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Research article

Error bounds for linear complementarity problems of strong SDD_1 matrices and strong SDD_1-B matrices

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Abstract: In this paper, an error bound for linear complementarity problems of strong SDD_1 matrices is given. By properties of SDD_1 matrices, a new subclass of P-matrices called SDD_1-B is presented, which contains B-matrices. A new error bound of linear complementarity problems for SDD_1-B is also provided, which improves the corresponding results. Numerical examples are given to illustrate the effectiveness of the obtained results.

Keywords: linear complementarity problems; error bound; strong SDD_1 matrices; strong SDD_1-B matrices; P-matrices
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1. Introduction

Many fundamental problems in optimization and mathematical programming can be described as linear complementarity problems, such as quadratic programming, nonlinear obstacle problems, invariant capital stock, the Nash eqilibrium point of a bimatrix game, optimal stopping, free boundary problems for journal bearing and so on, see for instance, [1–4].

Some basic definitions for the special matrices are given as follows: let n be an integer number, $N = \{1, 2, \ldots, n\}$, and let $R^{n\times n}$ be the set of all real matrices of order n. Matrix $A = (a_{ij}) \in R^{n\times n}$ is called a Z-matrix, if $a_{ij} \le 0$ for any $i \ne j$; a P-matrix, if all its principal minors are positive; an M-matrix, if A is a Z-matrix with eigenvalues whose real parts are non-negative; an H-matrix, if its comparison matrix $\langle A\rangle = (\bar{a}_{ij})$ is an M-matrix, where

$$\bar{a}_{ij} = \begin{cases} |a_{ij}|, & if \quad i = j, \\ -|a_{ij}|, & if \quad i \ne j. \end{cases}$$

Linear complementarity problem of matrix A, denoted by $LCP(A, q)$, is to find a vector $x \in R^n$ such that

$$Ax + q \geq 0, \qquad (Ax + q)^T x = 0, \qquad x \geq 0, \tag{1.1}$$

or to prove that no such vector x exists, where $A \in R^{n\times n}$ and $q \in R^n$. One of the essencial problems in $LCP(A, q)$ is to estimate

$$\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty,$$

where $D = diag(d_i)$, $d = (d_1, d_2, \cdots, d_n)$, $0 \leq d_i \leq 1$, $i = 1, 2, \cdots, n$. It is well known that when A is a P-matrix, there is a unique solution to linear complementarity problems.

In [4], Chen et al. gave the following error bound for $LCP(A, q)$,

$$\|x - x^*\|_\infty \leq \max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \|r(x)\|_\infty, \quad \forall x \in R^n, \tag{1.2}$$

where x^* is the solution of $LCP(A, q)$, $r(x) = \min\{x, Ax + q\}$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. It is well known that when real H-matrix A with positive diagonal entries is a subclass of P-matrices, error bound of $LCP(A, q)$ can be obtained by formula (2.4) in [4]. Furthermore, to avoid the high-cost computations of the inverse matrix in (2.4), some easily computable bounds for $LCP(A, q)$ are derived for the different subclass of H-matrices, such as *Ostrowski* matrices [5], QN-matrices [6], *Nekrasov* matrices [7], S-$SDDS$ matrices [8] and DZ-matrices [9], which only depends on the entries of the involved matrix A.

When the class of involved matrices is subclass of P-matrices that are not H-matrices, error bounds of $LCP(A, q)$ also need to be studied, such as, B_π^R-matrices [10], B-Nekrasov matrices [11] and CKV-type-B-matrices [12].

In this paper, we apply upper bound for infinity norm of the inverse of strong SDD_1 matrix to estimate the error for linear complementarity problems of strong SDD_1 matrices and strong SDD_1-B matrices. Numerical examples show that the obtained results can improve other existing bounds.

2. Preliminaries

In this section, some definitions and lemmas are given. Assume that S denotes a nonempty subset of N and $\overline{S} := N\backslash S$ the complement of S. For each $i \in N$, $r_i(A) := \sum_{j\neq i} |a_{ij}|$, $r_i^S(A) := \sum_{j\in S\backslash\{i\}} |a_{ij}|$.

Definition 1. [13] *Matrix* $A = (a_{ij}) \in R^{n\times n}$ *is called a strictly diagonally dominant* (SDD) *matrix if, for all* $i \in N$,

$$|a_{ii}| > r_i(A).$$

Definition 2. [14] *Matrix* $A = (a_{ij}) \in R^{n\times n}$ *is said a strong* SDD_1 *matrix if there exists a subset* S *of* N *such that*

- (i) $|a_{ii}| > r_i(A)$, for $i \in S$ satisfying $r_i^{\overline{S}}(A) = 0$,
- (ii) $|a_{jj}| > r_j(A)$, for $j \in \overline{S}$,
- (iii) $[|a_{ii}| - r_i^S(A)]|a_{jj}| > r_i^{\overline{S}}(A) r_j(A)$, for $i \in S$ and $j \in \overline{S}$ such that $a_{ij} \neq 0$.

