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Research article

Numerical simulation and analysis of fractional-order Phi-Four equation

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Abstract: This paper introduces a novel numerical approach for tackling the nonlinear fractional Phi-four equation by employing the Homotopy perturbation method (HPM) and the Adomian decomposition method (ADM), augmented by the Shehu transform. These established techniques are adept at addressing nonlinear differential equations. The equation's complexity is reduced by applying the Shehu Transform, rendering it amenable to solutions via HPM and ADM. The efficacy of this approach is underscored by conclusive results, attesting to its proficiency in solving the equation. With extensive ramifications spanning physics and engineering domains like fluid dynamics, heat transfer, and mechanics, the proposed method emerges as a precise and efficient tool for resolving nonlinear fractional differential equations pervasive in scientific and engineering contexts. Its potential extends to analogous equations, warranting further investigation to unravel its complete capabilities.

Keywords: Shehu transform; Adomian decomposition method; homotopy perturbation method; fractional Phi-four equation; Caputo operator
Mathematics Subject Classification: 33B15, 34A34, 35A20, 35A22, 44A10

Nomenclature

ω : Independent variable; ϕ : Time; $\upsilon(\omega, \phi)$: Dependent function representing the physical quantity; α : Fractional order; S : Shehu transform; S^{-1} : Inverse Shehu transform; ϵ : Perturbation parameter

1. Introduction

In recent years, the utilization of non-integer order derivatives, also known as fractional derivatives, has gained remarkable traction across various scientific and engineering disciplines due to its ability to model complex phenomena with more accuracy and flexibility [1, 2]. Fractional calculus provides an extended framework that goes beyond the restrictions of typical integer-order derivatives, allowing for a more comprehensive modeling of processes with memory effects and long-range interactions [3, 4]. Mathematicians that study fractional calculus extend the idea of integrals and derivatives of integer orders to non-integer orders. The creation of fractional differential equations is the result of this concept's multiple applications in a variety of scientific and technical fields. Differential equations involving fractional derivatives of an unknown function are known as fractional differential equations. Complex processes that cannot be modelled using integer-order differential equations can be described by these equations. Numerous physical, biological, and engineering systems, including viscoelastic materials, diffusion processes, wave propagation, and control systems, have been modelled using them [5–7].

Nonlinear fractional differential equations are a specific category of fractional differential equations that demonstrate nonlinear characteristics, wherein the unknown function is present in nonlinear expressions [8, 9]. The resolution of these equations presents a greater level of difficulty compared to their linear equivalents, necessitating the utilisation of advanced analytical and numerical methods [10, 11]. In recent years, there has been significant interest in the examination of nonlinear fractional differential equations, with numerous academics dedicating their efforts to exploring their properties, solutions, and applications [12, 13]. Various techniques are employed to solve these equations, encompassing numerical methods, analytical methods such as fractional calculus, and perturbation methods [14, 15]. In brief, fractional nonlinear differential equations play a crucial role in the modelling of intricate systems that demonstrate nonlinear dynamics. These entities possess a wide range of applications across diverse disciplines and want advanced methodologies for their examination and resolution [16–21].

The fractional Phi-four equation is a non-linear partial differential equation that characterises the temporal and spatial evolution of a field variable. The equation under consideration can be seen as a fractional extension of the widely recognised Phi-four equation, which holds significant importance in the field of mathematical physics due to its relevance in the analysis of solitons and nonlinear waves. The equation known as the fractional Phi-four equation is characterised by the inclusion of a fractional derivative of the field variable with respect to time, a second-order derivative of the field variable with respect to space, and a cubic nonlinear factor. The inclusion of the fractional derivative in the system introduces a memory component, so enabling the manifestation of non-local characteristics and long-range interactions [22].

The fractional Phi-four equation is very important in the study of mathematical physics because of its capacity to demonstrate many dynamics and its wide applicability in many physics disciplines. The equation demonstrates a number of dynamic phenomena that are dependent on the parameters and starting circumstances, such as the formation of solitons, chaotic behaviour, and turbulence. In recent years, there has been a significant amount of study done on the aforementioned equation, owing to its relevance in a variety of areas such as condensed matter physics, statistical mechanics, nonlinear dynamics, and fluid mechanics [23, 24]. Researchers are presently investigating the equation and its many versions, employing fresh analytical and numerical methodologies to examine its behaviour under various situations [25–28]. Furthermore, research into the fractional Phi-four equation has made substantial contributions to the advancement of fractional calculus, a solid mathematical framework used for the analysis of complex systems showing memory and non-local phenomena. Because of its numerous applications, fractional calculus is widely used in a variety of scientific and engineering areas, including physics, chemistry, biology, and finance [29]. Depending on the parameter values and beginning conditions, the fractional Phi-four equation exhibits a variety of dynamic phenomena such as soliton production, chaotic behaviour, and turbulence. Several analytical and numerical techniques, such as the homotopy analysis method [30] and others [31, 32] have been developed to explore the dynamics of the fractional Phi-four equation.

In the field of nonlinear differential equation solving, the Homotopy Perturbation Method (HPM) and the Adomian Decomposition Method (ADM) are widely used numerical techniques. Liao proposed the HPM (Homotopy Perturbation Method) as a mathematical technique in 1992 [33]. Adomian, on the other hand, developed the ADM (Adomian Decomposition Method) as an alternate approach in 1988 [34]. The HPM (Homotopy Perturbation Method) includes creating a homotopy, or continuous transformation, that connects a linear problem with a known solution to the nonlinear problem under inquiry [35–38]. Particular advantages of High Performance Computing (HPC) technology may be seen in scientific research. This method works well for handling nonlinear problems with precision while avoiding the requirement for linearization. Utilising a homotopy parameter, the Homotopy Perturbation Method (HPM) simplifies complex nonlinear problems into more manageable linear ones, making it applicable in a variety of situations. It's important to remember that researchers can alter the convergence of solution series, which expands their practical applications. Because of its versatility, HPM may be easily integrated with a variety of methods, which improves its accuracy and effectiveness. The solution is obtained by working through a series of linear issues while progressively raising the homotopy parameter from zero to one. Numerous nonlinear problems, such as Burgers' equation [39], heat transport in porous media [40], and fractional differential equations [41], have been solved using the homotropy perturbation method, or HPM.

Numerous nonlinear issues, such as the nonlinear Schrödinger equation, have been tackled with the ADM approach [42]. A useful method for making nonlinear differential equations easier to solve using the HPM and ADM processes is the Shehu Transform. Numerous nonlinear scenarios, including the Duffing equation [43], the nonlinear Black-Scholes equation [44], and the Boussinesq equation [45], have been solved using the Shehu Transform.

This work is summarised in the following. We start Section 2 by providing a definition and description of the Shehu transform, which we employ in this study. The solution mechanism of the Shehu transform decomposition method (STDM) is explained in Section 3. The solution approach for the homotopy perturbation transform technique (HPTM) is explained in Section 4. For our current study, the results of numerical simulations are shown and discussed in Section 5. Lastly, the results of our investigation are presented in Section 6.

2. Basic definitions

Definition 2.1. Fractional derivative of $f \in C_{-1}^{n}$ is given in the sense of Caputo as the following [45,46]:

$$D_{\phi}^{\alpha}\upsilon(\omega,\phi) = \begin{cases} \frac{d^{n}\upsilon(\omega,\phi)}{d\phi^{n}}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)}\int_{0}^{\phi}(\omega,\phi-\vartheta)^{n-\alpha-1}\upsilon^{(n)}(\omega,\vartheta)d\vartheta, & n-1<\alpha<n, n\in\mathbb{N}. \end{cases}$$

Definition 2.2. Shehu transform is defined as follows [45, 46]:

$$S\left[Q(\phi)\right] = \int_{0}^{\infty} \mathrm{e}^{\left(-\frac{\wp\phi}{\varpi}\right)}Q(\phi)\mathrm{d}\phi.$$

Shehu transform will be transformed into Laplace transform by considering $\varpi = 1$,
Shehu transform will be transformed into Yang transformed by considering $\wp = 1$, where S is considered a Shehu Transform operator.

