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Higher order hyperexpansivity and higher order hypercontractivity

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Abstract: As a natural extension of the concept of (m, p)-hyperexpansive and (m, p)-hypercontractive of a single operator, we introduce and study the concepts of (m, p)-hyperexpansivity and (m, p)-hypercontractivity for d-tuple of commuting operators acting on Banach spaces. These concepts extend the definitions of m-isometries and (m, p)-isometric tuples of bounded linear operators acting on Hilbert or Banach spaces, which have been introduced and studied by many authors.

Keywords: m-isometric tuple; (m, p)-isometric tuple; expansive operator; contractive operator
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1. Introduction

We establish the notations used throughout this paper. The symbol $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ refers to the set of nonnegative integers. Let $\mathcal{X}$ be a complex Banach space and $\mathcal{H}$ be a complex Hilbert space. $\mathcal{B}[\mathcal{X}]$ (resp. $\mathcal{B}[\mathcal{H}]$) denotes the set of bounded linear operator on $\mathcal{X}$ (resp. on $\mathcal{H}$). For $d \in \mathbb{N}$, let $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ be a tuple of commuting bounded linear operators. Let $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}_0^d$ and set $|\beta| := \sum_{1\leq j\leq d} |\beta_j|$, $\beta! := \beta_1! \cdots \beta_d!$, $\mathbf{N}^\beta := N_1^{\beta_1} \cdots N_d^{\beta_d} = \prod_{1\leq j\leq d} N_j^{\beta_j}$. Further, the Hilbert adjoint of the commuting d-tuple $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}(\mathcal{H})^d$ is the d-tuple $\mathbf{N}^* = (N_1^*, \cdots, N_d^*)$.

J. Agler and M. Stankus introduced the class of m-isometry on Hilbert space [1–3]. An operator $N \in \mathcal{B}[\mathcal{H}]$ is said to be m-isometric operator for some integer $m \geq 1$ if it satisfies

$$\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} N^{*m-j} N^{m-j} = 0. \tag{1.1}$$

Notice that the Eq (1.1) is equivalently to

$$\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} \|N^{m-j}x\|^2 = 0 \qquad \forall\ x \in \mathcal{H}.$$

Many authors have defined new concepts related to m-isometries, such as (m, p)-isometries, (m, ∞)-isometries, (m, C)-isometries, (m, p)-isometric tuples, (m, ∞)-isometric tuples and (m, C)-isometric tuples. For the basic theory of these families of operators, the reader is referred to [8–12, 15, 18, 21–25].

Given $m \in \mathbb{N}$ and $p \in (0, \infty)$, an operator $N \in \mathcal{B}[\mathcal{X}]$ is called an (m, p)-isometry if and only if

$$\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} \|N^{m-j}x\|^p = 0 \quad \forall\ x \in \mathcal{X},$$

(see [8, 22]).

The concepts of completely hyperexpansive and completely hypercontractive operators on Hilbert space have attracted much attention from various authors. For a detailed account on these classes of operators, the reader is referred to [4, 5, 7, 13, 19, 28].

The concept of (m, p)-expansive and (m, p)-contractive operators on a Banach space were independently introduced and studied in the papers [16, 26, 27].

Let $N \in \mathcal{B}[\mathcal{X}]$, and we denote

$$\beta_k^{(p)}(N, x) := \sum_{0\leq j\leq k} (-1)^j \binom{k}{j} \left\|N^j x\right\|^p, \qquad \forall\ \ x \in \mathcal{X},$$

where $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $p \in (0, \infty)$. The operator N is said to be

(i) (m, p)-expansive if $\beta_m^{(p)}(N, x) \leq 0$ for all $x \in \mathcal{X}$,

(ii) (m, p)-hyperexpansive if $\beta_k^{(p)}(N, x) \leq 0$ for all $x \in \mathcal{X}$ and $k \in \{1, 2, \cdots, m\}$.

(iii) (m, p)-contractive if $\beta_m^{(p)}(N, x) \geq 0$, for all $x \in \mathcal{X}$,

(iv) (m, p)-hypercontractive if $\beta_k^{(p)}(N, x) \geq 0$ for all $x \in \mathcal{X}$ and $k \in \{1, 2, \cdots, m\}$.

The study of tuples of commuting operators has attracted much attention from many authors. Recently, several papers have been published on the study of tuples of commuting operators [6, 14, 15, 18, 20, 21, 23, 24, 29–31].

The notion of an m-isometric tuple (resp. (m, p)-isometric tuple) is a natural higher-dimensional generalization of the notion of an m-isometry (resp (m, p)-isometry) in a single variable operator. J. Gleason and S. Richter in [15] extended the notion of m-isometric operators to the case of commuting d-tuples of bounded linear operators on a Hilbert space. The defining equation for an m-isometric tuple $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{H}]^d$ reads:

$$\mathcal{S}_m(\mathbf{N}) := \sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\beta|=k} \frac{k!}{\beta!} \mathbf{N}^{*\beta} \mathbf{N}^{\beta} \Big) = 0 \tag{1.2}$$

or equivalently

$$\langle \mathcal{S}_m(\mathbf{N})x \mid x \rangle = \sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\beta|=k} \frac{k!}{\beta!} \|\mathbf{N}^{\beta} x\|^2 \Big) = 0 \ \text{ for all } x \in \mathcal{H}. \tag{1.3}$$

