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Research article

Variable exponent Besov-Lipschitz and Triebel-Lizorkin spaces for the Gaussian measure

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Abstract: In this paper, we introduce variable Gaussian Besov-Lipschitz $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ and Triebel-Lizorkin spaces $F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, i.e., Gaussian Besov-Lipschitz and Triebel-Lizorkin spaces with variable exponents $p(\cdot)$ and $q(\cdot)$, under certain regularity conditions on the functions $p(\cdot)$ and $q(\cdot)$. The condition on $p(\cdot)$ is associated with the Gaussian measure and was introduced in [3]. Trivially, they include the Gaussian Besov-Lipschitz $B^{\alpha}_{p,q}(\gamma_d)$ and Triebel-Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$ for p, q constants, which were introduced and studied in [10]. We consider some inclusion relations of those spaces and finally prove some interpolation results for them.

Keywords: Ornstein-Uhlenbeck; variable exponent; Besov-Lipschitz; Triebel-Lizorkin; Gaussian measure
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1. Introduction

Gaussian harmonic analysis is basically the study of the notions of classical harmonic analysis (such as semigroups, covering lemmas, maximal functions, Littlewood-Paley functions, Spectral multipliers, fractional integral and derivatives, singular integrals, etc.), which are formulated in the Lebesgue measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx)$, in the probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \gamma_d(dx))$, where $\gamma_d(dx) = \frac{e^{-\|x\|^2}}{\pi^{d/2}}dx$, $x \in \mathbb{R}^d$, is the Gaussian probability measure in $\mathbb{R}^d$.

A second component of classical harmonic analysis is the Laplace operator, $\Delta_x = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$.

In Gaussian harmonic analysis is the Ornstein-Uhlenbeck second order differential operator, $L = \frac{1}{2}\triangle_x - \langle x, \nabla_x \rangle$, where $\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_d})$.

A third component of Gaussian harmonic analysis are the Hermite polynomials that are orthogonal with respect to the Gaussian measure and are eigenfunctions of the Ornstein-Uhlenbeck operator L.

Some differences between classical and Gaussian harmonic analysis, are: Lebesgue measure is a doubling, translation invariant measure. Semigroups asociated to Lebesgue measure are convolution semigroups. Gaussian measure does not satisfy any of these properties, which makes many of the proofs are completely different from the classical case. For a detailed study, see [16].

The structure and properties of general Lipschitz spaces in the classical case (with respect to the Lebesgue measure in $\mathbb{R}^d$) were studied in [12, 14, 15]. In analogous way, for $\alpha > 0$ and $1 \leq p, q \leq \infty$, in [10, 16], were introduced and studied the structure of Besov-Lipschitz $B^{\alpha}_{p,q}(\gamma_d)$ and Triebel-Lizorkin $F^{\alpha}_{p,q}(\gamma_d)$ spaces, with respect to the Gaussian measure γ_d in $\mathbb{R}^d$, that is, for expansions on Hermite polynomials. In particular, for $\alpha = 0, q = 2$, $F^{0}_{p,2}(\gamma_d) = L^p(\gamma_d)$ (Gaussian Lebesgue spaces) and for $p, q = \infty$, $B^{\alpha}_{\infty,\infty}(\gamma_d) = Lip_{\alpha}(\gamma_d)$ (Gaussian Lipschitz spaces) [16], i.e., these spaces generalize known spaces. All of this, in the constant exponent setting.

Lebesgue spaces with variable exponents have been widely studied in the last three decades, see [2] or [4]. These spaces arose for a purely theoretical interest, although a short time later applications began to emerge, [23]. Also, recent research has been stimulated by applications in various problems, for example, elasticity theory and fluid mechanics, where electrorheological fluids are of special interest, see [1, 11]. All of the above motivates us to define more general variable exponent spaces.

In this paper, following [10] or [16] and replacing the constants p and q by measurable functions $p(\cdot), q(\cdot)$ taking values in $[1, \infty]$ and satisfying suitable regularity conditions, we define and study the structure of Besov-Lipschitz spaces $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ and Triebel-Lizorkin spaces $F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ with variable exponents respect to the Gaussian measure, generalizing some of the results in [10,16] such as inclusion relations of those spaces and interpolation results for them. Therefore, for the study of variable exponent spaces, $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ and $F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, we present four sections:

- In section 2, we give the preliminaries in the Gaussian setting and some background on variable spaces with respect to a Borel measure μ.
- In section 3, we obtain some technical results for the Haar measure on $\mathbb{R}^+$ that will be key in the proof of the main results.
- In section 4, we define and study the structure of the spaces $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ and $F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$.
- In section 5, we give some conclusions.

Finally, there are some important references on variable Besov and Triebel-Lizorkin spaces in the context of Lebesgue measure, for example, [6, 13, 17–22].

On the other hand, based on the results of this work, we can now study the boundedness of Riesz Potentials, Bessel Potentials and Bessel Fractional Derivatives on $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ and $F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, in order to generalize the ones presented in [7].

2. Preliminaries

Let us consider the Gaussian measure

$$\gamma_d(dx) = \frac{e^{-\|x\|^2}}{\pi^{d/2}} dx,\ x \in \mathbb{R}^d \tag{2.1}$$

on $\mathbb{R}^d$ and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2}\triangle_x - \langle x, \nabla_x \rangle . \tag{2.2}$$

Let $\nu = (\nu_1, ..., \nu_d)$ be a multi-index such that $\nu_i \geq 0, i = 1, \cdots, d$, let $\nu! = \prod_{i=1}^d \nu_i!$, $|\nu| = \sum_{i=1}^d \nu_i$, $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$ and $\partial^\nu = \partial_1^{\nu_1} ... \partial_d^{\nu_d}$.

Consider the normalized Hermite polynomials of order ν in d variables,

$$h_\nu(x) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}} \prod_{i=1}^{d} (-1)^{\nu_i} e^{x_i^2} \partial_i^{\nu_i}(e^{-x_i^2}). \tag{2.3}$$

The *Ornstein-Uhlenbeck semigroup* on $\mathbb{R}^d$ is defined by

$$T_t f(x) \;=\; \frac{1}{(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy).$$

Using the *Bochner subordination formula*

$$e^{-\lambda} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\lambda^2/4u} du, \tag{2.4}$$

we introduce the *Poisson-Hermite semigroup* by

$$P_t f(x) \;=\; \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du. \tag{2.5}$$

Now, taking the change of variables $s = \dfrac{t^2}{4u}$, $P_t f(x)$ can be written as

$$P_t f(x) = \int_0^\infty T_s f(x) \mu_t^{(1/2)}(ds), \tag{2.6}$$

where $\mu_t^{(1/2)}(ds) = \dfrac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds$, is *the one-sided stable measure on* $(0, \infty)$ *of order* $1/2$, it is easy to see that $\mu_t^{(1/2)}$ is a probability measure on $(0, \infty)$.

It is well known, that Hermite polynomials are eigenfunctions of the operator L,

$$L h_\nu(x) = -|\nu|\, h_\nu(x). \tag{2.7}$$

In consequence

$$T_t h_\nu(x) = e^{-t|\nu|} h_\nu(x), \tag{2.8}$$

and

$$P_t h_\nu(x) = e^{-t\sqrt{|\nu|}} h_\nu(x), \tag{2.9}$$

i.e., Hermite polynomials are also eigenfunctions of T_t and P_t for any $t \geq 0$, for more details, see [16].

Next, we present some technical results for the measure $\mu_t^{(1/2)}$ needed in what follows.

As $\mu_t^{(1/2)}(ds) = \dfrac{t}{2\sqrt{\pi}} \dfrac{e^{-t^2/4s}}{s^{3/2}} ds = g(t,s)ds$, for any $k \in \mathbb{N}$, we use the notation $\dfrac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)$ for

$$\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds) := \frac{\partial^k g(t,s)}{\partial t^k} ds. \tag{2.10}$$

Lemma 2.1. *Given $k \in \mathbb{N}$,*

$$\frac{\partial^k \mu_t^{(1/2)}}{\partial t^k}(ds) = \left(\sum_{i,j} a_{i,j} \frac{t^i}{s^j} \right) \mu_t^{(1/2)}(ds), \tag{2.11}$$

where $\{a_{i,j}\}$ is a finite set of constants and the indexes $i \in \mathbb{Z}$, $j \in \mathbb{N}$ verifies the equation $2j - i = k$.

Lemma 2.2. *Given $k \in \mathbb{N}$ and $t > 0$,*

$$\int_0^{+\infty} \frac{1}{s^k} \mu_t^{(1/2)}(ds) = \frac{C_k}{t^{2k}}, \text{ where } C_k = \frac{2^{2k}\Gamma(k+\frac{1}{2})}{\pi^{\frac{1}{2}}}. \tag{2.12}$$

Corollary 2.1. *Given $k \in \mathbb{N}$ and $t > 0$,*

$$\int_0^{+\infty} \left| \frac{\partial^k \mu_t^{(1/2)}}{\partial t^k} \right| (ds) \leq \frac{C_k}{t^k}. \tag{2.13}$$

On the other hand, by considering the *maximal function of the Ornstein-Uhlenbeck semigroup*

$$T^* f(x) = \sup_{t>0} |T_t f(x)|,$$

we obtain:

Lemma 2.3. *Let $f \in L^1(\gamma_d)$, $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$*

$$\left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \leq C_k T^* f(x) t^{-k}, \forall t > 0. \tag{2.14}$$

For the proofs of the previous technical results, see [10] or [16].