Definition 2.3. The Inverse Shehu transform operator is defined as [45, 46]
Let $S\left[Q(\phi)\right] = J(\wp,\varpi)$ and $S^{-1}[J(\wp,\varpi)] = Q(\phi)$,
then $Q(\phi) = S^{-1}[J(\wp,\varpi)] = \lim_{x\to\infty}\frac{1}{2\pi i}\int_{\beta+ix}^{\beta-ix}\frac{\mathrm{e}^{-\wp\phi}}{\varpi}J(\wp,\varpi)\mathrm{d}\wp$,
where $\wp$ and ϖ are considered as Shehu transform variables and β is a real constant.

Lemma 2.4. *Linearity property of Shehu transform [45, 46]:*
If $S\left[Q_1(\phi)\right] = J_1(\wp,\varpi)$ *and* $S\left[Q_2(\phi)\right] = J_2(\wp,\varpi)$,
Then $S\left[\alpha_1 Q_1(\phi) + \alpha_2 Q_2(\phi)\right] = \alpha_1 S\left[Q_1(\phi)\right] + \alpha_2 S\left[Q_2(\omega,\phi)\right]$,
$S\left[\alpha_1 Q_1(\phi) + \alpha_2 Q_2(\phi)\right] = \alpha_1 J_1(\wp,\varpi) + \alpha_2 S J_2(\wp,\varpi)$,
where α_1 *and* α_2 *are the arbitrary constants.*

Lemma 2.5. *Linearity property of inverse Shehu transform [45, 46]:*
If $S^{-1}\left[J_1(\wp,\varpi)\right] = Q_1(\phi)$ *and* $S^{-1}\left[J_2(\wp,\varpi)\right] = Q_2(\phi)$*, then,*

$$S^{-1}\left[\alpha_1 J_1(\wp,\varpi) + \alpha_2 J_2(\wp,\varpi)\right] = \alpha_1 S^{-1}\left[J_1(\wp,\varpi)\right] + \alpha_2 S^{-1}\left[J_2(\wp,\varpi)\right],$$
$$S^{-1}\left[\alpha_1 J_1(\wp,\varpi) + \alpha_2 J_2(\wp,\varpi)\right] = \alpha_1 Q_1(\phi) + \alpha_2 Q_2(\phi).$$

Definition 2.6. Shehu transform of Caputo fractional derivative (C.F.D) [45, 46]

$$S\left[D_{\phi}^{\alpha}Q\left(\eta_1,\phi\right)\right] = \frac{\wp^{\alpha}}{\varpi^{\alpha}}S\left[Q\left(\eta_1,\phi\right)\right] - \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}Q^{r}\left(\eta_1,0\right),\quad \theta = 1,2,3\cdots.$$

Definition 2.7. Mittag-Leffler function considered for two parameters [45, 46]

$$E_{\mu,\varpi}(n) = \sum_{k=0}^{\infty}\frac{n^{k}}{\Gamma(k\mu+\varpi)},$$

where $E_{1,1}(n) = \exp(n)$ and $E_{2,1}\left(n^{2}\right) = \cos(n)$.

3. Solution procedure of Shehu transform decomposition method (STDM)

In this section, we consider the nonlinear FDEs to demonstrate the basic idea of the projected algorithm as given

$$D_{\phi}^{\alpha}\upsilon(\omega,\phi) = \mathcal{P}_1(\omega,\phi) + \mathcal{Q}_1(\omega,\phi), 1 < \alpha \leq 2, \tag{3.1}$$

and

$$\upsilon(\omega,0) = \xi(\omega), \quad \frac{\partial}{\partial\phi}\upsilon(\omega,0) = \zeta(\omega).$$

where $D_{\phi}^{\alpha} = \frac{\partial^{\alpha}}{\partial\phi^{\alpha}}$ signifies fractional Caputo operator, $\mathcal{P}_1$, $\mathcal{Q}_1$ are respectively linear and non-linear operators.

Now we apply ST to obtain

$$\begin{aligned} &S[D_{\phi}^{\alpha}\upsilon(\omega,\phi)] = S[\mathcal{P}_1(\omega,\phi) + \mathcal{Q}_1(\omega,\phi)], \\ &\frac{\wp^{\alpha}}{\varpi^{\alpha}}S[\upsilon(\omega,\phi)] - \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0) = S[\mathcal{P}_1(\omega,\phi) + \mathcal{Q}_1(\omega,\phi)]. \end{aligned} \tag{3.2}$$

Now by employing inverse ST, I get

$$\upsilon(\omega,\phi) = \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0) + S^{-1}[\frac{\varpi^{\alpha}}{\wp^{\alpha}}S[\mathcal{P}_1(\omega,\phi) + \mathcal{Q}_1(\omega,\phi)]. \tag{3.3}$$

Now the solution is as

$$\upsilon(\omega,\phi) = \sum_{m=0}^{\infty}\upsilon_m(\omega,\phi). \tag{3.4}$$

The nonlinear terms $\mathcal{Q}_1$ is discarded as

$$\begin{aligned} &\mathcal{Q}_1(\omega,\phi) = \sum_{m=0}^{\infty}\mathcal{A}_m, \\ &\mathcal{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\ell^m}\left\{\mathcal{Q}_1\left(\sum_{k=0}^{\infty}\ell^k\omega_k\right)\right\}\right]_{\ell=0}. \end{aligned} \tag{3.5}$$

Using Eqs (3.4) and (3.5) into (3.3), I have

$$\sum_{m=0}^{\infty}\upsilon_m(\omega,\phi) = \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0) + S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}S\left\{\mathcal{P}_1(\sum_{m=0}^{\infty}\omega_m) + \sum_{m=0}^{\infty}\mathcal{A}_m\right\}\right]. \tag{3.6}$$

Comparing both sides allows for a straightforward approximation to be obtained.

$$\upsilon_0(\omega,\phi) = \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0),$$

$$\upsilon_1(\omega,\phi) = S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}S^{+}\{\mathcal{P}_1(\omega_0) + \mathcal{A}_0\}\right],$$

The general recursive equation can be derived as follows:

$$\upsilon_{m+1}(\omega,\phi) = S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}S^{+}\{\mathcal{P}_1(\omega_m) + \mathcal{A}_m\}\right].$$

4. Solution procedure of homotopy perturbation transform method (HPTM)

The basic idea of the projected algorithm is demonstrated by considering the nonlinear FDEs in this section.

$$D_{\phi}^{\alpha}\upsilon(\omega,\phi) = \mathcal{P}_1[\omega]\upsilon(\omega,\phi) + \mathcal{Q}_1[\omega]\upsilon(\omega,\phi), \quad 1 < \alpha \leq 2, \tag{4.1}$$

and

$$\upsilon(\omega,0) = \xi(\omega), \quad \frac{\partial}{\partial\phi}\upsilon(\omega,0) = \zeta(\omega).$$

The given expression involves the fractional Caputo operator D_{ϕ}^{α}, where α represents the order of differentiation, and ϕ represents the variable with respect to which differentiation is performed. Additionally, $\mathcal{P}_1[\omega]$ and $\mathcal{Q}_1[\omega]$ are the linear and nonlinear operators, respectively.

