frac{1}{\Gamma_q(\eta)} \int_0^\kappa (\kappa - qp)^{\eta-1} \mathfrak{g}(p)\, \mathrm{d}_q p \\
&+ \frac{2\mathbf{N} + (\theta^2 - 2\mathcal{S})(\lambda - 1)}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)} \int_0^\theta \int_0^p (p - qm)^{\eta-1} \mathfrak{g}(m)\, \mathrm{d}_q m\, \mathrm{d}p \\
&+ \frac{(\theta^2 - 2\mathcal{S})}{2\mathbf{N}\Gamma_q(\eta)} \int_0^\lambda \int_0^p (p - qm)^{\eta-1} \mathfrak{g}(m)\, \mathrm{d}_q m\, \mathrm{d}p \\
&+ \frac{(2\mathcal{S} - \theta^2)}{2\mathbf{N}\Gamma_q(\eta)} \int_0^1 (1 - qp)^{\eta-1} \mathfrak{g}(p)\, \mathrm{d}_q p
\end{aligned}$$

$$+\frac{\mathcal{P}(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}\mathfrak{g}(p)\,\mathrm{d}_qp$$

$$+\frac{(1-\theta)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^\lambda\int_0^p(p-qm)^{\eta-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p$$

$$+\frac{(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}\mathfrak{g}(p)\,\mathrm{d}_qp$$

$$+\frac{\mathcal{P}(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}\mathfrak{g}(p)\,\mathrm{d}_qp$$

$$+\frac{(\lambda-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^\theta\int_0^p(p-qm)^{\eta-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p.$$

It can be concluded from this $t\in\mathcal{E}(\hbar)$, thus $\mathcal{G}$ is closed values. In addition, from the compactness of the value of $\mathcal{T}$, it follows that $\in\mathcal{E}(\hbar)$ is bounded.

Step II. Our goal at this step is to establish the inequality $\mathcal{HM}(\mathcal{E}(w),\mathcal{E}(z))\leq\Psi(\|w-z\|)$ holds true. To do this, let $w,z\in\mathcal{G}$, $\hbar_1\in\mathcal{E}(z)$, and choose $\mathfrak{g}_1\in\mathcal{S}^*_{\mathcal{T},w}$ such that for almost $\kappa\in\mathcal{K}$, we can write

$$t_n(\kappa)\to t(\kappa)=\frac{1}{\Gamma_q(\eta)}\int_0^\kappa(\kappa-qp)^{\eta-1}\mathfrak{g}(p)\,\mathrm{d}_qp$$

$$+\frac{2\mathbf{N}+(\theta^2-2\mathcal{S})(\lambda-1)}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)}\int_0^\theta\int_0^p(p-qm)^{\eta-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p$$

$$+\frac{(\theta^2-2\mathcal{S})}{2\mathbf{N}\Gamma_q(\eta)}\int_0^\lambda\int_0^p(p-qm)^{\eta-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p$$

$$+\frac{(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}\mathfrak{g}(p)\,\mathrm{d}_qp$$

$$+\frac{\mathcal{P}(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}\mathfrak{g}(p)\,\mathrm{d}_qp$$

$$+\frac{(1-\theta)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^\lambda\int_0^p(p-qm)^{\eta-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p$$

$$+\frac{(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}\mathfrak{g}(p)\,\mathrm{d}_qp$$

$$+\frac{\mathcal{P}(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}\mathfrak{g}(p)\,\mathrm{d}_qp$$

$$+\frac{(\lambda-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^\theta\int_0^p(p-qm)^{\eta-1}\mathfrak{g}(m)\,\mathrm{d}_qm\,\mathrm{d}p.$$

But, in view of hypothesis $\mathcal{A}_3$

$$\begin{aligned}
&\mathcal{H}_{\mathfrak{d}}\Big(\mathcal{T}(\kappa, w_1, w_2, w_3, w_4), \mathcal{T}(\kappa, z_1, z_2, z_3, z_4)\Big)\\
&\leq \frac{1}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\mathbf{\Omega}(\kappa)\Psi\Big(|w_1(\kappa)-z_1(\kappa)|+|w_2'(\kappa)-z_2'(\kappa)|\\
&+|{}^C\mathcal{D}_q^{\eta}w_3(\kappa)-{}^C\mathcal{D}_q^{\eta}z_3(\kappa)|+\Big|\int_0^{\kappa}w_4(v)\,\mathrm{d}v-\int_0^{\kappa}z_4(v)\,\mathrm{d}v\Big|\Big),
\end{aligned}$$

hence, $\exists s\in\mathcal{T}(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}'(\kappa), {}^C\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa), \int_0^{\kappa}\boldsymbol{w}(v)\,\mathrm{d}v)$, where $\forall\kappa\in\mathcal{K}$:

$$|\mathfrak{g}_1-s|\leq\frac{1}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\mathbf{\Omega}(\kappa)\Psi\Big(\sum_{j=1}^{4}|w_j-z_j|\Big).$$

