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Research article

Real hypersurfaces in complex space forms with special almost contact structures

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Abstract: In this paper, we prove that an almost contact metric structure of a real hypersurface in a complex space form is quasi-contact if and only if it is contact. We also classify real hypersurfaces whose associated almost contact metric structures are nearly Kenmotsu or cosymplectic, which gives several extensions of some earlier results in this field.

Keywords: real hypersurface; nonflat complex space form; almost contact metric structure
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1. Introduction

Let $M^n(c)$ be a complete and simply connected complex space form, which is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$,
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$,
- a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$,

where c denotes the constant holomorphic sectional curvature. In general, $M^n(c)$ is said to be a nonflat complex space form when $c \neq 0$. Let M be a real hypersurface of real dimension $2n-1$ immersed in $M^n(c)$, $n \geq 2$. On M, there exists a natural almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure on $M^n(c)$ and the normal vector field, where ξ and ϕ are called the structure vector field and the structure tensor field, respectively. The behavior of the almost contact metric structure reveals some important properties of the real hypersurfaces and this leads many authors to investigate geometry of a real hypersurface in a complex space form from the view points of the associated almost contact metric structures.

An almost contact metric manifold is called

- a contact metric manifold if $d\eta = \Phi$,

- an almost Kenmotsu manifold if $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta \wedge \Phi$,
- an almost cosymplectic manifold if $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 0$,

where the fundamental two-form Φ is determined by $\Phi = g(\cdot, \phi\cdot)$. According to [5], a contact metric manifold (resp. almost Kenmotsu and almost cosymplectic manifold) is said to be Sasakian (resp. Kenmotsu and cosymplectic manifold) if the associated almost contact metric structure is normal. There are some studies of a real hypersurface in a Kähler manifold whose associated almost contact metric structures are almost contact metric (see [21, 22]), contact metric (see [2, 13, 24, 27]), Sasakian (see [1, 19]), almost Kenmotsu (see [12]), almost cosymplectic (see [23, 25]) or normal (see [12]). Besides the above cases, the generalized Sasakian space forms on a real hypersurface in a nonflat complex space form were considered recently in [8]. All these results are nice characterizations of homogeneous, ruled or Hopf hypersurfaces in a complex space form. For example, it was proved in [1, Lemma 2] that a Sasakian real hypersurface in a nonflat complex space form must be one of Hopf and homogeneous real hypersurfaces. The studies of real hypersurfaces from view points of almost contact metric structure can also be seen in papers [30, 31]. Motivated by these results, in this paper, we aim to investigate the almost contact metric structures of real hypersurfaces being some other interesting types.

First, we show that an almost contact metric structure of a real hypersurface in a complex space form is quasi-contact if and only if it is contact, although they are not the same in general, and consequently, such hypersurfaces are classified completely. Second, we prove that there exist no real hypersurfaces whose associated almost contact metric structure is nearly Kenmotsu. However, when the almost contact metric structures are nearly cosymplectic or nearly Sasakian, the corresponding real hypersurfaces are classified completely. All these can be viewed as natural extensions of some previous results.

2. Preliminaries

Let M be a real hypersurface in a complex space form $M^n(c)$ and N be a unit normal vector field of M. Let $\overline{\nabla}$ be the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of the metric g, respectively. The Gauss and Weingarten formulas are given respectively by:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \overline{\nabla}_X N = -AX \tag{2.1}$$

for any vector fields X, Y, where A denotes the shape operator of M in $M^n(c)$. For any vector field X, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi. \tag{2.2}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi) \tag{2.4}$$

for any X, Y. If the structure vector field ξ is principal, that is, $A\xi = \alpha\xi$ at each point, where $\alpha = \eta(A\xi)$, then M is called a Hopf hypersurface. From the parallelism of the complex structure (i.e., $\overline{\nabla}J = 0$) of

$M^n(c)$ and using (2.1), (2.2) we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX \tag{2.6}$$

for any X, Y. Let R be the Riemannian curvature tensor of M. Because $M^n(c)$ is of constant holomorphic sectional curvature c, the Gauss and Codazzi equations of M in $M^n(c)$ are given respectively as the following:

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.8}$$

for any vector fields X, Y. All above mentioned basic knowledge regarding real hypersurfaces and almost contact metric manifolds can be found in [4, 9], respectively.