Using the ST, we obtain

$$S[D_{\phi}^{\alpha}\upsilon(\omega,\phi)] = S[\mathcal{P}_1[\omega]\upsilon(\omega,\phi) + \mathcal{Q}_1[\omega]\upsilon(\omega,\phi)], \tag{4.2}$$

$$\frac{\wp^{\alpha}}{\varpi^{\alpha}}S[\upsilon(\omega,\phi)] - \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0) = S[\mathcal{P}_1[\omega]\upsilon(\omega,\phi) + \mathcal{Q}_1[\omega]\upsilon(\omega,\phi)]. \tag{4.3}$$

By applying the inverse of ST, we get:

$$\upsilon(\omega,\phi) = \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0) + S^{-1}[\frac{\varpi^{\alpha}}{\wp^{\alpha}}S[\mathcal{P}_1[\omega]\upsilon(\omega,\phi) + \mathcal{Q}_1[\omega]\upsilon(\omega,\phi)]]. \tag{4.4}$$

By HPM, we get

$$\upsilon(\omega,\phi) = \sum_{k=0}^{\infty}\epsilon^{k}\upsilon_k(\omega,\phi). \tag{4.5}$$

The nonlinear terms are neglected, with ϵ serving as the homotopy parameter ranging from 0 to 1.

$$\mathcal{Q}_1[\omega]\upsilon(\omega,\phi) = \sum_{k=0}^{\infty}\epsilon^{k}H_n(\upsilon), \tag{4.6}$$

with $H_k(\upsilon)$ representing the He's polynomials

$$H_n(\upsilon_0,\upsilon_1,...,\upsilon_n) = \frac{1}{\Gamma(n+1)}D_{\epsilon}^{k}\left[\mathcal{Q}_1\left(\sum_{k=0}^{\infty}\epsilon^{i}\upsilon_i\right)\right]_{\epsilon=0}, \tag{4.7}$$

with $D_{\epsilon}^{k} = \frac{\partial^k}{\partial\epsilon^k}$.

Using Eqs (4.5) and (4.7) in Eq (4.4), we obtain

$$\sum_{k=0}^{\infty}\epsilon^{k}\upsilon_k(\omega,\phi) = \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0) + \epsilon\times\left(S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}S\{\mathcal{P}_1\sum_{k=0}^{\infty}\epsilon^{k}\upsilon_k(\omega,\phi) + \sum_{k=0}^{\infty}\epsilon^{k}H_k(\upsilon)\}\right]\right). \tag{4.8}$$

When we compare the ϵ coefficients, we arrive at the following conclusion:

$$
\begin{aligned}
&\epsilon^0 : \upsilon_0(\omega,\phi) = \upsilon(0) + \upsilon^{'}(0),\\
&\epsilon^1 : \upsilon_1(\omega,\phi) = S^{-1}\left[\frac{\varpi^\alpha}{\wp^\alpha} S\left(\mathcal{P}_1[\omega]\upsilon_0(\omega,\phi) + H_0(\upsilon)\right)\right],\\
&\epsilon^2 : \upsilon_2(\omega,\phi) = S^{-1}\left[\frac{\varpi^\alpha}{\wp^\alpha} S\left(\mathcal{P}_1[\omega]\upsilon_1(\omega,\phi) + H_1(\upsilon)\right)\right],\\
&.\\
&.\\
&.\\
&\epsilon^k : \upsilon_k(\omega,\phi) = S^{-1}\left[\frac{\varpi^\alpha}{\wp^\alpha} S\left(\mathcal{P}_1[\omega]\upsilon_{k-1}(\omega,\phi) + H_{k-1}(\upsilon)\right)\right], \quad k > 0, k \in N.
\end{aligned}
\tag{4.9}
$$

Thus, the analytical solution is

$$
\upsilon(\omega,\phi) = \lim_{M\to\infty} \sum_{k=1}^{M} \upsilon_k(\omega,\phi).
$$

Theorem 4.1. *Convergence analysis. Let X be a Banach space and let $\upsilon_m(\omega,\phi)$ and $\upsilon_((\mu,\psi)$ be in X. Suppose $\Theta \in (0,1)$, then the series solution $\{\upsilon_m(\omega,\phi)\}_{m=0}^{\infty}$ which is defined from $\sum_{m=0}^{\infty} \upsilon_m(\omega,\phi)$ converges to the solution of Eq. (7) whenever $\upsilon_m(\omega,\phi) \leq \Theta\upsilon_{m-1}(\omega,\phi)\forall m > \mathbb{N}$, that is for any given $\varepsilon > 0$ there exists a positive number $\mathbb{N}$ such that $\|\upsilon_{m+n}(\omega,\phi)\| \leq \varepsilon\forall m,n > \mathbb{N}$. Besides, the absolute error is [46]*

$$
\left\|\upsilon(\omega,\phi) - \sum_{n=0}^{m} \upsilon_n(\omega,\phi)\right\| \leq \frac{\Theta^{m+1}}{1-\Theta}\left\|\upsilon_0(\omega,\phi)\right\|.
$$

5. Numerical solutions

The phi-four equation, a nonlinear partial differential equation, finds practical applications across physics and mathematics. It is crucial in describing phenomena like phase transitions in particle physics and condensed matter, guiding insights into superfluidity and superconductivity. In fields such as nonlinear optics, it models optical pulse propagation through solitons in optical fibers. Additionally, the equation's relevance extends to fluid dynamics, cosmology, and mathematical modeling, showcasing its versatility in explaining phenomena ranging from wave behavior to early universe processes, making it an invaluable tool in understanding a wide array of complex systems.

5.1. Example 1

Suppose that we consider the Phi-four equation in fractional form [22]:

$$
\frac{\partial^\alpha \upsilon(\omega,\phi)}{\partial\phi^\alpha} = \frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi)) - \chi_1\upsilon(\omega,\phi) - \chi_2\upsilon^3(\omega,\phi), \quad 1 < \alpha \leq 2, \tag{5.1}
$$

with

$$
\upsilon(\omega,0) = \sqrt{\frac{-\chi_1^2}{\chi_2}} \tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right), \;\; \frac{\partial}{\partial\phi}\upsilon(\omega,0) = -\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}} \operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right).
$$

Implementation of the HPTM

On taking the ST, we have

$$S\left(\frac{\partial^{\alpha}\upsilon}{\partial\phi^{\alpha}}\right)=S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))-\chi_1\upsilon(\omega,\phi)-\chi_2\upsilon^3(\omega,\phi)\right], \tag{5.2}$$

$$\frac{\wp^{\alpha}}{\varpi^{\alpha}}S\left[\upsilon(\omega,\phi)\right]-\sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0)=S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))-\chi_1\upsilon(\omega,\phi)-\chi_2\upsilon^3(\omega,\phi)\right]. \tag{5.3}$$

Now by employing inverse ST, we get

$$\begin{aligned}
\upsilon(\omega,\phi)&=S^{-1}[\sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0)]+S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))-\chi_1\upsilon(\omega,\phi)-\chi_2\upsilon^3(\omega,\phi)\right]\right\}\right],\\
\upsilon(\omega,\phi)&=\left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)-\phi\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)\right)+\\
&S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))-\chi_1\upsilon(\omega,\phi)-\chi_2\upsilon^3(\omega,\phi)\right]\right\}\right].
\end{aligned} \tag{5.4}$$

By HPM, we attain

$$\begin{aligned}
\sum_{k=0}^{\infty}\epsilon^{k}\upsilon_k(\omega,\phi)&=\left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)-\phi\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)\right)+\\
\epsilon\Bigg(S^{-1}\Bigg[\frac{\varpi^{\alpha}}{\wp^{\alpha}}S\Bigg[&\left(\sum_{k=0}^{\infty}\epsilon^{k}\upsilon_k(\omega,\phi)\right)_{\omega\omega}-\chi_1\left(\sum_{k=0}^{\infty}\epsilon^{k}\upsilon_k(\omega,\phi)\right)-\chi_2\left(\sum_{k=0}^{\infty}\epsilon^{k}H_k(\upsilon)\right)+\Bigg]\Bigg]\Bigg).
\end{aligned} \tag{5.5}$$