Now consider the map $\mathcal{F}:\mathcal{K}\to\mathcal{P}(\mathbb{R})$, such that

$$\mathcal{F}(\kappa)=\left\{s\in\mathbb{R}:|\mathfrak{g}_1-s|\leq\frac{1}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\mathbf{\Omega}(\kappa)\Psi\Big(\sum_{j=1}^{4}|w_j-z_j|\Big)\right\}.$$

Forasmuch as $\frac{1}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\mathbf{\Omega}(\kappa)\Psi\Big(\sum_{j=1}^{4}|w_j-z_j|\Big)$, and $\mathfrak{g}_1$ are measurable, so the set-valued map $\mathcal{F}(.)\cap\mathcal{T}(., \boldsymbol{w}(.), \boldsymbol{w}'(.), {}^C\mathcal{D}_q^{\sigma}\boldsymbol{w}(.), \int_0^{.}\boldsymbol{w}(v)\,\mathrm{d}v)$ is measurable.

Take $\mathfrak{g}_2(\kappa)\in\mathcal{T}(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}'(\kappa), {}^C\mathcal{D}_q^{\sigma}\boldsymbol{w}(\kappa), \int_0^{\kappa}\boldsymbol{w}(v)\,\mathrm{d}v)$, which for all $\kappa\in\mathcal{K}$, we have

$$|\mathfrak{g}_1(\kappa)-\mathfrak{g}_2(\kappa)|\leq\frac{1}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\mathbf{\Omega}(\kappa)\Psi\Big(\sum_{j=1}^{4}|w_j-z_j|\Big).$$

Now, $\forall\kappa\in\mathcal{K}$, assume that $\mathfrak{g}_2\in\mathcal{E}(\hbar)$, with

$$\begin{aligned}
\hbar_2&=\frac{1}{\Gamma_q(\eta)}\int_0^{\kappa}(\kappa-qp)^{\eta-1}\mathfrak{g}_2(p)\,\mathrm{d}_qp\\
&+\frac{2\mathbf{N}+(\theta^2-2\mathcal{S})(\lambda-1)}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)}\int_0^{\theta}\int_0^{p}(p-qm)^{\eta-1}\mathfrak{g}_2(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{(\theta^2-2\mathcal{S})}{2\mathbf{N}\Gamma_q(\eta)}\int_0^{\lambda}\int_0^{p}(p-qm)^{\eta-1}\mathfrak{g}_2(m)\,\mathrm{d}_qm\,\mathrm{d}p\\
&+\frac{(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta)}\int_0^{1}(1-qp)^{\eta-1}\mathfrak{g}_2(p)\,\mathrm{d}_qp\\
&+\frac{\mathcal{P}(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^{1}(1-qp)^{\eta-\sigma-1}\mathfrak{g}_2(p)\,\mathrm{d}_qp
\end{aligned}$$

$$+\frac{(1-\theta)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^\lambda\int_0^p(p-qm)^{\eta-1}g_2(m)\,d_qm\,dp$$

$$+\frac{(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^1(1-qp)^{\eta-1}g_2(p)\,d_qp+\frac{\mathcal{P}(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta-\sigma)}\int_0^1(1-qp)^{\eta-\sigma-1}g_2(p)\,d_qp$$

$$+\frac{(\lambda-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\int_0^\theta\int_0^p(p-qm)^{\eta-1}g_2(m)\,d_qm\,dp.$$

Subsequently, let $\sup_{\kappa\in\mathcal{K}}|\mathbf{\Omega}(\kappa)|=\|\mathbf{\Omega}\|$, therefore

$$\begin{aligned}|\hbar_1-\hbar_2|\le{}&\frac{1}{\Gamma_q(\eta)}\int_0^\kappa(\kappa-qp)^{\eta-1}|g_1-g_2|(p)\,d_qp\\&+\Big|\frac{2\mathbf{N}+(\theta^2-2\mathcal{S})(\lambda-1)}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)}\Big|\int_0^\theta\int_0^p(p-qm)^{\eta-1}|g_1(m)-g_2(m)|\,d_qm\,dp\\&+\Big|\frac{(\theta^2-2\mathcal{S})}{2\mathbf{N}\Gamma_q(\eta)}\Big|\int_0^\lambda\int_0^p(p-qm)^{\eta-1}|g_1(m)-g_2(m)|\,d_qm\,dp\\&+\Big|\frac{(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta)}\Big|\int_0^1(1-qp)^{\eta-1}|g_1-g_2|(p)\,d_qp\\&+\Big|\frac{\mathcal{P}(2\mathcal{S}-\theta^2)}{2\mathbf{N}\Gamma_q(\eta-\sigma)}\Big|\int_0^1(1-qp)^{\eta-\sigma-1}|g_1-g_2|(p)\,d_qp\\&+\Big|\frac{(1-\theta)\kappa}{\mathbf{N}\Gamma_q(\eta)}\Big|\int_0^\lambda\int_0^p(p-qm)^{\eta-1}|g_1(m)-g_2(m)|\,d_qm\,dp\\&+\Big|\frac{(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\Big|\int_0^1(1-qp)^{\eta-1}|g_1-g_2|(p)\,d_qp\\&+\Big|\frac{\mathcal{P}(\theta-1)\kappa}{\mathbf{N}\Gamma_q(\eta-\sigma)}\Big|\int_0^1(1-qp)^{\eta-\sigma-1}|g_1-g_2|(p)\,d_qp\\&+\Big|\frac{(\lambda-1)\kappa}{\mathbf{N}\Gamma_q(\eta)}\Big|\int_0^\theta\int_0^p(p-qm)^{\eta-1}|g_1(m)-g_2(m)|\,d_qm\,dp\\&\le\frac{\mathfrak{X}_1}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\Psi(\|w-z\|),\end{aligned}$$

