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Research article

Global well-posedness for the 3D rotating Boussinesq equations in variable exponent Fourier-Besov spaces

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Abstract: We study the small initial data Cauchy problem for the three-dimensional Boussinesq equations with the Coriolis force in variable exponent Fourier-Besov spaces. Using the Fourier localization argument and Littlewood-Paley decomposition, we obtain the global well-posedness result for small initial data (u_0, θ_0) belonging to the critical variable exponent Fourier-Besov spaces $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}$.

Keywords: Boussinesq equations; Coriolis force; global well-posedness; variable exponent Fourier-Besov spaces
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1. Introduction

In this paper, we consider the three-dimensional Boussinesq equations with the Coriolis force:

$$\begin{cases} \partial_t u - \nu\Delta u + \Omega e_3 \times u + (u\cdot\nabla)u + \nabla P = g\theta e_3, & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ \partial_t \theta - \mu\Delta\theta + (u\cdot\nabla)\theta = 0, & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ \mathrm{div} u = 0, & \text{in} \quad \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u_0, \quad \theta(x,0) = \theta_0, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ denotes the velocity field of the fluid, θ is the fluctuation, P is the the pressure. The positive constants ν, μ and g are the kinetic viscosity, the thermal diffusivity and the gravity, respectively. $\Omega \in \mathbb{R}$ is the Coriolis parameter, which denotes twice the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$. The term $g\theta e_3$ represents buoyancy force using the Boussinesq approximation, which consists in neglecting the density dependence in all the terms but the one involving the gravity. The parameters ν and μ do not play any important role and we set $\nu = \mu = 1$ throughout the rest of this paper. Sun, Liu and Yang [31] proved that the three-dimensional Boussinesq equations with Coriolis force possessed a unique global solution in Besov space. Koba, Mahalov and Yoneda [37] obtained the global well-posedness to the rotating Boussinesq equations for $(u_0, \theta_0) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ when the the Prandtl number $P = 1$. Charve and Ngo [10] proved the global well-posedness to the rotating Boussinesq equations with the fading, anisotropic viscosities. For more detailed explanation, we refer to [6, 9, 12, 27, 30, 32].

When $\Omega = 0$, (1.1) reduces to the classical Boussinesq equations. The global well-posedness result for three-dimensional Navier-Stokes-Boussinesq system with axisymmetric initial data has been studied by many researchers, which can be referred to [2] and [20]. We also refer to [13, 14, 23, 29] for details on these results.

When $\Omega \neq 0$, but $\theta \equiv 0$, (1.1) reduces to the Navier-Stokes equations with the Coriolis force. Fang, Han and Hieber [16] proved the uniqueness of the global mild solution to the rotating Navier-Stokes equations with only horizontal dissipation in Fourier-Besov space $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p}}_{p,r}(\mathbb{R}^3)$ for $p \in [2, \infty], r \in [1, \infty)$. Hieber and Shibata [19] proved that the well-posedness of the Navier-Stokes equations with the Coriolis force. In addition, they also obtained the Navier-Stokes equations possess a unique global mild solution for arbitrary speed of rotation provided that the initial data u_0 is small enough in $H^{\frac{1}{2}}_{\sigma}(\mathbb{R}^3)$. We refer to [4–7, 17, 21, 22, 24, 25, 33, 35] for details.

When $\Omega = 0$, and $\theta \equiv 0$, (1.1) reduces to the classical Navier-Stokes equations. Sun and Liu [34] demonstrated uniqueness of the weak solution to the fractional anisotropic Navier-Stokes system with only horizontal dissipation. Bourgain and Pavlovic [8] proved the three-dimensional Navier-Stokes equations is ill-posed in $\dot{\mathcal{B}}^{-1,\infty}_{\infty}(\mathbb{R}^3)$. Ru and Abidin [28] obtained the global well-posedness for the fractional Navier-Stokes equations in variable exponent Fourier-Besov spaces $\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}(\mathbb{R}^3)$. There are many studies on the classical Navier-Stokes equations, which we can refer to [1, 26, 36] and the references therein.

There are many differences between variable exponent Fourier-Besov spaces and Fourier-Besov Spaces. Some classical theories such as Young's inequality and the multiplier theorem do not hold in variable exponent Fourier-Besov spaces. Because of this, it is difficult to consider the well-posedness of equations on such spaces. In this paper, we mainly use the properties introduced in Sections 2 and 3, and combine with the Banach's contraction mapping principle to consider the global well-posedness of the Boussinesq equations with the Coriolis force in variable exponent frequency spaces $\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^3)$. The major results are as follows.