Definition 3. *Let* $A = (a_{ij}) \in R^{n\times n}$ $(n \geq 2)$ *be a matrix with the form of* $A = B^+ + C$. *We say that* A *is a strong* SDD_1-B *matrix if* B^+ *is a strong* SDD_1 *matrix with positive diagonal entries, where*

$$B^+ = (b_{ij}) = \begin{pmatrix} a_{11} - r_1^+ & \cdots & a_{1n} - r_1^+ \\ \vdots & & \vdots \\ a_{n1} - r_n^+ & \cdots & a_{nn} - r_n^+ \end{pmatrix}, \quad C = \begin{pmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & & \vdots \\ r_n^+ & \cdots & r_n^+ \end{pmatrix}, \tag{2.1}$$

and $r_i^+ = max\left\{0, a_{ij} \mid j \neq i\right\}$.

There is an equivalence definition of B-matrices in [1, 15], which is closely related to strictly diagonally dominant matrices.

Definition 4. [15] *Matrix $A = (a_{ij}) \in R^{n\times n}$ is called a B-matrix if, for all $i \in N$,*

$$\sum_{k=1}^{n} a_{ik} > 0, \qquad \frac{1}{n}\left(\sum_{k=1}^{n} a_{ik}\right) > a_{ij}, \qquad \forall j \neq i.$$

Definition 5. [1] *Let $A = (a_{ij}) \in R^{n\times n}$ and $A = B^+ + C$, where B^+ is defined as in* (2.1). *We say that A is a B-matrix if B^+ is an SDD matrix.*

Next, we will introduce some useful lemmas.

Lemma 1. [14] *Let $A = (a_{ij}) \in R^{n\times n}$ be a strong SDD_1 matrix. Then,*

$$\|A^{-1}\|_\infty \leq \max\left\{ \max_{i\in S: r_i^{\overline{S}}(A)=0} \frac{1}{|a_{ii}| - r_i(A)},\ \max_{j\in\overline{S}} \frac{1}{|a_{jj}| - r_j(A)},\ \max_{i\in S, j\in\overline{S}: a_{ij}\neq 0} \frac{|a_{jj}| + r_i^{\overline{S}}(A)}{(|a_{ii}| - r_i^S(A))|a_{jj}| - r_i^{\overline{S}}(A) r_j(A)} \right\}.$$

Lemma 2. [14] *If matrix $A = (a_{ij}) \in R^{n\times n}$ is a strong SDD_1 matrix, then A is a nonsingular H-matrix.*

Lemma 3. [15] *Let $A = (a_{ij}) \in R^{n\times n}$ be a nonsingular M-matrix, and let P be a nonnegative matrix with rank* 1. *Then $A + P$ is a P-matrix.*

Lemma 4. *Let $A = (a_{ij}) \in R^{n\times n}$ $(n \geq 2)$ be a strong SDD_1-B matrix. Then A is a P-matrix.*

Proof. By Definition 3, we have that C in (2.1) is a nonnegative matrix with rank 1. By Lemma 2, we get that B^+ is a nonnegative M-matrix. We can conclude that A is a P-matrix from Lemma 3. □

Remark 1. *From Definitions* 1–5*, Lemmas* 2 *and* 4*, we have the following relationships:*

$$SDD \text{ matrices} \subseteq \text{ strong } SDD_1 \text{ matrices} \subseteq H\text{-matrices},$$

$$B\text{-matrices} \subseteq \text{ strong } SDD_1\text{-}B \text{ matrices} \subseteq P\text{-matrices}.$$

Lemma 5. *Let $A = (a_{ij}) \in R^{n\times n}$ be a strong SDD_1 matrix. Then $\widetilde{A} = (\tilde{a}_{ij}) = I - D + DA$ is also a strong SDD_1 matrix, where $D = diag\,(d_i)$ with $0 \leq d_i \leq 1$, $\forall i \in N$.*

Proof. Since $\widetilde{A} = I - D + DA = (\tilde{a}_{ij})$, then,

$$\tilde{a}_{ij} = \begin{cases} 1 - d_i + d_i a_{ij}, & i = j, \\ d_i a_{ij}, & i \neq j. \end{cases}$$

Based on A is a strong SDD_1 matrix and $D = diag(d_i)$, $0 \le d_i \le 1(\forall i \in N)$, by Lemma 3, we can get the following results.

1) For $i \in S$, satisfying $r_i^{\overline{S}}(\widetilde{A}) = 0$, by Definition 3, it holds that if $d_i = 0$, then

$$|\tilde{a}_{ii}| = 1 > 0 = |d_i a_{ii}| = d_i r_i(A) = r_i(\widetilde{A}).$$

If $d_i \neq 0$, then

$$|\tilde{a}_{ii}| = |1 - d_i + d_i a_{ii}| > |d_i a_{ii}| > d_i r_i(A) = r_i(\widetilde{A}).$$

2) For $j \in \overline{S}$, it follows that if $d_j = 0$, then

$$|\tilde{a}_{jj}| = 1 > 0 = |d_j a_{jj}| = d_j r_j(A) = r_j(\widetilde{A}).$$

If $d_j \neq 0$, then

$$|\tilde{a}_{jj}| = |1 - d_j + d_j a_{jj}| > |d_j a_{jj}| > d_j r_j(A) = r_j(\widetilde{A}).$$

