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Research article

New solutions of time-space fractional coupled Schrödinger systems

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Abstract: The current manuscript focuses on the solution and analysis of space and time fractional coupled Schrödinger system that belongs to a class of evolution equations. These systems encounter in different fields related to plasma waves, optics, and quantum physics. The fractional He-Laplace approach is proposed for the series form solutions of fractional systems. This approach contains hybrid of Laplace transform and homotopy perturbation along with Caputo fractional derivative. The current study provide new results on time and space fractional coupled Schrödinger systems which are not captured in existing literature. Reliability of proposed algorithm in both time and space fractional scenarios is observed through residual error concept throughout fractional domains. The effect of fractional parameters on wave profiles are analyzed numerically and graphically as 2D and 3D illustrations. Analysis reveals that proposed algorithm is suitable for non-linear time-space fractional systems encountering in different fields of sciences.

Keywords: time-space fractional; Schrödinger system; Laplace transform; homotopy perturbation
Mathematics Subject Classification: 35G50, 35C10

1. Introduction

Partial differential equations (PDEs) arise in various fields like fluid dynamics [1, 2], kinetics [3], biology [4, 5], quantum field theory [6], engineering [7], and physics [8, 9]. Many non-linear systems [10] also emerge in several biological and chemical applications. Since the order of differential equations describes their nature and extension, therefore, in recent researches a great deal of focus is on fractional order differential equations (FDEs). Fractional derivatives generalized the integer order derivatives to non-integer form. This enables the memory effect of models to be studied more accurately. The first concept of fractional calculus arises in 18th century. Many researchers that are Liouville [11], Miller [12], Riemann [13], and Caputo [14] assisted in this field. Various linear and non-linear phenomena have been studied through fractional calculus [15, 16]. Agriculture [17], education [18], health [19], ocean waves [20], construction [21], and robotics [22] are examples of a few fields that assimilate fractional order modeling. Moreover, various branches of fluid mechanics [23], mathematics [24], thermodynamics [25], computer science [26], and climatology [27] also utilize it.

Primary focus of scientific community is to find the exact solutions of these non-linear equations. Various powerful techniques that are new function method [28], sine-Gordon method [29], Hirota bilinear method [30], sub-equation method [31], and Riccati equation method [32] are applied in literature to find such results. But the complex nature of non-linear equations makes it challenging to calculate exact solution. So, it encourages people to find approximate solutions by using many numerical and semi-numerical algorithms. Homotopy analysis method (HAM) [33, 34], finite difference method (FDM) [35], Meshless method [36], homotopy perturbation method (HPM) [37], Haar wavelet method [38], variational iteration method [39], and Runge-Kutta (RK) methods [40] are some of these schemes. HPM is a significant tool to solve variety of linear and non-linear problems of higher order. Many alteration of it [41, 42] are introduced to make it more effective especially when dealing with fractional derivatives. One modification is the utilization of Laplace transform. This hybrid of HPM and Laplace transform [43] makes an excellent tool in tackling higher order fractional models.

Non-linear Schrödinger equations (NLS) [44] are partial differential equations that describes the phenomena of slow wave packets evolution in wave systems. They illustrate the energy and position of electrons in time and space. NLS equations belong to a principal class of evolution equations and are extended to optics [45], quantum mechanics [46], plasma theory [47], and various areas of science. These equations have been studied in fractional form by many researches. Li et al. [48] analyzed non-linear time-fractional Schrödinger equations through $L1$-Galerkin finite element method. Linearized fast time-stepping schemes were adopted by Yuan et al. [49] for time and space fractional Schrödinger equations. Hosseini et al. [50] examined the dynamics of generalized Schrödinger equation. Kudryashov [51] found the optical solitons of these equations via an extended approach. Yuan et al. [52] applied linearized transformed $L1$-Galerkin finite differential method to them. In this study, we will analyze both time and space fractional non-linear couple Schrödinger system [53, 54] through He-Laplace method. A (1+1)-dimensional, time-space fractional, coupled Schrödinger model is given as:

$$\iota\frac{\partial^{\alpha}\mathcal{M}}{\partial t^{\alpha}}+\iota\rho\frac{\partial^{\beta}\mathcal{M}}{\partial x^{\beta}}+\tau\frac{\partial^{2\beta}\mathcal{M}}{\partial x^{2\beta}}+\sigma(|\mathcal{M}|^2+\gamma|\mathcal{N}|^2)\mathcal{M}=0,$$
$$\iota\frac{\partial^{\alpha}\mathcal{N}}{\partial t^{\alpha}}-\iota\rho\frac{\partial^{\beta}\mathcal{N}}{\partial x^{\beta}}+\tau\frac{\partial^{2\beta}\mathcal{N}}{\partial x^{2\beta}}+\sigma(|\mathcal{N}|^2+\gamma|\mathcal{M}|^2)\mathcal{N}=0, \tag{1.1}$$

where, $\mathcal{M}$ and $\mathcal{N}$ are complex valued functions that represent the amplitudes of circularly-polarized waves. α and β are orders of fractional derivative with respect to time t and space x respectively. ρ, τ, σ, and γ are real constants. Application of Eq (1.1) extends to many fields of optics, plasma waves, hydrodynamics, non-linear acoustics, and acoustics.

Rest of the manuscript is divided as follows: section II contains preliminaries concerning time and space Caputo fractional derivative and their Laplace transform. Methodology of extended He-Laplace method for a time-space fractional system is given in Section III. Section IV is base on theorems regarding convergence and error analysis, whereas, application of He-Laplace method on Eq (1.1) is in section V. Discussion of obtained results and conclusion are in section VI and VII respectively.

2. Preliminaries

Definition 1. *[55] The Caputo time-fractional derivative* ${}^{C}\mathbb{D}_t^{\alpha}$ *of a function* $\mathcal{Z}(x,t)$ *is defined by:*

$${}^{C}\mathbb{D}_t^{\alpha}\{\mathcal{Z}(x,t)\}=\frac{1}{\Gamma(\vartheta-\alpha)}\int_0^t(t-\xi)^{\vartheta-\alpha-1}\mathcal{Z}^{(\vartheta)}(x,\xi)d\xi, \qquad \vartheta-1<\alpha\leq\vartheta, \tag{2.1}$$

where, α represents the time-fractional parameter.

Definition 2. *[56] The Caputo space-fractional derivative* ${}^{C}\mathbb{D}_x^{\beta}$ *of a function* $\mathcal{Z}(x,t)$ *is defined as:*

$${}^{C}\mathbb{D}_x^{\beta}\{\mathcal{Z}(x,t)\}=\frac{1}{\Gamma(\vartheta-\beta)}\int_0^x(x-\xi)^{\vartheta-\beta-1}\mathcal{Z}^{(\vartheta)}(\xi,t)d\xi, \qquad \zeta-1<\beta\leq\zeta, \tag{2.2}$$

where, β represents the space-fractional parameter.