and

$$\begin{aligned}|\hbar_1'-\hbar_2'|\le{}&\frac{1}{\Gamma_q(\eta-1)}\int_0^\kappa(\kappa-qp)^{\eta-2}|g_1-g_2|(p)\,d_qp\\&+\Big|\frac{(1-\theta)}{\mathbf{N}\Gamma_q(\eta)}\Big|\int_0^\lambda\int_0^p(p-qm)^{\eta-1}|g_1(m)-g_2(m)|\,d_qm\,dp\end{aligned}$$

$$
\begin{aligned}
&+ \Big|\frac{(\theta-1)}{\mathbf{N}\Gamma_q(\eta)}\Big| \int_0^1 (1-qp)^{\eta-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_q p \\
&+ \Big|\frac{\mathcal{P}(\theta-1)}{\mathbf{N}\Gamma_q(\eta-\sigma)}\Big| \int_0^1 (1-qp)^{\eta-\sigma-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_q p \\
&+ \Big|\frac{(\lambda-1)}{\mathbf{N}\Gamma_q(\eta)}\Big| \int_0^\theta \int_0^p (p-qm)^{\eta-1}|\mathfrak{g}_1(m)-\mathfrak{g}_2(m)|\,\mathrm{d}_q m\,\mathrm{d}p \\
&\le \frac{\mathfrak{X}_1}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\Psi(\|w-z\|).
\end{aligned}
$$

Also, one can write

$$
\begin{aligned}
|{}^C\mathcal{D}_q^\sigma \hbar_1 - {}^C\mathcal{D}_q^\sigma \hbar_2| \le{}& \frac{1}{\Gamma_q(\eta-\sigma)} \int_0^\kappa (\kappa-qp)^{\eta-\sigma-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_q p \\
&+ \Big|\frac{(1-\theta)\kappa^{1-\sigma}}{\mathbf{N}\Gamma_q(\eta)\Gamma_q(2-\sigma)}\Big| \int_0^\lambda \int_0^p (p-qm)^{\eta-1}|\mathfrak{g}_1(m)-\mathfrak{g}_2(m)|\,\mathrm{d}_q m\,\mathrm{d}p \\
&+ \Big|\frac{(\theta-1)\kappa^{1-\sigma}}{\mathbf{N}\Gamma_q(\eta)\Gamma_q(2-\sigma)}\Big| \int_0^1 (1-qp)^{\eta-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_q p \\
&+ \Big|\frac{\mathcal{P}(\theta-1)\kappa^{1-\sigma}}{\mathbf{N}\Gamma_q(\eta-\sigma)\Gamma_q(2-\sigma)}\Big| \int_0^1 (1-qp)^{\eta-\sigma-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_q p \\
&+ \Big|\frac{(\lambda-1)\kappa^{1-\sigma}}{\mathbf{N}\Gamma_q(\eta)\Gamma_q(2-\sigma)}\Big| \int_0^\theta \int_0^p (p-qm)^{\eta-1}|\mathfrak{g}_1(m)-\mathfrak{g}_2(m)|\,\mathrm{d}_q m\,\mathrm{d}p \\
&\le \frac{\mathfrak{X}_3}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\Psi(\|w-z\|),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_0^\kappa \hbar_1(v)\,\mathrm{d}v - \int_0^\kappa \hbar_2(v)\,\mathrm{d}v\Big| \\
&\le \frac{1}{\Gamma_q(\eta+1)} \int_0^\kappa (\kappa-qp)^{\eta}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_q p \\
&+ \Big|\frac{2\mathbf{N}+(\theta^2-2\mathcal{S})(\lambda-1)\kappa}{2\mathbf{N}(1-\theta)\Gamma_q(\eta)}\Big| \int_0^\theta \int_0^p (p-qm)^{\eta-1}|\mathfrak{g}_1(m)-\mathfrak{g}_2(m)|\,\mathrm{d}_q m\,\mathrm{d}p \\
&+ \Big|\frac{(\theta^2-2\mathcal{S})\kappa}{2\mathbf{N}\Gamma_q(\eta)}\Big| \int_0^\lambda \int_0^p (p-qm)^{\eta-1}|\mathfrak{g}_1(m)-\mathfrak{g}_2(m)|\,\mathrm{d}_q m\,\mathrm{d}p \\
&+ \Big|\frac{(2\mathcal{S}-\theta^2)\kappa}{2\mathbf{N}\Gamma_q(\eta)}\Big| \int_0^1 (1-qp)^{\eta-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_q p
\end{aligned}
$$