Now, for completeness, we need some background on variable Lebesgue spaces with respect to a Borel measure μ. A μ-measurable function $p(\cdot) : \Omega \subset \mathbb{R}^d \to [1,\infty]$ is said to be an *exponent function*. The set of all the exponent functions will be denoted by $\mathcal{P}(\Omega,\mu)$. For $E \subset \Omega$, we set $p_-(E) = \operatorname{ess\,inf}_{x\in E} p(x)$, $p_+(E) = \operatorname{ess\,sup}_{x\in E} p(x)$ and $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

Also, we use the abbreviations $p_+ = p_+(\Omega)$ and $p_- = p_-(\Omega)$.

Definition 2.1. *Let $E \subset \mathbb{R}^d$ and $p(\cdot) : E \to \mathbb{R}$ a function. We say that:*

i) $p(\cdot)$ is locally log-Hölder continuous, denote by $p(\cdot) \in LH_0(E)$, if there exists a constant $C_1 > 0$ such that

$$|p(x) - p(y)| \;\; \leq \;\; \frac{C_1}{log\left(e + \frac{1}{\|x-y\|}\right)}$$

for all $x, y \in E$.

ii) $p(\cdot)$ is log-Hölder continuous at infinity with base point at $x_0 \in \mathbb{R}^d$, and denote this by $p(\cdot) \in LH_\infty(E)$, if there exist constants $p_\infty \in \mathbb{R}$ and $C_2 > 0$ such that

$$|p(x) - p_\infty| \;\; \leq \;\; \frac{C_2}{log(e + \|x - x_0\|)}$$

for all $x \in E$.

iii) $p(\cdot)$ is log-Hölder continuous, and denote this by $p(\cdot) \in LH(E)$ if both conditions i) and ii) are satisfied.

The maximum, $\max\{C_1, C_2\}$ *is called the log-Hölder constant of $p(\cdot)$.*

Definition 2.2. *Let $E \subset \mathbb{R}^d$, $p(\cdot) \in \mathcal{P}_d^{log}(E)$, if $\frac{1}{p(\cdot)}$ is log-Hölder continuous and denote by $C_{log}(p)$ or C_{log} the log-Hölder constant of $\frac{1}{p(\cdot)}$.*

Definition 2.3. *Let $\Omega \subset \mathbb{R}^d$ and $p(\cdot) \in \mathcal{P}(\Omega, \mu)$. For a μ-measurable function $f : \Omega \to \overline{\mathbb{R}}$, we define the modular $\rho_{p(\cdot),\mu}$ as*

$$\rho_{p(\cdot),\mu}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\mu(dx) + \|f\|_{L^\infty(\Omega_\infty,\mu)}, \tag{2.15}$$

and the norm

$$\|f\|_{L^{p(\cdot)}(\Omega,\mu)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot),\mu}(f/\lambda) \leq 1\right\}. \tag{2.16}$$

Definition 2.4. *The variable exponent Lebesgue space on $\Omega \subset \mathbb{R}^d$, $L^{p(\cdot)}(\Omega, \mu)$ consists on those μ_measurable functions f for which there exists $\lambda > 0$ such that $\rho_{p(\cdot),\mu}\left(\frac{f}{\lambda}\right) < \infty$, i.e.,*

$$L^{p(\cdot)}(\Omega, \mu) = \left\{f : \Omega \to \overline{\mathbb{R}} : f \text{ is measurable and } \rho_{p(\cdot),\mu}\left(\frac{f}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}.$$

Remark 2.1. *When μ is the Lebesgue measure, we write $\rho_{p(\cdot)}$ and $\|f\|_{p(\cdot)}$ instead of $\rho_{p(\cdot),\mu}$ and $\|f\|_{p(\cdot),\mu}$.*

Theorem 2.1. (Norm conjugate formula) *Let ν a complete, σ-finite measure on Ω and $p(\cdot) \in \mathcal{P}(\Omega, \nu)$, then*

$$\frac{1}{2}\|f\|_{p(\cdot),\nu} \leq \|f\|'_{p(\cdot),\nu} \leq 2\|f\|_{p(\cdot),\nu}, \tag{2.17}$$

for all f ν-measurable on Ω, where

$$\|f\|'_{p(\cdot),\nu} = \sup\left\{\int_\Omega |f||g|d\nu : g \in L^{p'(\cdot)}(\Omega, \nu), \|g\|_{p'(\cdot),\nu} \leq 1\right\}.$$

Proof. See Corollary 3.2.14 in [4]. □

Theorem 2.2. (Hölder's inequality) *Let ν a complete, σ-finite measure on Ω and $r(\cdot), q(\cdot) \in \mathcal{P}(\Omega, \nu)$. Define $p(\cdot) \in \mathcal{P}(\Omega, \nu)$ by $\dfrac{1}{p(x)} = \dfrac{1}{q(x)} + \dfrac{1}{r(x)}, \quad a.e.$*

Then for all $f \in L^{q(\cdot)}(\Omega, \nu)$ and $g \in L^{r(\cdot)}(\Omega, \nu)$, $fg \in L^{p(\cdot)}(\Omega, \nu)$ and

$$\|fg\|_{p(\cdot),\nu} \leq 2\|f\|_{q(\cdot),\nu}\|g\|_{r(\cdot),\nu}. \tag{2.18}$$

Proof. See Lemma 3.2.20 in [4]. □

Theorem 2.3. (Minkowski's integral inequality for variable Lebesgue spaces) *Given μ and ν complete σ-finite measures on X and Y respectively, $p \in \mathcal{P}(X, \mu)$. Let $f : X \times Y \to \overline{\mathbb{R}}$ measurable with respect to the product measure on $X \times Y$, such that for almost every $y \in Y$, $f(\cdot, y) \in L^{p(\cdot)}(X, \mu)$. Then*

$$\left\| \int_Y f(\cdot, y) d\nu(y) \right\|_{p(\cdot),\mu} \leq 4 \int_Y \|f(\cdot, y)\|_{p(\cdot),\mu} d\nu(y). \tag{2.19}$$

Proof. It is completely analogous to the proof of Corollary 2.38 in [2] by interchanging the Lebesgue measure for complete σ-finite measures μ and ν on X and Y respectively, and by using (2.18), Fubini's theorem and then (2.17). □

In the rest of the paper μ represents the Haar measure $\mu(dt) = \dfrac{dt}{t}$ on $\mathbb{R}^+$.

3. Technical results

In this section we present some technical results regarding the Haar measure that will be key to the main results.

Remark 3.1. *For a μ-measurable function $f : \mathbb{R}^+ \to \overline{\mathbb{R}}$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^+, \mu)$, and any $\lambda > 0$:*

$$\begin{aligned}
\rho_{q(\cdot),\mu}\left(\frac{f}{\lambda}\right) &= \int_0^\infty \left|\frac{f(t)}{\lambda}\right|^{q(t)} \mu(dt) = \int_0^\infty \left|\frac{t^{-1/q(t)} f(t)}{\lambda}\right|^{q(t)} dt \\
&= \rho_{q(\cdot)}\left(\frac{t^{-1/q(\cdot)} f}{\lambda}\right).
\end{aligned}$$

Thus,

$$\|f\|_{q(\cdot),\mu} = \|t^{-1/q(\cdot)} f\|_{q(\cdot)}. \tag{3.1}$$

Next, we present a technical result for the Haar measure μ.

Lemma 3.1. *Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^+, \mu)$ and $\alpha, \beta > 0$*

i) If $q_+ < \infty$, then $\|t^\alpha e^{-t\beta}\|_{q(\cdot),\mu} < \infty$.
ii) $\|t^\alpha \chi_{(0,1]}\|_{q(\cdot),\mu} < \infty$.
iii) $\|t^{-\alpha} \chi_{(1,\infty)}\|_{q(\cdot),\mu} < \infty$.
iv) For any $t_0 > 0$, $(\ln 2)^{\frac{1}{q_-}} \leq \|\chi_{[t_0/2, t_0]}\|_{q(\cdot),\mu} \leq 1$.

Proof. Let us prove *i*). Set $f(t) = t^{\alpha} e^{-t\beta}$,

$$\rho_{q(\cdot),\mu}(f) = \int_0^{\infty} |f(t)|^{q(t)} \mu(dt) = \int_0^1 |t^{\alpha} e^{-t\beta}|^{q(t)} \frac{dt}{t} + \int_1^{\infty} |t^{\alpha} e^{-t\beta}|^{q(t)} \frac{dt}{t}.$$

Now,

$$\int_0^1 |t^{\alpha} e^{-t\beta}|^{q(t)} \frac{dt}{t} = \int_0^1 t^{\alpha q(t)-1} e^{-t\beta q(t)} dt \leq \int_0^1 t^{\alpha-1} dt < \infty,$$

since $\alpha, \beta > 0$ and $0 \leq t \leq 1$.