3) For $i \in S$, $j \in \overline{S}$, satisfying $\tilde{a}_{ij} \neq 0$, i.e., $d_i \neq 0$ and $a_{ij} \neq 0$, we can obtain that if $d_i \neq 0$, $d_j = 0$, then

$$\begin{aligned}(|\tilde{a}_{ii}| - r_i^S(\widetilde{A}))|\tilde{a}_{jj}| &= (|1 - d_i + d_i a_{ii}| - d_i r_i^S(A)) \\ &> d_i d_j(|a_{ii}| - r_i^S(A))|a_{jj}| \\ &= d_i d_j r_i^{\overline{S}}(A) r_j(A) = r_i^{\overline{S}}(\widetilde{A}) r_j(\widetilde{A}).\end{aligned}$$

If $d_i \neq 0$, $d_j \neq 0$, then

$$\begin{aligned}(|\tilde{a}_{ii}| - r_i^S(\widetilde{A}))|\tilde{a}_{jj}| &= (1 - d_j + d_j a_{jj}) - d_i(1 - d_j + d_j a_{jj}) + d_i[a_{ii} - r_i^S(A)] \\ &\quad - d_i d_j[a_{ii} - r_i^S(A)] + d_i d_j a_{ii} a_{jj} - d_i d_j r_i^S(A) \\ &\ge d_i d_j[|a_{ii}| - r_i^S(A)]|a_{jj}| \\ &> d_i d_j r_i^{\overline{S}}(A) r_j(A) = r_i^{\overline{S}}(\widetilde{A}) r_j(\widetilde{A}).\end{aligned}$$

Therefore, $\widetilde{A}$ is a strong SDD_1 matrix, the conclusion follows. □

Lemma 6. [16] *Let $\gamma > 0$ and $\eta \ge 0$. Then for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + x\gamma} \le \frac{1}{\min\{\gamma, 1\}}, \quad \frac{\eta x}{1 - x + x\gamma} \le \frac{\eta}{\gamma}.$$

Lemma 7. [17] *If $A = (a_{ij}) \in R^{n\times n}$ is an SDD matrix, then*

$$\max_{d\in[0,1]^n} \left\|(I - D + DA)^{-1}\right\|_\infty \le \max\left\{\frac{1}{\min_{i\in N}\{|a_{ii}| - r_i(A)\}}, 1\right\}.$$

Lemma 8. [1] *Let $A = (a_{ij}) \in R^{n\times n}$ be a B-matrix, and let B^+ be the matrix in* (3)*. Then*

$$\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \le \frac{n-1}{\min\{\beta, 1\}},$$

where $\beta = \min_{i\in N}\{\beta_i\}$, $\beta_i = |b_{ii}| - \sum_{j\neq i}^{n} |b_{ij}|$.

3. Error bound for linear complementarity problems involving strong SDD_1 matrices

In this section, new error bound of $LCP(A, q)$ is provided when A is a strong SDD_1 matrix.

Theorem 1. *Let $A = (a_{ij}) \in R^{n\times n}$, $n \geq 2$, be a strong SDD_1 matrix with poistive diagonal entries, and let $\widetilde{A} = (\tilde{a}_{ij}) = I - D + DA$, where $D = diag\,(d_i)$ with $0 \leq d_i \leq 1$. Then*

$$\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq \max\left\{ \max_{i\in S: r_i^{\overline{S}}(A)=0} \frac{1}{\min\{a_{ii} - r_i(A), 1\}}, \max_{j\in\overline{S}} \frac{1}{\min\left\{a_{jj} - r_j(A), 1\right\}}, \eta(A) \right\}, \tag{3.1}$$

where

$$\eta(A) = \max_{i\in S, j\in\overline{S}, a_{ij}\neq 0} \frac{a_{jj}\left(\frac{a_{ii}-r_i^S(A)}{\min\left\{a_{ii}-r_i^S(A),1\right\}} + \frac{r_i^{\overline{S}}(A)}{\min\left\{a_{jj},1\right\}}\right)}{\left(a_{ii} - r_i^S(A)\right) a_{jj} - r_i^{\overline{S}}(A) r_j(A)}.$$

Proof. Since $\widetilde{A} = (\tilde{a}_{ij}) = I - D + DA$, then from Lemma 5, we know that $\widetilde{A}$ is a strong SDD_1 matrix with positive diagonal entries. By Lemma 1, it holds that

$$\|\widetilde{A}^{-1}\|_\infty \leq \max\left\{ \max_{i\in S: r_i^{\overline{S}}(\widetilde{A})=0} \frac{1}{|\tilde{a}_{ii}| - r_i(\widetilde{A})}, \max_{j\in\overline{S}} \frac{1}{|\tilde{a}_{jj}| - r_j(\widetilde{A})}, \max_{i\in S, j\in\overline{S}: \tilde{a}_{ij}\neq 0} \frac{|\tilde{a}_{jj}| + r_i^{\overline{S}}(\widetilde{A})}{(|\tilde{a}_{ii}| - r_i^S(\widetilde{A}))|\tilde{a}_{jj}| - r_i^{\overline{S}}(\widetilde{A}) r_j(\widetilde{A})} \right\}. \tag{3.2}$$

Note that $r_i(\widetilde{A}) = d_i r_i(A)$, $r_j(\widetilde{A}) = d_j r_j(A)$ for all $i \in S$, $j \in \bar{S}$. Next, we divide into three cases to prove the result.