Theorem 1.1. *Let $p(\cdot) \in C_{\log}(\mathbb{R}^3) \cap \mathcal{P}_0(\mathbb{R}^3)$, $2 \leq p(\cdot) \leq 6$, $1 \leq q, \rho \leq \infty$, and there exist a sufficiently small $\epsilon > 0$, such that*

$$\|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} < \epsilon$$

for $\Omega \in \mathbb{R}$. Then problem (1.1) *has a unique global solution*

$$(u, \theta) \in \widetilde{L}^{\infty}(0, \infty; \mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}) \cap \widetilde{L}^{\rho}(0, \infty; \dot{\mathcal{B}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q}) \cap \widetilde{L}^{\infty}(0, \infty; \dot{\mathcal{B}}^{\frac{1}{2}}_{2,q}).$$

Moreover, let $p(\cdot) \in C_{\log}(\mathbb{R}^3) \cap \mathcal{P}_0(\mathbb{R}^3)$, $s_1(\cdot) \in C_{\log}(\mathbb{R}^3)$, and $s_1(\cdot) = \frac{2}{\rho} + 2 - \frac{3}{p_1(\cdot)}$, if there exist a constant $c > 0$ such that $2 \leq p_1(\cdot) \leq c \leq p(\cdot)$, then the above solution is still satisfied

$$(u, \theta) \in C(0, \infty; \mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}) \cap \widetilde{L}^{\rho}(0, \infty; \mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q}).$$

Remark 1.1. *The Fourier-Besov space* $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}$ *is critical for Eq* (1.1)*. In fact, if* $u(t,x)$ *is the solution of Eq* (1.1)*, then*

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

is also a solution of the same equation and

$$\|u_0(0,x)\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} \sim \|u_\lambda(0,x)\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.$$

Remark 1.2. *From the structure of variable exponent Fourier-Besov space, we can find that this kind of space is quite different from variable exponent Besov space. Comparing with variable exponent Besov space, this kind of space is more favorable for us to consider the boundedness of semigroup operators and the estimation of nonlinear terms. This kind of space has been applied to dynamic systems, image processing and partial differential equations. However, due to the special structure of this kind of space, it is limited in the local and global well-posedness of some equations.*

In Section 1, we mainly introduce some backgrounds and major results. We recall the known basic facts about Littlewood-Paley theory and function spaces in Section 2. In Section 3, we establish the linear estimates of the semigroup $\{T_\Omega(t)\}_{t>0}$ and we are devoted to the proof of Theorem 1.1 in Section 4.

2. Function spaces

$\mathcal{S}(\mathbb{R}^n)$ denotes the space of smooth rapidly decreasing functions on $\mathbb{R}^n$. $\mathcal{S}'(\mathbb{R}^n)$ denotes the topological dual space of the $\mathcal{S}(\mathbb{R}^n)$, also be called temperate distribution. For any $f \in X$, there exists a constant $c > 0$ such that $\|f\|_a \leq c\|f\|_b$, then it is written as $\|\cdot\|_a \lesssim \|\cdot\|_b$. We first recall the homogeneous Littlewood-Paley decomposition [18].

Let (χ, φ) be a couple of smooth functions with values in $[0,1]$, χ is supported in the ball $B(0,\frac{3}{4}) = \{\xi \in \mathbb{R}^3 | |\xi| \leq \frac{3}{4}\}$, φ is supported in the shell $C(0,\frac{3}{4},\frac{8}{3}) = \{\xi \in \mathbb{R}^3 | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. We use $\varphi_j(\xi)$ to denote $\varphi(2^{-j}\xi)$ and

$$\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \qquad \forall \xi \in \mathbb{R}^3 \backslash \{0\}.$$

The localization operators are defined by

$$\dot{\Delta}_j u = \varphi_j(D)u = 2^{3j}\int_{\mathbb{R}^3} \psi(2^j y)u(x-y)\mathrm{d}y, \qquad \forall j \in \mathbb{Z},$$

$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{3j}\int_{\mathbb{R}^3} h(2^j y)u(x-y)\mathrm{d}y, \qquad \forall j \in \mathbb{Z},$$

where $\psi = \mathcal{F}^{-1}\varphi$ and $h = \mathcal{F}^{-1}\chi$.

From the definition above there hold that

$$\dot{\Delta}_k\dot{\Delta}_j u = 0, \qquad \text{if} \quad |j-k| \geq 2,$$

$$\dot{\Delta}_k(\dot{S}_{j-1}u\dot{\Delta}_j u) = 0, \quad \text{if} \quad |j-k| \geq 5.$$

If $u \in \mathcal{S}_h'$, there holds that

$$\dot{S}_j u = \sum_{i \le j-1} \dot{\Delta}_i u.$$

Let $\mathcal{P}_0(\mathbb{R}^n)$ be the set of all measure functions $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ such that $p_- = \operatorname{essinf}_{x\in\mathbb{R}^n} p(x)$, $p_+ = \operatorname{esssup}_{x\in\mathbb{R}^n} p(x)$. For $p \in \mathcal{P}_0(\mathbb{R}^n)$, let $L^{p(\cdot)}(\mathbb{R}^n)$ be the set of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\begin{aligned}\|f\|_{L^{p(\cdot)}} &:= \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \le 1\} \\ &= \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1\right\}.\end{aligned}$$

We postulate the following standard conditions to ensure that the Hardy-Maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$:

1) p is said to satisfy the Locally log-Hölder's continuous condition if there exists a positive constant $C_{\log}(p)$ such that $|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e+|x-y|^{-1})}$, (for all $x, y \in \mathbb{R}^n, x \ne y$).