Definition 3. *[57] The Laplace transform $\mathbb{L}$ of Caputo time and space-fractional derivative is respectively given as:*

$$\mathbb{L}\{{}^{C}\mathbb{D}_t^{\alpha}\mathcal{Z}(x,t)\}=s^{\alpha}\mathbb{L}\{\mathcal{Z}\}-\sum_{q=0}^{\vartheta-1}s^{\alpha-q-1}\mathcal{Z}^{(q)}(x,0), \qquad \vartheta-1<\alpha\leq\vartheta. \tag{2.3}$$

$$\mathbb{L}\{{}^{C}\mathbb{D}_x^{\beta}\mathcal{Z}(x,t)\}=s^{\beta}\mathbb{L}\{\mathcal{Z}\}-\sum_{q=0}^{\zeta-1}s^{\beta-q-1}\mathcal{Z}^{(q)}(0,t), \qquad \zeta-1<\beta\leq\zeta. \tag{2.4}$$

3. Methodology of extended He-Laplace method for time and space fractional systems

Let us consider a general non-linear, time-space-fractional, differential system as:

$$\mathbb{D}_t^{\alpha}\mathcal{Z}i(x,t)+\mathbb{D}_x^{\beta}\mathcal{Z}i(x,t)+\mathcal{L}[\mathcal{Z}]+\mathcal{N}[\mathcal{Z}]=0, \qquad i=1,\ldots,n,\ \ t>0,\ \ \vartheta-1<\alpha\leq\vartheta,\ \ \zeta-1<\beta\leq\zeta, \tag{3.1}$$

with initial conditions given as

$$\begin{aligned} \mathcal{Z}i(x,0) &= \mathcal{A}i, \\ \mathcal{Z}i(0,t) &= \mathcal{B}i, \qquad\qquad i = 1,\ldots,n, \end{aligned} \tag{3.2}$$

where, $\mathbb{D}_t^\alpha$ and $\mathbb{D}_x^\beta$ are the temporal 't' and spacial 'x' fractional derivatives of $\mathcal{Z}i$ respectively. $\mathcal{L}$ is the linear operator whereas $\mathcal{N}$ represents the non-linear operator.

Case 1: Procedure will be initiated by taking Laplace transform w.r.t. time and considering the spacial derivative in integer order

$$\mathbb{L}_t\{\mathbb{D}_t^\alpha \mathcal{Z}i(x,t)\} + \mathbb{L}_t\{\mathbb{D}_x \mathcal{Z}i(x,t)\} + \mathbb{L}_t\{\mathcal{L}[\mathcal{Z}] + \mathcal{N}[\mathcal{Z}]\} = 0. \tag{3.3}$$

Definition 3 leads to

$$\mathbb{L}_t\{\mathcal{Z}i(x,t)\} - \left(\frac{1}{s^\alpha}\right)\sum_{q=0}^{\vartheta-1} s^{\alpha-q-1}\mathcal{Z}i^{(q)}(x,0) + \left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\{\mathbb{D}_x\mathcal{Z}i(x,t) + \mathcal{L}[\mathcal{Z}] + \mathcal{N}[\mathcal{Z}]\} = 0. \tag{3.4}$$

The general homotopy of the system is

$$\begin{aligned} \text{Homotopy} = (1-\kappa)(\mathbb{L}_t\{\mathcal{Z}i(x,t)\} - \mathcal{Z}i_0(x,t)) + \kappa\Big(\mathbb{L}_t\{\mathcal{Z}i(x,t)\} - \left(\frac{1}{s^\alpha}\right)\sum_{q=0}^{\vartheta-1} s^{\alpha-q-1}\mathcal{Z}i^{(q)}(x,0) \\ + \left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\{\mathbb{D}_x\mathcal{Z}i(x,t) + \mathcal{L}[\mathcal{Z}] + \mathcal{N}[\mathcal{Z}]\}\Big) = 0, \end{aligned} \tag{3.5}$$

with $\mathcal{Z}i_0$ as initial guess and $\kappa \in [0,1]$. Expansion of $\mathcal{Z}i(x,t)$ in power series w.r.t. κ gives

$$\mathcal{Z}i(x,t) = \mathcal{Z}i_0(x,t) + \kappa^1\mathcal{Z}i_1(x,t) + \kappa^2\mathcal{Z}i_2(x,t) + \cdots \tag{3.6}$$

Substituting Eq (3.6) into Eq (3.5) and after that comparing identical coefficients of κ gives At κ^1:

$$\mathbb{L}_t\{\mathcal{Z}i_1(x,t)\} + \mathcal{Z}i_0 - \left(\frac{1}{s^\alpha}\right)\sum_{q=0}^{\vartheta-1} s^{\alpha-q-1}\mathcal{Z}i^{(q)}(x,0) + \left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\{\mathbb{D}_x\mathcal{Z}i_0(x,t) + \mathcal{L}[\mathcal{Z}_0] + \mathcal{N}[\mathcal{Z}_0]\} = 0. \tag{3.7}$$

At κ^p:

$$\mathbb{L}_t\{\mathcal{Z}i_p(x,t)\} + \left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\{\mathbb{D}_x\mathcal{Z}i_p(x,t) + \mathcal{L}[\mathcal{Z}_p] + \mathcal{N}[\mathcal{Z}_p]\} = 0, \qquad p = 2,\ldots,k. \tag{3.8}$$

By taking inverse Laplace transform we have solutions At κ^1:

$$\mathcal{Z}i_1(x,t) + \mathbb{L}_t^{-1}\Big\{\mathcal{Z}i_0 - \left(\frac{1}{s^\alpha}\right)\sum_{q=0}^{\vartheta-1} s^{\alpha-q-1}\mathcal{Z}i^{(q)}(x,0) + \left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\{\mathbb{D}_x\mathcal{Z}i_0(x,t) + \mathcal{L}[\mathcal{Z}_0] + \mathcal{N}[\mathcal{Z}_0]\}\Big\} = 0. \tag{3.9}$$

At κ^p:

$$\mathcal{Z}i_p(x,t) + \mathbb{L}_t^{-1}\Big\{\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\{\mathbb{D}_x\mathcal{Z}i_p(x,t) + \mathcal{L}[\mathcal{Z}_p] + \mathcal{N}[\mathcal{Z}_p]\}\Big\} = 0, \qquad p = 2,\ldots,k. \tag{3.10}$$

The approximate series solution of Eq (3.3) is

$$\tilde{\mathcal{Z}i} = \mathcal{Z}i = \mathcal{Z}i_0(x,t) + \mathcal{Z}i_1(x,t) + \mathcal{Z}i_2(x,t) + \mathcal{Z}i_3(x,t) + \cdots \tag{3.11}$$

We can obtain residual function by substituting Eq (3.11) in Eq (3.1) while considering $\beta = 1$.

$$\mathcal{R}_{\mathcal{Z}i} = \mathbb{D}_t^{\alpha}\tilde{\mathcal{Z}i} + \mathbb{D}_x\tilde{\mathcal{Z}i} + \mathcal{L}[\tilde{\mathcal{Z}i}] + \mathcal{N}[\tilde{\mathcal{Z}i}]. \tag{3.12}$$

Case 2: Take Laplace transform w.r.t. space and consider the temporal derivative in integer order

$$\mathbb{L}_x\{\mathbb{D}_x^{\beta}\mathcal{Z}i(x,t)\} + \mathbb{L}_x\{\mathbb{D}_t\mathcal{Z}i(x,t)\} + \mathbb{L}_x\{\mathcal{L}[\mathcal{Z}] + \mathcal{N}[\mathcal{Z}]\} = 0. \tag{3.13}$$

Definition 3 gives

$$\mathbb{L}_x\{\mathcal{Z}i(x,t)\} - \left(\frac{1}{s^{\beta}}\right)\sum_{q=0}^{\zeta-1} s^{\beta-q-1}\mathcal{Z}i^{(q)}(0,t) + \left(\frac{1}{s^{\beta}}\right)\mathbb{L}_x\{\mathbb{D}_t\mathcal{Z}i(x,t) + \mathcal{L}[\mathcal{Z}] + \mathcal{N}[\mathcal{Z}]\} = 0. \tag{3.14}$$

System's homotopy is constructed as

$$\begin{aligned}\text{Homotopy} = (1-\kappa)(\mathbb{L}_x\{\mathcal{Z}i(x,t)\} - \mathcal{Z}i_0(x,t)) + \kappa\Big(\mathbb{L}_x\{\mathcal{Z}i(x,t)\} - \left(\frac{1}{s^{\beta}}\right)\sum_{q=0}^{\zeta-1} s^{\beta-q-1}\mathcal{Z}i^{(q)}(0,t)\\ + \left(\frac{1}{s^{\beta}}\right)\mathbb{L}_x\{\mathbb{D}_t\mathcal{Z}i(x,t) + \mathcal{L}[\mathcal{Z}] + \mathcal{N}[\mathcal{Z}]\}\Big) = 0.\end{aligned} \tag{3.15}$$