On the other hand, by making the change of variables $u = t\beta q_-$

$$\begin{aligned}
\int_1^{\infty} |t^{\alpha} e^{-t\beta}|^{q(t)} \frac{dt}{t} &= \int_1^{\infty} t^{\alpha q(t)} e^{-t\beta q(t)} \frac{dt}{t} \\
&\leq \int_1^{\infty} t^{\alpha q_+} e^{-t\beta q_-} \frac{dt}{t} \leq \int_0^{\infty} t^{\alpha q_+} e^{-t\beta q_-} \frac{dt}{t} \\
&= \int_0^{\infty} (\frac{u}{\beta q_-})^{\alpha q_+} e^{-u} \frac{du}{u} = \frac{1}{(\beta q_-)^{\alpha q_+}} \int_0^{\infty} u^{\alpha q_+ - 1} e^{-u} du \\
&= \frac{1}{(\beta q_-)^{\alpha q_+}} \Gamma(\alpha q_+) < \infty, \text{ since } \alpha, \beta > 0 \text{ and } q_+ < \infty.
\end{aligned}$$

Thus, $\rho_{q(\cdot),\mu}(f) < \infty$, and therefore $\|t^{\alpha} e^{-t\beta}\|_{q(\cdot),\mu} < \infty$. The proofs of *ii*) and *iii*) are immediate.

Now, in order to prove *iv*), set $g = \chi_{[t_0/2, t_0]}$,

$$\rho_{q(\cdot),\mu}(g) = \int_0^{\infty} |g(t)|^{q(t)} \mu(dt) = \int_{t_0/2}^{t_0} \frac{dt}{t} = \ln 2 < 1.$$

Then, $\lambda \geq 1$ implies $\rho_{q(\cdot),\mu}(\frac{g}{\lambda}) \leq \rho_{q(\cdot),\mu}(g) \leq 1$. Thus, $\|g\|_{q(\cdot),\mu} \leq 1$.

On the other hand, taking $0 < \lambda < 1$

$$\rho_{q(\cdot),\mu}\left(\frac{g}{\lambda}\right) = \int_{t_0/2}^{t_0} \lambda^{-q(t)} \frac{dt}{t} \geq \int_{t_0/2}^{t_0} \lambda^{-q_-} \frac{dt}{t} = \lambda^{-q_-} (\ln 2).$$

So, $\lambda < (\ln 2)^{1/q_-}$ implies $\rho_{q(\cdot),\mu}(\frac{g}{\lambda}) > 1$. Thus, $\rho_{q(\cdot),\mu}(\frac{g}{\lambda}) \leq 1$ implies $\lambda \geq (\ln 2)^{1/q_-}$.

Therefore, $\|g\|_{q(\cdot),\mu} \geq (\ln 2)^{1/q_-}$. □

In the case $\Omega = \mathbb{R}^+$, we denote $\mathcal{M}_{0,\infty}$ the set of all measurable functions $p(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions:

i) $0 \leq p_- \leq p_+ < \infty$.

ii$_0$) there exists $p(0) = \lim_{x \to 0} p(x)$ and $|p(x) - p(0)| \leq \frac{A}{\ln(1/x)}, 0 < x \leq 1/2$.

ii$_\infty$) there exists $p(\infty) = \lim_{x \to \infty} p(x)$ and $|p(x) - p(\infty)| \leq \frac{A}{\ln(x)}, x > 2$.

We denote $\mathcal{P}_{0,\infty}$ the subset of functions $p(\cdot)$ such that $p_- \geq 1$.

Let $\alpha(\cdot), \beta(\cdot) \in LH(\mathbb{R}^+)$, bounded with

$$\alpha(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) < \frac{1}{p'(\infty)} \tag{3.2}$$

and

$$\beta(0) > -\frac{1}{p(0)}, \quad \beta(\infty) > -\frac{1}{p(\infty)}. \tag{3.3}$$

Theorem 3.1. *Let $p(\cdot) \in \mathcal{P}_{0,\infty}$, $\alpha(\cdot), \beta(\cdot) \in LH(\mathbb{R}^+)$, bounded. Then Hardy-type inequalities*

$$\left\| x^{\alpha(x)-1} \int_0^x \frac{f(y)}{y^{\alpha(y)}} dy \right\|_{p(\cdot)} \leq C_{\alpha(\cdot),p(\cdot)} \|f\|_{p(\cdot)} \tag{3.4}$$

and

$$\left\| x^{\beta(x)} \int_x^\infty \frac{f(y)}{y^{\beta(y)+1}} dy \right\|_{p(\cdot)} \leq C_{\beta(\cdot),p(\cdot)} \|f\|_{p(\cdot)} \tag{3.5}$$

are valid, if and only if, $\alpha(\cdot), \beta(\cdot)$ satisfy conditions (3.2) *and* (3.3).

Proof. For the proof see Theorem 3.1 and Remark 3.2 in [5]. □

As a consequence, we obtain the Hardy's inequalities associated to the exponent function $q(\cdot) \in \mathcal{P}_{0,\infty}$ and the measure μ.

Corollary 3.1. *Let $q(\cdot) \in \mathcal{P}_{0,\infty}$ and $r > 0$, then*

$$\left\| t^{-r} \int_0^t g(y) dy \right\|_{q(\cdot),\mu} \leq C_{r,q(\cdot)} \left\| y^{-r+1} g \right\|_{q(\cdot),\mu}, \text{ for all } g \text{ such that } y^{-r+1} g \in L^{q(\cdot)}(\mu) \tag{3.6}$$

and

$$\left\| t^{r} \int_t^\infty g(y) dy \right\|_{q(\cdot),\mu} \leq C_{r,q(\cdot)} \left\| y^{r+1} g \right\|_{q(\cdot),\mu}, \text{ for all } g \text{ such that } y^{r+1} g \in L^{q(\cdot)}(\mu). \tag{3.7}$$

Proof. Let $\alpha(t) = -r + \frac{1}{q'(t)} = -r + 1 - \frac{1}{q(t)}$, for any $t \in \mathbb{R}^+$, $f(y) = y^{\alpha(y)} g(y)$, for any $y \in \mathbb{R}^+$ then $\alpha(\cdot) \in LH(\mathbb{R}^+)$ and bounded, $\alpha(0) = -r + \frac{1}{q'(0)} < \frac{1}{q'(0)}$ and $\alpha(\infty) = -r + \frac{1}{q'(\infty)} < \frac{1}{q'(\infty)}$. Then, using (3.1) and (3.4)

$$\begin{aligned} \left\| t^{-r} \int_0^t g(y) dy \right\|_{q(\cdot),\mu} &= \left\| t^{-r-\frac{1}{q(t)}} \int_0^t g(y) dy \right\|_{q(\cdot)} = \left\| t^{\alpha(t)-1} \int_0^t g(y) dy \right\|_{q(\cdot)} \\ &\leq C_{r,q(\cdot)} \left\| y^{\alpha(y)} g \right\|_{q(\cdot)} = C_{r,q(\cdot)} \left\| y^{-r+1-\frac{1}{q(y)}} g \right\|_{q(\cdot)} \\ &= C_{r,q(\cdot)} \left\| y^{-r+1} g \right\|_{q(\cdot),\mu}. \end{aligned}$$

On the other hand, by taking $\beta(t) = r - \frac{1}{q(t)}$, $f(y) = y^{\beta(y)+1} g(y)$, $t, y \in \mathbb{R}^+$ then $\beta(\cdot) \in LH(\mathbb{R}^+)$ and the proof of (3.7) is completely analogous. □

4. Main results

In this section we are going to introduce variable Gaussian Besov-Lipschitz spaces and variable Gaussian Triebel-Lizorkin spaces. In what follows we will consider only variable Lebesgue spaces with respect to the Gaussian measure γ_d. The next condition was introduced by E. Dalmasso and R. Scotto in [3].

Definition 4.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d)$, we say that $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d)$ if there exist constants $C_{\gamma_d} > 0$ and $p_\infty \geq 1$ such that*

$$|p(x) - p_\infty| \leq \frac{C_{\gamma_d}}{\|x\|^2}, \tag{4.1}$$

for $x \in \mathbb{R}^d \setminus \{(0, 0, \ldots, 0)\}$.

Example 4.1. *Consider* $p(x) = p_\infty + \dfrac{A}{(e+\|x\|)^q}$, $x \in \mathbb{R}^d$, *for any* $p_\infty \geq 1$, $A \geq 0$ *and* $q \geq 2$. *Then* $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)$.

Remark 4.1. *It can be proved that if* $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)$, *then* $p(\cdot) \in LH_\infty(\mathbb{R}^d)$.
In fact, by fixing $x_0 \in \mathbb{R}^d$, *such that* $\|x_0\| = 1$, *we get* $\log(e + \|x - x_0\|) \leq C\|x\|^2$, *for all* $x \in \mathbb{R}^d$.

Lipschitz spaces can be generalized of the following way (see, for example [10, 12, 14, 15]), using the Poisson-Hermite semigroup. We are ready to define variable Gaussian Besov-Lipschitz spaces $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, also called Gaussian Besov-Lipschitz spaces with variable exponents or variable Besov-Lipschitz spaces for expansions in Hermite polinomials.