2) p is said to satisfy the Globally log-Hölder's continuous condition if there exists a positive constant $C_{\log}(p)$ and p_∞, such that $|p(x) - p_\infty| \le \frac{C_{\log}(p)}{\log(e+|x|)}$, (for all $x \in \mathbb{R}^n$).

We use $C_{\log}(\mathbb{R}^n)$ as the set of all real valued functions $p : \mathbb{R}^n \to \mathbb{R}$ satisfying 1) and 2).

Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, we use $l^{q(\cdot)}(L^{p(\cdot)})$ to denote the space consisting of all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$ such that

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\mu > 0, \varrho_{l^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\mu}\right\}_{j\in\mathbb{Z}}\right) \le 1\right\} \le \infty,$$

where

$$\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f_j(x)|}{\lambda^{\frac{1}{q(x)}}}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

Since we assume that $q_+ < \infty$, $\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} \left\| |f_j|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}$ holds.

Definition 2.1. *[3] Let $p(\cdot), q(\cdot) \in C_{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in C_{\log}(\mathbb{R}^n)$. The homogeneous Besov space with variable exponents $\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)} = \{f \in \mathcal{S}' : \|f\|_{\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} < \infty\},$$

$$\|f\|_{\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\Delta_j f\}_{j\in\mathbb{Z}}\|_{\varrho_{l^{q(\cdot)}(L^{p(\cdot)})}} < \infty,$$

where $\mathcal{S}'$ denote the dual of $\mathcal{S}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : (\mathrm{D}^\alpha \hat{f})(0) = 0, \forall \alpha\}$.

For $T > 0$ and $\rho \in [1, \infty]$, we denote by $L^\rho(0, T; \dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$ the set of all tempered distribution u satisfying

$$\|u\|_{L^\rho(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} := \left\| \left(\sum_{j=0}^{\infty} \|2^{js(\cdot)}\Delta_j u\|^r_{L^{p(\cdot)}} \right)^{\frac{1}{r}} \right\|_{L^\rho_T} < \infty.$$

The mixed $\widetilde{L}^{\rho}(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$ is the set of all tempered distribution u satisfying

$$\|u\|_{\widetilde{L}^{\rho}(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} := \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\Delta_j u\|^r_{L^{\rho}_T L^{p(\cdot)}}\right)^{\frac{1}{r}} < \infty.$$

For simplicity, we denote

$$L^{\rho}_T\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r} := L^{\rho}(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r}) \quad \text{and} \quad \widetilde{L}^{\rho}_T\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r} := \tilde{L}^{\rho}(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r}).$$

By virtue of the Minkowski's inequality, we have

$$\|u\|_{\widetilde{L}^{\rho}(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} \le \|u\|_{L^{\rho}(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} \quad \text{if} \quad \rho \le r,$$

$$\|u\|_{L^{\rho}(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} \le \|u\|_{\widetilde{L}^{\rho}(0,T;\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} \quad \text{if} \quad r \le \rho.$$

To obtain the global well-posedness of the small initial data Cauchy problem for the three-dimensional Boussinesq equations with the Coriolis force in variable exponent Fourier-Besov spaces, we need to introduce the following spaces.

Definition 2.2. *[28] [Homogeneous Fourier-Besov spaces with variable exponents] Let $p(\cdot), q(\cdot) \in C_{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in C_{\log}(\mathbb{R}^n)$. The homogeneous Fourier-Besov space with variable exponents $\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)} = \{f \in \mathcal{S}' : \|f\|_{\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} < \infty\},$$

$$\|f\|_{\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j\hat{f}\}^{\infty}_{-\infty}\|_{l^{q(\cdot)}L^{p(\cdot)}} < \infty.$$

Similarly, we denote by $L^{\rho}(0,T;\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$ the set of all tempered distribution u satisfying

$$\|u\|_{L^{\rho}(0,T;\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} := \left\|\left(\sum_{j=0}^{\infty} \|2^{js(\cdot)}\varphi_j\hat{u}\|^r_{L^{p(\cdot)}}\right)^{\frac{1}{r}}\right\|_{L^{\rho}_T} < \infty.$$

The mixed $\widetilde{L}^{\rho}(0,T;\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})$ is the set of all tempered distribution u satisfying

$$\|u\|_{\widetilde{L}^{\rho}(0,T;\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),r})} := \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\varphi_j\hat{u}\|^r_{L^{\rho}_T L^{p(\cdot)}}\right)^{\frac{1}{r}} < \infty.$$