Similar procedure as of Case 1 gives
At κ^1:

$$\mathbb{L}_x\{\mathcal{Z}i_1(x,t)\} + \mathcal{Z}i_0 - \left(\frac{1}{s^{\beta}}\right)\sum_{q=0}^{\zeta-1} s^{\beta-q-1}\mathcal{Z}i^{(q)}(0,t) + \left(\frac{1}{s^{\beta}}\right)\mathbb{L}_x\{\mathbb{D}_t\mathcal{Z}i_0(x,t) + \mathcal{L}[\mathcal{Z}_0] + \mathcal{N}[\mathcal{Z}_0]\} = 0. \tag{3.16}$$

At κ^p:

$$\mathbb{L}_x\{\mathcal{Z}i_p(x,t)\} + \left(\frac{1}{s^{\beta}}\right)\mathbb{L}_x\{\mathbb{D}_t\mathcal{Z}i_p(x,t) + \mathcal{L}[\mathcal{Z}_p] + \mathcal{N}[\mathcal{Z}_p]\} = 0, \qquad p = 2,\ldots,k. \tag{3.17}$$

By taking inverse Laplace transform we have solutions
At κ^1:

$$\mathcal{Z}i_1(x,t) + \mathbb{L}_x^{-1}\Big\{\mathcal{Z}i_0 - \left(\frac{1}{s^{\beta}}\right)\sum_{q=0}^{\zeta-1} s^{\beta-q-1}\mathcal{Z}i^{(q)}(0,t) + \left(\frac{1}{s^{\beta}}\right)\mathbb{L}_x\{\mathbb{D}_t\mathcal{Z}i_0(x,t) + \mathcal{L}[\mathcal{Z}_0] + \mathcal{N}[\mathcal{Z}_0]\}\Big\} = 0, \tag{3.18}$$

At κ^p:

$$\mathcal{Z}i_p(x,t) + \mathbb{L}_x^{-1}\Big\{\left(\frac{1}{s^{\beta}}\right)\mathbb{L}_x\{\mathbb{D}_t\mathcal{Z}i_p(x,t) + \mathcal{L}[\mathcal{Z}_p] + \mathcal{N}[\mathcal{Z}_p]\}\Big\} = 0, \qquad p = 2,\ldots,k. \tag{3.19}$$

Adding them produce approximate series solution. Substitution of the obtained approximate solution in Eq (3.1) at $\alpha = 1$ generates residual function.

$$\mathcal{R}_{\mathcal{Z}i} = \mathbb{D}_x^{\beta}\tilde{\mathcal{Z}i} + \mathbb{D}_t\tilde{\mathcal{Z}i} + \mathcal{L}[\tilde{\mathcal{Z}i}] + \mathcal{N}[\tilde{\mathcal{Z}i}]. \tag{3.20}$$

System errors can be observed by adding the absolute residual errors and then dividing them by i where, $i \in [1, n]$.

$$\sum_{i=1}^{n} \frac{|\mathcal{R}|_{\mathcal{Z}i}}{i}. \tag{3.21}$$

4. Convergence and error analysis of extended He-Laplace algorithm for fractional systems

4.1. Convergence

Theorem 1. *Suppose a Banach space (B) has $\mathcal{Z}i_n(x,t)$ and $\mathcal{Z}i(x,t)$ defined in it for $i = 2, \ldots, n$. Then, the approximate solution (3.11) of a system for $Q \in (0,1)$ converges to the exact solution of (3.1).*

Proof. Let $\{Ai_n\}$ is the sequence of partial sums of Eq. (3.11). To prove that Ai_n is a Cauchy sequence in space (B), consider

$$\begin{aligned} \| Ai_{n+1} - Ai_n \| = \| \mathcal{Z}i_{n+1} \| & \\ \leq & Q \| \mathcal{Z}i_n \| \\ \leq & Q^2 \| \mathcal{Z}i_{n-1} \| \\ & \vdots \\ \leq & Q^{n+1} \| \mathcal{Z}i_0 \|, \end{aligned} \tag{4.1}$$

for partial sums Ai_n and Ai_m where $n,m \in \mathbf{N}$ and $n \geq m$, triangle inequality property gives

$$\begin{aligned} \| Ai_n - Ai_m \| =& \| (Ai_n - Ai_{n-1}) + (Ai_{n-1} - Ai_{n-2}) \\ & + \cdots + (Ai_{m+1} - Ai_m) \| \\ \leq & \| Ai_n - Ai_{n-1} \| + \| Ai_{n-1} - Ai_{n-2} \| \\ & + \cdots + \| Ai_{m+1} - Ai_m \|, \end{aligned} \tag{4.2}$$

from Eq (4.1) we have

$$\begin{aligned} \| Ai_n - Ai_m \| \leq & \, Q^n \| \mathcal{Z}i_0 \| + Q^{n-1} \| \mathcal{Z}i_0 \| + \cdots + Q^{m+1} \| \mathcal{Z}i_0 \| \\ \leq & \, (Q^n + Q^{n-1} + \cdots + Q^{m+1}) \| \mathcal{Z}i_0 \| \\ \leq & \, Q^{m+1}(Q^{n-m-1} + Q^{n-m-2} + \cdots + Q + 1) \| \mathcal{Z}i_0 \| \\ \leq & \, Q^{m+1} \left(\frac{1 - Q^{n-m}}{1 - Q} \right) \| \mathcal{Z}i_0 \|, \end{aligned} \tag{4.3}$$

since $0 < Q < 1$, therefore, $1 - Q^{n-m} < 1$. Thus

$$\| Ai_n - Ai_m \| \leq \frac{Q^{m+1}}{1 - Q} \max |\mathcal{Z}i_0|, \tag{4.4}$$

boundedness of $\mathcal{Z}i_0$ implies

$$\lim_{n,m \to \infty} \| Ai_n - Ai_m \| = 0. \tag{4.5}$$

Hence, Ai_n is a Cauchy sequence in Banach space (B). Thus the given statement of convergence holds. □

4.2. Error estimation

Theorem 2. *The maximum absolute truncation error of solution of fractional system (3.1) is*

$$\left|\mathcal{Z}i - \sum_{j=0}^{m} \mathcal{Z}i_j\right| \leq \frac{Q^{m+1}}{1-Q} \parallel \mathcal{Z}i_0 \parallel . \tag{4.6}$$

Proof. From Eq (4.3) we get

$$\parallel \mathcal{Z}i - Ai_m \parallel \leq Q^{m+1}\left(\frac{1-Q^{n-m}}{1-Q}\right) \parallel \mathcal{Z}i_0 \parallel, \tag{4.7}$$

$0 < Q < 1$ implies $1 - Q^{n-m} < 1$. Thus, we have

$$\left|\mathcal{Z}i - \sum_{j=0}^{m} \mathcal{Z}i_j\right| \leq \frac{Q^{m+1}}{1-Q} \parallel \mathcal{Z}i_0 \parallel . \tag{4.8}$$