Case 1. For $i \in S$, satisfying $r_i^{\overline{S}}(\widetilde{A}) = 0$, it follows that $d_i = 0$ or $r_i^{\overline{S}}(A) = 0$. If $d_i = 0$ and $r_i^{\bar{S}}(A) = 0$, $i \in S$, then

$$\frac{1}{|\tilde{a}_{ii}| - r_i(\widetilde{A})} = \frac{1}{1 - d_i + d_i a_{ii} - d_i r_i(A)} = 1 < \frac{1}{\min\{a_{ii} - r_i(A), 1\}}.$$

If $d_i \neq 0$ and $r_i^{\overline{S}}(A) = 0$, $i \in S$, by Lemma 6, we have

$$\frac{1}{\tilde{a}_{ii} - r_i(\widetilde{A})} = \frac{1}{1 - d_i + d_i a_{ii} - d_i r_i(A)} \leq \frac{1}{\min\{a_{ii} - r_i(A), 1\}}.$$

If $d_i = 0$ and $r_i^{\overline{S}}(A) \neq 0$, $i \in S$, then there exists $j \in \overline{S}$ such that $a_{ij} \neq 0$. Thus, by Lemma 6, we get

$$\begin{aligned} \frac{1}{\tilde{a}_{ii} - r_i(\widetilde{A})} &= \frac{1}{1 - d_i + d_i a_{ii} - d_i r_i(A)} = 1 \\ &= \frac{1 - d_j + d_j a_{jj} + d_i r_i^{\overline{S}}(A)}{(1 - d_i + d_i a_{ii} - d_i r_i^S(A))(1 - d_j + d_j a_{jj}) - d_i r_i^{\overline{S}}(A) d_j r_j(A)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{1-d_j+d_ja_{jj}+d_ir_i^{\overline{S}}(A)}{\left(1-d_i+d_ia_{ii}-d_ir_i^S(A)\right)(1-d_j+d_ja_{jj})}}{1-\frac{d_ir_i^{\overline{S}}(A)}{\left(1-d_i+d_ia_{ii}-d_ir_i^S(A)\right)(1-d_j+d_ja_{jj})}}\\
&\le \frac{\frac{1}{\min\left\{a_{ii}-r_i^S(A),1\right\}}+\frac{r_i^{\overline{S}}(A)}{\left(a_{ii}-r_i^S(A)\right)\min\{a_{jj},1\}}}{1-\frac{r_i^{\overline{S}}(A)r_j(A)}{\left(a_{ii}-r_i^S(A)\right)a_{jj}}}\\
&= \frac{a_{jj}\left(\frac{a_{ii}-r_i^S(A)}{\min\left\{a_{ii}-r_i^S(A),1\right\}}+\frac{r_i^{\overline{S}}(A)}{\min\{a_{jj},1\}}\right)}{\left(a_{ii}-r_i^S(A)\right)a_{jj}-r_i^{\overline{S}}(A)r_j(A)}.
\end{aligned}
$$

So, it holds that

$$
\max_{i\in S:r_i^{\overline{S}}(\widetilde{A})=0}\frac{1}{\tilde{a}_{ii}-r_i(\widetilde{A})}\le\max\left\{\max_{i\in S:r_i^{\overline{S}}(A)=0}\frac{1}{\min\{a_{ii}-r_i(A),1\}},\ \max_{i\in S,j\in\overline{S},a_{ij}\neq0}\frac{a_{jj}\left(\frac{a_{ii}-r_i^S(A)}{\min\left\{a_{ii}-r_i^S(A),1\right\}}+\frac{r_i^{\overline{S}}(A)}{\min\{a_{jj},1\}}\right)}{\left(a_{ii}-r_i^S(A)\right)a_{jj}-r_i^{\overline{S}}(A)r_j(A)}\right\}.
$$

Case 2. For $j\in\overline{S}$, if $d_j=0$, then

$$
\frac{1}{\tilde{a}_{jj}-r_j(\widetilde{A})}=\frac{1}{1-d_j+d_ja_{jj}-d_jr_j(A)}=1<\frac{1}{\min\left\{a_{jj}-r_j(A),1\right\}}.
$$

If $d_j\neq0$, by Lemma 6, we get

$$
\frac{1}{\tilde{a}_{jj}-r_j(\widetilde{A})}=\frac{1}{1-d_j+d_ja_{jj}-d_jr_j(A)}\le\frac{1}{\min\left\{a_{jj}-r_j(A),1\right\}}.
$$

