AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 26945–26967
DOI: 10.3934/math.20231379
Received: 07 August 2023
Revised: 03 September 2023
Accepted: 05 September 2023
Published: 22 September 2023

Research article

Topological visualization and graph analysis of rough sets via neighborhoods: A medical application using human heart data

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Abstract: In the field of medical applications, graph theory offers diverse topological models for representing the human heart. The key challenge is identifying the optimal structure as an effective diagnostic model. This paper explains the rationale behind using topological visualization, graph analysis, and rough sets via neighborhood systems. We introduce the novel 1-neighborhood system (1-NS) tools, enabling rough set generalization and a heart topological graph model. Exploring minimal and core minimal neighborhoods, vital for classifying subsets and accuracy computation, these approaches outperform existing methods while preserving Pawlak's properties. Multiple topologies are constructed and examined using these systems. The paper presents a real-world example showcasing innovative topological spaces through a human heart's vertex network. These spaces enhance understanding of the heart's structural organization. Two algorithms are introduced for decision-making and generating graph topologies, defining unique spaces. Beyond graph theory, these techniques apply to medical contexts like blood circulation and geographical scenarios such as community street mapping. Implemented using MATLAB, they are valuable tools.

Keywords: topology; neighborhood systems; rough set theory; graph theory; human heart
Mathematics Subject Classification: 54A05, 54B10, 54D30, 54G99

Nomenclature

$\mathbb{NS}$s: Neighborhood systems. $\mathcal{U}$: Finite universe set. $\aleph(\mathbf{m})$: Neighborhood of $\mathbf{m}$. $\mathbb{C}\aleph(\mathbf{m})$: Core neighborhood of $\mathbf{m}$. $\mathbb{M}(\mathbf{m})$: Minimal neighborhood of $\mathbf{m}$. $\mathbb{CM}(\mathbf{m})$: Core minimal neighborhood of $\mathbf{m}$. $\mathbb{K}$: Equivalence relation. $[\mathbf{m}]_{\mathbb{K}}$: Equivalence classes. $\underline{\mathbb{K}}$: Lower approximation. $\overline{\mathbb{K}}$: Upper approximation. $\mathbf{B}(\mathbf{A})$: Boundary of $\mathbf{A}$. $\mathbf{POS}(\mathbf{A})$: Positive region of $\mathbf{A}$. $\mathbf{NEG}(\mathbf{A})$: Negative region of $\mathbf{A}$. $\mu(\mathbf{A})$: Accuracy of $\mathbf{A}$.

1. Introduction

In recent years, the application of advanced mathematical techniques to medical models has provided invaluable insights into complex datasets. This paper aims to present a concise and lucid elucidation of the rationale driving the utilization of topological visualization and graph analysis in conjunction with rough sets via neighborhood systems. This study assumes significance against the backdrop of numerous medical models that exist, each presenting a unique challenge in terms of data complexity and interrelationships.

The motivation for employing topological visualization lies in its ability to represent intricate data structures in a visually intuitive manner, transcending the limitations of traditional numerical analysis. By transforming complex data points into topological spaces, patterns and relationships that might remain concealed in raw data become apparent. Graph analysis, on the other hand, offers a means to unravel interconnectedness within datasets, highlighting key nodes and revealing emergent properties. The integration of rough sets, a mathematical framework for dealing with uncertainty and imprecision, with neighborhood systems enhances the versatility of analysis. This synergy capitalizes on the strengths of both approaches, allowing for a more comprehensive understanding of intricate medical data. Through the lens of neighborhood systems, the granularity of relationships between elements can be adjusted, accommodating various levels of abstraction as required by medical applications. The context of medical models underscores the significance of this endeavor. Medical data are inherently intricate, often encompassing multifaceted interactions between variables. The utilization of topological visualization and graph analysis in tandem with rough sets acknowledges the need for innovative methodologies that can discern meaningful insights from complex medical datasets. The present paper underscores the potential of this combined approach to revolutionize medical data analysis, leading to enhanced diagnostic accuracy, treatment strategies, and ultimately, improved patient care.

Rough sets, a concept initially introduced by Pawlak [46, 47], offer invaluable mathematical tools for effectively managing vagueness within information systems and facilitating data analysis. Furthermore, they serve as robust instruments for tackling issues entailing imperfect knowledge. The foundational principle of rough sets involves the classification of objects grounded in an equivalence relation. This framework enables us to encompass the entirety of information derived from a given set. The theory employs approximation operators and precision metrics, equipping decision-makers with vital insights into the structure and dimensions of the boundary region.

However, the stringent requirement of an equivalence relation imposes certain limitations on the applications of conventional rough set theory. In response, various generalizations of the theory have surfaced, employing either arbitrary or specific relations. In 1996, Yao [54, 55] pioneered this avenue of research. Here, specific relations were considered to establish distinct types of generalized rough sets, encompassing tolerance [51], similarity [5, 15], quasi-order [48] and a general relation [7, 8, 10].

On a different note, in 2014, Abd El-Monsef et al. [3] introduced the concept of the j-neighborhood space (abbreviated as j-***NS***), presenting a generalized variant of the neighborhood space derived from a general binary relation. They laid the groundwork for expanding Pawlak's rough sets by leveraging diverse topologies induced by the j-***NS***.

The applications of rough sets span a wide range, encompassing diverse fields such as medical diagnosis, marketing and image analysis. A comprehensive understanding of the applications of rough set theory can be found in works related to decision-making problems [6, 7, 11, 20, 22], medical applications [21, 23, 28, 32], economic fields [27, 29], topological reductions of attributes for predicting COVID-19 [24, 31], Graph theory [18, 19] and biology [9, 32]. Topology plays a crucial role in generalizing rough sets [2, 4, 41], as well as in rough fuzzy approximation operators [1, 3, 22, 30]. Rough set theory characterizes any subset of the universe using two approximations: Lower and upper approximations. Since its inception, researchers have shown significant interest in both the theoretical enrichment and practical applications of rough set theory, as evidenced by works like [38, 42, 45, 56]. Moreover, decision-theoretic rough sets, as a generalized form of rough sets, warrant comprehensive explanation [64, 68, 69].

Moving on, neighborhood systems represent a versatile and generalized concept, encompassing rough sets and finding various applications, especially in real-life scenarios where qualitative information holds greater significance than numerical data. Previous studies by Lin [39, 40], Wu [53] and Yao [55–59] have delved into granular computing, utilizing neighborhood systems for interpreting granules. Additionally, Sierpiński and Krieger [50] introduced $\mathbb{NS}$ spaces to explore the generalization of Fréchet (V)-topology, characterizing spaces where all $\mathbb{NS}$s for all elements are nonempty. Concurrently, researchers have explored topological models leveraging graph neighborhoods in [12, 13, 17, 19, 26].

In a mathematical context, granules are referred to as neighborhoods, and the family of granules associated with an object x is denoted as a neighborhood system (NS) on $\mathcal{U}$, represented by $\mathbb{NS}(\mathcal{U})$. Various types of neighborhood systems exist; for instance, in a binary system [55], each element has at most one neighborhood defined by a binary relation. Furthermore, if the relation is an equivalence, the system becomes a rough set system. Rough sets can thus be seen as a special form of neighborhood systems [53, 55, 57], with Yao's study delving into them as 1-neighborhood systems (1-NS).

Additionally, Hung's work in 2008 explored core neighborhoods in general topology [33], while El Atik et al. [25] introduced the concept of minimal neighborhoods, investigating properties of finite topological spaces with an ordered relation between minimal neighborhoods. The concept of hesitant fuzzy linguistic term sets (HFLTSs) holds a significant role, enabling experts to assess diverse qualitative information using a limited set of linguistic terms. This proves particularly valuable in situations where these experts hesitate to express themselves with certainty. To address the challenges posed by information analysis and fusion within hesitant fuzzy linguistic (HFL) group decision-making, and in alignment with the multi-granularity three-way decisions paradigm. In [63], Chao et al. introduced the concept of multigranulation decision-theoretic rough sets (MG-DTRSs) within the HFL framework, all within the context of a two-universe framework. On the other hand, processing low-resource languages presents a challenge for intelligent decision systems due to the scarcity of available data and resources. In effectively tackling this issue, fuzzy linguistic approaches emerge as a promising solution. These approaches excel in converting initial vague linguistic information into well-structured data, and they facilitate the learning of accurate decision rules within intricate data frameworks. So, there are many important papering appearing to study these problems, such as "Collaborative Fuzzy Linguistic Learning to Low-Resource and Robust Decision System" [61, 66], and "Hesitant Fuzzy Linguistic Rough Sets" [62, 65, 67].

The concept of fuzzy β covering has captured substantial attention as an innovative approach. However, both the traditional fuzzy β covering-based rough set and many of its extended models struggle to accurately represent the distribution of samples within real-world data. This limitation undermines their effectiveness in applications such as classification learning and decision making. Addressing these challenges, Huang et al. [34, 36, 37] presented a noteworthy study that introduces a robust rough set model. This model combines fuzzy rough sets, covering-based rough sets, and multigranulation rough sets. In this context, the model reconstructs optimistic and pessimistic lower and upper approximations of a target concept by utilizing the fuzzy β neighborhood associated with a family of fuzzy coverings. This innovative approach results in a novel multigranulation fuzzy rough set model.

The granular structure inherent in intuitionistic fuzzy (IF) information embodies a dual perspective, capturing both sample similarity and diversity. However, despite its potential, this structural framework has yet to fully showcase its technical prowess in data mining and information processing. This is primarily because of the dearth of suitable construction methods and semantic interpretation techniques for effectively applying IF information to real-world data. To elevate the performance of IF-based techniques in real-world data mining, several researchers have introduced a range of methods designed to tackle these obstacles, as exemplified by [14, 35, 52].

Finally, to facilitate easy reference, we provide a nomenclature section listing the symbols along with their abbreviations.

2. Basic concepts

In this study, we consider a finite universal set $\mathcal{U} \neq \phi$. We will introduce key concepts related to $\mathbb{NS}$s and explore fundamental notions and properties of rough sets, with a specific focus on the approximation operators via $\mathbb{NS}$s. Moreover, throughout our research, we will refer to $(\mathcal{U}, \mathbb{K})$ as the approximation space.

2.1. Exploring neighborhood systems using 1-neighborhoods

Definition 2.1. *[55] For each $\mathbf{m} \in \mathcal{U}$, there exists a neighborhood $\aleph(\mathbf{m}) \subseteq \mathcal{U}$. Consequently, the neighborhood system $\mathbb{NS}(\mathbf{m}) = \{\aleph(\mathbf{m}) : \mathbf{m} \in \mathcal{U}\}$ represents the neighborhoods associated with each element $\mathbf{m}$ in the set $\mathcal{U}$. In other words, $\mathbb{NS} : \mathcal{U} \to \mathfrak{P}(\mathcal{U})$ is a mapping that associates each $\mathbf{m} \in \mathcal{U}$ with its respective neighborhood $\aleph(\mathbf{m})$, where $\mathfrak{P}(\mathcal{U})$ denotes the power set of $\mathcal{U}$. Importantly, the pair $(\mathcal{U}, \mathbb{NS}(\mathcal{U}))$ forms a Fréchet (V)-topology without imposing any additional conditions on $\mathbb{NS}$s.*

Definition 2.2. *[55] (i) if $\forall\ \mathbf{m} \in \mathcal{U}$, $\exists$ a $\mathbf{n} \in \mathcal{U}$ s.t. $\mathbf{n} \in \aleph(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$, $\aleph(\mathbf{m}) \neq \phi$, then $\aleph(\mathbf{m})$ is serial.*

(ii) if $\forall\ \mathbf{m} \in \mathcal{U}$, $\exists$ a $\mathbf{n} \in \mathcal{U}$ s.t. $\mathbf{m} \in \aleph(\mathbf{n})$, $\bigcup_{\mathbf{m}\in\mathcal{U}} \aleph(\mathbf{m}) = \mathcal{U}$, then $\aleph(\mathbf{m})$ is inverse serial.

(iii) if $\forall\ \mathbf{m} \in \mathcal{U}$, $\mathbf{m} \in \aleph(\mathbf{m})$, then $\aleph(\mathbf{m})$ is reflexive.

(iv) if $\forall\ \mathbf{m}, \mathbf{n} \in \mathcal{U}$, $\mathbf{m} \in \aleph(\mathbf{n}) \Rightarrow \mathbf{n} \in \aleph(\mathbf{m})$, then $\aleph(\mathbf{m})$ is symmetric.

(v) if $\forall\ \mathbf{m}, \mathbf{n}, \mathbf{o} \in \mathcal{U}$ s.t. $[\mathbf{m} \in \aleph(\mathbf{n}), \mathbf{n} \in \aleph(\mathbf{o})] \Rightarrow \mathbf{m} \in \aleph(\mathbf{o})$, then $\aleph(\mathbf{m})$ is transitive.

(vi) if $\forall\ \mathbf{m}, \mathbf{n}, \mathbf{o} \in \mathcal{U}$ s.t. $[\mathbf{m} \in \aleph(\mathbf{o}), \mathbf{n} \in \aleph(\mathbf{o})] \Rightarrow \mathbf{n} \in \aleph(\mathbf{m})$, then $\aleph(\mathbf{m})$ is Euclidean.