More recently, P. H. W. Hoffmann and M. Mackey [23] introduced the concept of (m, p)-isometric tuples on normed space. Given $m \in \mathbb{N}$ and $p \in (0, \infty)$, the commuting d-tuple $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ is an an (m, p)-isometry if

$$\sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} \|\mathbf{N}^{\beta} x\|^p = 0 \ \text{ for all } \ \ x \in \mathcal{X}. \tag{1.4}$$

Remark 1.1. We have the following particular cases.
(i) When $m = 1$, then $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ is a $(1,\ p)$-isometric tuple if

$$\|N_1 x\|^p + \cdots + \|N_d x\|^p = \|x\|^p, \quad \text{for all } x \in \mathcal{X}.$$

(ii) When $m = 2$, then $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ is a $(2, p)$-isometric tuple if

$$\sum_{1\leq j\leq d} \|N_j x\|^p - \Big(\sum_{1\leq j\leq d} \|N_j^2 x\|^p + 2 \sum_{1\leq j<k\leq d} \|N_j N_k x\|^p \Big) = \|x\|^p \text{ forall } x \in \mathcal{X}.$$

(iii) When $m = d = 2$, then $\mathbf{N} = (N_1, N_2) \in \mathcal{B}[\mathcal{X}]^2$ be a commuting 2-tuple is a $(2, p)$-isometric pair if

$$\|x\|^p - 2\|N_1 x\|^p - 2\|N_2 x\|^p + \|N_1^2 x\|^2 + \|N_2^2 x\|^p + \|N_1 N_2 x\|^p = 0 \text{ for all } x \in \mathcal{X}.$$

Our aim in this paper is to consider a generalization of the concepts of (m, p)-hyperexpansive and (m, p)-hypercontractive of a single operator as discussed in [17, 27] to the (m, p)-hyperexpansive, (m, p)-hypercontractive tuples of commutative operators on Banach spaces.

2. (m, p)-Hyperexpansive and (m, p)-hypercontractive tuples of commuting operators

For a d-tuple of commuting operators $\mathbf{N} := (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$, $m \in \mathbb{N}$ and $p > 0$ being a real number, we define

$$Q_m^{(p)}(\mathbf{N};\ x) := \sum_{0\leq k\leq m} (-1)^k \binom{m}{k} \Big(\sum_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| = k}} \frac{k!}{\beta!} \|\mathbf{N}^\beta x\|^p \Big).$$

Definition 2.1. For a commuting d-tuple $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$, integer $m \in \mathbb{N}$ and $p \in (0, \infty)$. We say:

(1) $\mathbf{N}$ is an (m, p)-expansive tuple if $Q_m^{(p)}(\mathbf{N}; x) \leq 0$ for all $x \in \mathcal{X}$,

(2) $\mathbf{N}$ is a (m, p)-hyperexpansive tuple if $Q_k^{(p)}(\mathbf{N};\ x) \leq 0$ for all $x \in \mathcal{X}$ and $k \in \{1, \cdots, m\}$,

(3) $\mathbf{N}$ is a completely p-hyperexpansive tuple if it is a (m, p)-hyperexpansive tuple for every integer $m \in \mathbb{N}$.

Definition 2.2. For a commuting d-tuple $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$, integer $m \in \mathbb{N}$ and $p \in (0, \infty)$. We say:

(1) $\mathbf{N}$ is an (m, p)-contractive tuple if $Q_m^{(p)}(\mathbf{N}; x) \geq 0$ for all $x \in \mathcal{X}$,

(2) $\mathbf{N}$ is a (m, p)-hypercontractive tuple if $Q_k^{(p)}(\mathbf{N};\ x) \geq 0$ for all $x \in \mathcal{X}$ and $k \in \{1, \cdots, m\}$,

(3) $\mathbf{N}$ is a completely p-hypercontractive tuple if it is a (m, p)-hypercontractive tuple for all $m \in \mathbb{N}$.

Remark 2.1. When $d = 1$, Definitions 2.1 and 2.2 coincide with [16, Definition 1.1].

Notice that for $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{H}]^d$,

$$\langle \mathcal{S}_m(\mathbf{N})x \mid x\rangle = Q_m^{(2)}(\mathbf{N}; x) \;\; ; \forall \;\; x \in \mathcal{H}.$$

Remark 2.2. When $p = 2$ and $\mathcal{X} = \mathcal{H}$, Definition 2.1 coincides with [5, Definition 2.1].