Case 3. For $i\in S$ and $j\in\overline{S}$, such that $\tilde{a}_{ij}\neq0$, it holds that $d_i\neq0$ and $a_{ij}\neq0$. Thus, by Lemma 6, it holds that

$$
\begin{aligned}
\frac{\tilde{a}_{jj}+r_i^{\overline{S}}(\widetilde{A})}{(\tilde{a}_{ii}-r_i^S(\widetilde{A}))\tilde{a}_{jj}-r_i^{\overline{S}}(\widetilde{A})r_j(\widetilde{A})}
&= \frac{1-d_j+d_ja_{jj}+d_ir_i^{\overline{S}}(A)}{(1-d_i+d_ia_{ii}-d_ir_i^S(A))(1-d_j+d_ja_{jj})-d_ir_i^{\overline{S}}(A)d_jr_j(A)}\\
&= \frac{\frac{1-d_j+d_ja_{jj}+d_ir_i^{\overline{S}}(A)}{\left(1-d_i+d_ia_{ii}-d_ir_i^S(A)\right)(1-d_j+d_ja_{jj})}}{1-\frac{d_ir_i^{\overline{S}}(A)d_jr_j(A)}{\left(1-d_i+d_ia_{ii}-d_ir_i^S(A)\right)(1-d_j+d_ja_{jj})}}\\
&\le \frac{\frac{1}{\min\left\{a_{ii}-r_i^S(A),1\right\}}+\frac{r_i^{\overline{S}}(A)}{\left(a_{ii}-r_i^S(A)\right)\min\{a_{jj},1\}}}{1-\frac{r_i^{\overline{S}}(A)r_j(A)}{\left(a_{ii}-r_i^S(A)\right)a_{jj}}}\\
&= \frac{a_{jj}\left(\frac{a_{ii}-r_i^S(A)}{\min\left\{a_{ii}-r_i^S(A),1\right\}}+\frac{r_i^{\overline{S}}(A)}{\min\{a_{jj},1\}}\right)}{\left(a_{ii}-r_i^S(A)\right)a_{jj}-r_i^{\overline{S}}(A)r_j(A)}.
\end{aligned}
$$

From Cases 1–3, the conclusion follows. □

Next, let's use the following two examples to illustrate the advantages of our results.

Example 1. Consider the matrix:

$$A = \begin{pmatrix} 4 & 0 & 3.5 \\ 5 & 7 & 1 \\ 0 & 0.1 & 6 \end{pmatrix}.$$

Then, A is not only an SDD matrix but also a strong SDD_1 matrix for $S = \{1, 2\}$. From Lemma 7, we have

$$\max_{d\in[0,1]^3} \|(I - D + DA)^{-1}\|_\infty \leq 2.$$

By Theorem 1, we get

$$\max_{d\in[0,1]^3} \|(I - D + DA)^{-1}\|_\infty \leq 1.28.$$

Example 2. Consider the tri-diagonal matrix $A \in R^{n\times n}$ arising from the finite difference method for free boundary problems [4], where

$$A = \begin{pmatrix} b + \alpha sin\left(\frac{1}{n}\right) & c & 0 & \cdots & 0 \\ a & b + \alpha sin\left(\frac{2}{n}\right) & c & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & a & b + \alpha sin\left(\frac{n-1}{n}\right) & c \\ 0 & \cdots & 0 & a & b + \alpha sin\,(1) \end{pmatrix}.$$

Take that $n = 500$, $a = -0.5$, $b = 3$, $c = -2.3$ and $\alpha = 0$. Then A is not only an SDD matrix but also a strong SDD_1 matrix for $S = \{2, \cdots, 499\}$. From Lemma 7, we get

$$\max_{d\in[0,1]^{500}} \|(I - D + DA)^{-1}\|_\infty \leq 5.$$

By Theorem 1, we have

$$\max_{d\in[0,1]^{500}} \|(I - D + DA)^{-1}\|_\infty \leq 2.2677.$$

Example 3. Consider the matrix:

$$A = \begin{pmatrix} 4 & 1 & 0 & 1 & 3 \\ 50 & 100 & 0 & 20 & 50 \\ 2 & 3 & 10 & 2 & 0 \\ 0 & 7 & 3 & 10 & 0 \\ 1 & 0 & 1 & 0 & 4 \end{pmatrix}.$$

It is easy to verify that A is a strong SDD_1 matrix for $S = \{1, 2, 4\}$, but not an SDD matrix and nor S-SDD matrix for any nonempty subset S of N. By Theorem 1, we have

$$\max_{d\in[0,1]^5} \|(I - D + DA)^{-1}\|_\infty \leq 16.$$

4. Error bound for linear complementarity problems involving strong SDD_1-B matrices

In this section, a new error bound of $LCP(A,q)$ is presented when A is a strong SDD_1-B matrix.

Theorem 2. *Let $A = (a_{ij}) \in R^{n\times n}$ be a strong SDD_1-B matrix with the form of $A = B^+ + C$, and let $B^+ = (b_{ij})$ be the matrix in* (2.1). *Denote $A_D = I - D + DA$, where $D = diag\,(d_i)$ with $0 \le d_i \le 1$. Then*

$$\begin{aligned} &\max_{d\in[0,1]^n} \left\|(I - D + DA)^{-1}\right\|_\infty \\ \le\ & \zeta(B^+) := (n-1)\max\left\{ \max_{i\in S: r_i^{\overline{S}}(B^+)=0} \frac{1}{\min\{b_{ii} - r_i(B^+), 1\}}, \right. \\ & \left. \max_{j\in\overline{S}} \frac{1}{\min\left\{b_{jj} - r_j(B^+), 1\right\}},\ \eta(B^+) \right\}, \end{aligned} \tag{4.1}$$

where

$$\eta(B^+) := \max_{i\in S, j\in\overline{S}: b_{ij}\ne 0} \frac{b_{jj}\left(\frac{b_{ii}-r_i^S(B^+)}{\min\left\{b_{ii}-r_i^S(B^+),1\right\}} + \frac{r_i^{\overline{S}}(B^+)}{\min\left\{b_{jj},1\right\}}\right)}{\left(b_{ii} - r_i^S(B^+)\right) b_{jj} - r_i^{\overline{S}}(B^+) r_j(B^+)}.$$