Example 2.3. *Let $\mathcal{U} = \{\lambda, \mu, \nu, \epsilon\}$. Define $\mathbb{NS}(\mathbf{m}), \forall\ \mathbf{m} \in \mathcal{U} : \mathbb{NS}(\lambda) = \{\{\lambda, \mu\}, \{\mu, \nu\}\}$, $\mathbb{NS}(\mu) = \{\{\mu\}\}$, $\mathbb{NS}(c) = \{\{\mu, \nu\}, \{\nu\}\}$ and $\mathbb{NS}(\epsilon) = \{\{\lambda, \epsilon\}, \{\mu, \nu, \epsilon\}\}$. Here, $\mathbf{m} \in \aleph(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$ and so $\aleph(\mathbf{m})$ is reflexive.*

Remark 2.4. *It is evident that for a reflexive neighborhood, both the serial and inverse serial criteria are satisfied. However, this is not generally true.*

Example 2.5. *(Continued for Example 2.3). Neighborhoods are* $\aleph_1(\lambda) = \{\mu, \nu\}$, $\aleph_1(\mu) = \{\lambda, \epsilon\}$, $\aleph_1(\nu) = \{\nu\}$ *and* $\aleph_1(\epsilon) = \phi$. *Also,* $\aleph_2(\lambda) = \{\lambda, \mu\}$, $\aleph_2(\mu) = \{\mu\}$, $\aleph_2(\nu) = \{\epsilon\}$ *and* $\aleph_2(\epsilon) = \{\lambda, \nu\}$. *For every* $\mathbf{m} \in \mathcal{U}$, *the neighborhood* $\aleph_1(\mathbf{m})$ *is serial, and* $\aleph_2(\mathbf{m})$ *is inverse serial. However, there are no reflexive elements in these neighborhoods.*

Definition 2.6. *[55] A neighborhood system* $\mathbb{NS}(\mathbf{m})$ *is considered a* 1-$\mathbb{NS}$ *of* $\mathbf{m}$ *if it contains only one neighborhood. Furthermore, the set* $NS(\mathcal{U}) = \{\mathbb{NS}(\mathbf{m}) : \forall \mathbf{m} \in \mathcal{U}\}$ *is defined as the* 1-$\mathbb{NS}$ *of* $\mathcal{U}$.

Definition 2.7. *[60] The core neighborhood of* $\mathbf{m}$ *is defined by* $\mathbb{C}\aleph(\mathbf{m}) = \{\mathbf{n} \in \mathcal{U} : \aleph(\mathbf{m}) = \aleph(\mathbf{n})\}$.

Some of the following conclusions are simple to demonstrate by Definition 2.7. As a result, their proofs are skipped.

Lemma 2.8. $\forall\ \mathbf{m}, \mathbf{n} \in \mathcal{U}$, $\mathbf{n} \in \mathbb{C}\aleph(\mathbf{m})$ *if and only if* $\mathbf{m} \in \mathbb{C}\aleph(\mathbf{n})$.

Corollary 2.9. *If* $\mathbf{n} \in \mathbb{C}\aleph(\mathbf{m})$, *then* $\mathbb{C}\aleph(\mathbf{n}) = \mathbb{C}\aleph(x)$, $\forall\ \mathbf{m}, \mathbf{n} \in \mathcal{U}$.

Lemma 2.10. $\forall\ \mathbf{m} \in \mathcal{U}$, *the following hold:*

(i) $\mathbf{m} \in \mathbb{C}\aleph(\mathbf{m})$.

(ii) The class $\{\mathbb{C}\aleph(\mathbf{m}) : \mathbf{m} \in \mathcal{U}\}$ *is a partition on* $\mathcal{U}$.

Lemma 2.11. *If the neighborhood* $\aleph(\mathbf{m})$ *is reflexive for all* $\mathbf{m} \in \mathcal{U}$, *then it follows that* $\mathbb{C}\aleph(\mathbf{m}) \subseteq \aleph(\mathbf{m})$.

Proof. By assumption, we have $\mathbf{m} \in \aleph(\mathbf{m})$. Now, if $\mathbf{n} \in \mathbb{C}\aleph(\mathbf{m})$, then $\mathbb{C}\aleph(\mathbf{n}) = \mathbb{C}\aleph(\mathbf{m})$ and hence $\mathbf{n} \in \aleph(\mathbf{m})$. Therefore, $\mathbb{C}\aleph(\mathbf{m}) \subseteq \aleph(\mathbf{m})$. □

2.2. Rough sets based on 1*-neighborhoods*

Definition 2.12. *[46] In* $(\mathcal{U}, \mathbb{K})$, *let* $\mathcal{U}/\mathbb{K} = \{[\mathbf{m}]_{\mathbb{K}} : \mathbf{m} \in \mathcal{U}\}$ *be the equivalence classes. The lower (resp. upper) approximation of* $\mathfrak{X} \subseteq \mathcal{U}$ *are* $\underline{\mathbb{K}}(\mathfrak{X}) = \{\mathbf{m} \in \mathfrak{X} : [\mathbf{m}]_{\mathbb{K}} \subseteq \mathfrak{X}\}$ *(resp.* $\overline{\mathbb{K}}(\mathfrak{X}) = \{\mathbf{m} \in \mathcal{U} : [\mathbf{m}]_{\mathbb{K}} \cap \mathfrak{X} \neq \phi\}$*).*

According to Pawlak's definition, set $\mathfrak{X}$ is classified as rough if $\underline{\mathbb{K}}(\mathfrak{X}) \neq \overline{\mathbb{K}}(\mathfrak{X})$.

Proposition 2.13. *[46] Let* $\mathfrak{X}^c$ *be the complement of* $\mathfrak{X}$ *w.r.t.* $\mathcal{U}$. *Then:*

(1) $\underline{\mathbb{K}}(\mathfrak{X}) \subseteq \mathfrak{X}$.	*(1*)* $\mathfrak{X} \subseteq \overline{\mathbb{K}}(\mathfrak{X})$.
(2) $\underline{\mathbb{K}}(\phi) = \phi$.	*(2*)* $\overline{\mathbb{K}}(\phi) = \phi$.
(3) $\underline{\mathbb{K}}(\mathcal{U}) = \mathcal{U}$.	*(3*)* $\overline{\mathbb{K}}(\mathcal{U}) = \mathcal{U}$.
(4) $\underline{\mathbb{K}}(\mathfrak{X} \cap \mathfrak{Y}) = \underline{\mathbb{K}}(\mathfrak{X}) \cap \underline{\mathbb{K}}(\mathfrak{Y})$	*(4*)* $\overline{\mathbb{K}}(\mathfrak{X} \cup \mathfrak{Y}) = \overline{\mathbb{K}}(\mathfrak{X}) \cup \overline{\mathbb{K}}(\mathfrak{Y})$
(5) If $\mathfrak{X} \subseteq \mathfrak{Y}$, *then* $\underline{\mathbb{K}}(\mathfrak{X}) \subseteq \underline{\mathbb{K}}(\mathfrak{Y})$	*(5*) If* $\mathfrak{X} \subseteq \mathfrak{Y}$, *then* $\overline{\mathbb{K}}(\mathfrak{X}) \subseteq \overline{\mathbb{K}}(\mathfrak{Y})$
(6) $\underline{\mathbb{K}}(\mathfrak{X}) \cup \underline{\mathbb{K}}(\mathfrak{Y}) \subseteq \underline{\mathbb{K}}(\mathfrak{X} \cup \mathfrak{Y})$	*(6*)* $\overline{\mathbb{K}}(\mathfrak{X}) \cap \overline{\mathbb{K}}(\mathfrak{Y}) \supseteq \overline{\mathbb{K}}(\mathfrak{X} \cap \mathfrak{Y})$
(7) $\underline{\mathbb{K}}(\mathfrak{X}^c) = (\overline{\mathbb{K}}(\mathfrak{X}))^c$	*(7*)* $\overline{\mathbb{K}}(\mathfrak{X}^c) = (\underline{\mathbb{K}}(\mathfrak{X}))^c$
(8) $\underline{\mathbb{K}}(\underline{\mathbb{K}}(\mathfrak{X})) = \underline{\mathbb{K}}(\mathfrak{X})$	*(8*)* $\overline{\mathbb{K}}(\overline{\mathbb{K}}(\mathfrak{X})) = \overline{\mathbb{K}}(\mathfrak{X})$
(9) $\underline{\mathbb{K}}((\underline{\mathbb{K}}(\mathfrak{X}))^c) = (\underline{\mathbb{K}}(\mathfrak{X}))^c$	*(9*)* $\overline{\mathbb{K}}((\overline{\mathbb{K}}(\mathfrak{X}))^c) = (\overline{\mathbb{K}}(\mathfrak{X}))^c$

Note that: Yao [55] expanded the equivalence classes for Pawlak's rough sets by utilizing binary relations.

Definition 2.14. *[55] Lower (resp. upper) approximations of* $\mathbf{A}$ *is defined by* $\underline{apr}_{\aleph}(\mathbf{A}) = \{\mathbf{m} \in \mathcal{U} : \aleph(\mathbf{m}) \subseteq \mathbf{A}\}$ *(resp.* $\overline{apr}_{\aleph}(\mathbf{A}) = \{\mathbf{m} \in \mathcal{U} : \aleph(\mathbf{m}) \cap \mathbf{A} \neq \phi\}$*), for* $\mathbf{A} \subseteq \mathcal{U}$.

As stated in [55, 60], the mentioned approximations fulfill properties (3–9) and (1*, 2*, 4*–9*). Moreover, various types of $\mathbb{NS}$s also adhere to other properties of Pawlak. Furthermore, the following approximations satisfy properties (1–9) and (1*–9*).

Definition 2.15. *[60] Let us define the core-lower (respectively, core-upper) approximations of* $\mathbf{A}$ *as follows:*

$$\underline{apr}_{c\aleph}(\mathbf{A}) = \bigcup\{\mathbf{m} \in \mathcal{U} : \mathbb{C}\aleph(\mathbf{m}) \subseteq \mathbf{A}\},\ (resp.\ \overline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) = \bigcup \mathbf{m} \in \mathcal{U} : \mathbb{C}\aleph(\mathbf{m}) \cap \mathbf{A} \neq \emptyset),\ for\ \mathbf{A} \subseteq \mathcal{U}.$$

3. Exploring minimal neighborhoods and their core neighborhoods via 1-neighborhoods

This section presents new variations of minimal neighborhoods. Furthermore, we conduct a comparative analysis between the proposed concepts and the methods studied in [54, 55, 60].

Definition 3.1. *The minimal neighborhood of* $\mathbf{m}$ *is* $\mathbb{M}(\mathbf{m}) = \bigcap\{\aleph(\mathbf{n}) : \mathbf{m} \in \aleph(\mathbf{n})\}$.

Remark 3.2. *It is clear that* $\forall\ \mathbf{m} \in \mathcal{U}$*:*

$$\mathbb{M}(\mathbf{m}) = \begin{cases} \bigcap_{\mathbf{m}\in\aleph(\mathbf{n})} \aleph(\mathbf{n}) : & if\ \exists\ \mathbf{n}\ s.t.\ \mathbf{m} \in \aleph(\mathbf{n}); \\ \phi : & Otherwise. \end{cases}$$

In the upcoming results, we will present some fundamental properties of a minimal neighborhood.

Lemma 3.3. *If* $\mathbf{n} \in \mathbb{M}(\mathbf{m})$*, then* $\mathbb{M}(\mathbf{n}) \subseteq \mathbb{M}(\mathbf{m})$*,* $\forall\ \mathbf{m}, \mathbf{n} \in \mathcal{U}$.

Proof. Firstly, if $\mathbf{n} \in \mathbb{M}(\mathbf{m})$, then $\mathbf{n} \in \aleph(\mathbf{m})$, $\forall\ \mathbf{m}$. Now, let $\mathbf{o} \in \mathbb{M}(\mathbf{n})$. Then, $\mathbf{o} \in \aleph(\mathbf{n})$, $\forall\ \mathbf{n}$ and so $\mathbf{o} \in \aleph(\mathbf{m})$, $\forall\ \mathbf{m}$. Hence, $\mathbf{o} \in \mathbb{M}(\mathbf{m})$, and then $\mathbb{M}(\mathbf{n}) \subseteq \mathbb{M}(\mathbf{m})$. □

Lemma 3.4. *Let* $\aleph(\mathbf{m})$ *be inverse serial. Then,* $\mathbb{M}(\mathbf{m}) \neq \phi$*,* $\forall\ \mathbf{m} \in \mathcal{U}$.

Proof. By assumption, we have $\mathbf{m} \in \aleph(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$. Now, let $\mathbf{n} \in \mathbb{M}(\mathbf{m})$. Then, $\mathbf{m} \in \aleph(\mathbf{n})$, which means that $\mathbf{n} \in \aleph(\mathbf{m})$. Therefore, $\mathbb{M}(\mathbf{m}) \subseteq \aleph(\mathbf{m})$. □

Lemma 3.5. *Let* $\aleph(\mathbf{m})$ *be reflexive. Then,* $\mathbb{M}(\mathbf{m}) \subseteq \aleph(\mathbf{m})$*,* $\forall\ \mathbf{m} \in \mathcal{U}$.

Proof. By assumption, we get $\mathbf{m} \in \aleph(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$. If $\mathbf{n} \in \mathbb{M}(\mathbf{m})$, then $\mathbf{m} \in \aleph(\mathbf{n})$, which means that $\mathbf{n} \in \aleph(\mathbf{m})$. Hence, $\mathbb{M}(\mathbf{m}) \subseteq \aleph(\mathbf{m})$. □

In general, the converse of Lemma 3.5 is false.

Example 3.6. *(Continued for Example 2.3). Take* $\aleph(\lambda) = \{\lambda, \mu\}$*,* $\aleph(\mu) = \{\lambda, \mu\}$*,* $\aleph(\nu) = \{\lambda, \mu, \nu\}$ *and* $\aleph(\epsilon) = \{\mu, \nu, \epsilon\}$*. Then,* $\mathbb{M}(\lambda) = \{\lambda\}$*,* $\mathbb{M}(\mu) = \{\lambda, \mu\}$*,* $\mathbb{M}(\nu) = \{\mu, \nu\}$ *and* $\mathbb{M}(\epsilon) = \{\mu, \nu, \epsilon\}$*. Clearly,* $(\mathbf{m}) \subseteq \aleph(\mathbf{m})$*,* $\forall\ \mathbf{m} \in \mathcal{U}$.

Definition 3.7. *A core minimal neighborhood of* $\mathbf{m}$ *is* $\mathbb{CM}(\mathbf{m}) = \{\mathbf{n} \in \mathcal{U} : \mathbb{M}(\mathbf{m}) = \mathbb{M}(\mathbf{n})\}$.

The proofs of some subsequent results are simple to examine, so we omit them.