The polynomial $H_k(\upsilon)$ is used to discard the nonlinear terms.

$$\sum_{k=0}^{\infty}\epsilon^{k}H_k(\upsilon)=\upsilon^3. \tag{5.6}$$

Certain terms are computed as

$$\begin{aligned}
H_0(\upsilon)&=\upsilon_0^3,\\
H_1(\upsilon)&=3\upsilon_0^2\upsilon_1.
\end{aligned}$$

Comparing the ϵ coefficients, we obtain

$$\epsilon^0 : \upsilon_0(\omega,\phi) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right) - \phi\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)\right),$$

$$\epsilon^1 : \upsilon_1(\omega,\phi) = \frac{\chi_1^2\rho^2\phi^\alpha}{8(-1+\rho^2)\Gamma(\alpha+1)}\operatorname{sech}^6\left(\frac{\chi_1\omega\sqrt{\frac{1}{-1+\rho^2}}}{\sqrt{2}}\right)\Bigg(-3\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} + 2\sqrt{2}\chi_1^3\rho\phi^3\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} +$$
$$2\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(\sqrt{2}\chi_1\phi\sqrt{\frac{1}{-1+\rho^2}}\right) + \sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(2\sqrt{2}\chi_1\phi\sqrt{\frac{1}{-1+\rho^2}}\right)$$
$$2\sqrt{\frac{-\chi_1^2}{\chi_2}}\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\omega\right) + 6\phi^2\left(-\frac{\chi_1}{\chi_2}\right)^{\frac{3}{2}}\chi_2\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\omega\right)$$
$$+\sqrt{-\frac{\chi_1^2}{\chi_2}}\sinh\left(2\sqrt{2}\chi_1\sqrt{\frac{1}{(-1+\rho^2)}}\omega\right)\Bigg)$$
$$\vdots$$

Therefore, the analytical solution is

$$\upsilon(\omega,\phi) = \upsilon_0(\omega,\phi) + \upsilon_1(\omega,\phi) + \cdots .$$
$$\upsilon(\omega,\phi) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right) - \phi\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)\right)+$$
$$\frac{\chi_1^2\rho^2\phi^\alpha}{8(-1+\rho^2)\Gamma(\alpha+1)}\operatorname{sech}^6\left(\frac{\chi_1\omega\sqrt{\frac{1}{-1+\rho^2}}}{\sqrt{2}}\right)\Bigg(-3\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} + 2\sqrt{2}\chi_1^3\rho\phi^3\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}} +$$
$$2\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(\sqrt{2}\chi_1\phi\sqrt{\frac{1}{-1+\rho^2}}\right) + \sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(2\sqrt{2}\chi_1\phi\sqrt{\frac{1}{-1+\rho^2}}\right)$$
$$2\sqrt{\frac{-\chi_1^2}{\chi_2}}\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\omega\right) + 6\phi^2\left(-\frac{\chi_1}{\chi_2}\right)^{\frac{3}{2}}\chi_2\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\omega\right)$$
$$+\sqrt{-\frac{\chi_1^2}{\chi_2}}\sinh\left(2\sqrt{2}\chi_1\sqrt{\frac{1}{(-1+\rho^2)}}\omega\right)\Bigg) + \cdots .$$

Implementation of the STDM

On taking the ST, we have

$$S\left\{\frac{\partial^\alpha \upsilon}{\partial\phi^\alpha}\right\} = S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi)) - \chi_1\upsilon(\omega,\phi) - \chi_2\upsilon^3(\omega,\phi)\right], \tag{5.7}$$

$$\frac{\wp^\alpha}{\varpi^\alpha}S[\upsilon(\omega,\phi)] - \sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^r(\omega,0) = S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi)) - \chi_1\upsilon(\omega,\phi) - \chi_2\upsilon^3(\omega,\phi)\right]. \tag{5.8}$$

Now by employing inverse ST, we get

$$
\begin{aligned}
\upsilon(\omega,\phi) &= S^{-1}[\sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0)] + S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi)) - \chi_1\upsilon(\omega,\phi) - \chi_2\upsilon^3(\omega,\phi)\right]\right\}\right],\\
\upsilon(\omega,\phi) &= \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right) - \phi\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)\right)\\
&+ S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi)) - \chi_1\upsilon(\omega,\phi) - \chi_2\upsilon^3(\omega,\phi)\right]\right\}\right].
\end{aligned}
\tag{5.9}
$$

Now the solution is

$$
\upsilon(\omega,\phi) = \sum_{m=0}^{\infty}\upsilon_m(\omega,\phi). \tag{5.10}
$$

The Adomian polynomials are discarded and the nonlinear term υ^3 is expressed as $\sum_{m=0}^{\infty}\mathcal{A}_m$.

$$
\begin{aligned}
\sum_{m=0}^{\infty}\upsilon_m(\omega,\phi) &= \upsilon(\omega,0) + S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi)) - \chi_1\upsilon(\omega,\phi) - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right],\\
\sum_{m=0}^{\infty}\upsilon_m(\omega,\phi) &= \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right) - \phi\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)\right)\\
&+ S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi)) - \chi_1\upsilon(\omega,\phi) - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right].
\end{aligned}
\tag{5.11}
$$

Some terms are calculated as

$$
\begin{aligned}
\mathcal{A}_0 &= \upsilon_0^3,\\
\mathcal{A}_1 &= 3\upsilon_0^2\upsilon_1.
\end{aligned}
$$

We can easily obtain the approximation by comparing both sides

$$
\upsilon_0(\omega,\phi) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right) - \phi\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)\right).
$$

On $m = 0$

$$\upsilon_1(\omega,\phi) = \frac{\chi_1^2\rho^2\phi^\alpha}{8(-1+\rho^2)\Gamma(\alpha+1)}\operatorname{sech}^6\left(\frac{\chi_1\omega\sqrt{\frac{1}{-1+\rho^2}}}{\sqrt{2}}\right)\Bigg(-3\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}+2\sqrt{2}\chi_1^3\rho\phi^3\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}+$$

$$2\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(\sqrt{2}\chi_1\phi\sqrt{\frac{1}{-1+\rho^2}}\right)+\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(2\sqrt{2}\chi_1\phi\sqrt{\frac{1}{-1+\rho^2}}\right)$$

$$2\sqrt{\frac{-\chi_1^2}{\chi_2}}\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\omega\right)+6\phi^2\left(-\frac{\chi_1}{\chi_2}\right)^{\frac{3}{2}}\chi_2\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\omega\right)$$

$$+\sqrt{-\frac{\chi_1^2}{\chi_2}}\sinh\left(2\sqrt{2}\chi_1\sqrt{\frac{1}{(-1+\rho^2)}}\omega\right)\Bigg)$$

$\vdots$

The series form STDM solution are as follows:

$$\upsilon(\omega,\phi) = \sum_{m=0}^{\infty}\upsilon_m(\omega,\phi) = \upsilon_0(\omega,\phi)+\upsilon_1(\omega,\phi)+\cdots.$$

$$\upsilon(\omega,\phi) = \left(\sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)-\phi\chi_1\rho\sqrt{\frac{-\chi_1^2}{2\chi_2(\rho^2-1)}}\operatorname{sech}^2\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}\omega\right)\right)+$$

$$\frac{\chi_1^2\rho^2\phi^\alpha}{8(-1+\rho^2)\Gamma(\alpha+1)}\operatorname{sech}^6\left(\frac{\chi_1\omega\sqrt{\frac{1}{-1+\rho^2}}}{\sqrt{2}}\right)\Bigg(-3\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}+2\sqrt{2}\chi_1^3\rho\phi^3\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}+$$

$$2\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(\sqrt{2}\chi_1\phi\sqrt{\frac{1}{-1+\rho^2}}\right)+\sqrt{2}\chi_1\rho\phi\sqrt{\frac{\chi_1^2}{\chi_2-\rho^2\chi_2}}\cosh\left(2\sqrt{2}\chi_1\phi\sqrt{\frac{1}{-1+\rho^2}}\right)$$

$$2\sqrt{\frac{-\chi_1^2}{\chi_2}}\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\omega\right)+6\phi^2\left(-\frac{\chi_1}{\chi_2}\right)^{\frac{3}{2}}\chi_2\sinh\left(\sqrt{2}\chi_1\sqrt{\frac{1}{2(-1+\rho^2)}}\omega\right)$$

$$+\sqrt{-\frac{\chi_1^2}{\chi_2}}\sinh\left(2\sqrt{2}\chi_1\sqrt{\frac{1}{(-1+\rho^2)}}\omega\right)\Bigg)+\cdots.$$