$$+\left|\frac{\mathcal{P}(2\mathcal{S}-\theta^2)\kappa}{2\mathbf{N}\Gamma_q(\eta-\sigma)}\right|\int_0^1(1-qp)^{\eta-\sigma-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_qp$$

$$+\left|\frac{(1-\theta)\frac{\kappa^2}{2}}{\mathbf{N}\Gamma_q(\eta)}\right|\int_0^\lambda\int_0^p(p-qm)^{\eta-1}|\mathfrak{g}_1(m)-\mathfrak{g}_2(m)|\,\mathrm{d}_qm\,\mathrm{d}p$$

$$+\left|\frac{(\theta-1)\frac{\kappa^2}{2}}{\mathbf{N}\Gamma_q(\eta)}\right|\int_0^1(1-qp)^{\eta-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_qp$$

$$+\left|\frac{\mathcal{P}(\theta-1)\frac{\kappa^2}{2}}{\mathbf{N}\Gamma_q(\eta-\sigma)}\right|\int_0^1(1-qp)^{\eta-\sigma-1}|\mathfrak{g}_1-\mathfrak{g}_2|(p)\,\mathrm{d}_qp$$

$$+\left|\frac{(\lambda-1)\frac{\kappa^2}{2}}{\mathbf{N}\Gamma_q(\eta)}\right|\int_0^\theta\int_0^p(p-qm)^{\eta-1}|\mathfrak{g}_1(m)-\mathfrak{g}_2(m)|\,\mathrm{d}_qm\,\mathrm{d}p$$

$$\leq\frac{\mathfrak{X}_4}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\Psi(\|w-z\|).$$

It can be inferred from the above relationships that

$$\begin{aligned}\|\hbar_1-\hbar_2\|&=\sup_{\kappa\in\mathcal{K}}|\hbar_1(\kappa)-\hbar_2(\kappa)|+\sup_{\kappa\in\mathcal{K}}|\hbar_1'(\kappa)-\hbar_2'(\kappa)|+\sup_{\kappa\in\mathcal{K}}|{}^C\mathcal{D}_q^\sigma\hbar_1(\kappa)-{}^C\mathcal{D}_q^\sigma\hbar_2(\kappa)|\\&+\sup_{\kappa\in\mathcal{K}}\left|\int_0^\kappa\hbar_1(s)ds-\int_0^\kappa\hbar_2(v)dv\right|\\&\leq\frac{1}{\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4}\Psi(\|w-z\|)(\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4)\\&=\Psi(\|w-z\|).\end{aligned}$$

Thus, for all $w,z\in\mathcal{G}$, we have

$$\mathcal{HM}(\mathcal{E}(w),\mathcal{E}(z))\leq\Psi(\|w-z\|).$$

Now, according to Lemma 2.12, and the endpoint property of $\mathcal{E}$, $\exists w^*\in\mathcal{G}$, where $\mathcal{E}(w^*)=\{w^*\}$. Hence, w^* is a solution for the fractional q-inclusion problem mentioned in (1.1) and (1.2). □

4. Examples

Example 4.1. *Consider the following fractional quantum integro-differential inclusion problem*

$$^c\mathcal{D}_q^{\frac{8}{5}}\boldsymbol{w}(\kappa)\in\mathcal{T}\Big[0,\frac{11(2+\sin(\boldsymbol{w}(\kappa)))}{49(33\kappa+\kappa^2)}+\frac{11e^{w'(\kappa)}}{49(7+e^{w'(\kappa)})}+\frac{11\kappa}{49}\int_0^\kappa(1+v)\boldsymbol{w}(v)\,\mathrm{d}v \tag{4.1}$$

$$+\frac{11}{49}e^{|^{c}\mathcal{D}_q^{\frac{7}{20}}\boldsymbol{w}(\kappa)|}\Big],$$

with the following boundary conditions:

$$\begin{cases}\boldsymbol{w}(0)+\mathcal{S}\boldsymbol{w}'(0)=\int_0^{\frac{4}{9}}\boldsymbol{w}(p)\,\mathrm{d}p,\\ \boldsymbol{w}(1)+\mathcal{P}\,{}^{C}\mathcal{D}_q^{\frac{7}{20}}\boldsymbol{w}(1)=\int_0^{\frac{5}{12}}\boldsymbol{w}(p)\,\mathrm{d}p,\end{cases} \tag{4.2}$$

where $\kappa\in\mathcal{K}=[0,1]$*. Here, we put:* $\eta=\frac{8}{5}$, $\sigma=\frac{7}{20}$, $\theta=\frac{4}{9}$, $\lambda=\frac{5}{12}$, $\mathcal{S}=\sum_{j=1}^{4}\nu_j=1.7$ *with* $\nu_1=\frac{3}{10}$, $\nu_2=\frac{1}{4}$, $\nu_3=\frac{2}{5}$, $\nu_4=\frac{3}{4}$*, and* $\mathcal{P}=\prod_{j=1}^{j=4}u_j=\frac{16}{625}$ *with* $u_j=\frac{2}{5}$*. We choose* $\boldsymbol{\Omega}:[0,1]\to[0,\infty)$ *by* $\boldsymbol{\Omega}(\kappa)=\frac{11}{49}\kappa$, $\|\boldsymbol{\Omega}\|=\frac{11}{49}$*, and* $\Psi(\kappa)=\frac{\kappa}{19}$*. Obviously* Ψ *is non-decreasing and (USC) on* $\mathcal{K}$*.*