Definition 2.3. *[18] Let $u, v \in \mathcal{S}'_h$, the product uv has the homogeneous Bony decomposition as follows*

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v),$$

where

$$\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v, \qquad \dot{T}_v u = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}v\dot{\Delta}_j u,$$

$$\dot{R}(u,v) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u\tilde{\dot{\Delta}}_j v, \quad \tilde{\dot{\Delta}}_j v = \sum_{|j-k|\le 1} \dot{\Delta}_k v.$$

Lemma 2.1. *The following inclusions hold for the variable exponent function spaces.*

(I) (Hölder inequality [11]) Given a measurable set A and exponent functions $r(\cdot), q(\cdot) \in \mathcal{P}_0(A)$ define $p(\cdot) \in \mathcal{P}_0(A)$ by

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Then there exists a constant C such that for all $f \in L^{q(\cdot)}(A)$ and $g \in L^{r(\cdot)}(A)$, $fg \in L^{p(\cdot)}(A)$ and

$$\|fg\|_{p(\cdot)} \leq C\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

In particular, given A and $p(\cdot) \in \mathcal{P}_0(A)$, for all $f \in L^{p(\cdot)}(A)$ and $g \in L^{p'(\cdot)}(A)$, $fg \in L^1(A)$ and

$$\int_A |f(x)g(x)|\mathrm{d}x \leq C_{p(\cdot)}\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

where the function p' is called the dual variable exponent of p and A_*, A_1, A_∞ are disjoint sets, i.e.,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad C_{p(\cdot)} = \left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\|\chi_{A_*}\|_\infty + \|\chi_{A_\infty}\|_\infty + \|\chi_{A_1}\|_\infty.$$

(II) (Sobolev inequality [3]) Let $p_0, p_1, q \in \mathcal{P}_0(\mathbb{R}^n)$ and $s_0, s_1 \in L^\infty(\mathbb{R}^n) \cap C_{log}(\mathbb{R}^n)$ with $s_0 > s_1$. If $\frac{1}{q}$ and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$$

are locally log-Hölder continuous, then

$$\dot{\mathcal{B}}^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

(III) ([3]) Let $p_0, p_1, q_0, q_1 \in \mathcal{P}_0(\mathbb{R}^n)$ and $s_0, s_1 \in L^\infty(\mathbb{R}^n) \cap C_{log}(\mathbb{R}^n)$ with $s_0 > s_1$. If $\frac{1}{q_0}, \frac{1}{q_1}$ and

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1} + \varepsilon(x)$$

are locally log-Hölder continuous and $\operatorname{essinf}_{x\in\mathbb{R}^n} \varepsilon(x) > 0$, *then*

$$\dot{\mathcal{B}}^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)} \hookrightarrow \dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}.$$

(IV) (Mollification inequality [15]) For $p(\cdot) \in C_{\log}(\mathbb{R}^n)$ and $\psi \in L^1(\mathbb{R}^n)$, assume that $\Psi(x) = \sup_{y\notin B(0,|x|)} |\psi(y)|$ is integrable. Then

$$\|f * \psi_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\Psi\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, where $\psi_\varepsilon = \frac{1}{\varepsilon^n}\psi(\frac{1}{\varepsilon})$ and C depends only on n.

Lemma 2.2. *[18] [Hausdorff-Young's inequality] Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. Then $\hat{f} \in L^{p'}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and*

$$\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}.$$

Lemma 2.3. *[18] A constant C exists such that for all $s \in \mathbb{R}$,*

$$r_1 \le r_2 \quad \Rightarrow \quad \|u\|_{\dot{B}^s_{p,r_2}} \le C\|u\|_{\dot{B}^s_{p,r_1}},$$
$$p_1 \le p_2 \quad \Rightarrow \quad \|u\|_{\dot{B}^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r}} \le C\|u\|_{\dot{B}^s_{p_1,r}}.$$

Lemma 2.4. *Let $s > 0$, $1 \le p, r \le \infty$, $p_1(\cdot), p_2(\cdot) \in C_{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, and $\frac{1}{p} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$. Then*

$$\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{\dot{B}^0_{p_1(\cdot),r}}\|v\|_{\dot{B}^s_{p_2(\cdot),r}} + \|v\|_{\dot{B}^0_{p_1(\cdot),r}}\|u\|_{\dot{B}^s_{p_2(\cdot),r}}.$$

Proof. According to Definition 2.3, for fixed $j \ge 0$, we have

$$\begin{aligned}\Delta_j(uv) &= \sum_{|k-j|\le 4} \Delta_j(S_{k-1}u\Delta_k v) + \sum_{|k-j|\le 4} \Delta_j(S_{k-1}v\Delta_k u) + \sum_{k\ge j-2} \Delta_j(\Delta_k u\widetilde{\Delta}_k v)\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