If we take $\alpha = 2$, we get the exact solution as

$$\upsilon(\omega,\phi) = \sqrt{\frac{-\chi_1^2}{\chi_2}}\tanh\left(\chi_1\sqrt{\frac{1}{2(\rho^2-1)}}(\omega-\rho\phi)\right). \tag{5.12}$$

5.2. *Example 2*

Assuming the fractional Phi-four equation with $\chi_1 = 1$ and $\chi_2 = -1$ [22]

$$\frac{\partial^\alpha\upsilon(\omega,\phi)}{\partial\phi^\alpha} = \frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi), \quad 1<\alpha\le 2, \tag{5.13}$$

with

$$\upsilon(\omega,0)=\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right),\ \ \frac{\partial}{\partial\phi}\upsilon(\omega,0)=\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right).$$

Implementation of the HPTM

On taking the ST, we have

$$S\left(\frac{\partial^\alpha\upsilon}{\partial\phi^\alpha}\right)=S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi)\right], \tag{5.14}$$

$$\frac{\wp^\alpha}{\varpi^\alpha}S\left[\upsilon(\omega,\phi)\right]-\sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^r(\omega,0)=S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi)\right]. \tag{5.15}$$

Now by employing inverse ST, we get

$$\begin{aligned}\upsilon(\omega,\phi)&=S^{-1}[\sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^r(\omega,0)]+S^{-1}\left[\frac{\varpi^\alpha}{\wp^\alpha}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi)\right]\right\}\right],\\ \upsilon(\omega,\phi)&=(1+\phi)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)+S^{-1}\left[\frac{\varpi^\alpha}{\wp^\alpha}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi)\right]\right\}\right].\end{aligned} \tag{5.16}$$

By HPM, we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}\epsilon^k\upsilon_k(\omega,\phi)=(1+\phi)\left(\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)\right)+\\ &\epsilon\left(S^{-1}\left[\frac{\varpi^\alpha}{\wp^\alpha}S\left[\left(\sum_{k=0}^{\infty}\epsilon^k\upsilon_k(\omega,\phi)\right)_{\omega\omega}+\left(\sum_{k=0}^{\infty}\epsilon^k\upsilon_k(\omega,\phi)\right)-\left(\sum_{k=0}^{\infty}\epsilon^kH_k(\upsilon)\right)+\right]\right]\right).\end{aligned} \tag{5.17}$$

The polynomial $H_k(\upsilon)$ is used to eliminate the nonlinear terms.

$$\sum_{k=0}^{\infty}\epsilon^kH_k(\upsilon)=\upsilon^3. \tag{5.18}$$

Certain terms are computed as

$$\begin{aligned}H_0(\upsilon)&=\upsilon_0^3,\\ H_1(\upsilon)&=3\upsilon_0^2\upsilon_1.\end{aligned}$$

By comparing the ϵ coefficients, we can derive the following.

$$\epsilon^0 : \upsilon_0(\omega,\phi) = (1+\phi)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right),$$

$$\epsilon^1 : \upsilon_1(\omega,\phi) = -\frac{\phi^\alpha}{\Gamma(\alpha+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)\left(-1-\phi-\frac{(1+\phi)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{\kappa^2-1}\right)+(1+\phi)^3$$

$$\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right),$$

$$\epsilon^2 : \upsilon_2(\omega,\phi) = \frac{\phi^{2\alpha}}{\Gamma(2\alpha+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)(1+\phi+\frac{1+\phi}{-1+\kappa^2}\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$\frac{2(1+\phi)(\kappa^2(1+\phi)^2-\phi(2+\phi))}{(-1+\kappa^2)^2}\times\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-$$

$$\frac{(1+\phi)(\kappa^2(1+\phi)^2-\phi(2+\phi))\left(-2+\cosh\left(\sqrt{2}\sqrt{\frac{1}{1-\kappa^2}}\right)\right)\omega}{(-1+\kappa^2)^2}$$

$$\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)+\frac{2\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{2-2\kappa^2}$$

$$(-1-\phi-\frac{(1+\phi)\operatorname{sech}\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)^2}{-1+\kappa^2}+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right))+3(1+\phi)^2\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$(-1-\phi-\frac{(1+\phi)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{-1+\kappa^2}+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)))$$

$$\vdots$$

Hence, the analytical solution is

$$\upsilon(\omega,\phi)=\upsilon_0(\omega,\phi)+\upsilon_1(\omega,\phi)+\upsilon_2(\omega,\phi)+\cdots.$$

$$\upsilon(\omega,\phi)=(1+\phi)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-\frac{\phi^{\alpha}}{\Gamma(\alpha+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$\left(-1-\phi-\frac{(1+\phi)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{\kappa^2-1}\right)+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$-\frac{\phi^{2\alpha}}{\Gamma(2\alpha+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)(1+\phi+\frac{1+\phi}{-1+\kappa^2}\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$\frac{2(1+\phi)(\kappa^2(1+\phi)^2-\phi(2+\phi))}{(-1+\kappa^2)^2}\times\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-$$

$$\frac{(1+\phi)(\kappa^2(1+\phi)^2-\phi(2+\phi))\left(-2+\cosh\left(\sqrt{2}\sqrt{\frac{1}{1-\kappa^2}}\right)\right)\omega}{(-1+\kappa^2)^2}$$

$$\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)+\frac{2\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{2-2\kappa^2}$$

$$(-1-\phi-\frac{(1+\phi)\operatorname{sech}\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)^2}{-1+\kappa^2}+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right))+3(1+\phi)^2\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$(-1-\phi-\frac{(1+\phi)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{-1+\kappa^2}+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)))$$

Implementation of the STDM

On taking the ST, we attain

$$S\left\{\frac{\partial^{\alpha}\upsilon}{\partial\phi^{\alpha}}\right\}=S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi)\right],\tag{5.19}$$

$$\frac{\wp^{\alpha}}{\varpi^{\alpha}}S[\upsilon(\omega,\phi)]-\sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0)=S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi)\right].\tag{5.20}$$

Now by employing inverse ST, we get

$$\begin{aligned}\upsilon(\omega,\phi)&=S^{-1}[\sum_{r=0}^{\theta-1}\left(\frac{\wp}{\varpi}\right)^{\alpha-r-1}\upsilon^{r}(\omega,0)]+S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi)\right]\right\}\right],\\ \upsilon(\omega,\phi)&=(1+\phi)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)+S^{-1}\left[\frac{\varpi^{\alpha}}{\wp^{\alpha}}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))+\upsilon(\omega,\phi)-\upsilon^3(\omega,\phi)\right]\right\}\right].\end{aligned}\tag{5.21}$$

Now the solution is

$$\upsilon(\omega,\phi)=\sum_{m=0}^{\infty}\upsilon_m(\omega,\phi). \tag{5.22}$$

The Adomian polynomials are discarded and the nonlinear term υ^3 is expressed as the summation of $\mathcal{A}_m$ from $m=0$ to ∞.