□

5. Application and solution of time-space-fractional coupled Schrödinger model

Consider the time-space-fractional Schrödinger system given in Eq (1.1). Suppose the complex valued functions $\mathcal{M}$ and $\mathcal{N}$ can be written as $\mathcal{M} = \mathcal{Z}1 + \iota\mathcal{Z}2$ and $\mathcal{N} = \mathcal{Z}3 + \iota\mathcal{Z}4$, then, an equivalent system of Eq (1.1) is

$$\begin{aligned}
&\frac{\partial^{\alpha}\mathcal{Z}1}{\partial t^{\alpha}} + \rho\frac{\partial^{\beta}\mathcal{Z}1}{\partial x^{\beta}} + \tau\frac{\partial^{2\beta}\mathcal{Z}2}{\partial x^{2\beta}} + \sigma((\mathcal{Z}1^2 + \mathcal{Z}2^2) + \gamma(\mathcal{Z}3^2 + \mathcal{Z}4^2))\mathcal{Z}2 = 0,\\
&\frac{\partial^{\alpha}\mathcal{Z}2}{\partial t^{\alpha}} + \rho\frac{\partial^{\beta}\mathcal{Z}2}{\partial x^{\beta}} - \tau\frac{\partial^{2\beta}\mathcal{Z}1}{\partial x^{2\beta}} - \sigma((\mathcal{Z}1^2 + \mathcal{Z}2^2) + \gamma(\mathcal{Z}3^2 + \mathcal{Z}4^2))\mathcal{Z}1 = 0,\\
&\frac{\partial^{\alpha}\mathcal{Z}3}{\partial t^{\alpha}} - \rho\frac{\partial^{\beta}\mathcal{Z}3}{\partial x^{\beta}} + \tau\frac{\partial^{2\beta}\mathcal{Z}4}{\partial x^{2\beta}} + \sigma((\mathcal{Z}3^2 + \mathcal{Z}4^2) + \gamma(\mathcal{Z}1^2 + \mathcal{Z}2^2))\mathcal{Z}4 = 0,\\
&\frac{\partial^{\alpha}\mathcal{Z}4}{\partial t^{\alpha}} - \rho\frac{\partial^{\beta}\mathcal{Z}4}{\partial x^{\beta}} - \tau\frac{\partial^{2\beta}\mathcal{Z}3}{\partial x^{2\beta}} - \sigma((\mathcal{Z}3^2 + \mathcal{Z}4^2) + \gamma(\mathcal{Z}1^2 + \mathcal{Z}2^2))\mathcal{Z}3 = 0,\\
&0 < \alpha \leq 1,\\
&0 < \beta \leq 1,\ t > 0,
\end{aligned} \tag{5.1}$$

with initial conditions

$$\begin{aligned}
&\mathcal{Z}1(x,0) = \sin x, && \mathcal{Z}2(x,0) = \cos x, && \mathcal{Z}3(x,0) = \sin x,\\
&\mathcal{Z}4(x,0) = \cos x, && \mathcal{Z}1(0,t) = \sin t, && \mathcal{Z}2(0,t) = \cos t,\\
&\mathcal{Z}3(0,t) = \sin t, && \mathcal{Z}4(0,t) = \cos t, && \mathcal{Z}1_x(0,t) = \cos t,\\
&\mathcal{Z}2_x(0,t) = -\sin t, && \mathcal{Z}3_x(0,t) = \cos t, && \mathcal{Z}4_x(0,t) = -\sin t.
\end{aligned} \tag{5.2}$$

Case 1: Consider $\beta = 1$ in Eq (5.1). Laplace transform w.r.t. time ($\mathbb{L}_t$) and utilizing Definition 3 gives homotopies of system as

$$
\begin{aligned}
\mathrm{H1}: (1-\kappa)(\mathbb{L}_t\{\mathcal{Z}1\}-\mathcal{Z}1_0)+\kappa\Big(\mathbb{L}_t\{\mathcal{Z}1\}-\left(\frac{1}{s}\right)\sin x+\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\Big\{\rho\frac{\partial \mathcal{Z}1}{\partial x}\\
+\tau\frac{\partial^2\mathcal{Z}2}{\partial x^2}+\sigma((\mathcal{Z}1^2+\mathcal{Z}2^2)+\gamma(\mathcal{Z}3^2+\mathcal{Z}4^2))\mathcal{Z}2\Big\}\Big)=0,\\
\mathrm{H2}: (1-\kappa)(\mathbb{L}_t\{\mathcal{Z}2\}-\mathcal{Z}2_0)+\kappa\Big(\mathbb{L}_t\{\mathcal{Z}2\}-\left(\frac{1}{s}\right)\cos x+\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\Big\{\rho\frac{\partial \mathcal{Z}2}{\partial x}\\
-\tau\frac{\partial^2\mathcal{Z}1}{\partial x^2}-\sigma((\mathcal{Z}1^2+\mathcal{Z}2^2)+\gamma(\mathcal{Z}3^2+\mathcal{Z}4^2))\mathcal{Z}1\Big\}\Big)=0,\\
\mathrm{H3}: (1-\kappa)(\mathbb{L}_t\{\mathcal{Z}3\}-\mathcal{Z}3_0)+\kappa\Big(\mathbb{L}_t\{\mathcal{Z}3\}-\left(\frac{1}{s}\right)\sin x+\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\Big\{-\rho\frac{\partial \mathcal{Z}3}{\partial x}\\
+\tau\frac{\partial^2\mathcal{Z}4}{\partial x^2}+\sigma((\mathcal{Z}3^2+\mathcal{Z}4^2)+\gamma(\mathcal{Z}1^2+\mathcal{Z}2^2))\mathcal{Z}4\Big\}\Big)=0,\\
\mathrm{H4}: (1-\kappa)(\mathbb{L}_t\{\mathcal{Z}4\}-\mathcal{Z}4_0)+\kappa\Big(\mathbb{L}_t\{\mathcal{Z}4\}-\left(\frac{1}{s}\right)\cos x+\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\Big\{-\rho\frac{\partial \mathcal{Z}4}{\partial x}\\
-\tau\frac{\partial^2\mathcal{Z}3}{\partial x^2}-\sigma((\mathcal{Z}3^2+\mathcal{Z}4^2)+\gamma(\mathcal{Z}1^2+\mathcal{Z}2^2))\mathcal{Z}3\Big\}\Big)=0,
\end{aligned}
\tag{5.3}
$$

for $\kappa \epsilon$ [0,1], expansion of $\mathcal{Z}$ in power series w.r.t. κ gives

$$
\mathcal{Z}=\mathcal{Z}_0+\kappa^1\mathcal{Z}_1+\kappa^2\mathcal{Z}_2+\kappa^3\mathcal{Z}_3+\cdots \tag{5.4}
$$

After substituting Eq (5.4) in Eq (5.3), and comparing alike coefficients we acquire
At κ^1:

$$
\begin{aligned}
&\mathbb{L}_t\{\mathcal{Z}1_1\}+\mathcal{Z}1_0-\left(\frac{1}{s}\right)\sin x+\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\Big\{\rho\frac{\partial \mathcal{Z}1_0}{\partial x}+\tau\frac{\partial^2\mathcal{Z}2_0}{\partial x^2}+\sigma((\mathcal{Z}1_0^2+\mathcal{Z}2_0^2)\\
&+\gamma(\mathcal{Z}3_0^2+\mathcal{Z}4_0^2))\mathcal{Z}2_0\Big\}=0,\\
&\mathbb{L}_t\{\mathcal{Z}2_1\}+\mathcal{Z}2_0-\left(\frac{1}{s}\right)\cos x+\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\Big\{\rho\frac{\partial \mathcal{Z}2_0}{\partial x}-\tau\frac{\partial^2\mathcal{Z}1_0}{\partial x^2}-\sigma((\mathcal{Z}1_0^2+\mathcal{Z}2_0^2)\\
&+\gamma(\mathcal{Z}3_0^2+\mathcal{Z}4_0^2))\mathcal{Z}1_0\Big\}=0,\\
&\mathbb{L}_t\{\mathcal{Z}3_1\}+\mathcal{Z}3_0-\left(\frac{1}{s}\right)\sin x+\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\Big\{-\rho\frac{\partial \mathcal{Z}3_0}{\partial x}+\tau\frac{\partial^2\mathcal{Z}4_0}{\partial x^2}+\sigma((\mathcal{Z}3_0^2\\
&+\mathcal{Z}4_0^2)+\gamma(\mathcal{Z}1_0^2+\mathcal{Z}2_0^2))\mathcal{Z}4_0\Big\}=0,\\
&\mathbb{L}_t\{\mathcal{Z}4_1\}+\mathcal{Z}4_0-\left(\frac{1}{s}\right)\cos x+\left(\frac{1}{s^\alpha}\right)\mathbb{L}_t\Big\{-\rho\frac{\partial \mathcal{Z}4_0}{\partial x}-\tau\frac{\partial^2\mathcal{Z}3_0}{\partial x^2}-\sigma((\mathcal{Z}3_0^2\\
&+\mathcal{Z}4_0^2)+\gamma(\mathcal{Z}1_0^2+\mathcal{Z}2_0^2))\mathcal{Z}3_0\Big\}=0.
\end{aligned}
\tag{5.5}
$$