Consider the set-valued map $\mathcal{T}:\mathcal{K}\times\mathbb{R}^4\to\mathcal{P}_{ct}(\mathbb{R})$ *as follows:*

$$\mathcal{T}(t,w_1,w_2,w_3,w_4)=\Big[0,\frac{11(2+\sin(\boldsymbol{w}(\kappa)))}{49(33\kappa+\kappa^2)}+\frac{11e^{\boldsymbol{w}'(\kappa)}}{49(7+e^{\boldsymbol{w}'(\kappa)})}+\frac{11\kappa}{49}\int_0^{\kappa}(1+v)\boldsymbol{w}(v)\,\mathrm{d}v$$
$$+\frac{11}{49}e^{|^{c}\mathcal{D}_q^{\frac{3}{5}}\boldsymbol{w}(\kappa)|}\Big].$$

Nevertheless, the values of $\mathfrak{X}_1,\mathfrak{X}_2,\mathfrak{X}_3,\mathfrak{X}_4$ *are calculated for* $q=0.2$, 0.35, 0.45, 0.69, 0.83 *and* 0.98 *in Table 3. Also, the heatmap of the data in Table 3 is presented in Figure 3. Noticte that for convenience, we set the value of* Θ *coefficient equal to* $(\mathfrak{X}_1+\mathfrak{X}_2+\mathfrak{X}_3+\mathfrak{X}_4)^{-1}\|\boldsymbol{\Omega}\|$.

Table 3. Numerical result of $\mathfrak{X}_1,\mathfrak{X}_2,\mathfrak{X}_3,\mathfrak{X}_4$, for different values of q.

	$q_1=0.2$	$q_2=0.45$	$q_3=0.52$	$q_4=0.69$	$q_5=0.89$	$q_6=0.98$
$\mathfrak{X}_1$	1.8422	1.9364	1.9641	2.0339	2.1211	2.1935
$\mathfrak{X}_2$	1.0622	1.1667	1.1953	1.2646	1.3466	1.4125
$\mathfrak{X}_3$	1.0166	1.0816	1.1003	1.1467	1.2032	1.2639
$\mathfrak{X}_4$	1.8041	1.8682	1.8900	1.9489	2.0276	2.0990
Θ	0.0392	0.0371	0.0365	0.0351	0.0335	0.0322

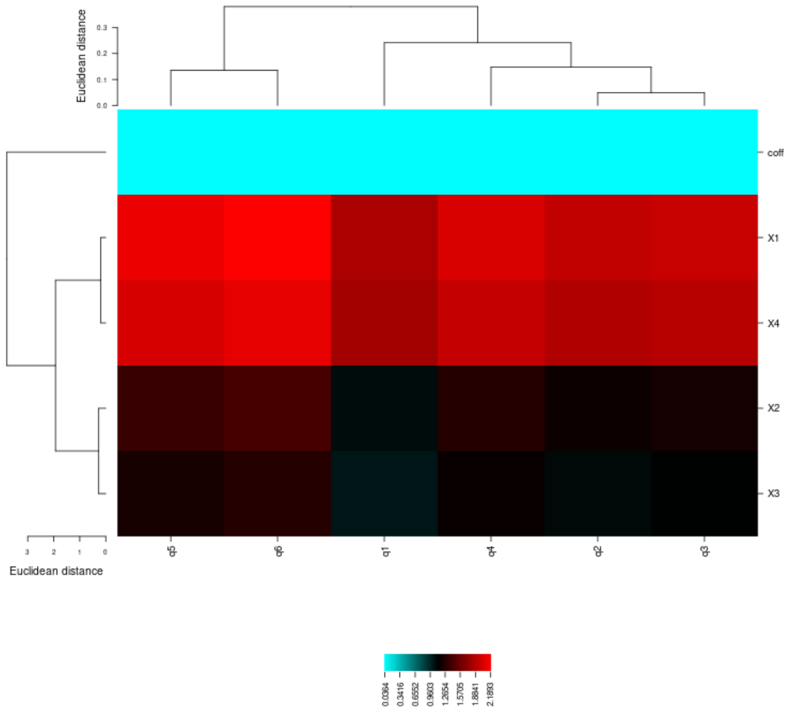

Figure 3. The heatmap of Table 3.