Lemma 3.8. $\forall\ \mathbf{m}, \mathbf{n} \in \mathcal{U}$*,* $\mathbf{n} \in \mathbb{CM}(\mathbf{m})$ *iff* $\mathbf{m} \in \mathbb{CM}(\mathbf{n})$.

Corollary 3.9. *If* $\mathbf{n} \in \mathbb{CM}(\mathbf{m})$*, then* $\mathbb{CM}(\mathbf{n}) = \mathbb{CM}(\mathbf{m})$*,* $\forall\ \mathbf{m}, \mathbf{n} \in \mathcal{U}$.

Lemma 3.10. *The following hold on* $\mathcal{U}$*:*

(i) $\mathbf{m} \in \mathbb{CM}(\mathbf{m})$*,* $\forall\ \mathbf{m} \in \mathcal{U}$*.*

(ii) $\{\mathbb{CM}(\mathbf{m}) : \mathbf{m} \in \mathcal{U}\}$ *is a partition.*

Lemma 3.11. *Let* $\aleph(\mathbf{m})$ *be inverse serial. Then,* $\forall\ \mathbf{m} \in \mathcal{U}$*,* $\mathbb{CM}(\mathbf{m}) \subseteq \mathbb{M}(\mathbf{m})$*.*

Proof. By assumption, we have $\mathbb{M}(\mathbf{m}) \neq \phi$ and $\mathbf{m} \in \mathbb{M}(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$. Now, let $\mathbf{o} \in \mathbb{CM}(\mathbf{m})$. So, $\mathbb{M}(\mathbf{m}) = \mathbb{M}(\mathbf{o})$ and since $\mathbf{o} \in \mathbb{M}(\mathbf{o})$, then $\mathbf{o} \in \mathbb{M}(\mathbf{m})$, and so $\mathbb{CM}(\mathbf{m}) \subseteq \mathbb{M}(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$. □

Corollary 3.12. *Let* $\aleph(\mathbf{m})$ *be reflexive. Then,* $\mathbb{CM}(\mathbf{m}) \subseteq \mathbb{M}(\mathbf{m})$*,* $\forall\ \mathbf{m} \in \mathcal{U}$*.*

Lemma 3.13. *Let* $\aleph(\mathbf{m})$ *be reflexive. Then,* $\mathbb{CM}(\mathbf{m}) \subseteq \mathbb{C}\aleph(\mathbf{m})$*,* $\forall\ \mathbf{m} \in \mathcal{U}$*.*

Proof. By assumption, we get $\mathbf{m} \in \aleph(\mathbf{m})$, $\mathbf{m} \in \mathbb{M}(\mathbf{m})$ and $\mathbb{M}(\mathbf{m}) \subseteq \mathbb{M}(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$. If $\mathbf{o} \in \mathbb{CM}(\mathbf{m})$, then $\mathbb{M}(\mathbf{m}) = \mathbb{M}(\mathbf{o})$. By reflexivity of $\mathbb{M}$, we have $\mathbf{o} \in \mathbb{M}(\mathbf{m})$ and so $\mathbf{o} \in \aleph(\mathbf{m})$. But, $\mathbf{m} \in \aleph(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$, and so $\aleph(\mathbf{m}) = \aleph(\mathbf{o})$. Therefore, $\mathbf{o} \in \mathbb{C}\aleph(x)$ and hence $\mathbb{CM}(x) \subseteq \mathbb{C}\aleph(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$. □

The connection between various neighborhoods in the case of reflexive neighborhoods is elucidated.

Proposition 3.14. *Let* $\aleph(\mathbf{m})$ *be reflexive. Then:*

(i) $\mathbb{CM}(\mathbf{m}) \subseteq \mathbb{C}\aleph(\mathbf{m}) \subseteq \aleph(\mathbf{m})$*; and*

(ii) $\mathbb{CM}(\mathbf{m}) \subseteq \mathbb{M}(\mathbf{m}) \subseteq \aleph(\mathbf{m})$*,* $\forall\ \mathbf{m} \in \mathcal{U}$*.*

4. Minimal rough sets via neighborhoods and their cores

In this section, we introduce a generalization of Pawlak approximations and conduct a comprehensive investigation into some of their properties.

Definition 4.1. *Let us define the minimal lower and upper approximations of* $\mathbf{A}$ *as follows:*

$$\underline{apr}_{\mathbb{M}}(\mathbf{A}) = \{\mathbf{m} \in \mathcal{U} : \mathbb{M}(\mathbf{m}) \subseteq \mathbf{A}\} \ (\textit{resp. } \overline{apr}_{\mathbb{M}}(\mathbf{A}) = \{\mathbf{m} \in \mathcal{U} : \mathbb{M}(\mathbf{m}) \cap \mathbf{A} \neq \phi\}).$$

Proposition 4.2. *The following are satisfied for* $\mathfrak{X}, \mathfrak{Y} \subseteq \mathcal{U}$*:*

(3) $\underline{apr}_{\mathbb{M}}(\mathcal{U}) = \mathcal{U}$; $\overline{apr}_{\mathbb{M}}(\phi) = \phi$.

(4) $\underline{apr}_{\mathbb{M}}(\mathfrak{X} \cap \mathfrak{Y}) = \underline{apr}_{\mathbb{M}}(\mathfrak{X}) \cap \underline{apr}_{\mathbb{M}}(\mathfrak{Y})$.

(5) If $\mathfrak{X} \subseteq \mathfrak{Y}$*, then* $\underline{apr}_{\mathbb{M}}(\mathfrak{X}) \subseteq \underline{apr}_{\mathbb{M}}(\mathfrak{Y})$.

(6) $\underline{apr}_{\mathbb{M}}(\mathfrak{X}) \cup \underline{apr}_{\mathbb{M}}(\mathfrak{Y}) \subseteq \underline{apr}_{\mathbb{M}}(\mathfrak{X} \cup \mathfrak{Y})$.

(7) $\underline{apr}_{\mathbb{M}}(\mathfrak{X}^c) = (\overline{apr}_{\mathbb{M}}(\mathfrak{X}))^c$.

(8) $\underline{apr}_{\mathbb{M}}(\underline{apr}_{\mathbb{M}}(\mathfrak{X})) = \underline{apr}_{\mathbb{M}}(\mathfrak{X})$.

(4)* $\overline{apr}_{\mathbb{M}}(\mathfrak{X} \cup \mathfrak{Y}) = \overline{apr}_{\mathbb{M}}(\mathfrak{X}) \cup \overline{apr}_{\mathbb{M}}(\mathfrak{Y})$.

(5) If* $\mathfrak{X} \subseteq \mathfrak{Y}$*, then* $\overline{apr}_{\mathbb{M}}(\mathfrak{X}) \subseteq \overline{apr}_{\mathbb{M}}(\mathfrak{Y})$.

(6)* $\overline{apr}_{\mathbb{M}}(\mathfrak{X}) \cap \overline{apr}_{\mathbb{M}}(\mathfrak{Y}) \supseteq \overline{apr}_{\mathbb{M}}(\mathfrak{X} \cap \mathfrak{Y})$.

(7)* $\overline{apr}_{\mathbb{M}}(\mathfrak{X}^c) = (\underline{apr}_{\mathbb{M}}(\mathfrak{X}))^c$.

(8)* $\overline{apr}_{\mathbb{M}}(\overline{apr}_{\mathbb{M}}(\mathfrak{X})) = \overline{apr}_{\mathbb{M}}(\mathfrak{X})$.

Proof. The properties (3–6) and (2*–6*) are straightforward. The other properties are proven as follows:

(7) $\underline{apr}_{\mathbb{M}}(\mathfrak{X}^c) = (\{\mathbf{m} \in \mathcal{U} : \mathbb{M}(\mathbf{m}) \cap \mathfrak{X} \neq \phi\})^c = \{\mathbf{m} \in \mathcal{U} : \mathbb{M}(\mathbf{m}) \cap \mathfrak{X} = \phi\} = \{\mathbf{m} \in \mathfrak{X}^c : \mathbb{M}(\mathbf{m}) \subseteq \mathfrak{X}^c\} = (\overline{apr}_{\mathbb{M}}(\mathfrak{X}))^c$.

(8) First, $\underline{apr}_{\mathbb{M}}(\underline{apr}_{\mathbb{M}}(\mathfrak{X})) \subseteq \underline{apr}_{\mathbb{M}}(\mathfrak{X})$, by ($L5$). Now, let $\mathbf{m} \in \underline{apr}_{\mathbb{M}}(\mathfrak{X})$, then $\mathbf{m} \in \mathfrak{X}$ and $\mathbb{M}(\mathbf{m}) \subseteq \mathfrak{X}$. It is needed to prove that $\mathbb{M}(\mathbf{m}) \subseteq \underline{apr}_{\mathbb{M}}(\mathfrak{X})$ as follows:

Let $\mathbf{n} \in \mathbb{M}(\mathbf{m})$, then $\mathbb{M}(\mathbf{n}) \subseteq \mathbb{M}(\mathbf{m})$ and thus $\mathbb{M}(\mathbf{n}) \subseteq \mathfrak{X}$. So, $\mathbf{n} \in \underline{apr}_{\mathbb{M}}(\mathfrak{X})$ and so $\mathbb{M}(\mathbf{n}) \subseteq \underline{apr}_{\mathbb{M}}(\mathfrak{X})$. Thus, $\mathbf{m} \in \underline{apr}_{\mathbb{M}}(\underline{apr}_{\mathbb{M}}(\mathfrak{X}))$ and hence $\underline{apr}_{\mathbb{M}}(\mathfrak{X}) \subseteq \underline{apr}_{\mathbb{M}}(\underline{apr}_{\mathbb{M}}(\mathfrak{X}))$.
Similarly, $(\overline{U}7)$ and $(\overline{U}8)$ are proven. □

As a general observation, properties (1), (2), (1*) and (3*) do not hold.

Example 4.3. *(Continued from Example 2.3). Put* $\aleph(\lambda) = \{\lambda, \mu\}$, $\aleph(\mu) = \aleph(\nu) = \{\mu, \nu\}$, $\aleph(\epsilon) = \phi$. *Directly,* $\mathbb{M}(\lambda) = \{\lambda, \mu\}$, $\aleph(\mu) = \{\mu\}$, $\aleph(\nu) = \{\mu, \nu\}$ *and* $\aleph(\epsilon) = \phi$. *Take* $\mathbf{A} = \{\lambda, \mu, \nu\}$ *and* $\mathbf{B} = \{\lambda, \nu, \epsilon\}$. *Then,* $\underline{apr}_{\mathbb{M}}(\mathbf{A}) = \mathcal{U} \nsubseteq \mathbf{A}$, *and* $\overline{apr}_{\mathbb{M}}(\mathbf{B}) = \{\lambda, \nu\} \nsupseteq \mathbf{B}$. *Furthermore,* $\underline{apr}_{\mathbb{M}}(\phi) = \{\epsilon\} \neq \phi$ *and* $\overline{apr}_{\mathbb{M}}(\mathcal{U}) = \{\lambda, \mu, \nu\} \neq \mathcal{U}$.

Proposition 4.4. *Let* $\aleph(\mathbf{m})$ *be inverse serial. Then, for* $\mathfrak{X} \subseteq \mathcal{U}$ *and* $\mathbf{m} \in \mathcal{U}$*:*
(1) $\underline{apr}_{\mathbb{M}}(\mathfrak{X}) \subseteq \mathfrak{X}$. *(1*)* $\mathfrak{X} \subseteq \overline{apr}_{\mathbb{M}}(\mathfrak{X})$.
(2) $\underline{apr}_{\mathbb{M}}(\phi) = \phi$. *(3*)* $\overline{apr}_{\mathbb{M}}(\mathcal{U}) = \mathcal{U}$.

Proof. Given the assumption that $\mathbf{m} \in \mathbb{M}(\mathbf{m})$ for all $\mathbf{m} \in \mathcal{U}$, it becomes necessary to focus on proving property (1). The other properties can be established using similar methods. Let $\mathbf{m} \in \underline{apr}_{\mathbb{M}}(\mathfrak{X})$. Then, $\mathbb{M}(\mathbf{m}) \subseteq \mathfrak{X}$ and so $\mathbf{m} \in \mathfrak{X}$. Therefore, $\underline{apr}_{\mathbb{M}}(\mathfrak{X}) \subseteq \mathfrak{X}$. □

Remark 4.5. *Proposition 4.4 demonstrates that only inverse serial neighborhoods satisfy all of Pawlak's properties. Consequently, minimal approximations are a more comprehensive generalization of rough approximations.*

Definition 4.6. *Let us define the core minimal lower (resp. upper) approximation of* $\mathbf{A}$ *as follows:*

$$\underline{apr}_{\mathbb{CM}}(\mathbf{A}) = \bigcup\{\mathbb{CM}(\mathbf{m}) : \mathbb{CM}(\mathbf{m}) \subseteq \mathbf{A}\}, \text{ and}$$
$$\overline{apr}_{\mathbb{CM}}(\mathbf{A}) = \bigcup\{\mathbb{CM}(\mathbf{m}) : \mathbb{CM}(\mathbf{m}) \cap \mathbf{A} \neq \phi\}, \text{ for } \mathbf{A} \subseteq \mathcal{U}.$$

The properties in Proposition 4.7 are simple to prove using Lemma 3.3, so we skip that step.