An application of Laplace transform inverse generate

$$
\begin{aligned}
\mathcal{Z}1_1 &= -\frac{t^{\alpha}\left(\gamma\sigma\cos^3(x)+\gamma\sigma\sin^2(x)\cos(x)+\rho\cos(x)+\sigma\cos^3(x)+\sigma\sin^2(x)\cos(x)-\tau\cos(x)\right)}{\Gamma(\alpha+1)},\\
\mathcal{Z}2_1 &= -\frac{t^{\alpha}\left(-\gamma\sigma\sin^3(x)-\gamma\sigma\sin(x)\cos^2(x)-\rho\sin(x)-\sigma\sin^3(x)-\sigma\sin(x)\cos^2(x)+\tau\sin(x)\right)}{\Gamma(\alpha+1)},\\
\mathcal{Z}3_1 &= -\frac{t^{\alpha}\left(\gamma\sigma\cos^3(x)+\gamma\sigma\sin^2(x)\cos(x)-\rho\cos(x)+\sigma\cos^3(x)+\sigma\sin^2(x)\cos(x)-\tau\cos(x)\right)}{\Gamma(\alpha+1)},\\
\mathcal{Z}4_1 &= -\frac{t^{\alpha}\left(-\gamma\sigma\sin^3(x)-\gamma\sigma\sin(x)\cos^2(x)+\rho\sin(x)-\sigma\sin^3(x)-\sigma\sin(x)\cos^2(x)+\tau\sin(x)\right)}{\Gamma(\alpha+1)}.
\end{aligned} \tag{5.6}
$$

At κ^2:

$$
\begin{aligned}
&\mathbb{L}_t\{\mathcal{Z}1_2\}+\left(\frac{1}{s^{\alpha}}\right)\mathbb{L}_t\Big\{\rho\frac{\partial\mathcal{Z}1_1}{\partial x}+\tau\frac{\partial^2\mathcal{Z}2_1}{\partial x^2}+\sigma((\mathcal{Z}1_1^2+\mathcal{Z}2_1^2)+\gamma(\mathcal{Z}3_1^2+\mathcal{Z}4_1^2))\mathcal{Z}2_1\Big\}=0,\\
&\mathbb{L}_t\{\mathcal{Z}2_2\}+\left(\frac{1}{s^{\alpha}}\right)\mathbb{L}_t\Big\{\rho\frac{\partial\mathcal{Z}2_1}{\partial x}-\tau\frac{\partial^2\mathcal{Z}1_1}{\partial x^2}-\sigma((\mathcal{Z}1_1^2+\mathcal{Z}2_1^2)+\gamma(\mathcal{Z}3_1^2+\mathcal{Z}4_1^2))\mathcal{Z}1_1\Big\}=0,\\
&\mathbb{L}_t\{\mathcal{Z}3_2\}+\left(\frac{1}{s^{\alpha}}\right)\mathbb{L}_t\Big\{-\rho\frac{\partial\mathcal{Z}3_1}{\partial x}+\tau\frac{\partial^2\mathcal{Z}4_1}{\partial x^2}+\sigma((\mathcal{Z}3_1^2+\mathcal{Z}4_1^2)+\gamma(\mathcal{Z}1_1^2+\mathcal{Z}2_1^2))\mathcal{Z}4_1\Big\}=0,\\
&\mathbb{L}_t\{\mathcal{Z}4_2\}+\left(\frac{1}{s^{\alpha}}\right)\mathbb{L}_t\Big\{-\rho\frac{\partial\mathcal{Z}4_1}{\partial x}-\tau\frac{\partial^2\mathcal{Z}3_1}{\partial x^2}-\sigma((\mathcal{Z}3_1^2+\mathcal{Z}4_1^2)+\gamma(\mathcal{Z}1_1^2+\mathcal{Z}2_1^2))\mathcal{Z}3_1\Big\}=0.
\end{aligned} \tag{5.7}
$$

By taking Laplace transform inverse we have

$$
\begin{aligned}
\mathcal{Z}1_2 &= -\frac{t^{2\alpha}\sin(x)(\gamma\sigma+\rho+\sigma-\tau)^2}{\Gamma(2\alpha+1)},\\
\mathcal{Z}2_2 &= -\frac{t^{2\alpha}\cos(x)(\gamma\sigma+\rho+\sigma-\tau)^2}{\Gamma(2\alpha+1)},\\
\mathcal{Z}3_2 &= -\frac{t^{2\alpha}\sin(x)(-(\gamma+1)\sigma+\rho+\tau)^2}{\Gamma(2\alpha+1)},\\
\mathcal{Z}4_2 &= -\frac{t^{2\alpha}\cos(x)(-(\gamma+1)\sigma+\rho+\tau)^2}{\Gamma(2\alpha+1)}.
\end{aligned} \tag{5.8}
$$

Hence, at fifth order the approximate solution in series form is

$$
\begin{aligned}
\mathcal{Z}1 &= \sum_{j=0}^{5}\mathcal{Z}1_j(x,t),\ \mathcal{Z}2=\sum_{j=0}^{5}\mathcal{Z}2_j(x,t),\\
\mathcal{Z}3 &= \sum_{j=0}^{5}\mathcal{Z}3_j(x,t),\ \mathcal{Z}4=\sum_{j=0}^{5}\mathcal{Z}4_j(x,t).
\end{aligned} \tag{5.9}
$$

Case 2: Take $\alpha = 1$ in Eq (5.1) and apply Laplace transform w.r.t. space ($\mathbb{L}_x$). By following the steps portrayed in section 3, the homotopies are