We will estimate each of the three above. Using Young's inequality and Hölder's inequality from the Lemma 2.1, we have

$$\|2^{js}\Delta_j(S_{k-1}u\Delta_k v)\|_{L^p} \lesssim \|S_{k-1}u\|_{L^{p_1(\cdot)}}\|2^{js}\Delta_k v\|_{L^{p_2(\cdot)}},$$

then

$$\|2^{js}I_1\|_{L^p} \lesssim \sum_{|k-j|\le 4} \|S_{k-1}u\|_{L^{p_1(\cdot)}}\|2^{js}\Delta_k v\|_{L^{p_2(\cdot)}}.$$

Similarly, for I_2 we have

$$\|2^{js}I_2\|_{L^p} \lesssim \sum_{|k-j|\le 4} \|S_{k-1}v\|_{L^{p_1(\cdot)}}\|2^{js}\Delta_k u\|_{L^{p_2(\cdot)}}.$$

Now, it remains to estimates I_3. Using Young's inequality, we have

$$\|\Delta_j(\Delta_k u\widetilde{\Delta}_k v)\|_{L^p} \lesssim \|\Delta_k u\|_{L^{p_1(\cdot)}}\|\widetilde{\Delta}_k v\|_{L^{p_2(\cdot)}}.$$

Hence,

$$\begin{aligned}\|2^{js}I_3\|_{L^p} &\lesssim \sum_{k\ge j-2} \|2^{js}\Delta_k u\|_{L^{p_1(\cdot)}}\|\widetilde{\Delta}_k v\|_{L^{p_2(\cdot)}}\\ &= \sum_{k\ge j-2} 2^{(j-k)s}\|2^{ks}\Delta_k u\|_{L^{p_1(\cdot)}}\|\widetilde{\Delta}_k v\|_{L^{p_2(\cdot)}}.\end{aligned}$$

Taking the norm $\|\cdot\|_{l^r}$ on both side of above inequality, there holds that

$$\|uv\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{\dot{B}^0_{p_1(\cdot),r}}\|v\|_{\dot{B}^s_{p_2(\cdot),r}} + \|v\|_{\dot{B}^0_{p_1(\cdot),r}}\|u\|_{\dot{B}^s_{p_2(\cdot),r}}.$$

□

Lemma 2.5. *Let $s > 0$, $1 \le p, r, \rho \le \infty$, $p_1(\cdot), p_2(\cdot) \in C_{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, and $\frac{1}{p} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$. Then*

$$\|uv\|_{\widetilde{L}^\rho_T\dot{B}^s_{p,r}} \lesssim \|u\|_{\widetilde{L}^{\rho_1}_T\dot{B}^0_{p_1(\cdot),r}}\|v\|_{\widetilde{L}^{\rho_2}_T\dot{B}^s_{p_2(\cdot),r}} + \|v\|_{\widetilde{L}^{\rho_1}_T\dot{B}^0_{p_1(\cdot),r}}\|u\|_{\widetilde{L}^{\rho_2}_T\dot{B}^s_{p_2(\cdot),r}}.$$

Proof. In the proof of the Lemma 2.4, replacing $L^{p(\cdot)}$ with $L^\rho_T L^{p(\cdot)}$, we can get that the conclusion holds. □

3. Linear estimates

We establish the linear estimates of the semigroup $\{T_\Omega(t)\}_{t>0}$ in this section, and see the specific introduction of the semigroup $\{T_\Omega(t)\}_{t>0}$ in Section 4.

Lemma 3.1. *Let $p(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathcal{P}_0(\mathbb{R}^3)$, $2\le p(\cdot)\le 6$, $2\le p_1(\cdot)\le c\le p(\cdot)$, $s_1(\cdot)=\frac{2}{\rho}+2-\frac{3}{p_1(\cdot)}$ and $1\le q,\rho\le\infty$. Then*

$$\|T_\Omega(t)f\|_{\widetilde{L^\rho}\left(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q}\right)}\lesssim\|f\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}$$

for $\Omega\in\mathbb{R}$ and $f\in\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}$.

Proof. By Definition 2.2, we have

$$\|T_\Omega(t)f\|_{\widetilde{L^\rho}\left(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q}\right)}=\left\|\left\{\|2^{js_1(\cdot)}\varphi_j\mathcal{F}[T_\Omega(t)f]\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}.$$