Remark 2.3. (i) Let $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ be a commuting tuple of operators. Then $\mathbf{N}$ is a $(1, p)$-expansive tuple if

$$\|x\|^p \leq \sum_{1\leq j\leq d} \|N_j x\|^p, \qquad (\forall \; x \in \mathcal{X}) \tag{2.1}$$

and it is a $(1, p)$-contractive tuple if

$$\|x\|^p \geq \sum_{1\leq j\leq d} \|N_j x\|^p, \qquad (\forall \; x \in \mathcal{X}). \tag{2.2}$$

(ii) If $d = 2$, let $\mathbf{N} = (N_1, N_2) \in \mathcal{B}[\mathcal{X}]^2$ be a commuting pair of operators. Then $\mathbf{N}$ is a $(2, p)$-expansive pair if

$$\|x\|^p \leq 2(\|N_1x\|^p + \|N_2x\|^p) - (\|N_1^2x\|^p + \|N_2^2x\|^p + 2\|N_1N_2x\|^p) \;\; \forall \; x \in \mathcal{X}, \tag{2.3}$$

and it is a $(2, p)$-contractive pair if

$$\|x\|^p \geq 2(\|N_1x\|^p + \|N_2x\|^p) - (\|N_1^2x\|^p + \|N_2^2x\|^p + 2\|N_1N_2x\|^p) \;\;\; \forall \; x \in \mathcal{X}. \tag{2.4}$$

(iii) Let $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ be a commuting tuple of operators. Then $\mathbf{N}$ is a $(2, p)$-expansive tuple if

$$\|x\|^p \leq 2\sum_{1\leq j\leq d} \|N_jx\|^p - \Big(\sum_{1\leq j\leq d} \|N_j^2x\|^p + 2\sum_{1\leq j<k\leq d} \|N_jN_kx\|^p\Big) \;\;\; \forall \;\; x \in \mathcal{X}, \tag{2.5}$$

and it is a $(2, p)$-contractive tuple if

$$\|x\|^p \geq 2\sum_{1\leq j\leq d} \|N_jx\|^p - \Big(\sum_{1\leq j\leq d} \|N_j^2x\|^p + 2\sum_{1\leq j<k\leq d} \|N_jN_kx\|^p\Big) \;\;\; \forall \;\; x \in \mathcal{X}. \tag{2.6}$$

Remark 2.4. Since the operators $N_1, \cdots, N_d$ are commuting, every permutation of an (m, p)-expansive tuple or a (m, p)-contractive tuple is also an (m, p)-expansive tuple or a (m, p)-contractive tuple .

Example 2.1. (1) Every (m, p)-isometric tuple of operators on a Banach space is an (m, p)-expansive tuple and a (m, p)-contractive tuple of operators.

(2) Every $(1, p)$-isometric tuple is a completely p-hyperexpansive tuple and it is also completely p-hypercontractive.

The following examples show that there exists a (m, p)-expansive tuple (resp. (m, p)-contractive tuple) of operators that is not an (m, p)-isometric tuple for some positive integer m.

Example 2.2. Let $\mathcal{X} = \mathbb{C}^3$ be equipped with the Euclidean norm $\|\ .\ \|_2$. Consider

$$N_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in \mathcal{B}[\mathbb{C}^3] \text{ and } N_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{B}[\mathbb{C}^3].$$

Then, the pair $\mathbf{N} = (N_1, N_2)$ is $(2, p)$-contractive pair for e $p \in (0,\ 1)$ on $(\mathcal{X} = \mathbb{C}^3, \|.\|_2)$.

In fact, it is easy to verify that $N_1N_2 = N_2N_1$. By direct computation, we have $N_1N_2 = N_1, N_1^2 = 3N_1$ and $N_2^2 = I_3$. Furthermore, for any vector $x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, direct computation yields

$$\|N_1x\|_2^p = \|N_1N_2x\|_2^p$$

$$\|N_2x\|_2^p = \|x\|_2^p = \|N_2^2x\|_2^p$$

$$\|N_1^2x\|_2^p = \|3N_1x\|_2^p = 3^p\|N_1x\|_2^p.$$

Hence,

$$\begin{aligned} & 2\Big(\|N_1x\|_2^p + \|N_2x\|_2^p\Big) - \Big(\|N_1^2x\|_2^p + \|N_2^2x\|_2^p + 2\|N_1N_2x\|_2^p\Big) \\ = \ & 2\Big(\|N_1x\|_2^p + \|x\|_2^p\Big) - \Big(3^p\|N_1x\|_2^p + \|x\|_2^p + 2\|N_1x\|_2^p\Big) \\ = \ & 2\|N_1x\|_2^p + 2\|x\|_2^p - \Big(3^p\|N_1x\|_2^p - \|x\|_2^p - 2\|N_1x\|_2^p\Big) \\ = \ & \|x\|_2^p - 3^p\|N_1x\|_2^p \\ \leq \ & \|x\|_2^p. \end{aligned}$$

Hence, $\mathbf{N} = (N_1, N_2)$ is $(2, p)$-contractive tuple for $p \in (0, 1)$.

Example 2.3. Let $\mathcal{X}$ be a normed space and $I_{\mathcal{X}}$ be the identity operator. Then, $(5I_{\mathcal{X}}, I_{\mathcal{X}}, I_{\mathcal{X}}) \in \mathcal{B}[\mathcal{X}]^3$ is a $(2, p)$-contractive tuple of operators which is not a $(2, p)$-isometric tuple.

Example 2.4. Let $p \in (0, \infty)$ and $N \in \mathcal{B}[\mathcal{X}]$ be an (m, p)-hyperexpansive (resp. (m, p)-hypercontractive) operator (see [29, Definition 1.3]) and $\gamma = (\gamma_1, \cdots, \gamma_d) \in (\mathbb{C}^d,\ \|.\|_p)$ with

$$\|\gamma\|_p^p = \sum_{1\leq j\leq d} |\gamma_j|^p = 1.$$

Then, the operator tuple $\mathbf{N} = (N_1, \cdots, N_d)$ with $N_j = \gamma_jN$ for $j = 1, 2, \cdots, d$ is an (m, p)-hyperexpansive tuple (resp. (m, p)-hypercontractive tuple).