Proposition 4.7. *For* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *belonging to* $\mathcal{U}$, *the following properties are satisfied:*

(1) $\underline{apr}_{\mathbb{CM}}(\mathfrak{X}) \subseteq \mathfrak{X}$.
(2) $\underline{apr}_{\mathbb{CM}}(\phi) = \phi$.
(3) $\underline{apr}_{\mathbb{CM}}(\mathcal{U}) = \mathcal{U}$.
(4) $\underline{apr}_{\mathbb{CM}}(\mathfrak{X} \cap \mathfrak{Y}) = \underline{apr}_{\mathbb{CM}}(\mathfrak{X}) \cap \underline{apr}_{\mathbb{CM}}(\mathfrak{Y})$.
(5) If $\mathfrak{X} \subseteq \mathfrak{Y}$, *then* $\underline{apr}_{\mathbb{CM}}(\mathfrak{X}) \subseteq \underline{apr}_{\mathbb{CM}}(\mathfrak{Y})$.
(6) $\underline{apr}_{\mathbb{CM}}(\mathfrak{X}) \cup \underline{apr}_{\mathbb{CM}}(\mathfrak{Y}) \subseteq \underline{apr}_{\mathbb{CM}}(\mathfrak{X} \cup \mathfrak{Y})$.
(7) $\underline{apr}_{\mathbb{CM}}(\mathfrak{X}^c) = (\overline{apr}_{\mathbb{CM}}(\mathfrak{X}))^c$
(8) $\underline{apr}_{\mathbb{CM}}(\underline{apr}_{\mathbb{CM}}(\mathfrak{X})) = \underline{apr}_{\mathbb{CM}}(\mathfrak{X})$.

(1)* $\mathfrak{X} \subseteq \overline{apr}_{\mathbb{CM}}(\mathfrak{X})$.
(2)* $\overline{apr}_{\mathbb{CM}}(\phi) = \phi$.
(3)* $\overline{apr}_{\mathbb{CM}}(\mathcal{U}) = \mathcal{U}$.
(4)* $\overline{apr}_{\mathbb{CM}}(\mathfrak{X} \cup \mathfrak{Y}) = \overline{apr}_{\mathbb{CM}}(\mathfrak{X}) \cup \overline{apr}_{\mathbb{CM}}(\mathfrak{Y})$.
(5) If* $\mathfrak{X} \subseteq \mathfrak{Y}$, *then* $\overline{apr}_{\mathbb{CM}}(\mathfrak{X}) \subseteq \overline{apr}_{\mathbb{CM}}(\mathfrak{Y})$.
(6)* $\overline{apr}_{\mathbb{CM}}(\mathfrak{X}) \cap \overline{apr}_{\mathbb{CM}}(\mathfrak{Y}) \supseteq \overline{apr}_{\mathbb{CM}}(\mathfrak{X} \cap \mathfrak{Y})$.
(7)* $\overline{apr}_{\mathbb{CM}}(\mathfrak{X}^c) = (\underline{apr}_{\mathbb{CM}}(\mathfrak{X}))^c$
(8)* $\overline{apr}_{\mathbb{CM}}(\overline{apr}_{\mathbb{CM}}(\mathfrak{X})) = \overline{apr}_{\mathbb{CM}}(\mathfrak{X})$.

Four different pairs of rough approximations via $\aleph(\mathbf{m})$, $\mathbb{M}(\mathbf{m})$, $c\aleph(\mathbf{m})$ and $c\mathbb{M}(\mathbf{m})$ are given. As shown in Definition 4.8, $\mathcal{U}$ can thus be partitioned into many regions.

Definition 4.8. $\forall\ \ell \in \{\aleph, \mathbb{M}, c\aleph, c\mathbb{M}\}$, *($\ell$-boundary, ℓ-positive and ℓ-negative) regions and ℓ-accuracy of the approximations of* $\mathbf{A} \subseteq \mathcal{U}$ *are defined by*

$$\mathbf{B}_{\ell}(\mathbf{A}) = \overline{apr}_{\ell}(\mathbf{A}) - \underline{apr}_{\ell}(\mathbf{A}),$$
$$\mathbf{POS}_{\ell}(\mathbf{A}) = \underline{apr}_{\ell}(\mathbf{A}),$$
$$\mathbf{NEG}_{\ell}(\mathbf{A}) = \mathcal{U} - \overline{apr}_{\ell}(\mathbf{A}), \text{ and}$$

$\mu_\ell(\mathbf{A}) = \frac{|\underline{apr}_\ell(\mathbf{A})|}{|\overline{apr}_\ell(\mathbf{A})|}$, *where* $|\overline{apr}_\ell(\mathbf{A})| \neq 0$, *respectively.*
It is evident that $0 \leq \mu_\ell(\mathbf{A}) \leq 1$. *When* $\mu_\ell(\mathbf{A}) = 1$, $\mathbf{A}$ *is referred to as* ℓ*-exact. On the other hand, if* $\mu_\ell(\mathbf{A}) \neq 1$, $\mathbf{A}$ *is classified as* ℓ*-rough.*

It is important to highlight that Definition 4.8 holds under the following conditions:

(a) For $\ell \in \{\mathbb{C}\aleph, \mathbb{CM}\}$.

(b) For $\ell = \aleph$*, provided the* $\aleph$*-neighborhood is reflexive and transitive.*

(c) For $\ell \in \mathbb{M}$*, in case the* $\mathbb{M}$*-neighborhood is inverse serial.*

Proposition 4.9. *Assume that* $\aleph(\mathbf{m})$ *is inverse serial. Thus, for all* $\mathbf{m} \in \mathcal{U}$:

$$\underline{apr}_{\mathbb{M}}(\mathbf{A}) \subseteq \underline{apr}_{\mathbb{CM}}(\mathbf{A}) \subseteq \mathbf{A} \subseteq \overline{apr}_{\mathbb{CM}}(\mathbf{A}) \subseteq \overline{apr}_{\mathbb{M}}(\mathbf{A}), \text{ for } \mathbf{A} \subseteq \mathcal{U}.$$

Proof. Let $\mathbf{m} \in \underline{apr}_{\mathbb{M}}(\mathbf{A})$. Then, $\mathbb{M}(\mathbf{m}) \subseteq \mathbf{A}$ and so $\mathbb{CM}(\mathbf{m}) \subseteq \mathbf{A}$, by Lemma 3.3. Hence, $\mathbf{m} \in \underline{apr}_{\mathbb{CM}}(\mathbf{A})$ and so $\underline{apr}_{\mathbb{M}}(\mathbf{A}) \subseteq \underline{apr}_{\mathbb{CM}}(\mathbf{A})$. Similarly, $\overline{apr}_{\mathbb{CM}}(\mathbf{A}) \subseteq \overline{apr}_{\mathbb{M}}(\mathbf{A})$ □

Corollary 4.10. *Assume that* $\aleph(\mathbf{m})$ *is inverse serial. Therefore,* $\forall\ \mathbf{m} \in \mathcal{U}$

(i) $\mu_{\mathbb{M}}(\mathbf{A}) \leq \mu_{\mathbb{CM}}(\mathbf{A})$.

(ii) $\mathbf{B}_{\mathbb{CM}}(\mathbf{A}) \subseteq \mathbf{B}_{\mathbb{M}}(\mathbf{A})$*, for* $\mathbf{A} \subseteq \mathcal{U}$.

Corollary 4.11. *Let* $\aleph(\mathbf{m})$ *be inverse serial. Then,* $\mathbf{A}$ *is* $\mathbb{CM}$*-exact if it is* $\mathbb{M}$*-exact, for* $\mathbf{A} \subseteq \mathcal{U}$ *and* $\forall$ $\mathbf{m} \in \mathcal{U}$*. The reverse does not hold, in general.*

Example 4.12. *(Continued from Example 2.3). Take* $\aleph(\lambda) = \{\lambda, \mu\}$, $\aleph(\mu) = \{\lambda\}$, $\aleph(\nu) = \{\nu, \epsilon\}$ *and* $\aleph(\epsilon) = \{\lambda, \mu\}$*. Then,* $\mathbb{M}(\lambda) = \{\lambda\}$, $\mathbb{M}(\mu) = \{\lambda, \mu\}$ *and* $\mathbb{M}(\nu) = \mathbb{M}(\epsilon) = \{\nu, \epsilon\}$*. So,* $\mathbb{CM}(\lambda) = \{\lambda\}$, $\mathbb{CM}(\mu) = \{\mu\}$ *and* $\mathbb{CM}(\nu) = \mathbb{CM}(\epsilon) = \{\nu, \epsilon\}$*. Take* $\mathbf{A} = \{\mu\}$ *and* $\mathbf{B} = \{\lambda, \mu\}$*. Thus,* $\underline{apr}_{\mathbb{M}}(\mathbf{B}) = \overline{apr}_{\mathbb{M}}(\mathbf{B}) = \{\lambda, \mu\}$ *and* $\underline{apr}_{\mathbb{CM}}(\mathbf{B}) = \overline{apr}_{\mathbb{CM}}(\mathbf{B}) = \{\lambda, \mu\}$*. Also,* $\underline{apr}_{\mathbb{M}}(\mathbf{A}) = \phi$ *and* $\overline{apr}_{\mathbb{M}}(\mathbf{A}) = \{\mu\}$*, but* $\underline{apr}_{\mathbb{M}}(\mathbf{A}) = \overline{apr}_{\mathbb{M}}(\mathbf{A}) = \{\mu\}$*. Clearly*

(i) $\mathbf{B}_{\mathbb{M}}(\mathbf{A}) = \mathbf{B}_{\mathbb{CM}}(\mathbf{A}) = \phi$ *and* $\mu_{\mathbb{M}}(\mathbf{B}) = \mu_{\mathbb{CM}}(\mathbf{B}) = 1$. $\mathbf{B}$ *is* $\mathbb{M}$ *(resp.* $\mathbb{CM}$*)-exact .*

(ii) $\mathbf{B}_{\mathbb{M}}(\mathbf{B}) = \{\lambda, \mu\}$ *and* $\mu_{\mathbb{M}}(\mathbf{B}) = 0$*. But,* $\mathbf{B}_{\mathbb{CM}}(\mathbf{B}) = \phi$ *and* $\mu_{\mathbb{CM}}(\mathbf{B}) = 1$*. So,* $\mathbf{A}$ *is* $\mathbb{CM}$*-exact, but not* $\mathbb{M}$*-exact.*

Proposition 4.13 can be readily established by utilizing Corollary 4.14, rendering its proof unnecessary.

Proposition 4.13. *Let* $\aleph(\mathbf{m})$ *be reflexive. Then,* $\forall$ $\mathbf{A} \subseteq \mathcal{U}$ *and* $\mathbf{m} \in \mathcal{U}$

(i) $\underline{apr}_{\aleph}(\mathbf{A}) \subseteq \underline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) \subseteq \underline{apr}_{\mathbb{CM}}(\mathbf{A}) \subseteq \mathbf{A} \subseteq \overline{apr}_{\mathbb{CM}}(\mathbf{A}) \subseteq \overline{apr}_{\aleph}(\mathbf{A}) \subseteq \overline{apr}_{\aleph}(\mathbf{A})$.

(ii) $\underline{apr}_{\aleph}(A) \subseteq \underline{apr}_{\mathbb{M}}(\mathbf{A}) \subseteq \underline{apr}_{\mathbb{CM}}(\mathbf{A}) \subseteq \mathbf{A} \subseteq \overline{apr}_{\mathbb{CM}}(\mathbf{A}) \subseteq \overline{apr}_{\mathbb{M}}(\mathbf{A}) \subseteq \overline{apr}_{\aleph}(\mathbf{A})$.

Corollary 4.14. *Let* $\aleph(\mathbf{m})$ *be reflexive. Then,* $\forall$ $\mathbf{A} \subseteq \mathcal{U}$ *and* $\mathbf{m} \in \mathcal{U}$

(i) $\mu_{\aleph}(\mathbf{A}) \leq \mu_{\mathbb{C}\aleph}(\mathbf{A}) \leq \mu_{\mathbb{CM}}(\mathbf{A})$. *(ii)* $\mu_{\aleph}(\mathbf{A}) \leq \mu_{\mathbb{M}}(\mathbf{A}) \leq \mu_{\mathbb{CM}}(\mathbf{A})$.

(iii) $\mathbf{B}_{\mathbb{CM}}(\mathbf{A}) \subseteq \mathbf{B}_{\mathbb{C}\aleph}(\mathbf{A}) \subseteq \mathbf{B}_{\aleph}(\mathbf{A})$. *(iv)* $\mathbf{B}_{\mathbb{CM}}(\mathbf{A}) \subseteq \mathbf{B}_{\mathbb{M}}(\mathbf{A}) \subseteq \mathbf{B}_{\aleph}(\mathbf{A})$.

Corollary 4.15. *Let* $\aleph(\mathbf{m})$ *be reflexive. Then, (i)* $\mathbb{CM}$*-exact* $\Longrightarrow$ $\mathbb{C}\aleph$*-exact* $\Longrightarrow$ $\aleph$*-exact.*
(ii) $\mathbb{CM}$*-exact* $\Longrightarrow$ $\mathbb{M}$*-exact* $\Longrightarrow$ $\aleph$*-exact.*

Generally, the reverse implications are not valid.