Proof. For $D = diag\,(d_i)$ with $0 \le d_i \le 1 (\forall i \in N)$, we have

$$A_D = I - D + DA = (I - D + DB^+) + DC = B_D^+ + C_D,$$

where $B_D^+ = (\tilde{b}_{ij}) = I - D + DB^+$ and $C_D = DC$. Notice that B^+ is a strong SDD_1 matrix with positive diagonal entries, then by Lemma 5, it follows that $B_D^+ = I - D + DB^+$ is also a strong SDD_1 matrix with positive diagonal entries. Similarly as the proof of Theorem 2.2 in [1], we can obtain:

$$\left\|A_D^{-1}\right\|_\infty \le \left\|\left[I + (B_D^+)^{-1} C_D\right]^{-1}\right\|_\infty \cdot \left\|(B_D^+)^{-1}\right\|_\infty \le (n-1)\left\|(B_D^+)^{-1}\right\|_\infty.$$

We now bound $\left\|(B_D^+)^{-1}\right\|_\infty$. By Lemma 1, it holds that

$$\left\|(B_D^+)^{-1}\right\|_\infty \le \max\left\{ \max_{i\in S: r_i^{\overline{S}}(B_D^+)=0} \frac{1}{|\tilde{b}_{ii}| - r_i(B_D^+)},\ \max_{j\in\overline{S}} \frac{1}{|\tilde{b}_{jj}| - r_j(B_D^+)},\ \mu_{ij}(B_D^+) \right\},$$

where

$$\mu_{ij}(B_D^+) := \max_{i\in S, j\in\overline{S}: \tilde{b}_{ij}\ne 0} \frac{|\tilde{b}_{jj}| + r_i^{\overline{S}}(B_D^+)}{\left(|\tilde{b}_{ii}| - r_i^S(B_D^+)\right)|\tilde{b}_{jj}| - r_i^{\overline{S}}(B_D^+) r_j(B_D^+)}.$$

Next, we divide into three cases to prove the conclusion.

Case 1. For $i \in S$, satisfying $r_i^{\overline{S}}(B_D^+) = 0$, then $r_i^{\overline{S}}(B^+) = 0$ or $d_i = 0$. If $d_i = 0$, $r_i^{\overline{S}}(B^+) \ne 0$, $i \in S$, then there exists $j \in \overline{S}$ such that $a_{ij} \ne 0$. Hence, by Lemmas 6 and 7, we get

$$\left\|(B_D^+)^{-1}\right\|_\infty \le \max_{i\in S: r_i^{\overline{S}}(B_D^+)=0} \frac{1}{|\tilde{b}_{ii}| - r_i(B_D^+)}$$

$$
\begin{aligned}
&= \frac{1}{1-d_i+d_ib_{ii}-d_ir_i(B^+)} = 1\\
&\le \max_{i\in S, j\in\overline{S}: b_{ij}\neq 0} \frac{\frac{1}{\min\left\{b_{ii}-r_i^S(B^+),1\right\}} + \frac{r_i^{\overline{S}}(B^+)}{\left(b_{ii}-r_i^S(B^+)\right)\min\left\{b_{jj},1\right\}}}{1-\frac{r_i^{\overline{S}}(B^+)r_j(B^+)}{\left(b_{ii}-r_i^S(B^+)\right)b_{jj}}}\\
&= \max_{i\in S, j\in\overline{S}: b_{ij}\neq 0} \frac{b_{jj}\left(\frac{b_{ii}-r_i^S(B^+)}{\min\left\{b_{ii}-r_i^S(B^+),1\right\}} + \frac{r_i^{\overline{S}}(B^+)}{\min\left\{b_{jj},1\right\}}\right)}{\left(b_{ii}-r_i^S(B^+)\right)b_{jj}-r_i^{\overline{S}}(B^+)r_j(B^+)} = \eta(B^+).
\end{aligned}
$$

If $d_i = 0$, $r_i^{\overline{S}}(B_D^+) = 0$, $i \in S$, we have

$$
\left\|(B_D^+)^{-1}\right\|_\infty \le \max_{i\in S: r_i^{\overline{S}}(B_D^+)=0} \frac{1}{|\tilde{b}_{ii}| - r_i(B_D^+)} \le \max_{i\in S: r_i^{\overline{S}}(B^+)=0} \frac{1}{\min\{b_{ii}-r_i(B^+),1\}}.
$$

Case 2. For $j \in \overline{S}$, it holds that

$$
\left\|(B_D^+)^{-1}\right\|_\infty \le \max_{j\in\overline{S}} \frac{1}{|\tilde{b}_{jj}| - r_j(B_D^+)} \le \max_{j\in\overline{S}} \frac{1}{\min\left\{b_{jj}-r_j(B^+),1\right\}}.
$$