$$
\begin{aligned}
&\text{H1}: (1-\kappa)(\mathbb{L}_x\{\mathcal{Z}1\} - \mathcal{Z}1_0) + \kappa\Big(\mathbb{L}_x\{\mathcal{Z}1\} - \left(\frac{1}{s}\right)\sin t + \left(\frac{1}{s^\beta \rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}1}{\partial t} \\
&\quad + \tau\frac{\partial^{2\beta}\mathcal{Z}2}{\partial x^{2\beta}} + \sigma((\mathcal{Z}1^2 + \mathcal{Z}2^2) + \gamma(\mathcal{Z}3^2 + \mathcal{Z}4^2))\mathcal{Z}2\Big\}\Big) = 0, \\
&\text{H2}: (1-\kappa)(\mathbb{L}_x\{\mathcal{Z}2\} - \mathcal{Z}2_0) + \kappa\Big(\mathbb{L}_x\{\mathcal{Z}2\} - \left(\frac{1}{s}\right)\cos t + \left(\frac{1}{s^\beta \rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}2}{\partial t} \\
&\quad - \tau\frac{\partial^{2\beta}\mathcal{Z}1}{\partial x^{2\beta}} - \sigma((\mathcal{Z}1^2 + \mathcal{Z}2^2) + \gamma(\mathcal{Z}3^2 + \mathcal{Z}4^2))\mathcal{Z}1\Big\}\Big) = 0, \\
&\text{H3}: (1-\kappa)(\mathbb{L}_x\{\mathcal{Z}3\} - \mathcal{Z}3_0) + \kappa\Big(\mathbb{L}_x\{\mathcal{Z}3\} - \left(\frac{1}{s}\right)\sin t - \left(\frac{1}{s^\beta \rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}3}{\partial t} \\
&\quad + \tau\frac{\partial^{2\beta}\mathcal{Z}4}{\partial x^{2\beta}} + \sigma((\mathcal{Z}3^2 + \mathcal{Z}4^2) + \gamma(\mathcal{Z}1^2 + \mathcal{Z}2^2))\mathcal{Z}4\Big\}\Big) = 0, \\
&\text{H4}: (1-\kappa)(\mathbb{L}_x\{\mathcal{Z}4\} - \mathcal{Z}4_0) + \kappa\Big(\mathbb{L}_x\{\mathcal{Z}4\} - \left(\frac{1}{s}\right)\cos t - \left(\frac{1}{s^\beta \rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}4}{\partial t} \\
&\quad - \tau\frac{\partial^{2\beta}\mathcal{Z}3}{\partial x^{2\beta}} - \sigma((\mathcal{Z}3^2 + \mathcal{Z}4^2) + \gamma(\mathcal{Z}1^2 + \mathcal{Z}2^2))\mathcal{Z}3\Big\}\Big) = 0.
\end{aligned} \tag{5.10}
$$

Expansion and comparison of κ in power series form gives
At κ^1:

$$
\begin{aligned}
&\mathbb{L}_x\{\mathcal{Z}1_1\} + \mathcal{Z}1_0 - \left(\frac{1}{s}\right)\sin t + \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}1_0}{\partial t} + \tau\frac{\partial^{2\beta}\mathcal{Z}2_0}{\partial x^{2\beta}} + \sigma((\mathcal{Z}1_0^2 + \mathcal{Z}2_0^2) + \gamma(\mathcal{Z}3_0^2 + \mathcal{Z}4_0^2))\mathcal{Z}2_0\Big\} = 0, \\
&\mathbb{L}_x\{\mathcal{Z}2_1\} + \mathcal{Z}2_0 - \left(\frac{1}{s}\right)\cos t + \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}2_0}{\partial t} - \tau\frac{\partial^{2\beta}\mathcal{Z}1_0}{\partial x^{2\beta}} - \sigma((\mathcal{Z}1_0^2 + \mathcal{Z}2_0^2) + \gamma(\mathcal{Z}3_0^2 + \mathcal{Z}4_0^2))\mathcal{Z}1_0\Big\} = 0, \\
&\mathbb{L}_x\{\mathcal{Z}3_1\} + \mathcal{Z}3_0 - \left(\frac{1}{s}\right)\sin t - \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}3_0}{\partial t} + \tau\frac{\partial^{2\beta}\mathcal{Z}4_0}{\partial x^{2\beta}} + \sigma((\mathcal{Z}3_0^2 + \mathcal{Z}4_0^2) + \gamma(\mathcal{Z}1_0^2 + \mathcal{Z}2_0^2))\mathcal{Z}4_0\Big\} = 0, \\
&\mathbb{L}_x\{\mathcal{Z}4_1\} + \mathcal{Z}4_0 - \left(\frac{1}{s}\right)\cos t - \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}4_0}{\partial t} - \tau\frac{\partial^{2\beta}\mathcal{Z}3_0}{\partial x^{2\beta}} - \sigma((\mathcal{Z}3_0^2 + \mathcal{Z}4_0^2) + \gamma(\mathcal{Z}1_0^2 + \mathcal{Z}2_0^2))\mathcal{Z}3_0\Big\} = 0.
\end{aligned} \tag{5.11}
$$

At κ^2:

$$
\begin{aligned}
&\mathbb{L}_x\{\mathcal{Z}1_2\} + \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}1_1}{\partial t} + \tau\frac{\partial^{2\beta}\mathcal{Z}2_1}{\partial x^{2\beta}} + \sigma((\mathcal{Z}1_1^2 + \mathcal{Z}2_1^2) + \gamma(\mathcal{Z}3_1^2 + \mathcal{Z}4_1^2))\mathcal{Z}2_1\Big\} = 0, \\
&\mathbb{L}_x\{\mathcal{Z}2_2\} + \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}2_1}{\partial t} - \tau\frac{\partial^{2\beta}\mathcal{Z}1_1}{\partial x^{2\beta}} - \sigma((\mathcal{Z}1_1^2 + \mathcal{Z}2_1^2) + \gamma(\mathcal{Z}3_1^2 + \mathcal{Z}4_1^2))\mathcal{Z}1_1\Big\} = 0, \\
&\mathbb{L}_x\{\mathcal{Z}3_2\} - \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}3_1}{\partial t} + \tau\frac{\partial^{2\beta}\mathcal{Z}4_1}{\partial x^{2\beta}} + \sigma((\mathcal{Z}3_1^2 + \mathcal{Z}4_1^2) + \gamma(\mathcal{Z}1_1^2 + \mathcal{Z}2_1^2))\mathcal{Z}4_1\Big\} = 0, \\
&\mathbb{L}_x\{\mathcal{Z}4_2\} - \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial \mathcal{Z}4_1}{\partial t} - \tau\frac{\partial^{2\beta}\mathcal{Z}3_1}{\partial x^{2\beta}} - \sigma((\mathcal{Z}3_1^2 + \mathcal{Z}4_1^2) + \gamma(\mathcal{Z}1_1^2 + \mathcal{Z}2_1^2))\mathcal{Z}3_1\Big\} = 0.
\end{aligned} \tag{5.12}
$$

At κ^3:

$$\begin{aligned}
&\mathbb{L}_x\{\mathcal{Z}1_2\} + \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial\mathcal{Z}1_2}{\partial t} + \tau\frac{\partial^{2\beta}\mathcal{Z}2_2}{\partial x^{2\beta}} + \sigma((\mathcal{Z}1_2^2 + \mathcal{Z}2_2^2) + \gamma(\mathcal{Z}3_2^2 + \mathcal{Z}4_2^2))\mathcal{Z}2_2\Big\} = 0,\\
&\mathbb{L}_x\{\mathcal{Z}2_2\} + \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial\mathcal{Z}2_2}{\partial t} - \tau\frac{\partial^{2\beta}\mathcal{Z}1_2}{\partial x^{2\beta}} - \sigma((\mathcal{Z}1_2^2 + \mathcal{Z}2_2^2) + \gamma(\mathcal{Z}3_2^2 + \mathcal{Z}4_2^2))\mathcal{Z}1_2\Big\} = 0,\\
&\mathbb{L}_x\{\mathcal{Z}3_2\} - \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial\mathcal{Z}3_2}{\partial t} + \tau\frac{\partial^{2\beta}\mathcal{Z}4_2}{\partial x^{2\beta}} + \sigma((\mathcal{Z}3_2^2 + \mathcal{Z}4_2^2) + \gamma(\mathcal{Z}1_2^2 + \mathcal{Z}2_2^2))\mathcal{Z}4_2\Big\} = 0,\\
&\mathbb{L}_x\{\mathcal{Z}4_2\} - \left(\frac{1}{s^\beta\rho}\right)\mathbb{L}_x\Big\{\frac{\partial\mathcal{Z}4_2}{\partial t} - \tau\frac{\partial^{2\beta}\mathcal{Z}3_2}{\partial x^{2\beta}} - \sigma((\mathcal{Z}3_2^2 + \mathcal{Z}4_2^2) + \gamma(\mathcal{Z}1_2^2 + \mathcal{Z}2_2^2))\mathcal{Z}3_2\Big\} = 0.
\end{aligned} \tag{5.13}$$