Example 4.16. *(Continued from Example 2.3). Assume that* $\aleph(\lambda) = \aleph(\mu) = \{\lambda, \mu\}$, $\aleph(\nu) = \{\lambda, \mu, \nu\}$ *and* $\aleph(\epsilon) = \{\mu, \nu, \epsilon\}$. *Then,* $\mathbb{C}\aleph(\lambda) = \mathbb{C}\aleph(\mu) = \{\lambda, \mu\}$, $\mathbb{C}\aleph(\nu) = \{\nu\}$ *and* $\mathbb{C}\aleph(\epsilon) = \{\epsilon\}$. *Also,* $\mathbb{M}(\lambda) = \{\lambda, \mu\}$, $\mathbb{M}(\mu) = \{\mu\}$, $\mathbb{M}(\nu) = \{\mu, \nu\}$ *and* $\mathbb{M}(\epsilon) = \{\mu, \nu, \epsilon\}$. *Thus,* $\mathbb{CM}(\lambda) = \{\lambda\}$, $\mathbb{CM}(\mu) = \{\mu\}$ *and* $\mathbb{CM}(\nu) = \{\nu\}$, $\mathbb{CM}(\epsilon) = \{\epsilon\}$. *Take* $\mathbf{A} = \{\lambda, \nu\}$. *Then,* $\underline{apr}_{\aleph}(\mathbf{A}) = \phi$, $\overline{apr}_{\aleph}(\mathbf{A}) = \mathcal{U}$, $\underline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) = \{\nu\}$ *and* $\overline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) = \{\lambda, \mu, \nu\}$. *But,* $\underline{apr}_{\mathbb{CM}}(\mathbf{A}) = \overline{apr}_{\mathbb{CM}}(\mathbf{A}) = \mathbf{A}$. *It is clear that* $\mathbf{B}_{\aleph}(\mathbf{A}) = \mathcal{U}$, $\mathbf{B}_{\mathbb{C}\aleph}(\mathbf{A}) = \{\lambda, \mu\}$, $\mu_{\aleph}(\mathbf{A}) = 0$ *and* $\mu_{\mathbb{C}\aleph}(\mathbf{A}) = \frac{1}{3}$. *Thus,* $\mathbf{A}$ *is* $\aleph$*-rough and also* $\mathbb{C}\aleph$*-rough. But,* $\mathbf{B}_{\mathbb{CM}}(\mathbf{A}) = \phi$, *and* $\mu_c\mathbb{M}(\mathbf{A}) = 1$. *Therefore,* $\mathbf{A}$ *is* $\mathbb{CM}$*-exact.*

Remark 4.17. *Proposition 4.13 demonstrates that the core minimal approach offers a more inclusive generalization compared to other approaches when considering a reflexive neighborhood condition. Additionally, it provides a broader generalization of Pawlak's rough approach without any limitations.*

5. A comparative study of constructed generalized rough sets using the 1-NS approach

In this study, we explore the connections between the proposed approximations in Section 4 and alternative methods discussed in [46,54,55,60]. To this end, we introduce four distinct approximation operators for rough sets denoted by $\underline{apr}_{\ell}$ and $\overline{apr}_{\ell}$ takes values from the set $\ell \in \{\aleph, \mathbb{M}, c\aleph, c\mathbb{M}\}$. Notably, when ℓ belongs to $\{c\aleph, c\mathbb{M}\}$, these approximations exhibit full compliance with Pawlak's properties.

5.1. The first approach of 1-NS rough sets

Definition 5.1. *We define the generalized lower (and upper) approximations of set* $\mathbf{A}$ *as follows:* $\mathbb{L}(\mathbf{A}) = \{\mathbf{m} \in \mathcal{U} : (\mathbb{C}\aleph(\mathbf{m}) \subseteq \mathbf{A}) \vee (\mathbb{CM}(\mathbf{m}) \subseteq \mathbf{A})\}$ *(and* $\mathbb{U}(\mathbf{A}) = \{\mathbf{m} \in \mathcal{U} : (\mathbb{C}\aleph(\mathbf{m}) \cap \mathbf{A} \neq \phi) \vee (\mathbb{CM}(\mathbf{m}) \cap \mathbf{A}) \neq \phi\}$*), for* $\mathbf{A} \subseteq \mathcal{U}$.

Definition 5.2. *We define the generalized boundary, positive region, negative region, and accuracy of set* $\mathbf{A}$ *as follows:*

- $\mathbb{BN}(\mathbf{A}) = \mathbb{U}(\mathbf{A}) - \mathbb{L}(\mathbf{A})$.
- $\mathbb{POS}(\mathbf{A}) = \mathbb{L}(\mathbf{A})$.
- $\mathbb{NEG}(\mathbf{A}) = \mathcal{U} - \mathbb{U}(\mathbf{A})$.
- $\sigma(\mathbf{A}) = \frac{|\mathbb{L}(\mathbf{A})|}{|\mathbb{U}(\mathbf{A})|}$, *where* $|\mathbb{U}(\mathbf{A})| \neq 0$, *respectively, for* $\mathbf{A} \subseteq \mathcal{U}$.

Theorem 5.3. *The following hold,* $\forall A \subseteq \mathcal{U}$:

(i) $\underline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) \subseteq \mathbb{L}(\mathbf{A}) \subseteq \mathbf{A} \subseteq \mathbb{U}(\mathbf{A}) \subseteq \overline{apr}_{\mathbb{C}\aleph}(\mathbf{A})$.

(ii) $\underline{apr}_{\mathbb{CM}}(\mathbf{A}) \subseteq \mathbb{L}(\mathbf{A}) \subseteq \mathbf{A} \subseteq \mathbb{U}(\mathbf{A}) \subseteq \overline{apr}_{\mathbb{CM}}(\mathbf{A})$.

(iii) $\mathbb{L}(\mathbf{A}) = \underline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) \cup \underline{apr}_{\mathbb{CM}}(\mathbf{A})$.

(iv) $\mathbb{U}(\mathbf{A}) = \overline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) \cap \overline{apr}_{\mathbb{CM}}(\mathbf{A})$.

Proof. **(i)** Let $\mathbf{m} \in \underline{apr}_{\mathbb{C}\aleph}(\mathbf{A})$. Then, $\mathbb{C}\aleph(\mathbf{m}) \subseteq \mathbf{A}$ and so $\mathbf{m} \in \mathbb{L}(\mathbf{A})$. Accordingly, $\underline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) \subseteq \mathbb{L}(\mathbf{A})$. If $\mathbf{m} \in \mathbb{L}(\mathbf{A})$, then $(\mathbb{C}\aleph(\mathbf{m}) \subseteq \mathbf{A}) \vee (\mathbb{CM}(\mathbf{m}) \subseteq \mathbf{A})$ and since $\mathbf{m} \in \mathbb{C}\aleph(\mathbf{m})$ and $\mathbf{m} \in \mathbb{CM}(\mathbf{m})$, $\forall\, \mathbf{m} \in \mathcal{U}$. Then, $\mathbf{m} \in \mathbf{A}$, and so $\mathbb{L}(\mathbf{A}) \subseteq \mathbf{A}$. Similarly, $\mathbf{A} \subseteq \mathbb{U}(\mathbf{A}) \subseteq \overline{apr}_{\mathbb{C}\aleph}(\mathbf{A})$ can be proven.
(ii) can be proven in a similar way to **(i)**.
(iii) and **(iv)** are clear. □

Corollary 5.4. *The following statements hold for all subsets* $\mathbf{A}$ *of* $\mathcal{U}$*:*
(i) $\mathbb{BN}(\mathbf{A}) \subseteq \mathbf{B}_{\mathbb{C}\aleph}(\mathbf{A})$. *(iii)* $\mu_{\mathbb{C}\aleph}(\mathbf{A}) \leq \sigma(\mathbf{A})$.
(ii) $\mathbb{BN}(\mathbf{A}) \subseteq \mathbf{B}_{\mathbb{CM}}(\mathbf{A})$. *(iv)* $\mu_{\mathbb{CM}}(\mathbf{A}) \leq \sigma(\mathbf{A})$.

5.2. The second approach of 1-*NS* rough sets

Definition 5.5. *We define the generalized lower (and upper) approximations of set* $\mathbf{A}$ *as follows:* $\underline{\underline{apr}}(\mathbf{A}) = \underline{apr}_{\aleph}(\mathbf{A}) \cup \underline{apr}_{\mathbb{M}}(\mathbf{A}) \cup \underline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) \cup \underline{apr}_{\mathbb{CM}}(\mathbf{A})$ *(resp.* $\overline{\overline{apr}}(\mathbf{A}) = \overline{apr}_{\aleph}(\mathbf{A}) \cap \overline{apr}_{\mathbb{M}}(\mathbf{A}) \cap \overline{apr}_{\mathbb{C}\aleph}(\mathbf{A}) \cap \overline{apr}_{\mathbb{CM}}(\mathbf{A})$*), for* $\mathbf{A} \subseteq \mathcal{U}$*.*

Definition 5.6. *We define the generalized boundary, positive region, negative region and accuracy of set* $\mathbf{A}$ *as follows:*

- $\mu(\mathbf{A}) = \overline{\overline{apr}}(\mathbf{A}) - \underline{\underline{apr}}(\mathbf{A})$.
- $\mathcal{POS}(\mathbf{A}) = \underline{\underline{apr}}(\mathbf{A})$.
- $\mathcal{NEG}(\mathbf{A}) = \mathcal{U} - \overline{\overline{apr}}(\mathbf{A})$.
- $\rho(\mathbf{A}) = \frac{|\underline{\underline{apr}}(\mathbf{A})|}{|\overline{\overline{apr}}(\mathbf{A})|}$, *respectively, such that* $|\overline{\overline{apr}}(\mathbf{A})| \neq 0$.

Theorem 5.7. *For each* $\ell \in \{\aleph, \mathbb{M}, \mathbb{C}\aleph, \mathbb{CM}\}$, $\underline{apr}_{\ell}(\mathbf{A}) \subseteq \underline{\underline{apr}}(\mathbf{A}) \subseteq \mathbf{A} \subseteq \overline{\overline{apr}}(\mathbf{A}) \subseteq \overline{apr}_{\ell}(\mathbf{A})$.

Proof. Straightforward. □

Corollary 5.8. $\forall\, \ell \in \{\aleph, \mathbb{M}, \mathbb{C}\aleph, \mathbb{CM}\}$ *and* $\mathbf{A} \subseteq \mathcal{U}$*, we get:*
(i) $\mu(\mathbf{A}) \subseteq \mathbf{B}_{\ell}(\mathbf{A})$. *(ii)* $\mu_{\ell}(\mathbf{A}) \leq \sigma(\mathbf{A})$.

In general, the converse of Corollary 5.8 does not hold.

Example 5.9. *(Continued from Example 4.3). Take* $\aleph(\lambda) = \aleph(\mu) = \{\lambda, \mu\}$, $\aleph(\nu) = \{\lambda, \mu, \nu\}$ *and* $\aleph(\epsilon) = \{\mu, \nu, \epsilon\}$*. So,* $\mathbb{C}\aleph(\lambda) = \mathbb{C}\aleph(\mu) = \{\lambda, \mu\}$, $\mathbb{C}\aleph(\nu) = \{\nu\}$ *and* $\mathbb{C}\aleph(\epsilon) = \{\epsilon\}$*. Also,* $\mathbb{M}(\lambda) = \{\lambda, \mu\}$, $\mathbb{M}(\mu) = \{\mu\}$, $\mathbb{M}(\nu) = \{\mu, \nu\}$ *and* $\mathbb{M}(\epsilon) = \{\mu, \nu, \epsilon\}$*. Hence,* $\mathbb{CM}(\lambda) = \{\lambda\}$, $\mathbb{CM}(\mu) = \{\mu\}$ *and* $\mathbb{CM}(\nu) = \{\nu\}$, $\mathbb{CM}(\epsilon) = \{\epsilon\}$*. In Table 1, we present comparisons among various types of generalized rough approximations, including Yao's method [54, 55] from Definition 2.14, Yu et al.'s approach [60] from Definition 2.15, and our proposed study.*

Table 1 provides the properties of our proposed approximations. To assess whether the approximations satisfy properties (1)–(9*), we have incorporated corresponding codes. In this table, a code of 1 indicates that the property is satisfied, while a code of 0 indicates that the property is not satisfied.

Table 1. Fundamental properties of various approximation methods.

	$\underline{apr}_{\aleph}$	$\underline{apr}_{\mathbb{M}}$	$\underline{apr}_{\mathbb{C}\aleph}$	$\underline{apr}_{\mathbb{C}\mathbb{M}}$	$\mathbb{L}(\mathbf{A})$	$\underline{Apr}(\mathbf{A})$		$\overline{apr}_{\aleph}$	$\overline{apr}_{\mathbb{M}}$	$\overline{apr}_{\mathbb{C}\aleph}$	$\overline{apr}_{\mathbb{C}\mathbb{M}}$	$\mathbb{U}(\mathbf{A})$	$\overline{Apr}(\mathbf{A})$
1	0	0	1	1	1	0	1*	0	0	1	1	1	0
2	0	0	1	1	1	0	2*	1	1	1	1	1	1
3	1	1	1	1	1	1	3*	0	0	1	1	1	0
4	1	1	1	1	1	1	4*	1	1	1	1	1	1
5	1	1	1	1	1	1	5*	1	1	1	1	1	1
6	1	1	1	1	1	1	6*	1	1	1	1	1	1
7	1	1	1	1	1	1	7*	1	1	1	1	1	1
8	1	1	1	1	1	1	8*	1	1	1	1	1	1
9	0	0	1	1	1	0	9*	0	0	1	1	1	0

Remark 5.10. *It is noteworthy, with reference to Table 2, that:*

(i) Example 5.9 exemplifies the contrasts between our approaches and those presented in previous studies.

(ii) The most effective techniques for determining the exactness and roughness of sets are Definitions 4.8 and 5.1. Due to the smaller boundary regions, our measurements are highly accurate. Therefore, compared to Yao's [54,55] and Yu's [60] approaches, which we have proposed, they are more accurate.

(iii) Employing a similar method, Definitions 5.2 and 5.1 demonstrate better outcomes than Yao and Yu's methods.

(iv) Definition 5.1 satisfies all of Pawlak's properties, whereas Definition 5.2 does not meet all of these criteria.

Table 2. Comparisons between different types of rough generalizations and current methods.