Case 3. For $i \in S$ and $j \in \overline{S}$, such that $\tilde{b}_{ij} \neq 0$, then $d_i \neq 0$ and $b_{ij} \neq 0$. Thus, by Lemma 6, we have

$$
\begin{aligned}
\left\|(B_D^+)^{-1}\right\|_\infty &\le \max_{i\in S, j\in\overline{S}: \tilde{b}_{ij}\neq 0} \frac{\tilde{b}_{jj} + r_i^{\overline{S}}(B_D^+)}{\left(\tilde{b}_{ii}-r_i^S(B_D^+)\right)\tilde{b}_{jj} - r_i^S(B_D^+)r_j(B_D^+)}\\
&= \max_{i\in S, j\in\overline{S}: b_{ij}\neq 0} \frac{1-d_j+d_jb_{jj}+d_ir_i^{\overline{S}}(B^+)}{(1-d_i+d_ib_{ii}-d_ir_i^S(B^+))(1-d_j+d_jb_{jj}) - d_ir_i^{\overline{S}}(B^+)d_jr_j(B^+)}\\
&= \max_{i\in S, j\in\overline{S}: b_{ij}\neq 0} \frac{\frac{1-d_j+d_jb_{jj}+d_ir_i^{\overline{S}}(B^+)}{\left(1-d_i+d_ib_{ii}-d_ir_i^S(B^+)\right)(1-d_j+d_jb_{jj})}}{1-\frac{d_ir_i^{\overline{S}}(B^+)d_jr_j(B^+)}{\left(1-d_i+d_ib_{ii}-d_ir_i^S(B^+)\right)(1-d_j+d_jb_{jj})}}\\
&\le \max_{i\in S, j\in\overline{S}: b_{ij}\neq 0} \frac{\frac{1}{\min\left\{b_{ii}-r_i^S(B^+),1\right\}} + \frac{r_i^{\overline{S}}(B^+)}{\left(b_{ii}-r_i^S(B^+)\right)\min\left\{b_{jj},1\right\}}}{1-\frac{r_i^{\overline{S}}(B^+)r_j(B^+)}{\left(b_{ii}-r_i^S(B^+)\right)b_{jj}}}\\
&= \max_{i\in S, j\in\overline{S}: b_{ij}\neq 0} \frac{b_{jj}\left(\frac{b_{ii}-r_i^S(B^+)}{\min\left\{b_{ii}-r_i^S(B^+),1\right\}} + \frac{r_i^{\overline{S}}(B^+)}{\min\left\{b_{jj},1\right\}}\right)}{\left(b_{ii}-r_i^S(B^+)\right)b_{jj}-r_i^{\overline{S}}(B^+)r_j(B^+)} = \eta(B^+).
\end{aligned}
$$

Consequently, from Cases 1–3, the conclusion follows. □

The bound in Theorem 2 also holds for B-matrix, because B-matrix is a subclass of strong SDD_1-B-matrix. Next, we will indicate that the bound in Theorem 2 is better than that in Lemma 8 in some cases.

Theorem 3. *Let* $A = (a_{ij}) \in R^{n\times n}$ *be a B-matrix with* $A = B^+ + C$*, and let* $B^+ = (b_{ij})$ *be the matrix in* (2.1)*. If* $0 < b_{ii} \le 1 (\forall i \in N)$*, then*

$$\zeta(B^+) \le \frac{1}{\min\limits_{i\in N}\{\beta, 1\}}, \tag{4.2}$$

where $\zeta(B^+)$ and β are defined as in Theorem 2 and Lemma 8, respectively.

Proof. By $0 < b_{ii} \le 1$ ($\forall i \in N$), we have

$$\max_{i\in S: r_i^{\overline{S}}(B^+)=0} \frac{1}{\min\{b_{ii} - r_i(B^+), 1\}} = \max_{i\in S: r_i^{\overline{S}}(B^+)=0} \frac{1}{b_{ii} - r_i(B^+)} \le \frac{1}{\min\limits_{i\in N}\{\beta, 1\}}$$

and

$$\max_{j\in\overline{S}} \frac{1}{\min\left\{b_{jj} - r_j(B^+), 1\right\}} = \max_{j\in\overline{S}} \frac{1}{b_{jj} - r_j(B^+)} \le \frac{1}{\min\limits_{i\in N}\{\beta, 1\}}.$$

For $i \in S$ and $j \in \overline{S}$, such that $b_{ij} \neq 0$, it follows that if $b_{ii} - r_i(B^+) \le b_{jj} - r_j(B^+)$, then

$$\eta(B^+) \le \frac{1}{b_{ii} - r_i(B^+)} \le \frac{1}{\min\limits_{i\in N}\{\beta, 1\}}.$$

If $b_{jj} - r_j(B^+) \le b_{ii} - r_i(B^+)$, then

$$\eta(B^+) \le \frac{1}{b_{jj} - r_j(B^+)} \le \frac{1}{\min\limits_{i\in N}\{\beta, 1\}}.$$

Therefore, the conclusion follows. □

The following numerical examples show the validity of the error bounds for strong SDD_1-B matrix.