$$\begin{aligned}
\sum_{m=0}^{\infty}\upsilon_m(\omega,\phi)&=\upsilon(\omega,0)+S^{-1}\left[\frac{\varpi^\alpha}{\wp^\alpha}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))-\chi_1\upsilon(\omega,\phi)-\sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right],\\
\sum_{m=0}^{\infty}\upsilon_m(\omega,\phi)&=(1+\phi)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)\\
&+S^{-1}\left[\frac{\varpi^\alpha}{\wp^\alpha}\left\{S\left[\frac{\partial^2}{\partial\omega^2}(\upsilon(\omega,\phi))-\chi_1\upsilon(\omega,\phi)-\sum_{m=0}^{\infty}\mathcal{A}_m\right]\right\}\right].
\end{aligned} \tag{5.23}$$

Some terms are calculated as

$$\mathcal{A}_0=\upsilon_0^3,$$
$$\mathcal{A}_1=3\upsilon_0^2\upsilon_1.$$

The approximation can be readily acquired by comparing each side of the equation.

$$\upsilon_0(\omega,\phi)=(1+\phi)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right).$$

On $m=0$

$$\upsilon_1(\omega,\phi)=-\frac{\phi^\alpha}{\Gamma(\alpha+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)\left(-1-\phi-\frac{(1+\phi)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{\kappa^2-1}\right)+(1+\phi)^3$$
$$\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right).$$

On $m=1$

$$\upsilon_2(\omega,\phi)=\frac{\phi^{2\alpha}}{\Gamma(2\alpha+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)(1+\phi+\frac{1+\phi}{-1+\kappa^2}\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$
$$\frac{2(1+\phi)(\kappa^2(1+\phi)^2-\phi(2+\phi))}{(-1+\kappa^2)^2}\times\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-$$
$$\frac{(1+\phi)(\kappa^2(1+\phi)^2-\phi(2+\phi))\left(-2+\cosh\left(\sqrt{2}\sqrt{\frac{1}{1-\kappa^2}}\right)\right)\omega}{(-1+\kappa^2)^2}$$
$$\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)+\frac{2\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{2-2\kappa^2}$$
$$(-1-\phi-\frac{(1+\phi)\operatorname{sech}\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)^2}{-1+\kappa^2}+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right))+3(1+\phi)^2\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$
$$(-1-\phi-\frac{(1+\phi)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{-1+\kappa^2}+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right))).$$

The series form STDM solution are as follows:

$$\upsilon(\omega,\phi)=\sum_{m=0}^{\infty}\upsilon_m(\omega,\phi)=\upsilon_0(\omega,\phi)+\upsilon_1(\omega,\phi)+\upsilon_2(\omega,\phi)+\cdots .$$

$$\upsilon(\omega,\phi)=(1+\phi)\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-\frac{\phi^{\alpha}}{\Gamma(\alpha+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$\left(-1-\phi-\frac{(1+\phi)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{\kappa^2-1}\right)+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$-\frac{\phi^{2\alpha}}{\Gamma(2\alpha+1)}\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)(1+\phi+\frac{1+\phi}{-1+\kappa^2}\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$\frac{2(1+\phi)(\kappa^2(1+\phi)^2-\phi(2+\phi))}{(-1+\kappa^2)^2}\times\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-$$

$$\frac{(1+\phi)(\kappa^2(1+\phi)^2-\phi(2+\phi))\left(-2+\cosh\left(\sqrt{2}\sqrt{\frac{1}{1-\kappa^2}}\right)\right)\omega}{(-1+\kappa^2)^2}$$

$$\operatorname{sech}^4\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)-(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)+\frac{2\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{2-2\kappa^2}$$

$$(-1-\phi-\frac{(1+\phi)\operatorname{sech}\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)^2}{-1+\kappa^2}+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right))+3(1+\phi)^2\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)$$

$$(-1-\phi-\frac{(1+\phi)\operatorname{sech}^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)}{-1+\kappa^2}+(1+\phi)^3\tanh^2\left(\sqrt{\frac{1}{2(1-\kappa^2)}}\omega\right)\Bigg)\Bigg).$$

If we take $\alpha=2$, we get the exact solution as

$$\upsilon(\omega,\phi)=\tanh\left(\sqrt{\frac{1}{2(1-\kappa^2)}}(\omega-\kappa\phi)\right). \tag{5.24}$$

6. Results and discussion

In Example 1 of the Phi-four equation, Figure 1 compares the precise solution to the answer determined by the supplied methodologies for the integer order situation with a value of $\alpha=2$ at $\chi_1=1$ and $\chi_1=-1$. Figure 1 depicts this comparison with graphs. The narrative structure allows for a visual assessment of the precision and usefulness of the supplied approaches in approximating the precise solution for this specific situation. The following study focuses on Figure 2, which depicts the solution produced from the adoption of the suggested approaches for fractional orders. The solutions for fractional orders of $\alpha = 1.8$ and $\alpha = 1.5$ at $\chi_1 = 1$ and $\chi_1 = -1$ are displayed in Example 1's pictures. This comparative research sheds light on how various fractional orders affect the dynamics of the solutions. This allows for the identification of solution trends or variations as the fractional order is changed. A more thorough analysis of the proposed techniques is presented in Figure 3, which focuses on a fractional order of $\alpha = 1.25$ in Example 1 at $\chi_1 = 1$ and $\chi_1 = -1$. It is possible to fully assess the accuracy and calibre of the solutions offered by the different approaches in this particular case by concentrating on a single fractional order. Table 1 displays the absolute error values of the solutions obtained using the suggested techniques in Example 1 for both the integer order $\alpha = 2$ and the multiple fractional orders $\alpha = 1.25$, $\alpha = 1.5$, and $\alpha = 1.8$. The discrepancy between the precise answer and the solutions obtained from the suggested methods is used to gauge how precise the solutions of the supplied strategies are. For every fractional order, the performance and accuracy of the supplied options may be evaluated using the absolute error numbers in Example 1.

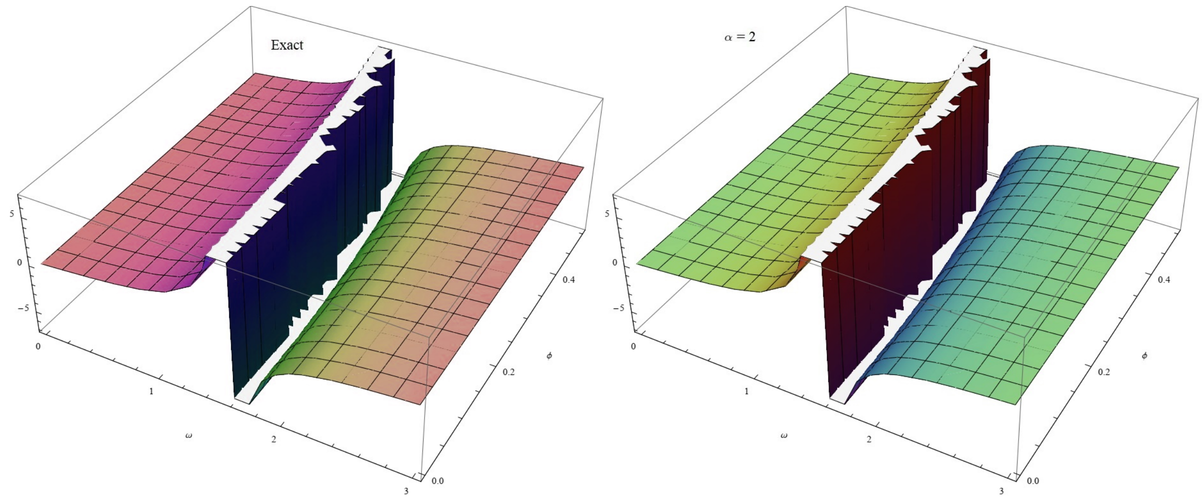

Figure 1. The exact solution and the solution obtained using the proposed methods for the integer order $\alpha = 2$ at $\chi_1 = 1$ and $\chi_1 = -1$ of Example 1.