3. Almost contact real hypersurfaces

Recently, Kim, Park and Sekigawa in [20] introduced a generalization of contact metric manifolds, which was said to be the quasi-contact metric manifolds, and such manifolds received many attentions in recent literature [3, 10, 15]. More precisely, an almost contact metric manifold (ϕ, ξ, η, g) is said to be quasi-contact if the corresponding almost Hermitian cone is a quasi Kähler manifold, or equivalently,

$$(\nabla_X \phi)Y + (\nabla_{\phi X}\phi)\phi Y = 2g(X,Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \eta(Y)hX, \tag{3.1}$$

where the $(1, 1)$-type tensor field h is defined by

$$g(hX, Y) = \frac{1}{2}g((\mathcal{L}_\xi \phi)X, Y) \tag{3.2}$$

for any vector fields X and Y (see [20, Theorem 4.2]). Here, we refer the reader to [29, 32, 33] for some recent results on h-operators of real hypersurfaces. Any contact metric manifold is a quasi-contact one, but the converse is not necessarily valid in general cases. It has been proposed as an open question in [20]:

Does there exist a quasi-contact metric manifold of dimension ≥ 5 that is not a contact metric manifold?

This problem has been considered in [3, 10, 15] under some reasonable restrictions. In this paper, we aim to answer this question from the view point of real hypersurfaces and we prove the following result.

Lemma 3.1. *The almost contact metric structure of a real hypersurface in a complex space form is quasi-contact if and only if it is contact.*

Proof. If the almost contact metric structure of a real hypersurface in a complex space form is quasi-contact, then equality (3.1) is valid. Substituting (2.5) into this equality we obtain

$$\begin{aligned} &\eta(Y)AX - g(AX,Y)\xi - g(A\phi X, \phi Y)\xi \\ =& 2g(X,Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \eta(Y)hX \end{aligned} \tag{3.3}$$

for any vector fields X, Y. By a direct calculation, we have

$$
\begin{aligned}
g(hX, Y) =& \frac{1}{2} g((\nabla_\xi \phi)X - \phi A \phi X + \phi^2 AX, Y) \\
=& \frac{1}{2}(\eta(X)A\xi - \phi A \phi X - AX, Y),
\end{aligned}
\tag{3.4}
$$

where we used again (2.5). Substituting (3.4) into (3.3) we obtain

$$
\begin{aligned}
& \eta(Y)AX - g(AX, Y)\xi - g(A\phi X, \phi Y)\xi \\
=& 2g(X, Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \frac{1}{2}\eta(X)\eta(Y)A\xi \\
&+ \frac{1}{2}\eta(Y)\phi A \phi X + \frac{1}{2}\eta(Y)AX.
\end{aligned}
\tag{3.5}
$$

In (3.5), setting $Y = \xi$ we get

$$
\frac{1}{2}AX - \eta(AX)\xi = \eta(X)\xi - X - \frac{1}{2}\eta(X)A\xi + \frac{1}{2}\phi A \phi X. \tag{3.6}
$$

In (3.6), setting $X = \xi$ we get $A\xi = \eta(A\xi)\xi$, which is used back in (3.6) giving

$$
\frac{1}{2}AX - \frac{1}{2}\eta(A\xi)\eta(X)\xi - \eta(X)\xi + X - \frac{1}{2}\phi A \phi X = 0. \tag{3.7}
$$

With the aid of (2.3), the action of ϕ on (3.7) yields that

$$
A\phi + \phi A = -2\phi. \tag{3.8}
$$

With the aid of (3.8), according to a direct calculation we obtain

$$
\begin{aligned}
(\mathrm{d}\eta - \Phi)(X, Y) =& \frac{1}{2}(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])) - g(X, \phi Y) \\
=& \frac{1}{2} g(\phi AX, Y) - \frac{1}{2} g(\phi AY, X) - g(X, \phi Y) = 0
\end{aligned}
$$

for any vector fields X, Y, where we used (2.6). Obviously, this equality implies that the almost contact structure is contact. The converse is trivial because any contact metric structure is quasi-contact. □

Theorem 3.1. *The almost contact metric structure of a real hypersurface M in a complex space form $M^n(c)$ is quasi-contact if and only if one of the following cases is valid.*

- *If $M^n(c) = \mathbb{C}P^n(c)$, M is locally congruent to a geodesic sphere or a tube around complex hyperquadric $\mathbb{C}Q^{n-1}$.*
- *If $M^n(c) = \mathbb{C}H^n(c)$, M is locally congruent to a horosphere, a geodesic hypersphere, a tube over a complex hyperbolic hyperplane $\mathbb{C}H^{n-1}(c)$ or a tube around totally real totally geodesic $\mathbb{R}H^n(c/4)$.*
- *If $M^n(c) = \mathbb{C}^n$, M is locally congruent to a sphere S^{2n-1} or a product of a sphere and an Euclidean space $S^{n-1} \times \mathbb{R}^n$.*

Proof. If the ambient space is a nonflat complex space form, the proof follows from Lemma 3.1 and [2, Lemma 2]. If the ambient space is the complex Euclidean space, the proof follows from Lemma 3.1 and [24, Theorem 6.3]. □

Remark 3.1. A quasi-contact metric manifold of dimension three is necessarily contact, and hence Lemma 3.1 and also Theorem 3.1 for the case $n = 2$ are valid naturally from [2, 24].