Now, it is easy to examine that

$$\mathcal{HM}\Big(\mathcal{T}(\kappa, w_1, w_2, w_3, w_4), \mathcal{T}(\kappa, z_1, z_2, z_3, z_4)\Big) \leq \frac{1}{\mathfrak{X}_1 + \mathfrak{X}_2 + \mathfrak{X}_3 + \mathfrak{X}_4}\mathbf{\Omega}(\kappa)\Psi\Big(\sum_{j=1}^{4} |w_j - z_j|\Big),$$

and $\inf_{w \in \mathcal{G}} \big(\sup_{z \in \mathcal{E}(z)} \|w - z\|\big) = 0$. *Now all the conditions of Theorem 3.2 are satisfied. Thanks to endpoint property and Theorem 3.2, our problem which formulated in* (4.1) *and* (4.2) *has a solution. Also the graphs of some functions presented in Figures 4 and 5.*

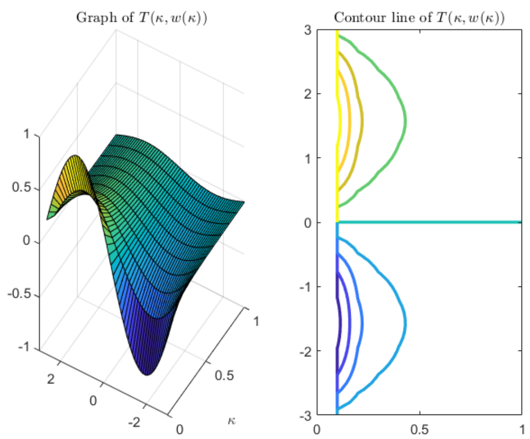

Figure 4. The graph of $\mathcal{T}(\kappa, \boldsymbol{w}(\kappa))$.

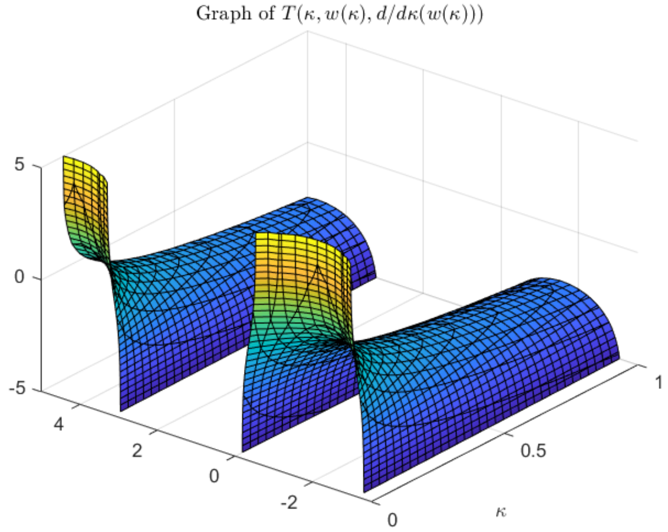

Figure 5. The graph of $\mathcal{T}(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}'(\kappa))$.

Example 4.2. *Consider the following fractional quantum integro-differential inclusion problem*

$$ {}^{c}\mathcal{D}_q^{\frac{5}{4}}\boldsymbol{w}(\kappa) \in \mathcal{T}\Big[0, \frac{8(2+\cos(\kappa))}{67(23\kappa^2+\kappa^3)} + \frac{8}{67(2+\sqrt{\kappa})}|\boldsymbol{w}(\kappa)| + \frac{8}{67}\sin(\boldsymbol{w}'(\kappa)) + \frac{8\kappa}{67}\int_0^{\kappa}\frac{\boldsymbol{w}(v)\,\mathrm{d}vs}{1+v} + \frac{8}{67}e^{|{}^{c}\mathcal{D}_q^{\frac{3}{5}}\boldsymbol{w}(\kappa)|}\Big], \tag{4.3} $$

with the following boundary conditions:

$$ \begin{cases} \boldsymbol{w}(0) + \mathcal{S}\boldsymbol{w}'(0) = \int_0^{\frac{2}{5}} \boldsymbol{w}(p)\,\mathrm{d}p, \\ \boldsymbol{w}(1) + \mathcal{P}\,{}^{C}\mathcal{D}_q^{\frac{3}{5}}\boldsymbol{w}(1) = \int_0^{\frac{3}{8}} \boldsymbol{w}(p)\,\mathrm{d}p, \end{cases} \tag{4.4} $$

where $\kappa \in \mathcal{K} = [0,1]$. *Here, we put:* $\eta = \frac{5}{4}$, $\sigma = \frac{3}{5}$, $\theta = \frac{2}{5}$, $\lambda = \frac{3}{8}$, $\mathcal{S} = \sum_{j=1}^{4} \nu_j = 2.4125$ *with* $\nu_1 = \frac{7}{10}$, $\nu_2 = \frac{9}{8}$, $\nu_3 = \frac{2}{5}$, $\nu_4 = \frac{3}{16}$, *and* $\mathcal{P} = \prod_{j=1}^{j=4} u_j = \frac{1}{81}$ *with* $u_j = \frac{1}{3}$. *We choose* $\boldsymbol{\Omega} : [0,1] \to [0,\infty)$ *by* $\boldsymbol{\Omega}(\kappa) = \frac{8}{67}\kappa$, $\|\boldsymbol{\Omega}\| = \frac{8}{67}$, *and* $\Psi(\kappa) = \frac{\kappa}{23}$. *Obviously* Ψ *is non-decreasing and (USC) on* $\mathcal{K}$.