Definition 4.2. *Let* $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ *and* $q(\cdot) \in \mathcal{P}_{0,\infty}$. *Let* $\alpha \geq 0$, k *the smallest integer greater than* α. *The variable Gaussian Besov-Lipschitz space* $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ *is defined as the set of functions* $f \in L^{p(\cdot)}(\gamma_d)$ *such that*

$$\left\| t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} < \infty. \tag{4.2}$$

The norm of $f \in B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ *is defined as*

$$\|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}} := \|f\|_{p(\cdot),\gamma_d} + \left\| t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu}. \tag{4.3}$$

The variable Gaussian Besov-Lipschitz space $B^{\alpha}_{p(\cdot),\infty}(\gamma_d)$ *is defined as the set of functions* $f \in L^{p(\cdot)}(\gamma_d)$ *for which there exists a constant A such that*

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq A t^{-k+\alpha}$$

and then the norm of $f \in B^{\alpha}_{p(\cdot),\infty}(\gamma_d)$ *is defined as*

$$\|f\|_{B^{\alpha}_{p(\cdot),\infty}} := \|f\|_{p(\cdot),\gamma_d} + A_k(f), \tag{4.4}$$

where $A_k(f)$ *is the smallest constant A in the above inequality.*

The following lemmas show that we could have replaced k with any other integer l greater than α and the resulting norms are equivalents. Next, we denote $u(\cdot, t) = P_t f$.

Lemma 4.1. *Let* $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$, $f \in L^{p(\cdot)}(\gamma_d), \alpha \geq 0$ *and* k, l *integers greater than* α, *then*

$$\left\| \frac{\partial^k u(\cdot,t)}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq A_k t^{-k+\alpha} \text{ if and only if } \left\| \frac{\partial^l u(\cdot,t)}{\partial t^l} \right\|_{p(\cdot),\gamma_d} \leq A_l t^{-l+\alpha}.$$

Moreover, if $A_k(f), A_l(f)$ *are the smallest constants in the inequalities above then there exist constants* $A_{k,l,\alpha,p(\cdot)}$ *and* $D_{k,l,\alpha}$ *such that*

$$A_{k,l,\alpha,p(\cdot)} A_k(f) \leq A_l(f) \leq D_{k,l,\alpha} A_k(f),$$

for all $f \in L^{p(\cdot)}(\gamma_d)$.

Proof. Let us suppose without loss of generality that $k \geq l$. We start by proving the direct implication. For this we use the representation of the Poisson-Hermite semigroup (2.6), this is,

$$P_t f(x) = \int_0^{+\infty} T_s f(x) \mu_t^{(1/2)}(ds).$$

Then, by differentiating k-times with respect to t and by using the dominated convergence theorem, we get

$$\frac{\partial^k P_t f(x)}{\partial t^k} = \int_0^{+\infty} T_s f(x) \frac{\partial^k \mu_t^{(1/2)}}{\partial t^k}(ds).$$

By using Lemma 2.3, it's easy to prove that for all $m \in \mathbb{N}$

$$\lim_{t\to+\infty} \frac{\partial^m P_t f(x)}{\partial t^m} = 0, \quad a.e. \quad x \in \mathbb{R}^d.$$

Now, given $n \in \mathbb{N}$, $n > \alpha$

$$\begin{aligned} -\int_t^{+\infty} \frac{\partial^{n+1} P_s f(x)}{\partial s^{n+1}} ds &= -\lim_{s\to+\infty} \frac{\partial^n P_s f(x)}{\partial s^n} + \frac{\partial^n P_t f(x)}{\partial t^n} \\ &= \frac{\partial^n P_t f(x)}{\partial t^n}, \quad a.e. \quad x \in \mathbb{R}^d. \end{aligned}$$

Thus, for Minkowski's integral inequality (2.19)

$$\begin{aligned} \left\| \frac{\partial^n u(\cdot, t)}{\partial t^n} \right\|_{p(\cdot),\gamma_d} &\leq 4 \int_t^{+\infty} \left\| \frac{\partial^{n+1} u(\cdot, s)}{\partial s^{n+1}} \right\|_{p(\cdot),\gamma_d} ds \\ &\leq 4 \int_t^{+\infty} A_{n+1}(f) s^{-(n+1)+\alpha} ds = 4 \frac{A_{n+1}(f)}{n-\alpha} t^{-n+\alpha}. \end{aligned}$$

Therefore

$$A_n(f) \leq 4 \frac{A_{n+1}(f)}{n-\alpha},$$

and since $n > \alpha$ is arbitrary, then, by using the above result $k-l$ times, we obtain

$$\begin{aligned} A_l(f) &\leq 4 \frac{A_{l+1}(f)}{l-\alpha} \leq 4^2 \frac{A_{l+2}(f)}{(l-\alpha)(l+1-\alpha)} \\ &\leq \ldots \leq 4^{k-l} \frac{A_k(f)}{(l-\alpha)(l+1-\alpha)\ldots(k-1-\alpha)} \\ &= D_{k,l,\alpha} A_k(f). \end{aligned}$$

To prove the converse implication, we use again the representation (2.6) and we obtain that

$$u(x, t_1 + t_2) = P_{t_1}(P_{t_2} f)(x) = \int_0^{+\infty} T_s(P_{t_2} f)(x) \mu_{t_1}^{\frac{1}{2}}(ds).$$

Thus, taking $t = t_1 + t_2$ and differentiating l times with respect to t_2 and $k-l$ times with respect to t_1, we get

$$\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{+\infty} T_s\left(\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\right) \frac{\partial^{k-l} \mu_{t_1}^{\frac{1}{2}}}{\partial t_1^{k-l}}(ds). \tag{4.5}$$

Then, by Corollary 2.1, Minkowski's integral inequality (2.19) and the $L^{p(\cdot)}$-boundedness of the Ornstein-Uhlenbeck semigroup (see [8]), we get

$$\begin{aligned}\left\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d} &\leq 4\int_0^{+\infty}\left\|T_s\left(\frac{\partial^l P_{t_2}f}{\partial t_2^l}\right)\right\|_{p(\cdot),\gamma_d}\left|\frac{\partial^{k-l}\mu_{t_1}^{\frac{1}{2}}}{\partial t_1^{k-l}}(ds)\right|\\ &\leq 4C_{p(\cdot)}\left\|\frac{\partial^l P_{t_2}f}{\partial t_2^l}\right\|_{p(\cdot),\gamma_d}\int_0^{+\infty}\left|\frac{\partial^{k-l}\mu_{t_1}^{\frac{1}{2}}}{\partial t_1^{k-l}}(ds)\right|\\ &\leq 4C_{p(\cdot)}\left\|\frac{\partial^l}{\partial t_2^l}P_{t_2}f\right\|_{p(\cdot),\gamma_d}C_{k-l}t_1^{l-k}\\ &\leq 4C_{p(\cdot)}A_l(f)C_{k-l}t_2^{-l+\alpha}t_1^{l-k}.\end{aligned}$$

Therefore, taking $t_1 = t_2 = \frac{t}{2}$,

$$\left\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d} \leq 4C_{p(\cdot)}A_l(f)C_{k-l}(\frac{t}{2})^{-k+\alpha}.$$

Thus, $A_k(f) \leq 4C_{p(\cdot)}\dfrac{C_{k-l}}{2^{-k+\alpha}}A_l(f)$. □

Lemma 4.2. *Let $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$. Let $\alpha \geq 0$ and k, l integers greater than α. Then*

$$\left\|t^{k-\alpha}\left\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu} < \infty$$

if and only if

$$\left\|t^{l-\alpha}\left\|\frac{\partial^l u(\cdot,t)}{\partial t^l}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha,p(\cdot)}$ and $D_{k,l,\alpha,q(\cdot)}$ such that

$$\begin{aligned}D_{k,l,\alpha,q(\cdot)}\left\|t^{l-\alpha}\left\|\frac{\partial^l u(\cdot,t)}{\partial t^l}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu} &\leq \left\|t^{k-\alpha}\left\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu}\\ &\leq A_{k,l,\alpha,p(\cdot)}\left\|t^{l-\alpha}\left\|\frac{\partial^l u(\cdot,t)}{\partial t^l}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu},\end{aligned}$$

for all $f \in L^{p(\cdot)}(\gamma_d)$.

Proof. Suppose without loss of generality that $k \geq l$. We prove first the converse implication; by proceeding as in Lemma 4.1, taking $t_1 = t_2 = \frac{t}{2}$, we have

$$\left\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d} \leq 4C_{p(\cdot)}\left\|\frac{\partial^l P_{t_2}f}{\partial t_2^l}\right\|_{p(\cdot),\gamma_d}\cdot C_{k-l}t_1^{l-k} = 4C_{p(\cdot)}\cdot C_{k-l}(\frac{t}{2})^{l-k}\left\|\frac{\partial^l P_{\frac{t}{2}}f}{\partial(\frac{t}{2})^l}\right\|_{p(\cdot),\gamma_d}.$$

Thus

$$\begin{aligned}\left\|t^{k-\alpha}\left\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu} &\leq 4C_{p(\cdot)}\frac{C_{k-l}}{2^{l-k}}\left\|t^{l-\alpha}\left\|\frac{\partial^l u(\cdot,\frac{t}{2})}{\partial(\frac{t}{2})^l}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu}\\ &= A_{k,l,\alpha,p(\cdot)}\left\|s^{l-\alpha}\left\|\frac{\partial^l u(\cdot,s)}{\partial s^l}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu},\end{aligned}$$

with $A_{k,l,\alpha,p(\cdot)} = 4C_{p(\cdot)}C_{k-l}2^{k-\alpha}$.