An almost contact metric manifold is called a nearly Kenmotsu manifold (see [26]) if

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X \tag{3.9}$$

for any vector fields X, Y. From [4, 16], an almost contact metric manifold is said to be a Kenmotsu manifold if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

So, a Kenmotsu manifold must be a nearly Kenmotsu manifold, but the converse is not necessarily true. For example, a warped product $\mathbb{R} \times_{e^t} N$ admits a nearly Kenmotsu but not Kenmotsu structure, where N denotes a non-Kähler nearly Kähler manifold. It was proved in [14, Theorem 3] that every normal nearly Kenmotsu manifold is Kenmotsu.

Theorem 3.2. *There exist no real hypersurfaces in a complex space form whose associated almost contact metric structure is nearly Kenmotsu.*

Proof. Let M be a real hypersurface in a complex space form whose associated almost contact metric structure is nearly Kenmotsu, (3.9) is valid. Substituting (2.5) into (3.9) we obtain

$$\eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi = -\eta(X)\phi Y - \eta(Y)\phi X \tag{3.10}$$

for any vector fields X, Y. In (3.10), setting $Y = \xi$ gives

$$AX + \eta(X)A\xi - 2\eta(AX)\xi + \phi X = 0. \tag{3.11}$$

In (3.11), setting $X = \xi$, with the aid of (2.3), we obtain $A\xi = \eta(A\xi)\xi$. Applying this back in (3.11) we obtain

$$AX = \eta(A\xi)\eta(X)\xi - \phi X.$$

Taking the inner product of the above equality with Y yields

$$g(AX, Y) = \eta(A\xi)\eta(X)\eta(Y) - g(\phi X, Y).$$

The interchange of the roles of X and Y in the above equality gives

$$g(AY, X) = \eta(A\xi)\eta(X)\eta(Y) - g(\phi Y, X).$$

Recall that the shape operator A is symmetric. Therefore, the subtraction of the above equality from the previous one gives

$$g(\phi X, Y) = 0.$$

However, this is impossible because we get a contradiction if we select $Y = \phi X$ being a unit vector field. This completes the proof. □

Remark 3.2. It was proved in [19, Theorem] that there exist no real hypersurfaces in a nonflat complex space form whose associated almost contact metric structure is Kenmotsu. Theorem 3.2 is an extension of this result.

An almost contact metric manifold is called a nearly cosymplectic manifold (see [4, 6]) if

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0 \tag{3.12}$$

for any vector fields X, Y. An almost contact metric manifold is called a cosymplectic manifold (see [4]) if

$$\nabla\phi = 0.$$

So, a cosymplectic manifold must be a nearly cosymplectic, but the converse is not necessarily true.

Theorem 3.3. *The almost contact metric structure of a real hypersurface M in a complex space form $M^n(c)$ is nearly cosymplectic if and only if $c = 0$ and M is cylinderical.*

Proof. Let M be a real hypersurface in a complex space form whose associated almost contact metric structure is nearly cosymplectic, (3.12) is valid. Substituting (2.5) into (3.12) we obtain

$$\eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi = 0 \tag{3.13}$$

for any vector fields X, Y. In (3.13), setting $Y = \xi$ gives

$$AX + \eta(X)A\xi - 2\eta(AX)\xi = 0. \tag{3.14}$$

In (3.14), setting $X = \xi$, with the aid of (2.3), we obtain $A\xi = \eta(A\xi)\xi$. Applying this back in (3.14) we obtain

$$A = \eta(A\xi)\eta \otimes \xi. \tag{3.15}$$

Obviously, this implies $A\phi + \phi A = 0$. Ki and Suh in [17, Lemma 2.1] proved that if a real hypersurface in a complex space form $M^n(c)$ satisfies $A\phi + \phi A = 0$, then $c = 0$. More precisely, it has been proved in [17, Proposition 2.2] that when (3.15) for real hypersurfaces in the complex Euclidean space is true, the real hypersurface is cylinderical. The converse is easy to check because of (2.5). □

Remark 3.3. Olszak in [25, Theorem] proved that there are no real hypersurfaces in a nonflat complex space form whose associated almost contact metric structure is cosympelctic (see also [12]). Theorem 3.3 and this result are both extensions of [23, Theorem 3.1] in which the author proved that there are no cosymplectic real hypersurfaces in a complex space form with positive constant holomorphic sectional curvatures.