Consider the set-valued map $\mathcal{T} : \mathcal{K} \times \mathbb{R}^4 \to \mathcal{P}_{ct}(\mathbb{R})$ *as follows:*

$$ \mathcal{T}(t, w_1, w_2, w_3, w_4) = \Big[0, \frac{8(2+\cos(\kappa))}{67(23\kappa^2+\kappa^3)} + \frac{8}{67(2+\sqrt{\kappa})}|\boldsymbol{w}(\kappa)| + \frac{8}{67}\sin(\boldsymbol{w}'(\kappa)) + \frac{8\kappa}{67}\int_0^{\kappa}\frac{\boldsymbol{w}(v)\,\mathrm{d}v}{1+v} + \frac{8}{67}e^{|{}^{c}\mathcal{D}_q^{\frac{3}{5}}\boldsymbol{w}(\kappa)|}\Big]. $$

Nevertheless, the values of $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4$ *are calculated for* $q = 0.2, 0.35, 0.59, 0.77, 0.89$ *and* 0.95 *in Table 4. Also, the heatmap of the data in Table 4 is presented in Figure 6. Noticte that for convenience, we set the value of* Θ *coefficient equal to* $(\mathfrak{X}_1 + \mathfrak{X}_2 + \mathfrak{X}_3 + \mathfrak{X}_4)^{-1}\|\mathbf{\Omega}\|$.

Table 4. Numerical result of $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4$, for different values of q.

	$q_1 = 0.2$	$q_2 = 0.31$	$q_3 = 0.49$	$q_4 = 0.77$	$q_5 = 0.83$	$q_6 = 0.95$
$\mathfrak{X}_1$	0.9536	0.9610	0.9736	0.9946	0.9993	1.0088
$\mathfrak{X}_2$	0.3348	0.3405	0.3487	0.3604	0.3627	0.3674
$\mathfrak{X}_3$	0.3270	0.3303	0.3353	0.3428	0.3444	0.3475
$\mathfrak{X}_4$	0.9332	0.9320	0.9337	0.9427	0.9454	0.9515
Θ	0.0469	0.0466	0.0461	0.0452	0.0450	0.0446

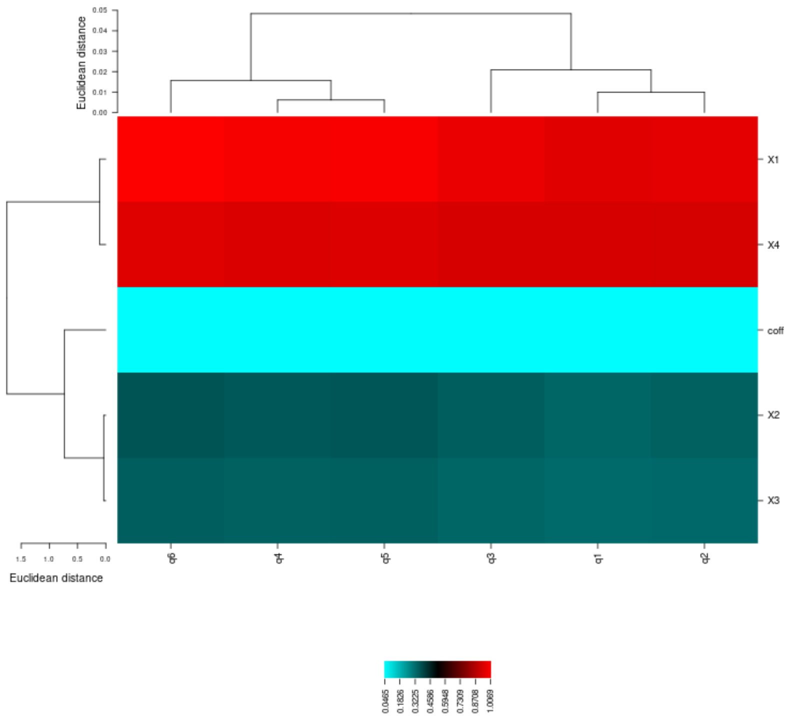

Figure 6. The heatmap of Table 4.