For the direct implication, given $n \in \mathbb{N}$, $n > \alpha$, again, as in the above lemma

$$\left\|\frac{\partial^n u(\cdot,t)}{\partial t^n}\right\|_{p(\cdot),\gamma_d} \leq 4\int_t^{+\infty}\left\|\frac{\partial^{n+1}u(\cdot,s)}{\partial s^{n+1}}\right\|_{p(\cdot),\gamma_d} ds.$$

Therefore, from this and by Hardy's inequality (3.7)

$$\begin{aligned}\left\|t^{n-\alpha}\left\|\frac{\partial^n u(\cdot,t)}{\partial t^n}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu} &\leq 4\left\|t^{n-\alpha}\int_t^{+\infty}\left\|\frac{\partial^{n+1}u(\cdot,s)}{\partial s^{n+1}}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu}\\ &\leq 4C_{n,\alpha,q(\cdot)}\left\|s^{n+1-\alpha}\left\|\frac{\partial^{n+1}u(\cdot,s)}{\partial s^{n+1}}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu}.\end{aligned}$$

Now, since $n > \alpha$ is arbitrary, by using the previous result $k-l$ times, we obtain

$$\begin{aligned}\left\|t^{l-\alpha}\left\|\frac{\partial^l u(\cdot,t)}{\partial t^l}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu} &\leq 4C_{l,\alpha,q(\cdot)}\left\|t^{l+1-\alpha}\left\|\frac{\partial^{l+1}u(\cdot,t)}{\partial t^{l+1}}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu}\\ &\leq 4^2C_{l,\alpha,q(\cdot)}C_{l+1,\alpha,q(\cdot)}\left\|t^{l+2-\alpha}\left\|\frac{\partial^{l+2}u(\cdot,t)}{\partial t^{l+2}}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu}\\ &\ \vdots\\ &\leq D_{k,l,\alpha,q(\cdot)}\left\|t^{k-\alpha}\left\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu},\end{aligned}$$

where $D_{k,l,\alpha,q(\cdot)} = 4^{k-l}C_{l,\alpha,q(\cdot)}\cdots C_{k-1,\alpha,q(\cdot)}$. □

Now, we define variable Gaussian Triebel-Lizorkin spaces $F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, which represent another way to measure regularity of functions, proceeding as in [10, 14, 15].

Definition 4.3. *Let $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$. Let $\alpha \geq 0$ and k the smallest integer greater than α. The variable Gaussian Triebel-Lizorkin space $F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ is the set of functions $f \in L^{p(\cdot)}(\gamma_d)$ such that*

$$\left\|\left\|t^{k-\alpha}\frac{\partial^k P_t f}{\partial t^k}\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} < \infty, \tag{4.6}$$

the norm of $f \in F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ is defined as

$$\|f\|_{F^{\alpha}_{p(\cdot),q(\cdot)}} := \|f\|_{p(\cdot),\gamma_d} + \left\|\left\|t^{k-\alpha}\frac{\partial^k P_t f}{\partial t^k}\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d}. \tag{4.7}$$

The following lemma shows that the definition of $F^{\alpha}_{p(\cdot),q(\cdot)}$ is independent of the integer $k > \alpha$ chosen and the resulting norms are equivalents.

Lemma 4.3. *Let $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$. Let $\alpha \geq 0$ and k, l integers greater than α. Then*

$$\left\|\left\|t^{k-\alpha}\left|\frac{\partial^k}{\partial t^k}P_t f\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} < \infty$$

if and only if

$$\left\|\left\|t^{l-\alpha}\left|\frac{\partial^l}{\partial t^l}P_t f\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha,p(\cdot)}, D_{k,l,\alpha,q(\cdot)}$ such that

$$\begin{aligned} D_{k,l,\alpha,q(\cdot)}\left\|\left\|t^{l-\alpha}\left|\frac{\partial^l}{\partial t^l}P_t f\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} &\leq \left\|\left\|t^{k-\alpha}\left|\frac{\partial^k}{\partial t^k}P_t f\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} \\ &\leq A_{k,l,\alpha,p(\cdot)}\left\|\left\|t^{l-\alpha}\left|\frac{\partial^l}{\partial t^l}P_t f\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d}, \end{aligned}$$

for all $f \in L^{p(\cdot)}(\gamma_d)$.

Proof. Suppose without loss of generality that $k \geq l$. Let $n \in \mathbb{N}$ such that $n > \alpha$, we can prove that

$$\left|\frac{\partial^n}{\partial t^n}P_t f(x)\right| \leq \int_t^{+\infty}\left|\frac{\partial^{n+1}}{\partial s^{n+1}}P_s f(x)\right| ds.$$

Then, by the Hardy's inequality (3.7),

$$\begin{aligned} \left\|t^{n-\alpha}\left|\frac{\partial^n}{\partial t^n}P_t f(x)\right|\right\|_{q(\cdot),\mu} &\leq \left\|t^{n-\alpha}\int_t^{+\infty}\left|\frac{\partial^{n+1}}{\partial s^{n+1}}P_s f(x)\right| ds\right\|_{q(\cdot),\mu} \\ &\leq C_{n,\alpha,q(\cdot)}\left\|s^{n+1-\alpha}\left|\frac{\partial^{n+1}}{\partial s^{n+1}}P_s f(x)\right|\right\|_{q(\cdot),\mu}. \end{aligned}$$

Now, since $n > \alpha$ is arbitrary, by iterating the previous argument $k - l$ times, we obtain

$$\begin{aligned} \left\|t^{l-\alpha}\left|\frac{\partial^l}{\partial t^l}P_t f(x)\right|\right\|_{q(\cdot),\mu} &\leq C_{l,\alpha,q(\cdot)}\left\|t^{l+1-\alpha}\left|\frac{\partial^{l+1}}{\partial t^{l+1}}P_t f(x)\right|\right\|_{q(\cdot),\mu} \\ &\leq C_{l,\alpha,q(\cdot)}C_{l+1,\alpha,q(\cdot)}\left\|t^{l+2-\alpha}\left|\frac{\partial^{l+2}}{\partial t^{l+2}}P_t f(x)\right|\right\|_{q(\cdot),\mu} \\ &\vdots \\ &\leq C_{k,l,\alpha,q(\cdot)}\left\|t^{k-\alpha}\left|\frac{\partial^k}{\partial t^k}P_t f(x)\right|\right\|_{q(\cdot),\mu}, \end{aligned}$$

where $C_{k,l,\alpha,q(\cdot)} = C_{l,\alpha,q(\cdot)}C_{l+1,\alpha,q(\cdot)}\cdots C_{k-1,\alpha,q(\cdot)}$.

Thus, $D_{k,l,\alpha,q(\cdot)}\left\|\left\|t^{l-\alpha}\left|\frac{\partial^l}{\partial t^l}P_tf\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} \leq \left\|\left\|t^{k-\alpha}\left|\frac{\partial^k}{\partial t^k}P_tf\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d}$, with, $D_{k,l,\alpha,q(\cdot)} = 1/C_{k,l,\alpha,q(\cdot)}$.

The other inequality is obtained from the case $k = l+1$ by an inductive argument. Let $t_1, t_2 > 0$ and take $t = t_1 + t_2$, from (4.5) we get

$$\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{+\infty} T_s\left(\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right)\frac{\partial^{k-l}}{\partial t_1^{k-l}}\mu_{t_1}^{(1/2)}(ds),$$

and since, $\frac{\partial}{\partial t_1}\mu_{t_1}^{(1/2)}(ds) = \left(t_1^{-1} - \frac{t_1}{2s}\right)\mu_{t_1}^{(1/2)}(ds)$, we obtain

$$\begin{aligned}\left|\frac{\partial^k u(x,t)}{\partial t^k}\right| &\leq \int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right)\cdot\left|\left(t_1^{-1} - \frac{t_1}{2s}\right)\right|\mu_{t_1}^{(1/2)}(ds)\\ &\leq t_1^{-1}\int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right)\mu_{t_1}^{(1/2)}(ds)\\ &\quad + \frac{t_1}{2}\int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right)\frac{1}{s}\mu_{t_1}^{(1/2)}(ds).\end{aligned}$$

Therefore

$$\begin{aligned}\left\|t_2^{k-\alpha}\left|\frac{\partial^k u(x,t)}{\partial t^k}\right|\right\|_{q(\cdot),\mu} &\leq \left\|t_2^{k-\alpha}t_1^{-1}\int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right)\mu_{t_1}^{(1/2)}(ds)\right\|_{q(\cdot),\mu}\\ &\quad + \left\|t_2^{k-\alpha}\frac{t_1}{2}\int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right)\frac{1}{s}\mu_{t_1}^{(1/2)}(ds)\right\|_{q(\cdot),\mu}\\ &= (I) + (II).\end{aligned}$$

Now, by using Minkowski's integral inequality twice (2.19) (since T_s is an integral transformation with positive kernel) and the fact that $\mu_{t_1}^{(1/2)}(ds)$ is a probability measure, we get

$$\begin{aligned}(I) &= \left\|t_2^{k-\alpha}t_1^{-1}\int_0^{+\infty} T_s\left(\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right)\mu_{t_1}^{(1/2)}(ds)\right\|_{q(\cdot),\mu}\\ &\leq 4\int_0^{+\infty}\left\|t_2^{k-\alpha}t_1^{-1}T_s\left(\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right)\right\|_{q(\cdot),\mu}\mu_{t_1}^{(1/2)}(ds)\\ &\leq 16\int_0^{+\infty} T_s\left(\left\|t_2^{k-\alpha}t_1^{-1}\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right\|_{q(\cdot),\mu}\right)\mu_{t_1}^{(1/2)}(ds)\\ &\leq 16T^*\left(\left\|t_2^{k-\alpha}t_1^{-1}\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right\|_{q(\cdot),\mu}\right).\end{aligned}$$