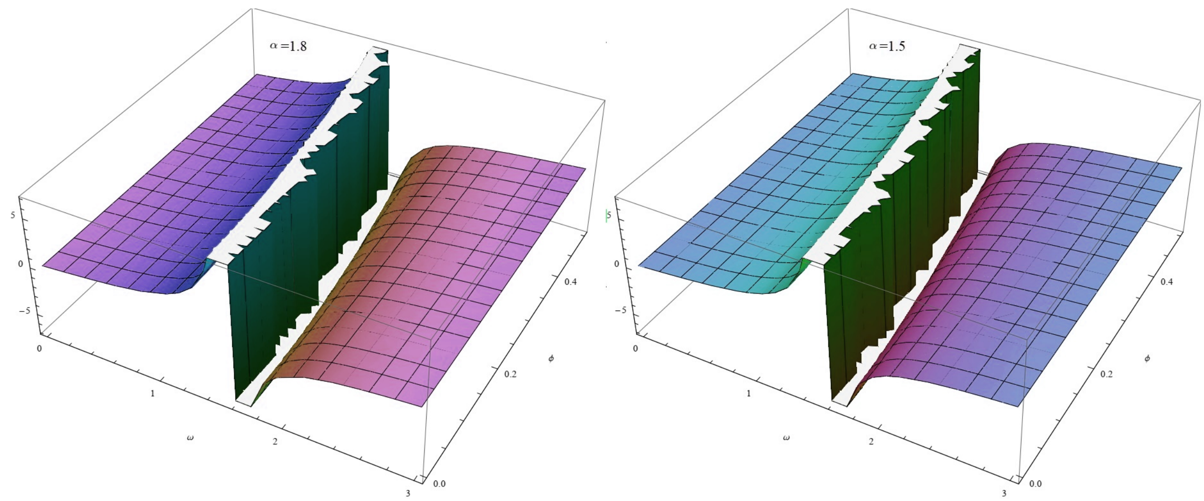

Figure 2. The solutions obtained using the proposed methods for the fractional orders $\alpha = 1.8$ and 1.5 at $\chi_1 = 1$ and $\chi_1 = -1$ of Example 1.

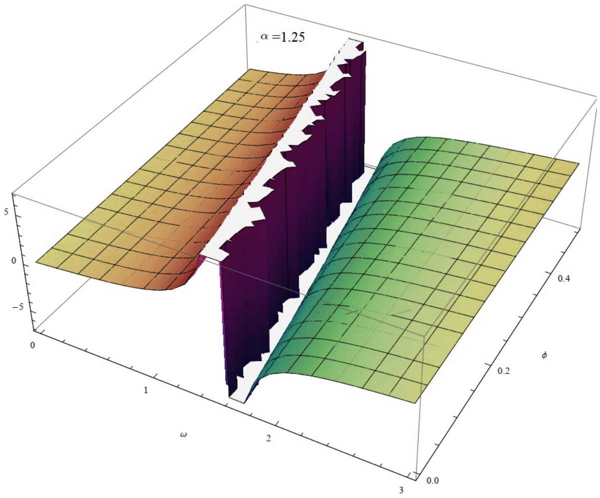

Figure 3. The solution obtained using the proposed methods for the fractional order $\alpha = 1.25$ at $\chi_1 = 1$ and $\chi_1 = -1$ of Example 1.

Table 1. The absolute error values of the solutions obtained using the proposed methods for different fractional order α in Example 1.

ϕ	ω	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.8$	$\alpha = 2$	$\alpha = 2$ q-HATM [22]
0.20	1	2.2345×10^{-3}	1.7601×10^{-5}	1.2023×10^{-7}	7.8224×10^{-9}	7.8224×10^{-9}
	2	2.7821×10^{-2}	3.3233×10^{-5}	2.3367×10^{-6}	1.5192×10^{-8}	1.5192×10^{-8}
	3	1.0279×10^{-2}	4.78018×10^{-5}	3.3360×10^{-6}	2.1684×10^{-8}	2.1684×10^{-8}
	4	1.7946×10^{-2}	6.07350×10^{-5}	4.1491×10^{-6}	2.6965×10^{-8}	2.6965×10^{-8}
	5	1.2796×10^{-2}	6.84409×10^{-5}	4.7420×10^{-6}	3.0816×10^{-8}	3.0816×10^{-8}
0.50	1	1.8597×10^{-3}	2.4849×10^{-5}	2.0189×10^{-6}	1.5626×10^{-8}	1.5626×10^{-8}
	2	2.7649×10^{-2}	4.72858×10^{-5}	3.9274×10^{-6}	3.0371×10^{-8}	3.0371×10^{-8}
	3	2.7946×10^{-2}	6.8238×10^{-5}	5.6089×10^{-6}	4.3360×10^{-8}	4.3360×10^{-8}
	4	1.2379×10^{-2}	8.5130×10^{-5}	6.9773×10^{-6}	5.3928×10^{-8}	5.3928×10^{-8}
	5	2.8970×10^{-2}	9.7255×10^{-5}	7.9755×10^{-6}	6.1634×10^{-8}	6.1634×10^{-8}
0.80	1	1.2790×10^{-2}	3.0393×10^{-5}	2.7327×10^{-6}	2.3412×10^{-8}	2.3412×10^{-8}
	2	1.7462×10^{-2}	5.8129×10^{-5}	5.3202×10^{-6}	4.5535×10^{-8}	4.5535×10^{-8}
	3	2.2561×10^{-3}	8.3556×10^{-5}	7.6004×10^{-6}	6.5026×10^{-8}	6.5026×10^{-8}
	4	1.2473×10^{-2}	1.0426×10^{-4}	9.4564×10^{-7}	8.0887×10^{-8}	8.0887×10^{-8}
	5	1.2479×10^{-2}	1.1037×10^{-4}	1.0810×10^{-6}	9.2455×10^{-8}	9.2455×10^{-8}
1.0	1	3.9863×10^{-3}	3.5056×10^{-5}	3.3865×10^{-7}	3.1180×10^{-8}	3.1180×10^{-8}
	2	6.2736×10^{-3}	6.7260×10^{-5}	6.4868×10^{-7}	6.0684×10^{-8}	6.0684×10^{-8}
	3	9.7456×10^{-3}	9.6633×10^{-5}	9.3174×10^{-7}	8.6683×10^{-8}	8.6683×10^{-8}
	4	1.5496×10^{-2}	1.1255×10^{-4}	1.0823×10^{-6}	1.0784×10^{-7}	1.0784×10^{-7}
	5	1.7123×10^{-2}	1.2821×10^{-4}	1.2314×10^{-6}	1.2327×10^{-7}	1.2327×10^{-7}

Figure 4 displays the graphs of the precise solution and the solution found using the proposed approaches for the integer order $\alpha = 2$ at $\chi_1 = 1$ and $\chi_1 = -1$ in Example 2 of the Phi-four equation. The map facilitates a visual evaluation of the precision and efficacy of the recommended methodologies in approximating the precise solution for this particular case. In Figure 5, the focus is on the solutions produced using the provided approaches for fractional orders of $\alpha = 1.8$ and $\alpha = 1.5$ at $\chi_1 = 1$ and $\chi_1 = -1$ in Example 2. This comparative analysis offers valuable insights into the impact of varying fractional orders on the dynamics of the solutions. This enables the observation of solution variations or trends in response to changes in the fractional order.