In fact, it is clair that $N_iN_j = N_jN_i$ for all $1 \leq i; j \leq d$. Furthermore, by the multinomial expansion, we get

$$\Big(|\gamma_1|^p + |\gamma_2|^p + \cdots + |\gamma_d|^p\Big)^j = \sum_{\beta_1+\beta_2+\cdots+\beta_d=j} \binom{j}{\beta_1, \beta_2, \cdots, \beta_d} \prod_{1\leq i\leq d} |\gamma_i|^{p\beta_i}$$

$$= \sum_{|\beta|=j} \frac{j!}{\beta!} |\gamma^\beta|^p.$$

On the other hand, we have for all $k \in \{1, \cdots, m\}$ and $x \in \mathcal{X}$

$$\begin{aligned} Q_k^{(p)}(\mathbf{N};\ x) &= \sum_{0\le j\le k} (-1)^j \binom{k}{j} \Big(\sum_{|\beta|=j} \frac{j!}{\beta!} \|\mathbf{N}^\beta x\|^p \Big) \\ &= \sum_{0\le j\le k} (-1)^j \binom{k}{j} \Big(\sum_{|\beta|=j} \frac{j!}{\beta!} \|\gamma^\beta N^{|\beta|} x\|^p \Big) \\ &= \sum_{0\le j\le k} (-1)^j \binom{k}{j} \|N^j x\|^p. \end{aligned}$$

It follows that if N is (m, p)-hyperexpansive then $Q_k^{(p)}(\mathbf{N},\ x) \le 0$ and if N is (m, p)-hypercontractive then $Q_k^{(p)}(\mathbf{N},\ x) \ge 0$.

The following Proposition generalizes [16, Lemma 2.1].

Proposition 2.1. *Let* $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a commuting tuple of operators. The following identities hold for all* $x \in \mathcal{X}$ *and* $m \in \mathbb{N}$.

$$Q_{m+1}^{(p)}(\mathbf{N},\ x) = Q_m^{(p)}(\mathbf{N},\ x) - \sum_{1\le j\le d} Q_m^{(p)}(\mathbf{N},\ N_j x). \tag{2.7}$$

$$Q_m^{(p)}(\mathbf{N}; x) = (-1)^m \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|\mathbf{N}^\alpha x\|^p - \sum_{0\le k\le m-1} (-1)^{m-k} \binom{m}{k} Q_k^{(p)}(\mathbf{N}, x). \tag{2.8}$$

$$\begin{aligned} &\sum_{1\le j\le d} \Big(\sum_{0\le k\le m-1} (-1)^k \binom{n}{k} Q_k^{(p)}(\mathbf{N},\ N_j x) \Big) \\ = &\sum_{0\le k\le m-1} (-1)^k \binom{n+1}{k} Q_k^{(p)}(\mathbf{N},\ x) + (-1)^m \binom{n}{m-1} Q_m^{(p)}(\mathbf{N},\ x). \end{aligned} \tag{2.9}$$

Proof. The identity in (2.7) follows from [23, Proposition 3.1] after noting the slight differences in notation, and so its proof is omitted.

We prove the equality (2.8) by induction on $m \ge 1$. For $m = 1$, it is true, since

$$Q_m^{(p)}(\mathbf{N},\ x) = \sum_{0\le k\le m} (-1)^k \binom{m}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} \|\mathbf{N}^\alpha x\|^p, \quad \text{where } \ Q_0^{(p)}(\mathbf{N},\ x) = \|x\|^p.$$

Assume that the induction hypothesis for some integer $m \ge 1$. By (2.7) we have

$$Q_{m+1}^{(p)}\mathbf{N},\ x) = Q_m^{(p)}(\mathbf{N},\ x) - \sum_{1\le j\le d} Q_m^{(p)}(\mathbf{N},\ N_j x) \ \text{ for all integers } \ m \ge 1.$$