For (II), we proceed in analogous way, and by using Lemma 2.2 we get

$$(II) \leq \frac{16}{2}T^*\left(\left\|t_2^{k-\alpha}t_1\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right\|_{q(\cdot),\mu}\right)\int_0^{+\infty}\frac{1}{s}\mu_{t_1}^{(1/2)}(ds)$$

$$= \quad 8T^*\left(\left\|t_2^{k-\alpha}t_1\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right\|_{q(\cdot),\mu}\right)C_1\frac{1}{t_1^2}.$$

Now, since T^* is defined as a supremum, we get

$$(II) \quad \leq \quad 8C_1T^*\left(\left\|t_2^{k-\alpha}t_1^{-1}\left|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\right|\right\|_{q(\cdot),\mu}\right).$$

Then, taking $t_1 = t_2 = \frac{t}{2}$ and the change of variable $s = \frac{t}{2}$, we have

$$(I) \quad \leq \quad 16T^*\left(\left\|s^{l-\alpha}\left|\frac{\partial^l P_s f(x)}{\partial s^l}\right|\right\|_{q(\cdot),\mu}\right)$$

and

$$(II) \quad \leq \quad 8C_1T^*\left(\left\|s^{l-\alpha}\left|\frac{\partial^l P_s f(x)}{\partial s^l}\right|\right\|_{q(\cdot),\mu}\right).$$

Thus, by the $L^{p(\cdot)}(\gamma_d)$-boundedness of T^* (see [8]),

$$\begin{aligned}
\left\|\left\|t^{k-\alpha}\left|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} &\leq 2^{k-\alpha}16\left\|T^*\left(\left\|s^{l-\alpha}\left|\frac{\partial^l P_s f}{\partial s^l}\right|\right\|_{q(\cdot),\mu}\right)\right\|_{p(\cdot),\gamma_d} \\
&+ 2^{k-\alpha}8C_1\left\|T^*\left(\left\|s^{l-\alpha}\left|\frac{\partial^l P_s f}{\partial s^l}\right|\right\|_{q(\cdot),\mu}\right)\right\|_{p(\cdot),\gamma_d} \\
&\leq 2^{k-\alpha}C_{p(\cdot)}(16+8C_1)\left\|\left\|s^{l-\alpha}\left|\frac{\partial^l P_s f}{\partial s^l}\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d}.
\end{aligned}$$

Therefore,

$$\left\|\left\|t^{k-\alpha}\left|\frac{\partial^k u(\cdot,t)}{\partial t^k}\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} \leq C_{p(\cdot),k,\alpha}\left\|\left\|s^{l-\alpha}\left|\frac{\partial^l P_s f}{\partial s^l}\right|\right\|_{q(\cdot),\mu}\right\|_{p(\cdot),\gamma_d}.$$

□

Next, we need a technical result for the $L^{p(\cdot)}(\gamma_d)$-norms of the derivatives of the Poisson-Hermite semigroup:

Lemma 4.4. *Let* $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$. *Suppose that* $f \in L^{p(\cdot)}(\gamma_d)$*, then for any integer k,* $\left\|\frac{\partial^k}{\partial t^k}P_t f\right\|_{p(\cdot),\gamma_d} \leq C_{p(\cdot)}\left\|\frac{\partial^k}{\partial s^k}P_s f\right\|_{p(\cdot),\gamma_d}$*, for whatever* $0 < s \leq t < +\infty$*. Moreover,*

$$\left\|\frac{\partial^k}{\partial t^k}P_t f\right\|_{p(\cdot),\gamma_d} \leq \frac{C_{k,p(\cdot)}}{t^k}\|f\|_{p(\cdot),\gamma_d}, \quad t > 0. \tag{4.8}$$

Proof. First, let us consider the case $k = 0$. Fixed $t_1, t_2 > 0$, by using the semigroup property of $\{P_t\}$, we get

$$P_{t_1+t_2}f(x) = P_{t_1}(P_{t_2}f(x)).$$

Thus, by the $L^{p(\cdot)}$-boundedness of $\{P_t\}$ (see [8]),

$$\|P_{t_1+t_2}f\|_{p(\cdot),\gamma_d} \leq C_{p(\cdot)}\|P_{t_2}f\|_{p(\cdot),\gamma_d}.$$

In order to prove the general case, $k > 0$, using the dominated convergence theorem and differentiating the identity $u(x, t_1 + t_2) = P_{t_1}(u(x, t_2))$ k-times with respect to t_2, we obtain

$$\frac{\partial^k u(x, t_1 + t_2)}{\partial(t_1 + t_2)^k} = P_{t_1}\left(\frac{\partial^k u(x, t_2)}{\partial t_2^k}\right),$$

and then we proceed as in the previous argument.

Finally, to prove (4.8), we use again the representation (2.6) of the Poisson-Hermite semigroup and differentiating k-times with respect to t, we get

$$\frac{\partial^k}{\partial t^k}u(x, t) = \int_0^{+\infty} T_s f(x)\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds).$$

Thus, by the Minkowski's integral inequality, the $L^{p(\cdot)}$-boundedness of the Ornstein-Uhlenbeck semigroup (see [8]) and the Corollary 2.1,

$$\begin{aligned}
\left\|\frac{\partial^k u(\cdot, t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d} &\leq 4\int_0^{+\infty}\left\|T_s f\frac{\partial^k \mu_t^{(1/2)}}{\partial t^k}(ds)\right\|_{p(\cdot),\gamma_d} \\
&= 4\int_0^{+\infty}\|T_s f\|_{p(\cdot),\gamma_d}\left|\frac{\partial^k \mu_t^{(1/2)}}{\partial t^k}(ds)\right| \\
&\leq 4C_{p(\cdot)}\|f\|_{p(\cdot),\gamma_d}\int_0^{+\infty}\left|\frac{\partial^k \mu_t^{(1/2)}}{\partial t^k}(ds)\right| \leq \frac{C_{k,p(\cdot)}}{t^k}\|f\|_{p(\cdot),\gamma_d}.
\end{aligned}$$

Hence, $\left\|\frac{\partial^k u(\cdot, t)}{\partial t^k}\right\|_{p(\cdot),\gamma_d} \leq \frac{C_{k,p(\cdot)}}{t^k}\|f\|_{p(\cdot),\gamma_d},\ t > 0.$ □

Now, let us study some inclusion relations between variable Gaussian Besov-Lipschitz spaces. The next result is analogous to Proposition 10, page 153 in [12] (see also [10] or Proposition 7.36 in [16]).

Proposition 4.1. *Let* $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ *and* $q_1(\cdot), q_2(\cdot) \in \mathcal{P}_{0,\infty}$. *The inclusion* $B^{\alpha_1}_{p(\cdot),q_1(\cdot)}(\gamma_d) \subset B^{\alpha_2}_{p(\cdot),q_2(\cdot)}(\gamma_d)$ *holds i.e.* $\|f\|_{B^{\alpha_2}_{p(\cdot),q_2(\cdot)}(\gamma_d)} \leq C\,\|f\|_{B^{\alpha_1}_{p(\cdot),q_1(\cdot)}(\gamma_d)}$ *if:*

i) $\alpha_1 > \alpha_2 > 0$ ($q_1(\cdot)$ and $q_2(\cdot)$ not need to be related)*, or*

ii) If $\alpha_1 = \alpha_2$ *and* $q_1(t) \leq q_2(t)$ *a.e.*

Proof. To prove part *ii*), let us take α the common value of α_1 and α_2.

Let $f \in B^{\alpha}_{p(\cdot),q_1(\cdot)}$ and set $A = \left\|t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right\|_{q_1(\cdot),\mu}$.

Fixed $t_0 > 0$,

$$\left\|\chi_{[\frac{t_0}{2},t_0]}t^{k-\alpha}\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right\|_{q_1(\cdot),\mu} \leq A.$$

However, by Lemma 4.4,

$$\left\|\frac{\partial^k P_{t_0} f}{\partial t_0^k}\right\|_{p(\cdot),\gamma_d} \leq C_{p(\cdot)} \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}, t \in [\frac{t_0}{2}, t_0].$$

Thus, we obtain

$$\begin{aligned}\left\|\frac{\partial^k P_{t_0} f}{\partial t_0^k}\right\|_{p(\cdot),\gamma_d} \left\|\chi_{[\frac{t_0}{2},t_0]} t^{k-\alpha}\right\|_{q_1(\cdot),\mu} &\leq C_{p(\cdot)} \left\|\chi_{[\frac{t_0}{2},t_0]} t^{k-\alpha} \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right\|_{q_1(\cdot),\mu} \\ &\leq C_{p(\cdot)} A.\end{aligned}$$

Therefore,

$$\begin{aligned}(\frac{t_0}{2})^{k-\alpha} \left\|\frac{\partial^k P_{t_0} f}{\partial t_0^k}\right\|_{p(\cdot),\gamma_d} \left\|\chi_{[\frac{t_0}{2},t_0]}\right\|_{q_1(\cdot),\mu} &\leq \left\|\frac{\partial^k P_{t_0} f}{\partial t_0^k}\right\|_{p(\cdot),\gamma_d} \left\|\chi_{[\frac{t_0}{2},t_0]} t^{k-\alpha}\right\|_{q_1(\cdot),\mu} \\ &\leq C_{p(\cdot)} A,\end{aligned}$$

and by Lemma 3.1

$$\begin{aligned}(\frac{t_0}{2})^{k-\alpha} \left\|\frac{\partial^k P_{t_0} f}{\partial t_0^k}\right\|_{p(\cdot),\gamma_d} (\ln 2)^{1/q_1^-} &\leq (\frac{t_0}{2})^{k-\alpha} \left\|\frac{\partial^k P_{t_0} f}{\partial t_0^k}\right\|_{p(\cdot),\gamma_d} \left\|\chi_{[\frac{t_0}{2},t_0]}\right\|_{q_1(\cdot),\mu} \\ &\leq C_{p(\cdot)} A.\end{aligned}$$

Then,

$$\left\|\frac{\partial^k P_{t_0} f}{\partial t_0^k}\right\|_{p(\cdot),\gamma_d} \leq \frac{C_{p(\cdot)} 2^{k-\alpha}}{(\ln 2)^{1/q_1^-}} A t_0^{-k+\alpha},$$

and since t_0 is arbitrary

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d} \leq C_{k,\alpha,p(\cdot)q_1(\cdot)} A t^{-k+\alpha}, \quad \text{for all } t > 0.$$

In other words, $f \in B^\alpha_{p(\cdot),q_1(\cdot)}$ implies that $f \in B^\alpha_{p(\cdot),\infty}$.