An almost contact metric manifold is called a nearly Sasakian manifold (see [4, 7]) if

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X \tag{3.16}$$

for any vector fields X, Y. An almost contact metric manifold is called a Sasakian manifold (see [4]) if

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X.$$

So, a Sasakian manifold is a nearly Sasakian manifold, but the converse is not necessarily true. It was proved in [18, Theorem 1] that the almost contact metric structure of a real hypersurface in a nonflat complex space form is Sasakian if and only if it is nearly Sasakian.

Theorem 3.4. *[18] The almost contact metric structure of a real hypersurface in a nonflat complex space form is nearly Sasakian if and only if the hypersurface is locally congruent to a geodesic sphere in $\mathbb{C}P^n$ or $\mathbb{C}H^n$, a horosphere in $\mathbb{C}H^n$ or a tube around totally geodesic $\mathbb{C}H^{n-1}$.*

Since the case of nonflat ambient spaces was considered, in view of this next we only consider the nearly Sasakian hypersurfaces in a flat complex space form.

Theorem 3.5. *The almost contact metric structure of a real hypersurface in a complex Euclidean space is nearly Sasakian if and only if it is locally isometric to a sphere S^{2n-1}.*

Proof. Let M be a real hypersurface in a complex Euclidean space whose associated almost contact metric structure is nearly Sasakian, (3.16) is valid. Substituting (2.5) into (3.16) we obtain

$$\eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X \tag{3.17}$$

for any vector fields X, Y. In (3.17), setting $Y = \xi$ gives

$$AX + \eta(X)A\xi - 2\eta(AX)\xi - \eta(X)\xi + X = 0. \tag{3.18}$$

In (3.18), setting $X = \xi$, with the aid of (2.3), we obtain $A\xi = \eta(A\xi)\xi$. Applying this back in (3.18) we obtain

$$AX = -X + (\eta(A\xi) + 1)\eta(X)\xi. \tag{3.19}$$

With the aid of (2.6), taking the covariant derivative of (3.19) we obtain

$$\begin{aligned}&(\nabla_Y A)X\\ =&Y(\eta(A\xi))\eta(X)\xi + (\eta(A\xi) + 1)g(\phi AY, X)\xi + (\eta(A\xi) - 1)\eta(X)\phi AY.\end{aligned} \tag{3.20}$$

Note that in this case, the holomorphic sectional curvature of the ambient space is zero and hence (2.8) becomes

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{3.21}$$

which is combined with (3.20) yielding

$$\begin{aligned}&Y(\eta(A\xi))\eta(X)\xi + (\eta(A\xi) + 1)g(\phi AY, X)\xi + (\eta(A\xi) + 1)\eta(X)\phi AY\\ =&X(\eta(A\xi))\eta(Y)\xi + (\eta(A\xi) + 1)g(\phi AX, Y)\xi + (\eta(A\xi) + 1)\eta(Y)\phi AX.\end{aligned} \tag{3.22}$$

In (3.22), setting $Y = \xi$ and using $A\xi = \eta(A\xi)\xi$, with the aid of (2.3), we obtain

$$\xi(\eta(A\xi))\eta(X)\xi = X(\eta(A\xi))\xi + (\eta(A\xi) + 1)\phi AX \tag{3.23}$$

for any vector field X. In view of $A\xi = \eta(A\xi)\xi$ and (2.3), the action of ϕ on the above equality gives

$$(\eta(A\xi) + 1)(AX - \eta(A\xi)\eta(X)\xi) = 0. \tag{3.24}$$

If there exists a point at which $\eta(A\xi) \neq -1$, it follows from (3.24) that $AX = \eta(A\xi)\eta(X)\xi$ at this point. Now putting this into (3.19) we obtain

$$X = \eta(X)\xi$$

for any vector field X. According to this, we arrive at a contradiction if we select X being a vector field orthogonal to ξ. Therefore, if follows immediately from (3.24) that

$$\eta(A\xi) = -1, \tag{3.25}$$

everywhere, and hence from (3.19) we obtain $A = -\mathrm{Id}$. Obviously, according to (2.7), we observe that the hypersurface is of constant sectional curvature 1. The converse is easy to check. □

4. Conclusions and perspectives

In this paper, we classified real hypersurfaces in a complex space form whose associated almost contact metric structures are special. As seen from Theorems 3.2–3.5, nearly Kenmotsu, nearly cosymplectic and nearly Sasakian structures have different effects on geometry of real hypersurfaces in a complex space form. However, unlike Theorems 3.3 and 3.5, Theorem 3.2 implies that there exists no real hypersurface in complex space forms whose associated almost contact metric structure is nearly Kenmotsu. According to this one may state that the nearly Kenmotsu structure is too strong for a real hypersurface in complex space forms. Therefore, it is very interesting to investigate the existence and classification problems for certain more weaker almost contact metric structures (for some other types of almost contact metric structures we refer the reader to [11]).

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conficts of interest.

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