By the induction hypothesis and in view of (2.8) we have that

$$Q_{m+1}^{(p)}(\mathbf{N},\ x)$$

$$
\begin{aligned}
&= (-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} \|\mathbf{N}^\beta x\|^p - \sum_{0\le k\le m-1} (-1)^{m-k} \binom{m}{k} Q_k^{(p)}(\mathbf{N},\ x) \\
&\quad - \sum_{1\le j\le d} \left((-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} \|\mathbf{N}^\beta N_j x\|^p - \sum_{0\le k\le m-1} (-1)^{m-k} \binom{m}{k} Q_k^{(p)}(\mathbf{N},\ N_j x) \right) \\
&= (-1)^{m+1} \sum_{|\alpha|=m+1} \frac{(m+1)!}{\beta!} \|\mathbf{N}^\beta x\|^p \\
&\quad +(-1)^m \sum_{|\beta|=m} \frac{m!}{\beta!} \|\mathbf{N}^\beta x\|^p + \sum_{1\le k\le m} (-1)^{m-k} \binom{m}{k-1} Q_k^{(p)}(\mathbf{N},\ x) \\
&= (-1)^{m+1} \sum_{|\alpha|=m+1} \frac{(m+1)!}{\beta!} \|\mathbf{N}^\beta x\|^p + Q_m^{(p)}(\mathbf{N},\ x) + \sum_{0\le k\le m-1} (-1)^{m-k} \binom{m}{k} Q_k^{(p)}(\mathbf{N},\ x) \\
&\quad + \sum_{1\le k\le m} (-1)^{m-k} \binom{m}{k-1} Q_k^{(p)}(\mathbf{N},\ x) \\
&= (-1)^{m+1} \sum_{|\beta|=m+1} \frac{(m+1)!}{\beta!} \|\mathbf{N}^\beta x\|^p + Q_m^{(p)}(\mathbf{N},\ x) + \sum_{1\le k\le m-1} (-1)^{m-k} (\binom{m}{k-1} \\
&\quad + \binom{m}{k}) Q_k^{(p)}(\mathbf{N},\ x) \\
&\quad +(-1)^m Q_0^{(p)}(\mathbf{N},\ x) + \binom{m}{m-1} Q_m^{(p)}(\mathbf{N},\ x) \\
&= (-1)^{m+1} \sum_{|\beta|=m+1} \frac{(m+1)!}{\beta!} \|\mathbf{N}^\beta x\|^p - \sum_{0\le k\le m} (-1)^{m+1-k} \binom{m+1}{k} Q_k^{(p)}(\mathbf{N},\ x).
\end{aligned}
$$

The conclusion of (2.8) for $(m+1)$ is now immediate.

To prove (2.9), we have by (2.7)

$$
\begin{aligned}
&\sum_{1\le k\le m} (-1)^k \binom{n}{k-1} Q_k^{(p)}(\mathbf{N},\ x) \\
&= \sum_{1\le k\le m} (-1)^k \binom{n}{k-1} \left(Q_{k-1}^{(p)}(\mathbf{N},\ x) - \sum_{1\le j\le d} Q_{k-1}^{(p)}(\mathbf{N},\ N_j x) \right) \\
&= -\sum_{0\le k\le m-1} (-1)^k \binom{n}{k} Q_k^{(p)}(\mathbf{N},\ x) + \sum_{1\le j\le d} \sum_{0\le k\le m-1} (-1)^k \binom{n}{k} Q_k^{(p)}(\mathbf{N},\ N_j x)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\sum_{1\le j\le d} \sum_{0\le k\le m-1} (-1)^k \binom{n}{k} Q_k^{(p)}(\mathbf{N};\ N_j x) \\
&= \sum_{1\le k\le m} (-1)^k \binom{n}{k-1} Q_k^{(p)}(\mathbf{N};\ x) + \sum_{0\le k\le m-1} (-1)^k \binom{n}{k} Q_k^{(p)}(\mathbf{N},\ x) \\
&= (-1)^m \binom{n}{m-1} Q_m^{(p)}(\mathbf{N},\ x) + \sum_{1\le k\le m-1} (-1)^k \left(\binom{n}{k-1} + \binom{n}{k} \right) Q_k^{(p)}(\mathbf{N};\ x)
\end{aligned}
$$

$$
\begin{aligned}
&+Q_0^{(p)}(\mathbf{N},\ x) \\
= &\sum_{0\leq k\leq m-1}(-1)^k\binom{n+1}{k}Q_k^{(p)}(\mathbf{N},\ x)+(-1)^m\binom{n}{m-1}Q_m^{(p)}(\mathbf{N};\ x).
\end{aligned}
$$

This completes the proof of the proposition. □

It is well-known that the class of (m, p)-isometric tuples is a subset of the class of $(m+1, p)$-isometric tuples. The following example shows that the class of (m, p)-expansive tuples and $(m+1, p)$-expansive tuples are independent.

Example 2.5. Let $\mathbf{N} = (I_{\mathcal{X}}, I_{\mathcal{X}}, I_{\mathcal{X}}) \in \mathcal{B}[\mathcal{X}]^3$. A simple computation shows that

(1) $\mathbf{N}$ is a $(1, p)$-expansive tuple but not a $(2, p)$-expansive tuple.

(2) $\mathbf{N}$ is a $(2, p)$-contractive but not a $(1, p)$-contractive.

The following Lemma generalizes [17, Proposition 5.3].

Lemma 2.1. *Let* $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a commuting tuple that is a* $(2, p)$*-expansive tuple. Then the following statements hold.*

$$\sum_{|\beta|=n}\frac{n!}{\beta!}\|\mathbf{N}^\beta x\|_{\mathcal{X}}^p \leq (1-n)\|x\|_{\mathcal{X}}^p + n\Big(\sum_{1\leq j\leq d}\|N_jx\|_{\mathcal{X}}^p\Big),\quad \forall\ \ x\in\mathcal{X},\ \forall\ n\in\mathbb{N}. \tag{2.10}$$

$$\sum_{1\leq j\leq d}\|N_jx\|^p \geq \frac{n}{n-1}\|x\|^p \quad \forall\ x\in\mathcal{X},\ n\in\mathbb{N}, n\neq 1. \tag{2.11}$$

$$\sum_{1\leq j\leq d}\|N_jx\|^p \geq \|x\|^p \quad \forall\ x\in\mathcal{X}. \tag{2.12}$$