Now, it is easy to examine that

$$\mathcal{HM}\Big(\mathcal{T}(\kappa, w_1, w_2, w_3, w_4), \mathcal{T}(\kappa, z_1, z_2, z_3, z_4)\Big) \leq \frac{1}{\mathfrak{X}_1 + \mathfrak{X}_2 + \mathfrak{X}_3 + \mathfrak{X}_4}\mathbf{\Omega}(\kappa)\Psi\Big(\sum_{j=1}^{4}|w_j - z_j|\Big),$$

and $\inf_{w \in \mathcal{G}}(\sup_{z \in \mathcal{E}(z)} \|w - z\|) = 0$*. Now all the conditions of Theorem 3.2 are satisfied. Thanks to endpoint* *property and Theorem 3.2, our problem which formulated in* (4.3) *and* (4.4) *has a solution. Also the graphs of some functions presented in Figures 7–9.*

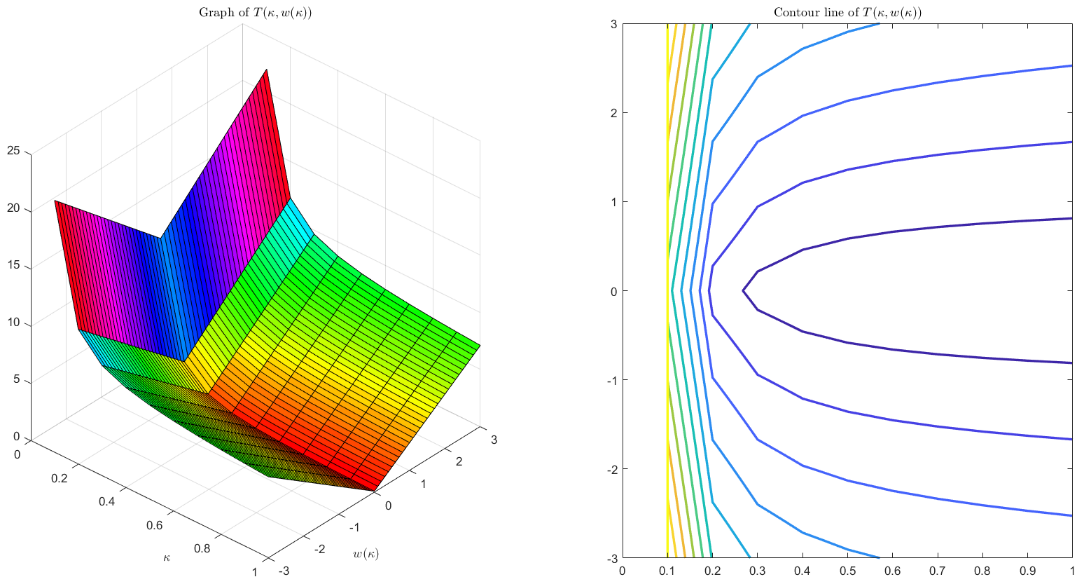

Figure 7. The graph of $\mathcal{T}(\kappa, \boldsymbol{w}(\kappa))$.

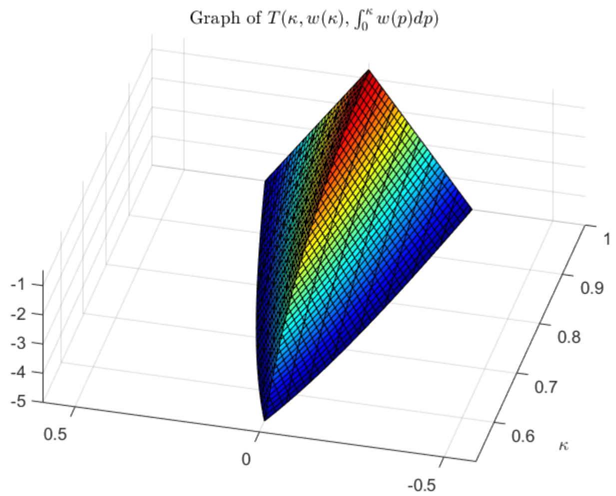

Figure 8. The graph of $\mathcal{T}(\kappa, \boldsymbol{w}(\kappa), \int_0^\kappa \boldsymbol{w}(p)dp)$.

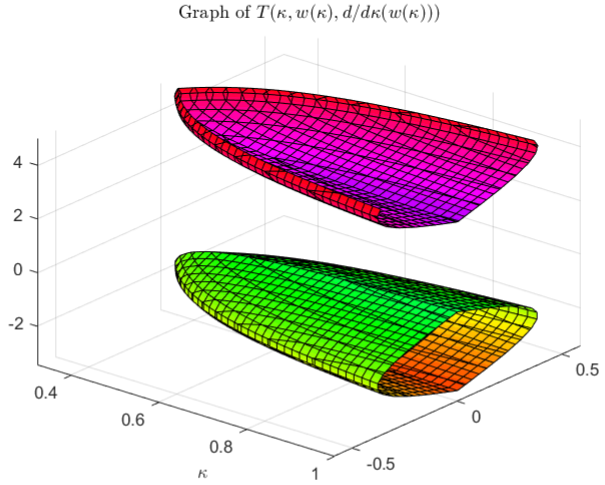

Figure 9. The graph of $\mathcal{T}(\kappa, w(\kappa), w'(\kappa))$.

5. Conclusions

In this paper, we study the numerical and analytical solutions for an integrodifferential inclusion problem by fractional Caputo q-derivative. We proposed an algorithm for computing the gamma function of quantum numbers. This enabled us to solve numerical problems using computers. The problem addressed and the numerical techniques used are more general than previous research. For the first time, we utilized heatmaps to simplify the interpretation of quantum values. Researchers can use our method to investigate other inclusions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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