Since $T_\Omega(t)f$ is bounded Fourier multiplier, we estimate by a positive constant. Using Lemma 2.1, we have

$$\begin{aligned}
&\|T_\Omega(t)f\|_{\widetilde{L^\rho}\left(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q}\right)}\\
&=\left\|\left\{\|2^{js_1(\cdot)}\varphi_j\mathcal{F}[T_\Omega(t)f]\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\
&\lesssim\|\{\|2^{js_1(\cdot)}\varphi_je^{-t|\cdot|^2}\hat{f}\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\}_{j\in\mathbb{Z}}\|_{l^q(\mathbb{Z})}\\
&\lesssim\left\|\left\{\sum_{l=0,\pm1}\left\|2^{j(2-\frac{3}{c})}\right\|_{L^c}\left\|2^{j(\frac{2}{\rho}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}e^{-t2^{2(j+l)}}\right\|_{L^\rho(0,\infty;L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\
&\lesssim\|f\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},
\end{aligned}$$

where the second norm in the second line above is estimated as follows

$$\begin{aligned}
&\left\|2^{j(\frac{2}{\rho}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}e^{-t2^{2(j+l)}}\right\|_{L^\rho(0,\infty;L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}})}\\
&=\|2^{j\frac{2}{\rho}}e^{-t2^{2(j+l)}}\|_{L^\rho(0,\infty)}\|2^{j(\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+l}\|_{L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}}}\\
&=\|2^{j\frac{2}{\rho}}e^{-t2^{2(j+l)}}\|_{L^\rho(0,\infty)}\inf\{\lambda>0:\int_{\mathbb{R}^3}|\frac{2^{j(\frac{3}{c}-\frac{3}{p_1(x)})}\varphi_{j+l}}{\lambda}|^{\frac{cp_1(x)}{c-p_1(x)}}\mathrm{d}x\le1\}\\
&\lesssim\inf\{\lambda>0:\int_{\mathbb{R}^3}|\frac{2^{j(\frac{3}{c}-\frac{3}{p_1(x)})}\varphi_{j+l}}{\lambda}|^{\frac{cp_1(x)}{c-p_1(x)}}\mathrm{d}x\le1\}\\
&\lesssim\inf\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_{j+l}}{\lambda}|^{\frac{cp_1(x)}{c-p_1(x)}}2^{-3j}\mathrm{d}x\le1\}\\
&\lesssim\inf\{\lambda>0:\int_{\mathbb{R}^3}|\frac{\varphi_l}{\lambda}|^{\frac{cp_1(2^jx)}{c-p_1(2^jx)}}\mathrm{d}x\le1\}\\
&\lesssim C.
\end{aligned}$$

□

Lemma 3.2. *Let $p(\cdot)\in C_{\log}(\mathbb{R}^3)\cap\mathcal{P}_0(\mathbb{R}^3)$, $2\le p(\cdot)\le 6$, $2\le p_1(\cdot)\le c\le p(\cdot)$, $s_1(\cdot)=\frac{2}{\rho}+2-\frac{3}{p_1(\cdot)}$ and $1\le q,\rho\le\infty$. Then*

$$\left\|\int_0^tT_\Omega(t-\tau)\mathbb{P}f\mathrm{d}\tau\right\|_{\widetilde{L^\rho}(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\lesssim\|f\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}$$

for $\Omega\in\mathbb{R}$ and $f\in\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}$.

Proof. Using Lemmas 2.1 and 2.2 and Young' inequality, we obtain

$$
\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P} f \mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})} \\
=& \left\|\left\{\left\|2^{js_1(\cdot)}\varphi_j \mathcal{F}[\int_0^t T_\Omega(t-\tau)\mathbb{P} f \mathrm{d}\tau]\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})} \\
\lesssim& \left\|\left\{\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}\widehat{f}\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})} \\
\lesssim& \|f\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})},
\end{aligned}
$$

where the inner norm of the second line above is estimated as follows

$$
\begin{aligned}
&\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}\widehat{f}\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})} \\
\lesssim& \left\|\int_0^t \|2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}\|_{L^{\frac{2p_1(\cdot)}{2-p_1(\cdot)}}}\|\varphi_j\widehat{f}\|_{L^2}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)} \\
\lesssim& \left\|\int_0^t \|2^{j(s_1(\cdot)+1)}\varphi_j e^{-(t-\tau)|\cdot|^2}\|_{L^{\frac{2p_1(\cdot)}{2-p_1(\cdot)}}}\|\Delta_j f\|_{L^2}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)} \\
\lesssim& \left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac{1}{2})}e^{-(t-\tau)2^{2j}}\|2^{-3j\frac{2-p_1(\cdot)}{2p_1(\cdot)}}\varphi_j\|_{L^{\frac{2p(\cdot)}{2-p(\cdot)}}}\|\Delta_j f\|_{L^2}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)} \\
\lesssim& \left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac{1}{2})}e^{-(t-\tau)2^{2j}}\|\Delta_j f\|_{L^2}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)} \\
\lesssim& \left\|2^{j(\frac{2}{\rho}+\frac{1}{2})}\|\Delta_j f\|_{L^2}\right\|_{L^\rho(0,\infty)}\|e^{-t2^{2j}}\|_{L^1(0,\infty)} \\
\lesssim& \left\|2^{j(\frac{2}{\rho}+\frac{1}{2})}\|\Delta_j f\|_{L^2}\right\|_{L^\rho(0,\infty)}.
\end{aligned}
$$