Proof. We shall prove the inequality (2.10) by induction on n. For $n = 0$ or $n = 1$ it is clear. Assume that (2.10) is true for n and prove it for $n + 1$. Indeed, in view of [23, Lemma 2.1], it follows that

$$\sum_{|\beta|=n+1}\frac{(n+1)!}{\beta!}\|\mathbf{N}^\alpha x\|^p = \sum_{1\leq k\leq d}\Big(\sum_{|\beta|=n}\frac{n!}{\beta!}\|\mathbf{N}^\beta N_kx\|^p\Big).$$

Therefore, by the induction hypothesis, we get

$$
\begin{aligned}
&\sum_{|\beta|=n+1}\frac{(n+1)!}{\beta!}\|\mathbf{N}^\beta x\|^p \\
\leq\ &(1-n)\sum_{1\leq k\leq d}\|N_kx\|^p + n\sum_{1\leq k\leq d}\Big(\sum_{1\leq j\leq d}\|N_jN_kx\|^p\Big) \\
=\ &(1-n)\sum_{1\leq k\leq d}\|N_kx\|^p + n\sum_{1\leq j\leq d}\|N_j^2x\|^p + 2n\Big(\sum_{1\leq j<k\leq d}\|N_jN_kx\|^p\Big).
\end{aligned}
$$

Since $\mathbf{N}$ is a $(2, p)$-expansive tuple, it follows from (2.5)

$$\sum_{|\beta|=n+1}\frac{(n+1)!}{\beta!}\|\mathbf{N}^\beta x\|^p$$

$$\leq \ (1-n)\sum_{1\leq k\leq d}\|N_kx\|_{\mathcal{X}}^p + n\Big(-\|x\|^p - 2\sum_{1\leq k\leq d}\|N_kx\|^p\Big)$$
$$\leq \ -n\|x\|^p + (n+1)\Big(\sum_{1\leq k\leq d}\|N_kx\|^p\Big),$$

so that (2.10) holds for $n+1$.

The inequality (2.11) follows from (2.10) and the inequality (2.12) follows from (2.11) by taking $n \longrightarrow \infty$. □

Remark 2.5. There is an immediate related consequence of this result. If **N** is a $(2, p)$-expansive tuple, then **N** is a $(1, p)$-expansive tuple i.e., **N** is a $(2, p)$-hyperexpansive tuple.

Lemma 2.2. *Let* $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a commuting tuple that is a* $(2, p)$*-contractive tuple. Then*

$$\sum_{|\beta|=n}\frac{n!}{\beta!}\|\mathbf{N}^{\beta}x\|^p \geq (1-n)\|x\|^p + n\Big(\sum_{1\leq j\leq d}\|N_jx\|^p\Big), \ \ \forall \ \ x\in\mathcal{X}, \ \forall \ n\in\mathbb{N}. \tag{2.13}$$

Proof. We omit the proof since it is similar to the one of Lemma 2.1. □

Remark 2.6. Let $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ be a commuting tuple of operators. The null space of **N** is defined by

$$\mathcal{N}(\mathbf{N}) := \{x\in\mathcal{X} \ / \ N_1x = ... = N_dx = 0\} = \bigcap_{1\leq j\leq d}\mathcal{N}(N_j).$$

The rang of **N** is given by

$$\mathcal{R}(\mathbf{N}) := \{z\in\mathcal{X} \ /\exists \ \ x_1,\cdots,x_d\in\mathcal{X}: \ z = N_1x_1+\cdots+N_dx_d \ \} = \sum_{1\leq j\leq d}\mathcal{R}(N_j).$$

We discuss below several consequences of Proposition 2.1.

Proposition 2.2. *Let* $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ *be commuting tuple of operators such that* $\mathbf{N}_{/\overline{\mathcal{R}(\mathbf{N})}} := (N_{1/\overline{\mathcal{R}(\mathbf{N})}}, \cdots, N_{d/\overline{\mathcal{R}(\mathbf{N})}})$*-is an* $(m-1, p)$*-isometric tuple. Then following properties hold.*
(1) **N** *is an* (m, p)*-expansive tuple if and only if* **N** *is a* $(m-1, p)$*-expansive tuple on* $\mathcal{X}$*.*
(2) **N** *is a* (m, p)*-contractive tuple if and only if* **N** *is a* $(m-1, p)$*-contractive tuple.*

Proof. In the first step, we note that $\overline{\mathcal{R}(N_j)} \subset \overline{\mathcal{R}(\mathbf{N})}$ for all $j = 1, \cdots, d$. In view of (2.7), we get

$$Q_m^{(p)}(\mathbf{N};x) = Q_{m-1}^{(p)}(\mathbf{N};x) - \sum_{1\leq j\leq d}Q_{m-1}^{(p)}(\mathbf{N}; \ N_jx), \ \ \forall \ x\in\mathcal{X}.$$

Since **N** is an $(m-1, \ p)$-isometric tuple on $\mathbf{N}_{/\overline{\mathcal{R}(\mathbf{N})}}$, we deduce that

$$Q_m^{(p)}(\mathbf{N};x) = Q_{m-1}^{(p)}(\mathbf{N};x), \ \forall \ x\in\mathcal{X}.$$