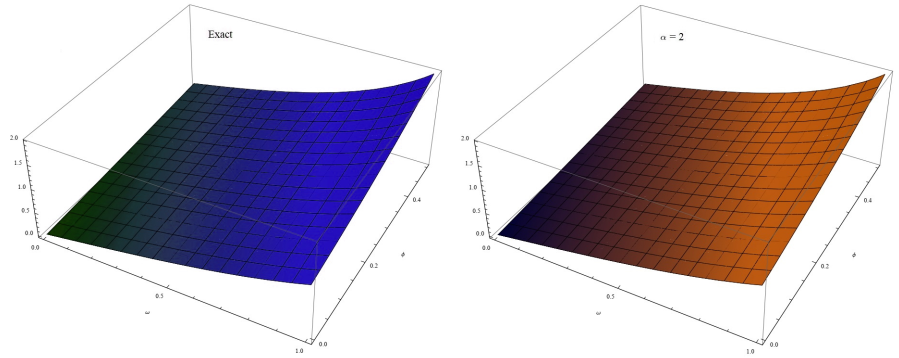

Figure 4. The exact solution and the solution obtained using the proposed methods for the integer order $\alpha = 2$ at $\chi_1 = 1$ and $\chi_1 = -1$ of Example 2.

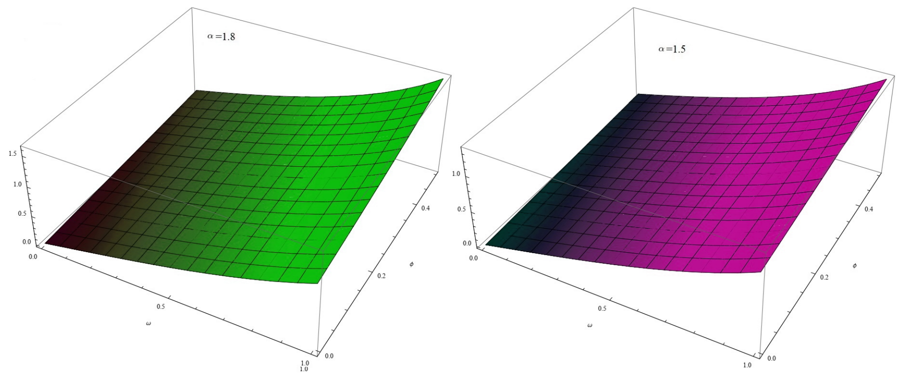

Figure 5. The solutions obtained using the proposed methods for the fractional orders $\alpha =$ 1.8 and 1.5 at $\chi_1 = 1$ and $\chi_1 = -1$ of Example 2.

In Example 2, Figure 6 provides a more detailed analysis of the proposed approaches, specifically focusing on a specific fractional order of $\alpha = 1.25$ at $\chi_1 = 1$ and $\chi_1 = -1$. By directing attention towards a certain fractional order, one may thoroughly examine the precision and attributes of the solutions generated by the suggested methodologies in this specific scenario. The utilisation of numerous graphs depicting different fractional orders enables the examination and juxtaposition of solution behaviour and accuracy in response to alterations in the fractional order.

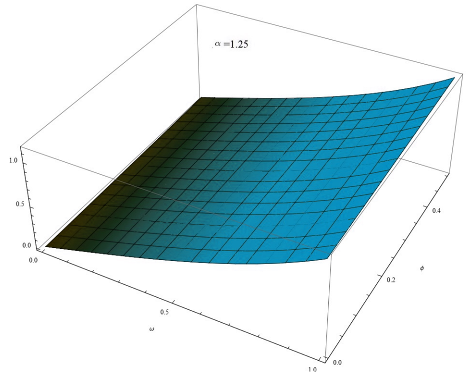

Figure 6. The solution obtained using the proposed methods for the fractional order $\alpha = 1.25$ at $\chi_1 = 1$ and $\chi_1 = -1$ of Example 2.

The measurement of the accuracy of the solutions produced through the proposed approaches is achieved by assessing the disparity between the precise solution and the solutions derived from the proposed methods. Table 2 displays the absolute error values of the solutions derived using the given approaches for several fractional orders, namely $\alpha = 1.25$, $\alpha = 1.5$, $\alpha = 1.8$, and the integer order $\alpha = 2$, in Example 2. The performance and precision of the offered approaches for each fractional order discussed in Example 2 can be evaluated by analysing the absolute error numbers.

Table 2. The absolute error of the proposed methods of different fractional order of Example 2.

ϕ	ω	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.8$	$\alpha = 2$
0.20	1	2.5462×10^{-2}	1.7216×10^{-4}	6.7701×10^{-8}	7.0832×10^{-13}
	2	3.5120×10^{-2}	2.7439×10^{-4}	1.2595×10^{-8}	4.4031×10^{-13}
	3	4.1456×10^{-2}	3.0123×10^{-4}	1.7018×10^{-8}	1.1304×10^{-13}
	4	5.8721×10^{-2}	4.5789×10^{-4}	1.9756×10^{-8}	1.6642×10^{-13}
	5	7.2546×10^{-2}	4.1587×10^{-4}	2.0842×10^{-7}	3.3639×10^{-13}
0.50	1	1.1596×10^{-2}	1.7290×10^{-4}	7.5341×10^{-8}	1.1326×10^{-11}
	2	1.4218×10^{-2}	2.5197×10^{-4}	1.3864×10^{-8}	7.0326×10^{-12}
	3	4.2586×10^{-2}	3.8712×10^{-4}	1.8653×10^{-8}	1.7976×10^{-12}
	4	5.5200×10^{-2}	4.6523×10^{-4}	2.1600×10^{-8}	2.6707×10^{-12}
	5	3.7824×10^{-2}	4.8745×10^{-4}	2.2749×10^{-7}	5.3857×10^{-12}
0.80	1	1.1485×10^{-2}	1.3578×10^{-4}	6.6788×10^{-8}	5.7308×10^{-11}
	2	3.0148×10^{-2}	2.2794×10^{-4}	1.2112×10^{-8}	3.5546×10^{-11}
	3	7.1458×10^{-2}	2.7228×10^{-4}	1.6203×10^{-8}	9.0450×10^{-12}
	4	2.7892×10^{-2}	3.1272×10^{-4}	1.8701×10^{-7}	1.3560×10^{-11}
	5	2.1755×10^{-2}	3.8742×10^{-4}	1.9650×10^{-7}	2.7284×10^{-11}
1	1	7.0178×10^{-2}	7.2330×10^{-5}	4.8542×10^{-7}	1.8101×10^{-10}
	2	2.7824×10^{-2}	1.28452×10^{-4}	8.5846×10^{-7}	1.1217×10^{-10}
	3	7.0796×10^{-2}	1.5424×10^{-4}	1.1366×10^{-8}	2.8412×10^{-11}
	4	2.7021×10^{-2}	1.8168×10^{-4}	1.3039×10^{-7}	4.2984×10^{-11}
	5	2.1679×10^{-2}	1.8784×10^{-4}	1.3644×10^{-7}	8.6290×10^{-11}

7. Conclusions

In conclusion, my study focused on the nonlinear fractional Phi-four equation, employing the homotopy perturbation and Adomian decomposition methods with the Shehu transform. These analytical techniques are potent tools, particularly for solving fractional differential equations. Comparing the solutions obtained from both methods, we observed a high level of agreement, validating their effectiveness as illustrated in the previous figures and tables. The homotopy Perturbation and Adomian decomposition methods have consistently proven their efficiency and reliability in solving various nonlinear differential equations across disciplines like physics, engineering, and finance. The rising popularity of fractional calculus in modeling complex physical phenomena underscores the significance of the nonlinear fractional Phi-four equation we examined. The solutions obtained by both methods enhance our understanding and predictive capabilities for such systems. In summary, the application of the homotopy perturbation method and the Adomian decomposition method, with the Shehu transform, offers an efficient and accurate approach to solving the Nonlinear fractional Phi-four equation. Furthermore, these methods can be extended to address other nonlinear fractional differential equations, contributing significantly to the advancement of mathematical modeling in science and engineering.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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