Now, let us take $g(t) = t^{k-\alpha} \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}$, then $\rho_{q_1(\cdot),\mu}(g) < \infty$, since $f \in B^\alpha_{p(\cdot),q_1(\cdot)}$.

Thus, as $q_2(t) \geq q_1(t)$ *a.e.*,

$$\begin{aligned}\rho_{q_2(\cdot),\mu}(g) &= \int_0^{+\infty} \left(t^{k-\alpha} \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right)^{q_2(t)} \frac{dt}{t} \\ &= \int_0^{+\infty} \left(t^{k-\alpha} \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right)^{q_2(t)-q_1(t)} \left(t^{k-\alpha} \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right)^{q_1(t)} \frac{dt}{t} \\ &\leq (C_{k,\alpha,p(\cdot)q_1(\cdot)} A)^{q_2^+-q_1^-} \int_0^{+\infty} \left(t^{k-\alpha} \left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p(\cdot),\gamma_d}\right)^{q_1(t)} \frac{dt}{t} \\ &= (C_{k,\alpha,p(\cdot)q_1(\cdot)} A)^{q_2^+-q_1^-} \rho_{q_1(\cdot),\mu}(g) < +\infty.\end{aligned}$$

Hence, $f \in B^{\alpha}_{p(\cdot),q_2(\cdot)}$.

In order to prove part *i*), by Lemma 4.4, we obtain

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq C_{k,p(\cdot)} \|f\|_{p(\cdot),\gamma_d} t^{-k},\ t > 0.$$

Now, given $f \in B^{\alpha_1}_{p(\cdot),q_1(\cdot)}$, again by setting

$$A = \left\| t^{k-\alpha_1} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q_1(\cdot),\mu},$$

we obtain, as in part *ii*),

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \leq C_{k,\alpha_1,p(\cdot)q_1(\cdot)} A t^{-k+\alpha_1},\ \text{for all } t > 0.$$

Therefore,

$$\begin{aligned} \left\| t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q_2(\cdot),\mu} &\leq \left\| \chi_{(0,1]} t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q_2(\cdot),\mu} \\ &\quad + \left\| \chi_{(1,\infty)} t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q_2(\cdot),\mu} \\ &= (I) + (II). \end{aligned}$$

Now, again by Lemma 3.1 we get,

$$\begin{aligned} (I) &= \left\| \chi_{(0,1]} t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q_2(\cdot),\mu} \leq \left\| \chi_{(0,1]} t^{k-\alpha_2} C_{k,\alpha_1,p(\cdot)q_1(\cdot)} A t^{-k+\alpha_1} \right\|_{q_2(\cdot),\mu} \\ &= C_{k,\alpha_1,p(\cdot)q_1(\cdot)} A \left\| \chi_{(0,1]} t^{\alpha_1-\alpha_2} \right\|_{q_2(\cdot),\mu} < \infty, \end{aligned}$$

and also by Lemma 3.1,

$$\begin{aligned} (II) &= \left\| \chi_{(1,\infty)} t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q_2(\cdot),\mu} \leq \left\| \chi_{(1,\infty)} t^{k-\alpha_2} C_{k,p(\cdot)} t^{-k} \right\|_{q_2(\cdot),\mu} \\ &= C_{k,p(\cdot)} \left\| \chi_{(1,\infty)} t^{-\alpha_2} \right\|_{q_2(\cdot),\mu} < \infty. \end{aligned}$$

Hence, $\left\| t^{k-\alpha_2} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q_2(\cdot),\mu} < +\infty$, and then $f \in B^{\alpha_2}_{p(\cdot),q_2(\cdot)}$. □

Remark 4.2. *Variable Gaussian Besov-Lipschitz and variable Gaussian Triebel-Lizorkin spaces are, by construction, subspaces of* $L^{p(\cdot)}(\gamma_d)$. *Moreover, since trivially* $\|f\|_{p(\cdot),\gamma_d} \leq \|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}}$ *and* $\|f\|_{p(\cdot),\gamma_d} \leq \|f\|_{F^{\alpha}_{p(\cdot),q(\cdot)}}$, *the inclusions are continuous.*

On the other hand, from (2.9) it is clear that for all $t > 0$ *and* $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial t^k} P_t h_\beta(x) = (-1)^k |\beta|^{k/2} e^{-t\sqrt{|\beta|}} h_\beta(x),$$

and again by Lemma 3.1,

$$\left\| t^{k-\alpha} \left\| \frac{\partial^k}{\partial t^k} P_t h_\beta \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} = \left\| t^{k-\alpha} \left\| (-|\beta|^{1/2})^k e^{-t\sqrt{|\beta|}} h_\beta \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu}$$
$$= |\beta|^{k/2} \|h_\beta\|_{p(\cdot),\gamma_d} \left\| t^{k-\alpha} e^{-t\sqrt{|\beta|}} \right\|_{q(\cdot),\mu}$$
$$= C_{k,\alpha,\beta,q(\cdot)} \|h_\beta\|_{p(\cdot),\gamma_d} < \infty.$$

Thus, $h_\beta \in B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$ and

$$\|h_\beta\|_{B^\alpha_{p(\cdot),q(\cdot)}} = (1 + C_{k,\alpha,\beta,q(\cdot)}) \|h_\beta\|_{p(\cdot),\gamma_d}.$$

In a similar way, $h_\beta \in F^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$ and

$$\|h_\beta\|_{F^\alpha_{p(\cdot),q(\cdot)}} = \|h_\beta\|_{p(\cdot),\gamma_d} + \left\| \left\| t^{k-\alpha} \left| \frac{\partial^k}{\partial t^k} P_t h_\beta \right| \right\|_{q(\cdot),\mu} \right\|_{p(\cdot),\gamma_d}$$
$$= \|h_\beta\|_{p(\cdot),\gamma_d} + |\beta|^{k/2} \left\| t^{k-\alpha} e^{-t\sqrt{|\beta|}} \right\|_{q(\cdot),\mu} \|h_\beta\|_{p(\cdot),\gamma_d}$$
$$= (1 + C_{k,\alpha,\beta,q(\cdot)}) \|h_\beta\|_{p(\cdot),\gamma_d} = \|h_\beta\|_{B^\alpha_{p(\cdot),q(\cdot)}}.$$

Hence, the set of all polynomials $\mathcal{P}$ is contained in $B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$ and in $F^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$.

Also, we have an inclusion result for variable Gaussian Triebel-Lizorkin spaces, which is analogous to Proposition 4.1, see also [10] or Proposition 7.40 in [16].

Proposition 4.2. *Let $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q_1(\cdot), q_2(\cdot) \in \mathcal{P}_{0,\infty}$. The inclusion $F^{\alpha_1}_{p(\cdot),q_1(\cdot)}(\gamma_d) \subset F^{\alpha_2}_{p(\cdot),q_2(\cdot)}(\gamma_d)$ holds i.e., $\|f\|_{F^{\alpha_2}_{p(\cdot),q_2(\cdot)}(\gamma_d)} \leq C\|f\|_{F^{\alpha_1}_{p(\cdot),q_1(\cdot)}(\gamma_d)}$, for $\alpha_1 > \alpha_2 > 0$ and $q_1(t) > q_2(t)$ a.e.*

Proof. Let us consider $f \in F^{\alpha_1}_{p(\cdot),q_1(\cdot)}$, then

$$\left\| t^{k-\alpha_2} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right\|_{q_2(\cdot),\mu} \leq \left\| t^{k-\alpha_2} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \chi_{(0,1]} \right\|_{q_2(\cdot),\mu}$$
$$+ \left\| t^{k-\alpha_2} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \chi_{(1,\infty)} \right\|_{q_2(\cdot),\mu}$$
$$= (I) + (II).$$