The desired results in the statements (1) and (2) follow immediately. □

Proposition 2.3. *Let* $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ *be commuting tuple of operators such that* $\mathbf{N}_{/\overline{\mathcal{R}(\mathbf{N})}} := (N_{1\,/\overline{\mathcal{R}(\mathbf{N})}}, \cdots, N_{d\,/\overline{\mathcal{R}(\mathbf{N})}})$ *is a* $(1,\ p)$*-isometric tuple. The following properties hold.*
(1) *If* $\mathbf{N}$ *is an* (m, p)*-expansive tuple, then* $\mathbf{N}$ *is an* (m, p)*-hyperexpansive tuple.*
(2) *If* $\mathbf{N}$ *is an* (m, p)*-contractive tuple, then* $\mathbf{N}$ *is an* (m, p)*-hypercontractive tuple.*

Proof. By (2.7), we have for all $k \in \{1, 2, \cdots, m\}$

$$Q_k^{(p)}(\mathbf{N}; x) = Q_{k-1}^{(p)}(\mathbf{N}; x) - \sum_{1\le j\le d} Q_{k-1}^{(p)}(\mathbf{N};\ N_j x), \quad \forall\ x \in \mathcal{X}.$$

If $\mathbf{N}$ is an $(1, p)$-isometric tuple on $\overline{\mathcal{R}(\mathbf{N})}$, it is well known that $\mathbf{N}$ is an (k, p)-isometric tuple on $\overline{\mathcal{R}(\mathbf{N})}$ for $k = 1, \cdots, m$. Consequently,

$$Q_1^{(p)}(\mathbf{N}; x) = Q_2^{(p)}(\mathbf{N}; x) = ... = Q_{m-1}^{(p)}(\mathbf{N}; x) = Q_m^{(p)}(\mathbf{N}; x).$$

If $\mathbf{N}$ is a (m, p)-expansive tuple, it follows that (1) is valid.
By the same argument as above, (2) is obtained. □

The next theorem shows that certain (m, p)-expansive (resp. (m, p)-contractive) tuples are (m, p)-hyperexpansive (resp. (m, p)-hypercontractive) tuples.

Theorem 2.1. *Let* $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}[\mathcal{X}]^d$ *be a tuple of commuting operators. The following statements hold.*
(1) *If* $\mathbf{N}$ *is an* (m, p)*-expansive tuple and* $\sup_n \Big(\sum_{|\beta|=n} \frac{n!}{\beta!} \|\mathbf{N}^\beta x\|^p \Big) < \infty$ *for all* $x \in \mathcal{X}$*, then* $\mathbf{N}$ *is an* (m, p)*-hyperexpansive tuple.*
(2) *If* $\mathbf{N}$ *is an* (m, p)*-contractive tuple and* $\sup_n \Big(\sum_{|\alpha|=n} \frac{n!}{\alpha!} \|\mathbf{N}^\alpha x\|^p \Big) < \infty$ *for all* $x \in \mathcal{X}$*, then* $\mathbf{N}$ *is a* (m, p)*-hypercontractive tuple.*

Proof. Assume that $\mathbf{N}$ is an (m, p)-expansive tuple. From (2.7), it is clear that

$$Q_m^{(p)}(\mathbf{N}, x) \le 0 \ \ \forall\ x \in \mathcal{X} \iff Q_{m-1}^{(p)}(\mathbf{N}, x) \le \sum_{1\le j\le d} Q_{m-1}^{(p)}(\mathbf{N}, N_j x) \quad \forall\ x \in \mathcal{X}.$$

It can easily be established that

$$\begin{aligned} Q_{m-1}^{(p)}(\mathbf{N}, x) \le \sum_{1\le k_1\le d} Q_{m-1}^{(p)}(\mathbf{N}, N_{k_1} x) \le \cdots \quad &\le \sum_{1\le k_1,\cdots,k_d\le d} Q_{m-1}^{(p)}(\mathbf{N}, N_{k_d}\cdots N_{k_1} x) \\ &\le \cdots\cdots \\ &\le \cdots\cdots \\ &\le \sum_{1\le k_1,\cdots,k_d\le d} Q_{m-1}^{(p)}(\mathbf{N}, N_{k_d}^{n_d}\cdots N_{k_1}^{n_1} x). \end{aligned}$$

Since for any $\mathbf{N} = (N_1, \cdots, N_d) \in \mathcal{B}(\mathcal{X})^d$,

$$Q_{m-1}^{(p)}(\mathbf{N}, N_{k_d}^{n_d}\cdots N_{k_1}^{n_1} x) = Q_{m-2}^{(p)}(\mathbf{N}, N_{k_d}^{n_d}\cdots N_{k_1}^{n_1} x) - \sum_{1\le j\le d} Q_{m-2}^{(p)}(\mathbf{N}, N_j N_{k_d}^{n_d}\cdots N_{k_1}^{n_1} x),$$