Example 4. Consider the matrix:

$$A = \begin{pmatrix} 0.7 & -0.2 & -0.2 & -0.2 \\ 0 & 0.5 & 0.1 & 0.1 \\ 0 & 0.1 & 0.4 & 0.1 \\ 0 & 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

We can write it as $A = B^+ + C$, where

$$B^+ = \begin{pmatrix} 0.7 & -0.2 & -0.2 & -0.2 \\ -0.1 & 0.4 & 0 & 0 \\ -0.1 & 0 & 0.3 & 0 \\ -0.2 & 0 & 0 & 0.4 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix}.$$

It is easy to check that A is a B-matrix, consequently, a strong SDD_1-B matrix. By Lemma 8, we have

$$\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty \leq 30.$$

When $S = \{1\}$, by Theorem 2, we have

$$\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty \leq 25.$$

It is shown by Figure 1, in which the first 1000 matrices are given by the following MATLAB codes, that 25 is better than 30 for $\max \|(I-D+DA)^{-1}\|_\infty$. Blue stars in Figure 1 represent the $\|(I-D+DA)^{-1}\|_\infty$ when matrices D come from 1000 different random matrices in $[0, 1]$.

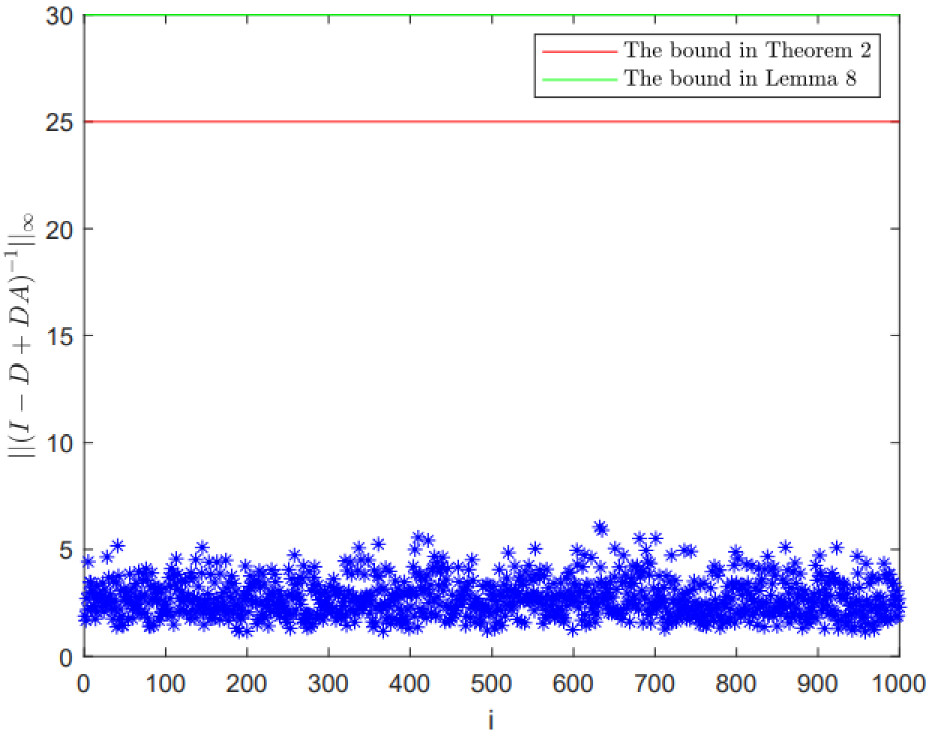

Figure 1. $\|(I - D + DA)^{-1}\|_\infty$ for the first 1000 matrices D generated by diag(rand (5,1)). MATLAB codes: for i = 1:1000; D=diag(rand (5,1)); end.

Example 5. Consider the matrix:

$$A = \begin{pmatrix} 6 & -4 & -1 & 0 \\ -2 & 4 & -0.5 & -2 \\ -2 & 0 & 3 & 0 \\ -2 & 0 & 0 & 6 \end{pmatrix}.$$

Matrix A can be split into $A = B^+ + C$, where

$$B^+ = \begin{pmatrix} 6 & -4 & -1 & 0 \\ -2 & 4 & -0.5 & -2 \\ -2 & 0 & 3 & 0 \\ -2 & 0 & 0 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that A is neither an SDD matrix nor a B-matrix. On the other hand, A is a strong SDD_1-B matrix for $S = \{1, 2, 3\}$. By Theorem 2, we get

$$\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty \leq 4.2.$$

5. Conclusions

Based on the properties strong SDD_1 matrices, we introduce a new subclass of P-matrices called strong SDD_1-B matrices. Moreover, we apply upper bound for the infinity norm of the inverse of SDD_1 matrices to estimate error bounds for linear complementarity problems of SDD_1 matrices and SDD_1-B matrices, which is useful for free boundary problems. Numerical examples are given to show the sharpness of the proposed bounds. In the future, based on the proposed infinity norm bound, we will explore the computable global error bounds of extended vertical linear complementarity problems for SDD_1 matrices and SDD_1-B matrices.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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