Following similar working as that of Case 1, we can have fifth order approximation as

$$\begin{aligned}
\mathcal{Z}1 = \sum_{j=0}^{5}\mathcal{Z}1_j(x,t), \mathcal{Z}2 = \sum_{j=0}^{5}\mathcal{Z}2_j(x,t),\\
\mathcal{Z}3 = \sum_{j=0}^{5}\mathcal{Z}3_j(x,t), \mathcal{Z}4 = \sum_{j=0}^{5}\mathcal{Z}4_j(x,t).
\end{aligned} \tag{5.14}$$

6. Results and Discussion

In this research article, a time-space fractional non-linear coupled Schrödinger system is solved via a semi-numerical algorithm titled as He-Laplace method. The time and space fractional derivatives are taken in Caputo form which are further simplified with the help of Laplace transform. While initiating the process, the complex valued functions of Schrödinger system are separated in real and imaginary parts. Thus, it is further modified in the system of four equations. Since the range of fractional parameters α and β are between 0 to 1, two cases are considered. For Case 1, the Laplace transform is applied with respect to time t while taking the fractional parameter $\beta = 1$. Similarly, for Case 2 Laplace transform in space x is employed while keeping $\alpha = 1$.

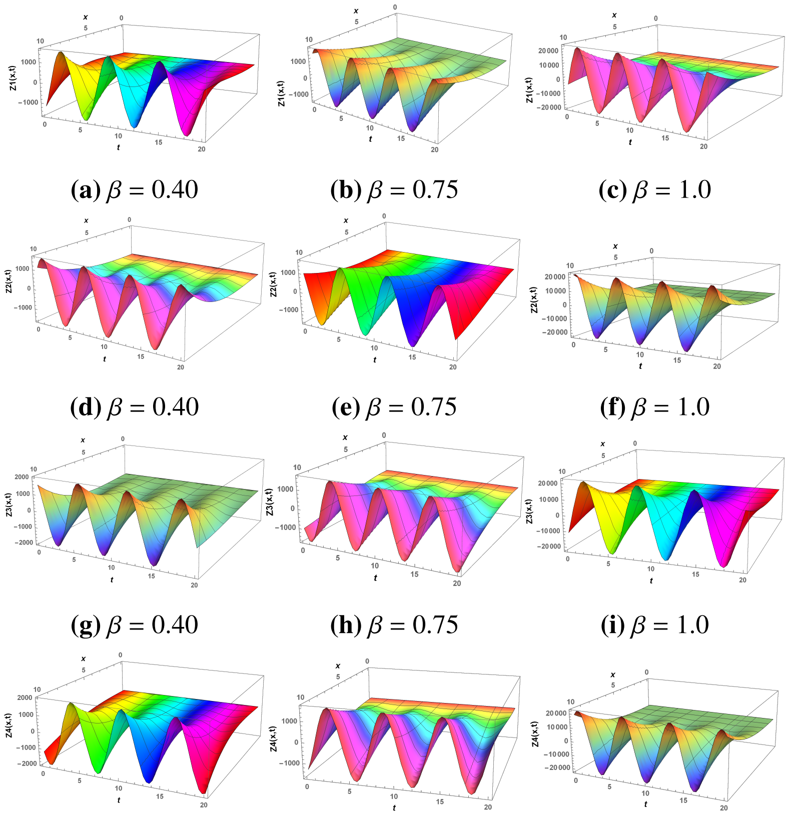
Efficiency and productivity of the applied method for a complex non-linear system throughout the fractional domain is elaborated through absolute residual and system errors in Tables 1 and 2. At distinct values of t and x, errors are determined for $\alpha = 0.30, 0.55, 0.80, \& 1.0$. Analysis demonstrates that the errors keep improving as fractional parameter increase in its domain and approaches the integer order value 1.0. The system errors indicate that the obtained solution is consistent even for a larger value of time and space. It can be concluded from these tables that the calculated approximate solution converges towards the exact solution by increasing the number of iterations of the proposed methodology. Three dimensional graphical presentation of solutions for $\beta = 0.40, 0.75 \& 1.0$ can be seen through Figure 1. The up and down motion of the solution curves indicate sinusoidal behaviour of waves for the whole system. It is noticed that at bigger value of β, the crest and trough of waves are large compared to a smaller value of β. Moreover, in the beginning the system's surface is flat, but as space expanded waves started to form and gradually increased. Figures 2 and 3 display the wave profiles for distinct fractional parameter values in two dimension. The rise and fall of waves can be observed with the help of arrows. For half cycle the waves profiles are showing decreasing behaviour with the increase of fractional value, whereas, it shows opposite behaviour in other half for larger fractional values. It is illustrated that for varying time and fixed x, the real function plots (3a) and (3c) of Figure 3 has opposite behaviour for same β value. Similarly, the imaginary function plots (3b) and (3d) also depict contrasting behaviour.

Table 1. Error analysis at $x = 2$, $\rho = \tau = 0.1$ and $\sigma = \gamma = 0.01$.

	t	$\|\mathcal{R}\|_{\mathcal{Z}1}$	$\|\mathcal{R}\|_{\mathcal{Z}2}$	$\|\mathcal{R}\|_{\mathcal{Z}3}$	$\|\mathcal{R}\|_{\mathcal{Z}4}$	$\|\mathcal{R}\|_{system}$
	0.1	3.59×10^{-11}	5.62×10^{-11}	6.85×10^{-7}	3.04×10^{-7}	2.47×10^{-7}
$\alpha = 0.30$	0.6	3.83×10^{-10}	1.15×10^{-9}	1.10×10^{-5}	4.40×10^{-6}	3.63×10^{-6}
	1.1	7.88×10^{-10}	3.24×10^{-9}	2.51×10^{-5}	1.08×10^{-5}	9.00×10^{-6}
	1.6	1.18×10^{-9}	6.15×10^{-9}	4.42×10^{-5}	1.89×10^{-5}	1.58×10^{-5}
	0.1	6.28×10^{-13}	5.83×10^{-13}	1.22×10^{-8}	5.05×10^{-9}	4.07×10^{-9}
$\alpha = 0.55$	0.6	6.19×10^{-11}	1.36×10^{-10}	1.55×10^{-6}	6.82×10^{-7}	5.60×10^{-7}
	1.1	2.51×10^{-10}	8.99×10^{-10}	8.28×10^{-6}	3.56×10^{-6}	2.96×10^{-6}
	1.6	5.41×10^{-10}	2.90×10^{-9}	2.32×10^{-6}	9.86×10^{-6}	8.29×10^{-6}
	0.1	3.80×10^{-15}	3.03×10^{-15}	1.28×10^{-10}	5.86×10^{-11}	4.68×10^{-11}
$\alpha = 0.80$	0.6	2.97×10^{-12}	8.29×10^{-12}	1.67×10^{-7}	7.50×10^{-8}	6.05×10^{-8}
	1.1	1.65×10^{-11}	1.32×10^{-10}	1.89×10^{-6}	8.39×10^{-7}	6.83×10^{-7}
	1.6	9.81×10^{-12}	7.47×10^{-10}	8.48×10^{-6}	3.72×10^{-6}	3.05×10^{-6}
	0.1	0.0	1.73×10^{-18}	2.77×10^{-17}	2.77×10^{-17}	1.43×10^{-17}
$\alpha = 1.0$	0.6	5.72×10^{-17}	1.26×10^{-16}	8.53×10^{-12}	3.92×10^{-12}	3.11×10^{-12}
	1.1	7.17×10^{-15}	1.61×10^{-14}	5.93×10^{-10}	2.73×10^{-10}	2.16×10^{-10}
	1.6	1.39×10^{-13}	3.17×10^{-13}	8.16×10^{-9}	3.77×10^{-9}	2.98×10^{-9}