□

4. Proof of Theorem 1.1

In order to solve the Boussinesq equations with Coriolis force, we consider the following linear generalized problem

$$
\begin{cases}
\partial_t u - \Delta u + \Omega e_3 \times u + \nabla P = 0, & \text{in} \quad \mathbb{R}^3\times(0,\infty), \\
\mathrm{div} u = 0, & \text{in} \quad \mathbb{R}^3\times(0,\infty), \\
u|_{t=0} = u_0, & \text{in} \quad \mathbb{R}^3.
\end{cases} \tag{4.1}
$$

The solution of (4.1) can be given by the generalized Stokes-Coriolis semigroup $T_\Omega(t)$, which has the following explicit expression

$$
\begin{aligned}
T_\Omega(t)u &= \mathcal{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^2}I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-t|\xi|^2}R(\xi)\right] * u \\
&= \mathcal{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)R(\xi)\right] * (e^{t\Delta}u),
\end{aligned}
$$

where divergence free vector field $u \in \mathcal{S}(\mathbb{R}^3)$, I is the unit matrix in $M_{3\times3}(\mathbb{R})$ and $R(\xi)$ is skew-symmetric matrix defined by

$$
R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ \xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}, \quad \xi \in \mathbb{R}^3\backslash\{0\}.
$$

Hence, the solution of the Eq (1.1) can be rewritten as

$$\begin{cases} u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau + \int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau, \\ \theta(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\theta]\mathrm{d}\tau. \end{cases}$$

For the derivation of explicit form of $T_\Omega(\cdot)$, we refer to [4, 17, 19].

Proof of Theorem 1.1. Let $M > 0$, $\delta > 0$ to be determined. Set

$$X = \Bigg\{(u,\theta) : \|u\|_{\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})} + \|\theta\|_{\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})} \le M,$$
$$\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} + \|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} \le \delta\Bigg\},$$

which is equipped with the metric

$$d((u,\theta),(w,\upsilon)) = \|u-w\|_{\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})\cap\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}$$
$$+ \|\theta-\upsilon\|_{\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})\cap\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}.$$

It is easy to see that (X,d) is a complete metric space. Next we consider the following mapping

$$\Phi : (u,\theta) \to (T_\Omega(t)u_0, e^{t\Delta}\theta_0) - \left(\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau, \int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\theta]\mathrm{d}\tau\right)$$
$$+ \left(\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau, 0\right),$$

where $\mathbb{P} := I - \nabla(-\Delta)^{-1}$ denotes the Helmholtz projection onto the divergence free vector fields.

We shall prove there exist $M, \delta > 0$ such that $\Phi : (X,d) \to (X,d)$ is a strict contraction mapping.

First, we establish that the estimate of $(T_\Omega(t)u_0, e^{t\Delta}\theta_0)$. According to Lemma 3.1, it follows that

$$\|T_\Omega(t)u_0\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q}\right)} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},$$

and we have

$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q}\right)} \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}$$

when $\Omega = 0$.

Similarly we can obtain

$$\|T_\Omega(t)u_0\|_{\widetilde{L}^\rho(0,\infty;\dot{\mathcal{B}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},$$
$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^\rho(0,\infty;\dot{\mathcal{B}}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})} \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.$$

It is easy to show that the estimate for $T_\Omega(t)u_0$ and $e^{t\Delta}\theta_0$ also hold for $\rho = \infty$ and $p_1(\cdot) = p(\cdot)$, i.e.,

$$\|T_\Omega(t)u_0\|_{\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},$$
$$\|T_\Omega(t)u_0\|_{\widetilde{L}^\infty(0,\infty;\dot{\mathcal{B}}^{\frac{1}{2}}_{2,q})} \lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},$$
$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})} \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}},$$
$$\|e^{t\Delta}\theta_0\|_{\widetilde{L}^\infty(0,\infty;\dot{\mathcal{B}}^{\frac{1}{2}}_{2,q})} \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.$$

Next we show that the estimate of the remaining terms. Using Lemmas 2.1–2.3 and 2.5, we can show that

$$\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\\
&=\left\|\left\{\left\|2^{js_1(\cdot)}\varphi_j\mathcal{F}[\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau]\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\
&\lesssim\left\|\left\{\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}[\widehat{(u\cdot\nabla)}u]\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\right\}_{j\in\mathbb{Z}}\right\|_{l^q(\mathbb{Z})}\\
&\lesssim\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^0_{3,q})}\\
&\lesssim\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})},
\end{aligned}$$