Now, since $q_1(t) > q_2(t)$ *a.e.*, by taking $r(t) = \dfrac{q_1(t)q_2(t)}{q_1(t) - q_2(t)}$, we obtain that $r(\cdot) \geq 1$ *a.e.* and $\dfrac{1}{r(\cdot)} + \dfrac{1}{q_1(\cdot)} = \dfrac{1}{q_2(\cdot)}$. Thus, by Hölder's inequality (2.18) and Lemma 3.1

$$(I) = \left\| t^{\alpha_1-\alpha_2} \chi_{(0,1]} t^{k-\alpha_1} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right\|_{q_2(\cdot),\mu} \leq 2 \left\| t^{\alpha_1-\alpha_2} \chi_{(0,1]} \right\|_{r(\cdot),\mu} \left\| t^{k-\alpha_1} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right\|_{q_1(\cdot),\mu}$$
$$= C_{\alpha_1,\alpha_2,q_1(\cdot),q_2(\cdot)} \left\| t^{k-\alpha_1} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right\|_{q_1(\cdot),\mu}.$$

Now, for the second term (II), by using Lemmas 3.1 and 2.3, we get

$$\begin{aligned}(II) &= \left\| t^{k-\alpha_2}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right| \chi_{(1,\infty)}\right\|_{q_2(\cdot),\mu} \leq C_k T^* f(x)\left\|\chi_{(1,\infty)} t^{k-\alpha_2} t^{-k}\right\|_{q_2(\cdot),\mu} \\ &= C_k T^* f(x)\left\|\chi_{(1,\infty)} t^{-\alpha_2}\right\|_{q_2(\cdot),\mu} = C_{k,\alpha_2,q_2(\cdot)} T^* f(x).\end{aligned}$$

Then, by using the $L^{p(\cdot)}(\gamma_d)$ boundedness of T^* (see [8]),

$$\begin{aligned}\left\|\left\| t^{k-\alpha_2}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right\|_{q_2(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} &\leq C_{\alpha_1,\alpha_2,q_1(\cdot),q_2(\cdot)}\left\|\left\| t^{k-\alpha_1}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right\|_{q_1(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} \\ &+ C_{k,\alpha_2,q_2(\cdot)}\|T^* f\|_{p(\cdot),\gamma_d} \\ &\leq C_{\alpha_1,\alpha_2,q_1(\cdot),q_2(\cdot)}\left\|\left\| t^{k-\alpha_1}\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right\|_{q_1(\cdot),\mu}\right\|_{p(\cdot),\gamma_d} \\ &+ C_{k,\alpha_2,p(\cdot),q_2(\cdot)}\|f\|_{p(\cdot),\gamma_d} < +\infty.\end{aligned}$$

Therefore, $f \in F^{\alpha_2}_{p(\cdot),q_2(\cdot)}$. □

Finally, we are going to consider some interpolation results for the Gaussian variable Besov-Lipschitz and the variable Triebel-Lizorkin spaces. We will use the following results for general variable Lebesgue spaces $L^{p(\cdot)}(X,\nu)$.

Lemma 4.5. *Let $p(\cdot) \in \mathcal{P}(\Omega,\nu)$ and $s > 0$ such that $sp^- \geq 1$. Then* $\||f|^s\|_{p(\cdot),\nu} = \|f\|^s_{sp(\cdot),\nu}$.

Proof. It is the same proof of Lemma 3.2.6 in [4]. □

Lemma 4.6. *Let ν a complete σ-finite measure on X. $r_j(\cdot) \in \mathcal{P}(X,\nu)$, $1 < r_j^-, r_j^+ < \infty, j = 0,1$. For all $0 < \lambda < 1$, if $f \in L^{r_j(\cdot)}(X,\nu)$, $j = 0,1$ then $f \in L^{r(\cdot)}(X,\nu)$ where $\frac{1}{r(y)} = \frac{1-\lambda}{r_0(y)} + \frac{\lambda}{r_1(y)}$, a.e. $y \in X$ and*

$$\|f\|_{r(\cdot),\nu} \leq 2\|f\|^{1-\lambda}_{r_0(\cdot),\nu}\|f\|^{\lambda}_{r_1(\cdot),\nu}. \tag{4.9}$$

Proof. It is a consequence of Hölder's inequality (2.18) and Lemma 4.5. □

Now, we present the interpolation result.

Theorem 4.1. *Let $p_j(\cdot) \in \mathcal{P}(\mathbb{R}^d,\gamma_d)$, $q_j \in \mathcal{P}(\mathbb{R}^+,\mu)$, with $j = 0,1$. Suppose that $1 < p_j^-, q_j^-, p_j^+, q_j^+ < +\infty$ and $\alpha_j \geq 0$. For all $0 < \theta < 1$, let us take*

$$\alpha = \alpha_0(1-\theta) + \alpha_1\theta,$$
$$\frac{1}{p(x)} = \frac{1-\theta}{p_0(x)} + \frac{\theta}{p_1(x)},\ a.e.\ x \in \mathbb{R}^d,$$
$$and\quad \frac{1}{q(t)} = \frac{1-\theta}{q_0(t)} + \frac{\theta}{q_1(t)},\ a.e.\ t \in \mathbb{R}^+.$$

i) *if* $f \in B^{\alpha_j}_{p_j(\cdot),q_j(\cdot)}(\gamma_d)$, $j = 0, 1$, *then* $f \in B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$.

ii) *if* $f \in F^{\alpha_j}_{p_j(\cdot),q_j(\cdot)}(\gamma_d)$, $j = 0, 1$, *then* $f \in F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$.

Proof. *i)* Let k be any integer greater than α_0 and α_1, by using Lemma 4.6, we obtain for $\alpha = \alpha_0(1-\theta)+\alpha_1\theta$,

$$\begin{aligned}
&\left\| t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} \\
&\leq \left\| t^{k-(\alpha_0(1-\theta)+\alpha_1\theta)} 2 \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|^{1-\theta}_{p_0(\cdot),\gamma_d} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|^{\theta}_{p_1(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} \\
&= 2 \left\| t^{(1-\theta)(k-\alpha_0)+\theta(k-\alpha_1)} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|^{1-\theta}_{p_0(\cdot),\gamma_d} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|^{\theta}_{p_1(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} \\
&= 2 \left\| \left(t^{k-\alpha_0} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_0(\cdot),\gamma_d} \right)^{1-\theta} \left(t^{k-\alpha_1} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_1(\cdot),\gamma_d} \right)^{\theta} \right\|_{q(\cdot),\mu}.
\end{aligned}$$

Thus, by Hölder's inequality (2.18) and Lemma 4.5,

$$\begin{aligned}
&\left\| t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} \\
&\leq 4 \left\| t^{k-\alpha_0} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_0(\cdot),\gamma_d} \right\|^{1-\theta}_{q_0(\cdot),\mu} \left\| t^{k-\alpha_1} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p_1(\cdot),\gamma_d} \right\|^{\theta}_{q_1(\cdot),\mu} < +\infty,
\end{aligned}$$

that is, $f \in B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$.

ii) Analogously, by Hölder's inequality (2.18) and Lemma 4.5, we obtain for $\alpha = \alpha_0(1-\theta)+\alpha_1\theta$,

$$\begin{aligned}
\left\| t^{k-\alpha} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right\|_{q(\cdot),\mu} &= \left\| \left(t^{k-\alpha_0} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right)^{1-\theta} \left(t^{k-\alpha_1} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right)^{\theta} \right\|_{q(\cdot),\mu} \\
&\leq 2 \left\| t^{k-\alpha_0} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right\|^{1-\theta}_{q_0(\cdot),\mu} \left\| t^{k-\alpha_1} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \right\|^{\theta}_{q_1(\cdot),\mu}, \ \text{a.e.} \quad x \in \mathbb{R}^d.
\end{aligned}$$

Therefore

$$\left\| \left\| t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right\|_{q(\cdot),\mu} \right\|_{p(\cdot),\gamma_d} \leq 2 \left\| \left\| t^{k-\alpha_0} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right\|^{1-\theta}_{q_0(\cdot),\mu} \left\| t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right\|^{\theta}_{q_1(\cdot),\mu} \right\|_{p(\cdot),\gamma_d},$$

and again by Hölder's inequality and Lemma 4.5,

$$\left\| \left\| t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right\|_{q(\cdot),\mu} \right\|_{p(\cdot),\gamma_d} \leq 4 \left\| \left\| t^{k-\alpha_0} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right\|_{q_0(\cdot),\mu} \right\|^{1-\theta}_{p_0(\cdot),\gamma_d} \left\| \left\| t^{k-\alpha_1} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right\|_{q_1(\cdot),\mu} \right\|^{\theta}_{p_1(\cdot),\gamma_d} < +\infty,$$

that is, $f \in F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$. □

In a forthcoming paper [9], we establish boundedness properties on $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ for some operators associated with the Gaussian measure, such as Riesz Potentials, Bessel Potentials and Bessel Fractional Derivatives.

5. Conclusions

i) Lemmas 4.1–4.3 showed that the definitions of $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ and $F^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ are independent of the integer k greater than α considered and the corresponding norms are equivalent.

ii) Lemma 3.1 was the key in the proof of Proposition 4.1.

iii) The boundedness of the maximal function of the Ornstein-Uhlenbeck semigroup T^* on $L^{p(\cdot)}(\gamma_d)$ (see [8]) was crucial in the proof of Lemma 4.4 and Proposition 4.2.

iv) The structure and properties of the Besov-Lipschitz and Triebel-Lizokin Gaussian spaces are preserved when we go from constant exponent to variable exponent setting if the exponent functions $p(\cdot)$, $q(\cdot)$ satisfy the regularity conditions $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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