Table 2. Error analysis at $t = 0.3$, $\rho = \tau = 0.1$ and $\sigma = \gamma = 0.01$.

	x	$\|\mathcal{R}\|_{\mathcal{Z}1}$	$\|\mathcal{R}\|_{\mathcal{Z}2}$	$\|\mathcal{R}\|_{\mathcal{Z}3}$	$\|\mathcal{R}\|_{\mathcal{Z}4}$	$\|\mathcal{R}\|_{system}$
	-2.0	3.73×10^{-10}	1.13×10^{-10}	3.52×10^{-6}	1.67×10^{-6}	1.29×10^{-6}
	-1.5	2.73×10^{-10}	2.78×10^{-10}	3.89×10^{-6}	2.17×10^{-7}	1.02×10^{-6}
$\alpha = 0.30$	0.0	2.58×10^{-10}	2.92×10^{-10}	5.81×10^{-8}	3.89×10^{-6}	9.89×10^{-7}
	1.5	3.09×10^{-10}	2.37×10^{-10}	3.88×10^{-6}	3.33×10^{-7}	1.05×10^{-6}
	2.0	1.58×10^{-10}	3.57×10^{-10}	3.56×10^{-6}	1.56×10^{-6}	1.28×10^{-6}
	-2.0	1.94×10^{-11}	2.17×10^{-12}	2.28×10^{-7}	1.07×10^{-7}	8.40×10^{-8}
	-1.5	1.59×10^{-11}	1.12×10^{-11}	2.52×10^{-7}	1.50×10^{-8}	6.68×10^{-8}
$\alpha = 0.55$	0.0	1.00×10^{-11}	1.67×10^{-11}	2.82×10^{-9}	2.52×10^{-7}	6.38×10^{-8}
	1.5	1.74×10^{-11}	8.84×10^{-12}	2.51×10^{-7}	2.06×10^{-8}	6.81×10^{-8}
	2.0	1.10×10^{-11}	1.61×10^{-11}	2.30×10^{-7}	1.02×10^{-7}	8.33×10^{-8}
	-2.0	4.43×10^{-13}	4.77×10^{-14}	1.03×10^{-8}	4.80×10^{-9}	3.80×10^{-9}
	-1.5	3.65×10^{-13}	2.54×10^{-13}	1.14×10^{-8}	7.64×10^{-10}	3.04×10^{-9}
$\alpha = 0.80$	0.0	2.27×10^{-13}	3.83×10^{-13}	4.55×10^{-11}	1.14×10^{-8}	2.87×10^{-9}
	1.5	3.98×10^{-13}	2.00×10^{-13}	1.14×10^{-8}	8.55×10^{-10}	3.06×10^{-9}
	2.0	2.53×10^{-13}	3.66×10^{-13}	1.04×10^{-8}	4.72×10^{-9}	3.78×10^{-9}



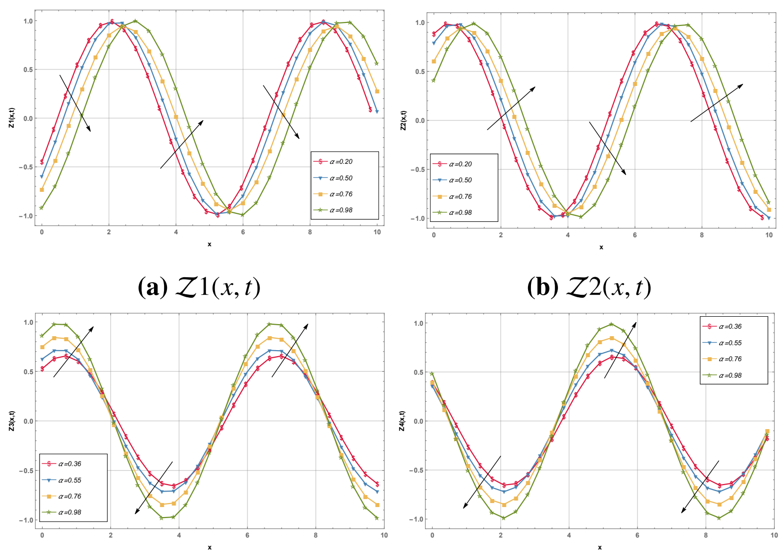
(a) $\beta = 0.40$ **(b)** $\beta = 0.75$ **(c)** $\beta = 1.0$

(d) $\beta = 0.40$ **(e)** $\beta = 0.75$ **(f)** $\beta = 1.0$

(g) $\beta = 0.40$ **(h)** $\beta = 0.75$ **(i)** $\beta = 1.0$

(j) $\beta = 0.40$ **(k)** $\beta = 0.75$ **(l)** $\beta = 1.0$

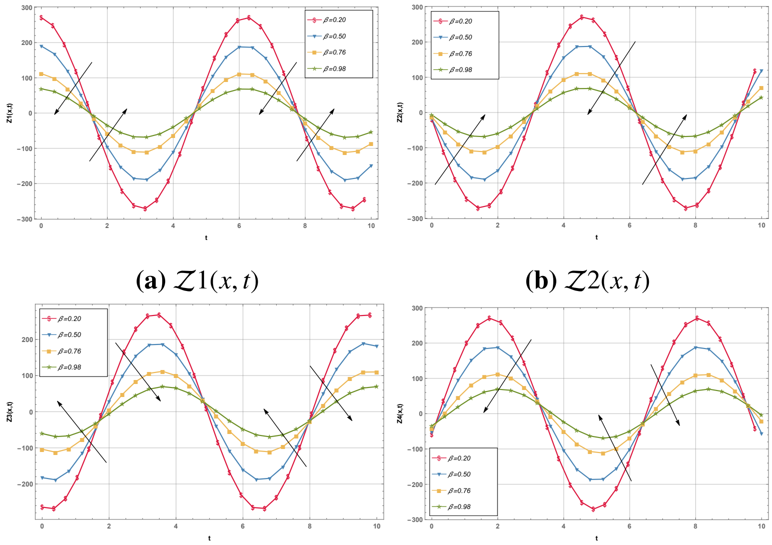
Figure 1. Three dimensional solutions of Schrödinger system at different value of fractional parameter when $\rho = \tau = \sigma = 2.0$ and $\gamma = 1.0$.



(a) $\mathcal{Z}1(x, t)$ **(b)** $\mathcal{Z}2(x, t)$

(c) $\mathcal{Z}3(x, t)$ **(d)** $\mathcal{Z}4(x, t)$

Figure 2. Effect of time-fractional parameter α on the wave profiles of Schrödinger system (Case 1) when $\rho = \tau = 0.2$, $\gamma = \sigma = 0.3$ and $t = 3$.



(a) $\mathcal{Z}1(x, t)$ **(b)** $\mathcal{Z}2(x, t)$

(c) $\mathcal{Z}3(x, t)$ **(d)** $\mathcal{Z}4(x, t)$

Figure 3. Effect of spacial-fractional parameter β on the wave profiles of Schrödinger system (Case 2) when $\rho = \tau = 0.2$, $\gamma = \sigma = 0.1$ and $x = 1$.

7. Conclusions

In this research article, a hybrid of Laplace transform and homotopy perturbation is successfully applied to time-space fractional Schrodinger system. The obtained solutions are analyzed theoretically by proving convergence theorem and finding error estimates. Efficiency of the proposed methodology is tested by finding residual errors throughout the fractional domain. These obtained numerical results endorse the convergence of proposed algorithm. Physical behavior of different fractional parameters on the wave profile are observed through 2D and 3D plots. Analysis concludes that proposed method is effective in relation to highly non-linear complex system in time-space fractional environment. Hence, in future the proposed scheme can be extended to other complex systems having uncertain fractional space.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest regarding the publication of this paper.

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