where the inner norm of the third line is estimated as follows

$$\begin{aligned}
&\left\|\int_0^t 2^{js_1(\cdot)}\varphi_j e^{-(t-\tau)|\cdot|^2}[\widehat{(u\cdot\nabla)}u]\mathrm{d}\tau\right\|_{L^\rho(0,\infty;L^{p_1(\cdot)})}\\
&\lesssim\left\|\int_0^t \|2^{j(s_1(\cdot)+1)}\varphi_j e^{-(t-\tau)|\cdot|^2}\|_{L^{\frac{6p_1(\cdot)}{6-p_1(\cdot)}}}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac65}}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
&\lesssim\left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac52)}e^{-(t-\tau)2^{2j}}\|2^{-3j\frac{6-p_1(\cdot)}{6p_1(\cdot)}}\varphi_j\|_{L^{\frac{6p_1(\cdot)}{6-p_1(\cdot)}}}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac65}}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
&\lesssim\left\|\int_0^t 2^{j(\frac{2}{\rho}+\frac52)}e^{-(t-\tau)2^{2j}}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac65}}\mathrm{d}\tau\right\|_{L^\rho(0,\infty)}\\
&\lesssim\left\|2^{j(\frac{2}{\rho}+\frac52)}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac65}}\right\|_{L^\rho(0,\infty)}\|e^{-t2^{2j}}\|_{L^1(0,\infty)}\\
&\lesssim\left\|2^{j(\frac{2}{\rho}+\frac12)}\|\dot{\Delta}_j(u\otimes u)\|_{L^{\frac65}}\right\|_{L^\rho(0,\infty)}.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\lesssim\|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})},\\
&\left\|\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\theta]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q})}\lesssim\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})}.
\end{aligned}$$

In addition, we can also get

$$\begin{aligned}
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})}\\
&=\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}[(u\cdot\nabla)u]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}^{\frac12}_{2,q})},\\
&\lesssim\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})},\\
&\left\|\int_0^t T_\Omega(t-\tau)\mathbb{P}g\theta e_3\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})}\\
&\lesssim\|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})},\\
&\left\|\int_0^t e^{(t-\tau)\Delta}[(u\cdot\Delta)\theta]\mathrm{d}\tau\right\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})}\\
&\lesssim\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac12}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac12}_{2,q})}.
\end{aligned}$$

We finally prove that the existence and uniqueness.

Let $Y = \widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}) \bigcap \widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q}) \bigcap \widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})$, then

$$\begin{aligned}
&\|\Phi(u,\theta)\|_Y \\
&= \|\Phi(u)\|_Y + \|\Phi(\theta)\|_Y \\
&\lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} + \|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} \\
&\quad+\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} + \|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}.
\end{aligned}$$

Denote $\delta = M = 2\left(\|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}}\right) < 2C_\epsilon$, if ϵ is small enough, then we have

$$\|\Phi(u,\theta)\|_Y \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

and

$$d(\Phi(u,\theta),\Phi(w,\upsilon)) \leq \frac{1}{2}d((u,\theta),(w,\upsilon)).$$

It follows from the Banach's contraction mapping principle that the rotating Boussinesq equation has a unique global solution and satisfies

$$(u,\theta) \in \widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}) \cap \widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q}) \cap \widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})$$

when ϵ is small enough.

On the other hand, let

$$\begin{aligned}
Z = &\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}^{s_1(\cdot)}_{p_1(\cdot),q}) \cap \widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q}) \\
&\cap \widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q}) \cap \widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}),
\end{aligned}$$

then we have

$$\begin{aligned}
&\|\Phi(u,\theta)\|_Z \\
&= \|\Phi(u)\|_Z + \|\Phi(\theta)\|_Z \\
&\lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}\cap\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p_1(\cdot)}}_{p_1(\cdot),q}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}\cap\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p_1(\cdot)}}_{p_1(\cdot),q}} + \|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}\|u\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} \\
&\quad+\|u\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}\|\theta\|_{\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})} + \|\theta\|_{\widetilde{L}^\rho(0,\infty;\dot{B}^{\frac{2}{\rho}+\frac{1}{2}}_{2,q})\cap\widetilde{L}^\infty(0,\infty;\dot{B}^{\frac{1}{2}}_{2,q})}.
\end{aligned}$$

Set $\delta = M = 2\left(\|u_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}\cap\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p_1(\cdot)}}_{p_1(\cdot),q}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q}\cap\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{p_1(\cdot)}}_{p_1(\cdot),q}}\right) < 2C_\epsilon$, if ϵ is small enough, then we have

$$\|\Phi(u,\theta)\|_Z \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

and

$$d(\Phi(u,\theta),\Phi(w,\upsilon)) \leq \frac{1}{2}d((u,\theta),(w,\upsilon)).$$

According to the Banach's contraction mapping principle, it follows that the rotating Boussinesq equations has a unique global solution and satisfies

$$(u,\theta) \in \widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{B}^{s_1(\cdot)}_{p_1(\cdot),q}) \cap \widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{B}^{2-\frac{3}{p(\cdot)}}_{p(\cdot),q})$$

when ϵ is small enough. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that they have no conflict of interest.

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