	Yao's technique (Definition 2.14)				The suggested method (Definition 4.1)				The suggested method (Definition 2.15)			
	$\underline{apr}_{\aleph}(\mathbf{A})$	$\overline{apr}_{\aleph}(\mathbf{A})$	$\mathbf{B}_{\aleph}(\mathbf{A})$	$\mu_{\aleph}(\mathbf{A})$	$\underline{apr}_{\mathbb{M}}(\mathbf{A})$	$\overline{apr}_{\mathbb{M}}(\mathbf{A})$	$\mathbf{B}_{\mathbb{M}}(\mathbf{A})$	$\mu_{\mathbb{M}}(\mathbf{A})$	$\underline{apr}_{\mathbb{C}\aleph}(\mathbf{A})$	$\overline{apr}_{\mathbb{C}\aleph}(\mathbf{A})$	$\mathbf{B}_{\mathbb{C}\aleph}(\mathbf{A})$	$\mu_{\mathbb{C}\aleph}(\mathbf{A})$
$\{\lambda\}$	ϕ	$\{\lambda,\mu,\nu\}$	$\{\lambda,\mu,\nu\}$	0	ϕ	$\{\lambda\}$	$\{\lambda\}$	0	ϕ	$\{\lambda,\mu\}$	$\{\lambda,\mu\}$	0
$\{\mu\}$	ϕ	$\mathcal{U}$	$\mathcal{U}$	0	$\{\mu\}$	$\mathcal{U}$	$\{\lambda,\nu,\epsilon\}$	$\frac{1}{4}$	ϕ	$\{\lambda,\mu\}$	$\{\lambda,\mu\}$	0
$\{\nu\}$	ϕ	$\{\nu,\epsilon\}$	$\{\nu,\epsilon\}$	0	ϕ	$\{\nu,\epsilon\}$	$\{\nu,\epsilon\}$	0	$\{\nu\}$	$\{\nu\}$	ϕ	1
$\{\epsilon\}$	ϕ	$\{\epsilon\}$	$\{\epsilon\}$	0	ϕ	$\{\epsilon\}$	$\{\epsilon\}$	0	$\{\epsilon\}$	$\{\epsilon\}$	ϕ	1
$\{\lambda,\mu\}$	$\{\lambda,\mu\}$	$\mathcal{U}$	$\{\nu,\epsilon\}$	$\frac{1}{2}$	$\{\lambda,\mu\}$	$\mathcal{U}$	$\{\nu,\epsilon\}$	$\frac{1}{2}$	$\{\lambda,\mu\}$	$\{\lambda,\mu\}$	ϕ	1
$\{\lambda,\nu\}$	ϕ	$\mathcal{U}$	$\mathcal{U}$	0	ϕ	$\{\lambda,\nu,\epsilon\}$	$\{\lambda,\nu,\epsilon\}$	0	$\{\nu\}$	$\{\lambda,\mu,\nu\}$	$\{\lambda,\mu,\nu\}$	$\frac{1}{3}$
$\{\lambda,\epsilon\}$	ϕ	$\mathcal{U}$	$\mathcal{U}$	0	ϕ	$\{\lambda,\epsilon\}$	$\{\lambda,\epsilon\}$	0	$\{\epsilon\}$	$\{\lambda,\mu,\epsilon\}$	$\{\lambda,\mu\}$	$\frac{1}{3}$
$\{\mu,\nu\}$	ϕ	$\mathcal{U}$	$\mathcal{U}$	0	$\{\mu,\nu\}$	$\mathcal{U}$	$\{\lambda,\epsilon\}$	$\frac{1}{2}$	$\{\nu\}$	$\{\lambda,\mu,\nu\}$	$\{\lambda,\mu\}$	$\frac{1}{3}$
$\{\mu,\epsilon\}$	ϕ	$\mathcal{U}$	$\mathcal{U}$	0	$\{\mu\}$	$\mathcal{U}$	$\{\lambda,\mu,\epsilon\}$	$\frac{1}{3}$	$\{\epsilon\}$	$\{\lambda,\mu,\epsilon\}$	$\{\lambda,\mu\}$	$\frac{1}{3}$
$\{\nu,\epsilon\}$	ϕ	$\{\nu,\epsilon\}$	$\{\nu,\epsilon\}$	0	ϕ	$\mathcal{U}$	$\mathcal{U}$	0	$\{\nu,\epsilon\}$	$\{\nu,\epsilon\}$	ϕ	1
$\{\lambda,\mu,\nu\}$	$\{\lambda,\mu,\nu\}$	$\mathcal{U}$	$\{\epsilon\}$	$\frac{3}{4}$	$\{\lambda,\mu,\nu\}$	$\mathcal{U}$	$\{\epsilon\}$	$\frac{3}{4}$	$\{\lambda,\mu,\nu\}$	$\{\lambda,\mu,\nu\}$	ϕ	1
$\{\lambda,\mu,\epsilon\}$	$\{\lambda,\mu\}$	$\mathcal{U}$	$\mathcal{U}$	$\frac{1}{2}$	$\{\lambda,\mu\}$	$\mathcal{U}$	$\{\nu,\epsilon\}$	$\frac{1}{2}$	$\{\lambda,\mu,\epsilon\}$	$\{\lambda,\mu,\epsilon\}$	ϕ	1
$\{\lambda,\nu,\epsilon\}$	ϕ	$\mathcal{U}$	$\mathcal{U}$	0	ϕ	$\{\lambda,\nu,\epsilon\}$	$\{\lambda,\nu,\epsilon\}$	0	$\{\nu,\epsilon\}$	$\mathcal{U}$	$\{\lambda,\mu\}$	$\frac{1}{2}$
$\{\mu,\nu,\epsilon\}$	ϕ	$\mathcal{U}$	$\mathcal{U}$	0	$\{\mu,\nu,\epsilon\}$	$\mathcal{U}$	$\{\lambda\}$	$\frac{3}{4}$	$\{\nu,\epsilon\}$	$\mathcal{U}$	$\{\lambda,\mu\}$	$\frac{1}{2}$
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	ϕ	1	$\mathcal{U}$	$\mathcal{U}$	ϕ	1	$\mathcal{U}$	$\mathcal{U}$	ϕ	1

Table 2. Continued.

	The current method (Definition 4.8)				The current method (Definitions 5.1,5.2)			
	$\underline{apr}_{\mathbb{CM}}(\mathbf{A})$	$\overline{apr}_{\mathbb{CM}}(\mathbf{A})$	$\mathbf{B}_{\mathbb{CM}}(\mathbf{A})$	$\mu_{\mathbb{CM}}(\mathbf{A})$	$\mathbb{L}(\mathbf{A})$	$\mathbb{U}(\mathbf{A})$	$\mathbb{BN}(\mathbf{A})$	$\sigma(\mathbf{A})$
$\{\lambda\}$	$\{\lambda\}$	$\{\lambda\}$	ϕ	1	$\{\lambda\}$	$\{\lambda\}$	ϕ	1
$\{\mu\}$	$\{\mu\}$	$\{\mu\}$	ϕ	1	$\{\mu\}$	$\{\mu\}$	ϕ	1
$\{\nu\}$	$\{\nu\}$	$\{\nu\}$	ϕ	1	$\{\nu\}$	$\{\nu\}$	ϕ	1
$\{\epsilon\}$	$\{\epsilon\}$	$\{\epsilon\}$	ϕ	1	$\{\epsilon\}$	$\{\epsilon\}$	ϕ	1
$\{\lambda,\mu\}$	$\{\lambda,\mu\}$	$\{\lambda,\mu\}$	ϕ	1	$\{\lambda,\mu\}$	$\{\lambda,\mu\}$	ϕ	1
$\{\lambda,\nu\}$	$\{\lambda,\nu\}$	$\{\lambda,\nu\}$	ϕ	1	$\{\lambda,\nu\}$	$\{\lambda,\nu\}$	ϕ	1
$\{\lambda,\epsilon\}$	$\{\lambda,\epsilon\}$	$\{\lambda,\epsilon\}$	ϕ	1	$\{\lambda,\epsilon\}$	$\{\lambda,\epsilon\}$	ϕ	1
$\{\mu,\nu\}$	$\{\mu,\nu\}$	$\{\mu,\nu\}$	ϕ	1	$\{\mu,\nu\}$	$\{\mu,\nu\}$	ϕ	1
$\{\mu,\epsilon\}$	$\{\mu,\epsilon\}$	$\{\mu,\epsilon\}$	ϕ	1	$\{\mu,\epsilon\}$	$\{\mu,\epsilon\}$	ϕ	1
$\{\nu,\epsilon\}$	$\{\nu,\epsilon\}$	$\{\nu,\epsilon\}$	ϕ	1	$\{\nu,\epsilon\}$	$\{\nu,\epsilon\}$	ϕ	1
$\{\lambda,\mu,\nu\}$	$\{\lambda,\mu,\nu\}$	$\{\lambda,\mu,\nu\}$	ϕ	1	$\{\lambda,\mu,\nu\}$	$\{\lambda,\mu,\nu\}$	ϕ	1
$\{\lambda,\mu,\epsilon\}$	$\{\lambda,\mu,\epsilon\}$	$\{\lambda,\mu,\epsilon\}$	ϕ	1	$\{\lambda,\mu,\epsilon\}$	$\{\lambda,\mu,\epsilon\}$	ϕ	1
$\{\lambda,\nu,\epsilon\}$	$\{\lambda,\nu,\epsilon\}$	$\{\lambda,\nu,\epsilon\}$	ϕ	1	$\{\lambda,\nu,\epsilon\}$	$\{\lambda,\nu,\epsilon\}$	ϕ	1
$\{\mu,\nu,\epsilon\}$	$\{\mu,\nu,\epsilon\}$	$\{\mu,\nu,\epsilon\}$	ϕ	1	$\{\mu,\nu,\epsilon\}$	$\{\mu,\nu,\epsilon\}$	ϕ	1
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	ϕ	1	$\mathcal{U}$	$\mathcal{U}$	ϕ	1

6. Exploring various topologies utilizing neighborhood systems

Abd El-Monsef et al. [3] construct topological spaces by utilizing $\mathbb{NS}$s derived from binary relations. A noteworthy extension of their work involves considering four neighborhoods, which allows us to generate four distinct topologies and determine their respective minimal topology. This technique holds immense significance for a wide range of applications, particularly in the fields of rough sets and graphs.

Theorem 6.1. *$\forall\ p \in \mathcal{U}$, $\exists\ \mathcal{N}(p) \subseteq \mathcal{U}$. Then, $\tau = \{\mathbf{A} \subseteq \mathcal{U} : \forall\ p \in \mathbf{A}, \mathcal{N}(p) \subseteq \mathbf{A}\}$ is a topology on $\mathcal{U}$.*

Proof. **(T1)** Clearly, $\mathcal{U}, \phi \in \tau$.
(T2) Let $\{\mathbf{A}_i : i \in I\} \in \tau$ and $p \in \bigcup_{i\in I} \mathbf{A}_i$. Then, $\exists\ i_0 \in I$ s.t. $p \in \mathbf{A}_{i_0}$. Thus, $\mathcal{N}(p) \subseteq \mathbf{A}_{i_0}$ and so $\mathcal{N}(p) \subseteq \bigcup_{i\in I} \mathbf{A}_i$. So, $\bigcup_{i\in I} \mathbf{A}_i \in \tau$.
(T3) Let $\mathbf{A}_1$, $\mathbf{A}_2 \in \tau$ and $p \in \mathbf{A}_1 \cap \mathbf{A}_2$. Then, $p \in \mathbf{A}_1$ and $p \in \mathbf{A}_2$ and so $\mathcal{N}(p) \subseteq \mathbf{A}_1$ and $\mathcal{N}(p) \subseteq \mathbf{A}_2$. Thus, $\mathcal{N}(p) \subseteq \mathbf{A}_1 \cap \mathbf{A}_2$ and so $\mathbf{A}_1 \cap \mathbf{A}_2 \in \tau$. □

According to Theorem 6.1, four topologies from 1-$\mathbb{NS}$s can be generated.

Corollary 6.2. *The classes $\tau_\ell = \{\mathbf{A} \subseteq \mathcal{U} : \forall\ \mathbf{m} \in \mathbf{A}, \aleph_\ell(\mathbf{m}) \subseteq \mathbf{A}\}$, where $\ell \in \{\aleph, \mathbb{M}, \mathbb{C}\aleph, \mathbb{CM}\}$, form topologies on $\mathcal{U}$. Notably, the topologies $\tau_\aleph$, $\tau_{\mathbb{C}\aleph}$, $\tau_{\mathbb{M}}$ and $\tau_{\mathbb{CM}}$ are independent of each other.*

Example 6.3. *(Continued from Example 2.3). Let $\aleph(\lambda) = \{\lambda\}$, $\aleph(\mu) = \{\lambda,\mu\}$, $\aleph(\nu) = \{\lambda\}$ and $\aleph(\epsilon) = \{\lambda,\mu\}$. Then, $\mathbb{C}\aleph(\lambda) = \mathbb{C}\aleph(\mu) = \{\mu,\epsilon\}$ and $\mathbb{C}\aleph(\nu) = \mathbb{C}\aleph(\epsilon) = \{\lambda,\nu\}$. Also, $\mathbb{M}(\lambda) = \{\lambda\}$, $\mathbb{M}(\mu) = \{\lambda,\mu\}$ and $\mathbb{M}(\nu) = \mathbb{M}(\epsilon) = \phi$. Hence, $\mathbb{CM}(\lambda) = \{\lambda\}$, $\mathbb{CM}(\mu) = \{\mu\}$ and $\mathbb{CM}(\nu) = \mathbb{CM}(\epsilon) = \{\nu,\epsilon\}$. Therefore, the topologies that induced by neighborhoods are* *(i)* $\tau_{\aleph} = \{\mathcal{U}, \phi, \{\lambda\}, \{\lambda,\mu\}, \{\lambda,\nu\}, \{\lambda,\mu,\nu\}, \{\lambda,\mu,\epsilon\}\}$.

(ii) $\tau_{\mathbb{C}\aleph} = \{\mathcal{U}, \phi, \{\lambda,\nu\}, \{\mu,\epsilon\}\}$.

(iii) $\tau_{\mathbb{M}} = \{\mathcal{U}, \phi, \{\lambda\}, \{\nu\}, \{\epsilon\}, \{\lambda,\mu\}, \{\lambda,\nu\}, \{\lambda,\epsilon\}, \{\nu,\epsilon\}, \{\lambda,\mu,\nu\}, \{\lambda,\mu,\epsilon\}, \{\lambda,\nu,\epsilon\}\}$.

(iv) $\tau_{\mathbb{CM}} = \{\mathcal{U}, \phi, \{\lambda\}, \{\mu\}, \{\lambda,\mu\}, \{\nu,\epsilon\}, \{\lambda,\nu,\epsilon\}, \{\mu,\nu,\epsilon\}\}$.

Proposition 6.4. *Suppose $\aleph(\mathbf{m})$ represents an inverse serial neighborhood. It follows that $\tau_{\mathbb{M}} \subseteq \tau_{c\mathbb{M}}$.*

Proof. With reference to Lemma 3.5, the proof is evident. □

In general, the converse of Proposition 6.4 does not hold.