it is now easy to see that for $1 \leq k_1, \cdots ; k_d \leq d$,

$$\begin{aligned}
& Q^{(p)}_{m-1}(\mathbf{N}, N_{k_d}^{n_d} \cdots N_{k_1}^{n_1} x) \\
= & \left\{ \sum_{0\leq k\leq m-2} (-1)^k \binom{m-2}{k} \Big(\sum_{|\beta|=k} \frac{k!}{\beta!} \Big(\|\mathbf{N}^\beta N_{k_d}^{n_d} \cdots N_{k_1}^{n_1} x\|^p - \sum_{1\leq j\leq d} \|\mathbf{N}^\beta N_j N_{k_d}^{n_d} \cdots N_{k_1}^{n_1} x\|^p \Big) \Big) \right\} \\
= & \left\{ \sum_{0\leq k\leq m-2} (-1)^k \binom{m-2}{k} \Big(\sum_{|\beta|=k} \frac{k!}{\beta!} \|\mathbf{N}^\beta N_{k_d}^{n_d} \cdots N_{k_1}^{n_1} x\|^p - \sum_{1\leq j\leq d} \sum_{|\beta|=k} \frac{k!}{\beta!} \|\mathbf{N}^\beta N_j N_{k_d}^{n_d} \cdots N_{k_1}^{n_1} x\|^p \Big) \right\}.
\end{aligned}$$

Set

$$a_{n_1,\cdots,n_d} = \sum_{|\beta|=k} \frac{k!}{\beta!} \|\mathbf{N}^\beta N_{k_d}^{n_d} \cdots N_{k_1}^{n_1} x\|^p - \sum_{1\leq j\leq d} \sum_{|\beta|=k} \frac{k!}{\beta!} \|\mathbf{N}^\beta N_j N_{k_d}^{n_d} \cdots N_{k_1}^{n_1} x\|^p.$$

Under the assumption that $\sup_n \Big(\sum_{|\beta|=n} \frac{n!}{\alpha!} \|\mathbf{N}^\beta x\|^p \Big) < \infty$, it follows that the sequence $(a_{n_1,\cdots,n_d})_{n_1,\cdots n_d}$ is bounded. Hence, there is a subsequence $(a_{n_{k_1},\cdots,n_{n_{k_d}}})_{n_{k_1},\cdots n_{k_d}}$ which converges. By a direct calculation we get

$$Q^{(p)}_{m-1}(\mathbf{N}, N_{k_d}^{n_{k_d}} \cdots N_{k_1}^{n_{k_1}} x) \longrightarrow 0 \ \text{ as } \ n_{k_j} \longrightarrow \infty, \ j = 1, \cdots, d.$$

This means that $Q^{(p)}_{m-1}(\mathbf{N}, x) \leq 0$. Consequently, $\mathbf{N}$ is an $(m-1, p)$-expansive tuple.
By repeating this process, we reach the following inequalities $Q^{(p)}_k(\mathbf{N}, x) \leq 0$ for $k = 1, \cdots, m$, from which $\mathbf{N}$ is a (m, p)-hyperexpansive tuple as desired.
(2) Using the fact that $\mathbf{N}$ is an (m, p)-contractive tuple and together with (2.7), we obtain

$$Q^{(p)}_m(\mathbf{N}, x) \geq 0 \ \ \forall\, x \in \mathcal{X} \iff Q^{(p)}_{m-1}(\mathbf{N}, x) \geq \sum_{1\leq j\leq d} Q^{(p)}_{m-1}(\mathbf{N}, N_j x) \quad \forall\, x \in \mathcal{X}.$$

It can easily be established that

$$\begin{aligned}
Q^{(p)}_{m-1}(\mathbf{N}, x) \geq \sum_{1\leq k_1\leq d} Q^{(p)}_{m-1}(\mathbf{N}, N_{k_1} x) \geq \cdots & \geq \sum_{1\leq k_1, \cdots, k_d\leq d} Q^{(p)}_{m-1}(\mathbf{N}, N_{k_d} \cdots N_{k_1} x) \\
& \geq \cdots \cdots \\
& \geq \cdots \cdots \\
& \geq \sum_{1\leq k_1, \cdots, k_d\leq d} Q^{(p)}_{m-1}(\mathbf{N}, N_{k_d}^{n_d} \cdots N_{k_1}^{n_1} x).
\end{aligned}$$

Now, using the line of argument from the proof of statement (1), one can prove that

$$Q^{(p)}_{m-1}(\mathbf{N}, N_{k_d}^{n_{k_d}} \cdots N_{k_1}^{n_{k_1}} x) \longrightarrow 0 \ \text{ as } \ n_{k_j} \longrightarrow \infty, \ j = 1, \cdots, d.$$

Thus, $Q^{(p)}_{m-1}(\mathbf{N}, x) \geq 0$, and hence, $\mathbf{N}$ is an $(m-1, p)$-contractive tuple. By repeating this process, we reach the following inequalities

$$Q^{(p)}_k(\mathbf{N}, x) \geq 0, \qquad 1 \leq k \leq m,$$

from which $\mathbf{N}$ is an (m, p)-hypercontractive tuple. □

3. Conclusions

In the work, we have introduced a new classes of operators known as (m, p)-hyperexpensive tuple and (m, p)-hypercontractive tuple. Several properties are proved by exploiting the special kind of structure of single operator. In the course of our investigation, we find some properties of (m, p)-hyperexpensive and (m, p)-hypercontractive for single operators which are retained by (m, p)-hyperexpensive tuple and (m, p)-hypercontractive tuple.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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