Example 6.5. *(Continued from Example 2.3). Suppose that $\aleph(\lambda) = \{\lambda\}$, $\aleph(\mu) = \aleph(\nu) = \{\mu,\nu,\epsilon\}$ and $\aleph(\epsilon) = \{\mu\}$. So, $\mathbb{M}(\lambda) = \{\lambda\}$, $\mathbb{M}(b) = \{\mu\}$ and $\mathbb{M}(\nu) = m(\epsilon) = \{\mu,\nu,\epsilon\}$. Hence, $\mathbb{CM}(\lambda) = \{\lambda\}$, $\mathbb{CM}(\mu) = \{\mu\}$ and $\mathbb{CM}(\nu) = \mathbb{CM}(\epsilon) = \{\nu,\epsilon\}$. Therefore, the topologies that are generated by $\mathbb{NS}$s are:*

(i) $\tau_{\aleph} = \{\mathcal{U}, \phi, \{\lambda\}, \{\mu,\nu,\epsilon\}\}$.

(ii) $\tau_{\mathbb{M}} = \{\mathcal{U}, \phi, \{\lambda\}, \{\mu\}, \{\lambda,\mu\}, \{\mu,\nu,\epsilon\}\}$.

(iii) $\tau_{\mathbb{CM}} = \{\mathcal{U}, \phi, \{\lambda\}, \{\mu\}, \{\lambda,\mu\}, \{\nu,\epsilon\}, \{\lambda,\nu,\epsilon\}, \{\mu,\nu,\epsilon\}\}$, $\forall\ \mathbf{m} \in \mathcal{U}$.

Proposition 6.6. *Suppose $\aleph(\mathbf{m})$ represents a reflexive neighborhood. Thus,*

(i) $\tau_{\aleph} \subseteq \tau_{\mathbb{C}\aleph} \subseteq \tau_{\mathbb{CM}}$.

(i) $\tau_{\aleph} \subseteq \tau_{\mathbb{M}} \subseteq \tau_{\mathbb{CM}}$.

Proof. With reference to Theorem 5.7, the proof is evident. □

In general, the reverse implications of Proposition 6.6 do not hold.

Example 6.7. *(Continued from Example 4.16). From topologies*

(i) $\tau_{\aleph} = \{\mathcal{U}, \phi, \{\lambda,\mu\}, \{\lambda,\mu,\nu\}\}$.

(ii) $\tau_{c\aleph} = \{\mathcal{U}, \phi, \{\nu\}, \{\epsilon\}, \{\lambda,\nu\}, \{\nu,\epsilon\}, \{\lambda,\mu,\nu\}, \{\lambda,\mu,\epsilon\}, \{\lambda,\nu,\epsilon\}\}$.

(iii) $\mathbb{M}(\mathbf{m})$ *is* $\tau_{\mathbb{M}} = \{\mathcal{U}, \phi, \{\mu\}, \{\lambda,\mu\}, \{\mu,\nu\}, \{\lambda,\mu,\nu\}\}$.

(iv) $\tau_{c\mathbb{M}} = \{\mathcal{U}, \phi, \{\lambda\}, \{\mu\}, \{\nu\}, \{\epsilon\}, \{\lambda,\mu\}, \{\lambda,\nu\}, \{\lambda,\epsilon\}, \{\mu,\nu\}, \{\mu,\epsilon\}, \{\nu,\epsilon\}, \{\lambda,\mu,\nu\}, \{\lambda,\mu,\epsilon\}, \{\lambda,\nu,\epsilon\}, \{\mu,\nu,\epsilon\}\}$.
It is noted that the reverse inclusions of Proposition 6.6 are not satisfied.

Proposition 6.8. *Consider $\mathcal{N}(\mathbf{m}) = \mathcal{N}_{\aleph}(\mathbf{m}) \cap \mathcal{N}_{\mathbb{C}\aleph}(\mathbf{m}) \cap \mathcal{N}_{\mathbb{M}}(\mathbf{m}) \cap \mathcal{N}_{\mathbb{CM}}(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$. Then, $\mathcal{T} = \{\mathbf{A} \subseteq \mathcal{U} : \forall\ \mathbf{m} \in \mathbf{A}, \mathcal{N}(\mathbf{m}) \subseteq \mathbf{A}\}$ is a minimal topology on $\mathcal{U}$ that contains each of τ_ℓ, $\forall\ \ell \in \{\aleph, \mathbb{C}\aleph, \mathbb{M}, \mathbb{CM}\}$.*

Proof. Directly using Theorem 6.1. □

Corollary 6.9. *Suppose $\mathcal{T}$ is a minimal topology. Thus, $\tau_{\aleph}(\mathbf{m}) \cap \tau_{\mathbb{C}\aleph}(\mathbf{m}) \cap \tau_{\mathbb{M}}(\mathbf{m}) \cap \tau_{\mathbb{CM}} \subseteq \mathcal{T}$.*

The reverse inclusion does not hold, in general.

Example 6.10. *(Continued from Example 5.9). The neighborhood $\mathcal{N}(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathcal{U}$ that is generated by neighborhoods $\mathcal{N}_\ell(\mathbf{m})$, $\forall\ \ell \in \{\aleph, \mathbb{C}\aleph, \mathbb{M}, \mathbb{CM}\}$: $\mathcal{N}(\lambda) = \{\lambda\}$, $\mathcal{N}(\mu) = \{\mu\}$ and $\mathcal{N}(\nu) = \mathcal{N}(\epsilon) = \phi$. Thus, a minimal topology $\mathcal{T}$ is given by: $\mathcal{T} = \{\mathcal{U}, \phi, \{\lambda\}, \{\mu\}, \{\nu\}, \{\epsilon\}, \{\lambda,\mu\}, \{\lambda,\nu\}, \{\lambda,\epsilon\}, \{\mu,\nu\}, \{\mu,\epsilon\}, \{\nu,\epsilon\}, \{\lambda,\mu,\nu\}, \{\lambda,\mu,\epsilon\}, \{\lambda,\nu,\epsilon\}, \{\mu,\nu,\epsilon\}\}$. Clearly, $\forall\ \ell \in \{\aleph, \mathbb{C}\aleph, \mathbb{M}, \mathbb{CM}\}$: $\tau_\ell \subseteq \mathcal{T}$, but $\mathcal{T} \neq \tau_\ell$. In addition, $\tau_{\aleph} \cap \tau_{\mathbb{C}\aleph} \cap \tau_{\mathbb{M}} \cap \tau_{\mathbb{CM}} = \{\mathcal{U}, \phi, \{\lambda\}, \{\mu\}, \{\nu\}, \{\epsilon\}, \{\lambda,\mu\}, \{\lambda,\nu\}, \{\lambda,\epsilon\}, \{\mu,\epsilon\}, \{\nu,\epsilon\}, \{\lambda,\mu,\nu\}, \{\lambda,\mu,\epsilon\}, \{\lambda,\nu,\epsilon\}, \{\mu,\nu,\epsilon\}\}$. Therefore, $\tau_{\aleph} \cap \tau_{\mathbb{C}\aleph} \cap \tau_{\mathbb{M}} \cap \tau_{\mathbb{CM}} \subseteq \mathcal{T}$, but $\mathcal{T} \neq \tau_{\aleph} \cap \tau_{\mathbb{C}\aleph} \cap \tau_{\mathbb{M}} \cap \tau_{\mathbb{CM}}$.*

7. Medical applications: Exploring various topologies in human blood circulation

The human body relies on the efficient distribution of oxygen and nutrients to every cell. This essential process is achieved through the circulation of oxygen-rich blood, nourishing body organs, tissues and cells, while also facilitating the removal of waste products from the system. The coordination between the heart, lungs and blood vessels ensures the seamless flow of this vital process. At the center of this circulation is the cardiopulmonary system, involving the movement of blood from the heart to the lungs and back. Initially, blood is directed to the right side of the heart, where it absorbs waste materials like carbon dioxide. It is then propelled to the lungs to acquire oxygen for the body's cells while simultaneously releasing carbon dioxide. The oxygenated blood subsequently returns to the left side of the heart.

Systemic circulation follows, with the left side of the heart pumping oxygenated blood to nourish bodily tissues. The blood travels through the entire body via the aorta, reaching all organs and tissues to deliver vital oxygen. The cells utilize this oxygen for their proper functioning, generating waste materials, such as carbon dioxide, that need to be eliminated.

Blood that has released its oxygen content collects waste materials from the cells and is transported back to the heart, where it is sent to the lungs for expulsion. This efficient circulation system ensures that all cells receive the necessary nutrients and oxygen while eliminating waste, thereby contributing to overall health and optimal organ function.

In the study of human heart topology, operators on graphs were employed [44, 49]. Nada et al. [43] further advanced this research by classifying the heart into vertices and edges, resulting in a graph model as shown in Figure 1. They established a topological structure denoted as τ_G based on this graph.

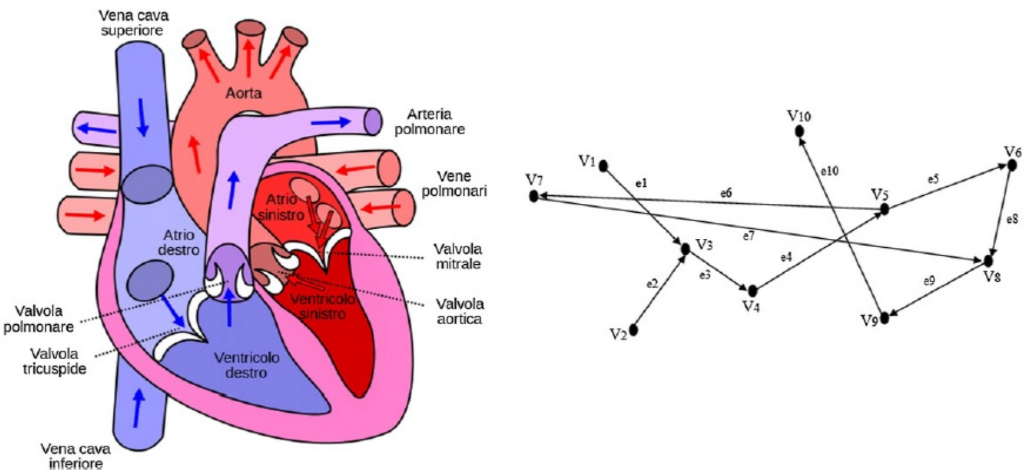

Figure 1. A digraph model of a human heart.

We explore additional classifications of the heart using neighborhoods, core neighborhoods, minimal neighborhoods, and core minimal neighborhoods. Through these classifications, several topological spaces are induced. In this context, the vertices of the graph $G = (V, E)$ represent different regions where blood flows, while the edges represent the pathways of blood circulation within the heart. Specifically, vertices $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6, \upsilon_7$ represent deoxygenated blood from the superior vena cavae, inferior vena cavae, right atrium, right ventricle, pulmonary trunk, right lung, and left lung, respectively. On the other hand, vertices $\upsilon_8, \upsilon_9, \upsilon_{10}$ represent oxygenated blood from the left atrium, left ventricle, and aorta, respectively.

To delve deeper, we consider the set $\mathcal{U}_G = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5, \upsilon_6, \upsilon_7, \upsilon_8, \upsilon_9, \upsilon_{10}\}$ and the right (core, minimal, core minimal) neighborhoods of each vertex, representing in-blood properties. These neighborhoods are presented in Tables 3–5. Moreover, we examine the topologies on a subgraph $H_G = \{\upsilon_5, \upsilon_6, \upsilon_7, \upsilon_8\}$ of $\mathcal{U}_G$, $\forall$ $\mathbf{m} \in \mathcal{U}$. The topologies are as follows:

Table 3. Right neighborhoods of $\mathbf{m} \in \mathcal{U}_G$.

$\mathbf{m} \in \mathcal{U}_G$	$\aleph_r(\mathbf{m})$	$\mathbb{M}_r(\mathbf{m})$	$\mathbb{C}\aleph_r(\mathbf{m})$	$\mathbb{CM}_r(\mathbf{m})$
υ_1	$\{\upsilon_3\}$	ϕ	$\{\upsilon_1, \upsilon_2\}$	$\{\upsilon_1, \upsilon_2\}$
υ_2	$\{\upsilon_3\}$	ϕ	$\{\upsilon_1, \upsilon_2\}$	$\{\upsilon_1, \upsilon_2\}$
υ_3	$\{\upsilon_4\}$	$\{\upsilon_3\}$	$\{\upsilon_3\}$	$\{\upsilon_3\}$
υ_4	$\{\upsilon_5\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$	$\{\upsilon_4\}$
υ_5	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_5\}$	$\{\upsilon_5\}$	$\{\upsilon_5\}$
υ_6	$\{\upsilon_8\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$
υ_7	$\{\upsilon_8\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$
υ_8	$\{\upsilon_9\}$	$\{\upsilon_8\}$	$\{\upsilon_8\}$	$\{\upsilon_8\}$
υ_9	$\{\upsilon_{10}\}$	$\{\upsilon_9\}$	$\{\upsilon_9\}$	$\{\upsilon_9\}$
υ_{10}	ϕ	$\{\upsilon_{10}\}$	$\{\upsilon_{10}\}$	$\{\upsilon_{10}\}$

Table 4. Right neighborhoods for $\mathbf{m} \in \mathcal{U}_G$.

$\mathbf{m} \in \mathcal{U}_G$	$\aleph_r(\mathbf{m})$	$\mathbb{M}_r(\mathbf{m})$	$\mathbb{C}\aleph_r(\mathbf{m})$	$\mathbb{CM}_r(\mathbf{m})$
υ_5	$\{\upsilon_6, \upsilon_7\}$	ϕ	$\{\upsilon_5\}$	$\{\upsilon_5\}$
υ_6	$\{\upsilon_8\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$
υ_7	$\{\upsilon_8\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$
υ_8	ϕ	$\{\upsilon_8\}$	$\{\upsilon_8\}$	$\{\upsilon_8\}$

Table 5. Left neighborhoods for $\mathbf{m} \in \mathcal{U}_G$.

$\mathbf{m} \in \mathcal{U}_G$	$\aleph_l(\mathbf{m})$	$\mathbb{M}_l(\mathbf{m})$	$\mathbb{C}\aleph_l(\mathbf{m})$	$\mathbb{CM}_l(\mathbf{m})$
υ_5	ϕ	$\{\upsilon_5\}$	$\{\upsilon_5\}$	$\{\upsilon_5\}$
υ_6	$\{\upsilon_5\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$
υ_7	$\{\upsilon_5\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$	$\{\upsilon_6, \upsilon_7\}$
υ_8	$\{\upsilon_6, \upsilon_7\}$	ϕ	$\{\upsilon_8\}$	$\{\upsilon_8\}$

(i) Topology using $\aleph_r(\mathbf{m})$ is $\tau_{\aleph_r} = \{\mathcal{U}_G, \phi, \{\upsilon_8\}, \{\upsilon_6, \upsilon_8\}, \{\upsilon_7, \upsilon_8\}, \{\upsilon_6, \upsilon_7, \upsilon_8\}\}$.

(ii) Topology using $\mathbb{M}_r(\mathbf{m})$ is $\tau_{\mathbb{M}_r} = \{\mathcal{U}_G, \phi, \{\upsilon_5\}, \{\upsilon_8\}, \{\upsilon_5, \upsilon_8\}, \{\upsilon_6, \upsilon_7\}, \{\upsilon_5, \upsilon_6, \upsilon_7\}, \{\upsilon_6, \upsilon_7, \upsilon_8\}\}$.

(iii) Topology using $\mathbb{C}\aleph_r(\mathbf{m})$ is $\tau_{\mathbb{C}\aleph_r} = \{\mathcal{U}_G, \phi, \{\upsilon_5\}, \{\upsilon_8\}, \{\upsilon_5, \upsilon_8\}, \{\upsilon_6, \upsilon_7\}, \{\upsilon_5, \upsilon_6, \upsilon_7\}, \{\upsilon_6, \upsilon_7, \upsilon_8\}\}$.

(iv) Topology using $c\mathbb{M}_r(\mathbf{m})$ is $\tau_{\mathbb{CM}_r} = \{\mathcal{U}_G, \phi, \{\upsilon_5\}, \{\upsilon_8\}, \{\upsilon_5, \upsilon_8\}, \{\upsilon_6, \upsilon_7\}, \{\upsilon_5, \upsilon_6, \upsilon_7\}, \{\upsilon_6, \upsilon_7, \upsilon_8\}\}$.

(v) Topology induced by $\aleph_l(\mathbf{m})$ is $\tau_{\aleph_l} = \{\mathcal{U}_G, \phi, \{\upsilon_5\}, \{\upsilon_5, \upsilon_6\}, \{\upsilon_5, \upsilon_7\}, \{\upsilon_5, \upsilon_6, \upsilon_7\}\}$.
(vi) Topology using $\mathbb{M}_l(\mathbf{m})$ is $\tau_{\mathbb{M}_l} = \{\mathcal{U}_G, \phi, \{\upsilon_5\}, \{\upsilon_8\}, \{\upsilon_5, \upsilon_8\}, \{\upsilon_6, \upsilon_7\}, \{\upsilon_5, \upsilon_6, \upsilon_7\}, \{\upsilon_6, \upsilon_7, \upsilon_8\}\}$.
(vii) Topology using $\mathbb{C}\aleph_l(\mathbf{m})$ is $\tau_{\mathbb{C}\aleph_l} = \{\mathcal{U}_G, \phi, \{\upsilon_5\}, \{\upsilon_8\}, \{\upsilon_5, \upsilon_8\}, \{\upsilon_6, \upsilon_7\}, \{\upsilon_5, \upsilon_6, \upsilon_7\}, \{\upsilon_6, \upsilon_7, \upsilon_8\}\}$.
(viii) Topology using $\mathbb{CM}_l(\mathbf{m})$ is $\tau_{\mathbb{CM}_l} = \{\mathcal{U}_G, \phi, \{\upsilon_5\}, \{\upsilon_8\}, \{\upsilon_5, \upsilon_8\}, \{\upsilon_6, \upsilon_7\}, \{\upsilon_5, \upsilon_6, \upsilon_7\}, \{\upsilon_6, \upsilon_7, \upsilon_8\}\}$.

These results for blood circulation can be analyzed as follows:

(i) The topologies $\tau_{\aleph_r}$ and $\tau_{\aleph_l}$ are independent.
(ii) Within the human heart circulation, we find a set of distinct topological structures represented by $\tau_{\mathbb{M}_r}$, $\tau_{\mathbb{M}_l}$, $\tau_{\mathbb{C}\aleph_r}$, $\tau_{\mathbb{C}\aleph_l}$, $\tau_{\mathbb{CM}_r}$ and $\tau_{\mathbb{CM}_l}$, all of which are equivalent to each other.
(iii) The core topology $\tau_{\mathbb{CM}_l}$ encompasses all parts of the heart, making it the most suitable structure for any subgraph of G. This implies that experts can utilize it for decision-making in their diagnoses. Additionally, it serves as the optimal choice for topologists in their studies.

Making an accurate choice is one of the most difficult areas of clinical diagnosis. Therefore, with this application, we suggested many topological tools that could assist an expert in the diagnosis of heart problems. A generating topology is a mathematical fundamental basis for applying several topology theories, such as connectivity, compactness, and separation axioms, which are essential features of solutions in the medical fields.

8. Algorithms

This section presents Algorithms 1 and 2 that are used in decision-making problems and for generating various topological structures in graphs. Each topological structure on a graph defines a topological space. These algorithms can be applied not only in graph theory and topology but also in medical scenarios, such as blood circulation within the human body, and geographical scenarios, such as the street system of a community. Therefore, these techniques serve as valuable tools that can be implemented using MATLAB.

9. Discussion and conclusions

Our exploration of the latest medical discoveries and cutting-edge application methods has unveiled intricate graph theory structures, presenting promising trends for practical applications. By generalizing rough set theory using neighborhood systems ($\mathbb{NS}$s), we introduced and thoroughly examined novel neighborhoods based on the concept of 1-$\mathbb{NS}$. These neighborhoods illuminated relationships among different types and were reinforced by several insightful counterexamples.

Drawing on these neighborhoods, we devised new models for rough sets and delved into their essential characteristics, encompassing both lower and upper approximations. Through their application, we achieved accurate subset classification and computed precise measures of accuracy. Remarkably, our proposed methods demonstrated significantly higher accuracy compared to the existing approaches documented in the published literature.

Algorithm 1: A frame work to use ℓ-approximations in decision-making problems.

Input: Initiate an information table generating from the given data such that the first column contains the set of objects $\mathcal{U}$, and the set of attributes A_t as a first row.

Output: An accurate decision for exact and rough sets.

Define the binary relations $q_m \mathbb{K}_{a_k} q_n \Rightarrow \nu_{a_m}(q_m) \subseteq \nu_{a_n}(q_n)$, for each $a_k \in \mathbf{A}$. $m, n \in \{1, 2, 3, \cdots\}$, and $k \in \{1, 2, 3, \cdots\}$.

for $\ell \in \{\mathbb{M}, \mathbb{CM}\}$ **do**

  Compute all neighborhoods, as follows:

  $\Omega_{q_m} = \bigcup_k \mathbb{K}_{a_k}$, for each k, m.

  Compute all ℓ-neighborhoods, using Definitions 3.1 and 3.7.

end

for $Q \subseteq \mathcal{U}$ **do**

  Compute the ℓ-lower approximation $\underline{\mathbb{K}}_\ell(Q) = \{m \in \mathcal{U} : \Omega_\ell(m) \subseteq Q\}$.

  if $\underline{\mathbb{K}}_\ell(Q) = \phi$ **then**

    return Q is a rough set.

    else

      Compute the ℓ-upper approximation $\overline{\mathbb{K}}_\ell(Q) = \{m \in \mathcal{U} : \Omega_\ell(m) \cap Q \neq \phi\}$.

      Compute the ℓ-accuracy $\mu_\ell(Q) = \frac{|\underline{\mathbb{K}}_\ell(Q)|}{|\overline{\mathbb{K}}_\ell(Q)|}$.

      if $\mu_\ell(Q) = 1$ **then**

        return Q is an exact set. **else**

          return Q is a rough set.

        end

      end

    end

  end

end

Algorithm 2: A topological analysis for a human heart

Input: A digraph $G = (V, E)$ of a human heart.

Output: A core topology $\tau_{\mathbb{CM}_l}$.

Step 1: Insert each vertex $\upsilon_i \in \mathcal{U}_G$, where $\mathcal{U}_G$ is the universal set of vertices.

Step 2: Determine the adjacent vertices for each $\upsilon_i \in \mathcal{U}_G$.

Step 3: Each adjacent, topologically, is called a right neighborhood for $\upsilon_i \in \mathcal{U}_G$.

Step 4: Each neighborhood $\aleph_r(\upsilon)$, $\mathbb{M}_r(\upsilon)$, $\mathbb{C}\aleph_r(\upsilon)$, $c\mathbb{M}_r(\upsilon)$, $c\mathbb{M}_r(\upsilon)$, $\aleph_l(\upsilon)$, $\mathbb{M}_l(\upsilon)$, $\mathbb{C}\aleph_l(\upsilon)$, and $\mathbb{CM}_l(\upsilon)$ is a base for topology $\tau_{\aleph_r(\upsilon)}$, $\tau_{\mathbb{M}_r(\upsilon)}$, $\tau_{\mathbb{C}\aleph_r(\upsilon)}$, $\tau_{c\mathbb{M}_r(\upsilon)}$, $\tau_{c\mathbb{M}_r(\upsilon)}$, $\tau_{\aleph_l(\upsilon)}$, $\tau_{\mathbb{M}_l(\upsilon)}$, $\tau_{\mathbb{C}\aleph_l(\upsilon)}$, and $\tau_{\mathbb{CM}_l(\upsilon)}$, respectively, by its arbitrary unions.

Step 5: Compare between topologies on each subgraph of G.

Step 6: $\tau_{\mathbb{CM}_l}$ is the core topology.

Furthermore, these methods broadened the scope of practical problems that can be addressed, all while upholding the core properties of Pawlak approximation spaces under fewer constraints. The validity of these statements was established through Propositions 3–6, along with their respective corollaries, as well as Theorems 1 and 2, along with their corollaries.

Furthermore, we have constructed various types of topological structures using neighborhood systems ($\mathbb{NS}$s). The introduction of minimal and core minimal neighborhoods has facilitated a comprehensive exploration of their characteristics, as demonstrated in Theorem 3, Corollaries 10 and 11 and Propositions 8–10. To establish the superiority of our methods, we conducted a comparative analysis against other established approaches, employing a variety of examples for validation. As a result, our research empowers experts to make informed decisions and assists topologists in selecting the most suitable topological model for studying blood circulation in the human heart. Notably, among the clinical diagnosis areas, core minimal neighborhoods emerged as the optimal choice. By scrutinizing the topological graph structure of blood circulation, topologists can glean invaluable insights into connectivity—A crucial aspect of medicine.

At the end of the paper, two algorithms have been introduced for decision-making and generating graph topologies, defining unique spaces. Beyond graph theory, these techniques apply to medical contexts like blood circulation and geographical scenarios, such as community street mapping. Implemented using MATLAB, they are valuable tools.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors affirm that they do not have any conflicts of interest regarding its publication.

References

1. M. E. Abd El-Monsef, M. A. El-Gayar, R. M. Aqeel, On relationships between revised rough fuzzy approximation operators and fuzzy topological spaces, *Int. J. Granul. Comput. Rough Sets Intell. Syst.*, **3** (2014), 257–271. https://doi.org/10.1504/IJGCRSIS.2014.068022
2. M. E. Abd El-Monsef, M. A. El-Gayar, R. M. Aqeel, A comparison of three types of rough fuzzy sets based on two universal sets, *Int. J. Mach. Learn. Cybern.* **8** (2017), 343–353. https://doi.org/10.1007/s13042-015-0327-8
3. M. E. Abd El-Monsef, O. A. Embaby, M. K. El-Bably, Comparison between rough set approximations based on different topologies, *Int. J. Granul. Comput. Rough Sets Intell. Syst.*, **3** (2014), 292–305. https://doi.org/10.1504/IJGCRSIS.2014.068032
4. M. E. Abd El-Monsef, A. M. Kozae, A. L. El Maghrabi, Some semi topological applications on rough sets, *J. Egypt Math. Soc.*, **12** (2004), 45–53.
5. E. A. Abo-Tabl, Rough sets and topological spaces based on similarity, *Int. J. Mach. Learn. Cybern.*, **4** (2013), 451–458. https://doi.org/10.1007/s13042-012-0107-7
6. E. A. Abo-Tabl, M. K. El-Bably, Rough topological structure based on reflexivity with some applications, *AIMS Mathematics*, **7** (2022), 9911–9922. https://doi.org/10.3934/math.2022553

7. R. Abu-Gdairi, M. A. El-Gayar, T. M. Al-shami, A. S. Nawar, M. K. El-Bably, Some topological approaches for generalized rough sets and their decision-making applications, *Symmetry*, **14** (2022), 95. https://doi.org/10.3390/sym14010095

8. R. Abu-Gdairi, M. A. El-Gayar, M. K. El-Bably, K. K. Fleifel, Two different views for generalized rough sets with applications, *Mathematics*, **9** (2022), 2275. https://doi.org/10.3390/math9182275

9. M. I. Ali, M. K. El-Bably, E. A. Abo-Tabl, Topological approach to generalized soft rough sets via near concepts, *Soft Comput.*, **26** (2022), 499–509. https://doi.org/10.1007/s00500-021-06456-z

10. A. A. Allam, M. Y. Bakeir, E. A. Abo-Tabl, New approach for basic rough set concepts, In: *Rough sets, fuzzy sets, data mining, and granular computing*, Berlin, Heidelberg: Springer, 2005, 64–73. https://doi.org/10.1007/11548669_7

11. W. S. Amer, M. I. Abbas, M. K. El-Bably, On j-near concepts in rough sets with some applications, *J. Intell. Fuzzy Syst.*, **32** (2017), 1089–1099. https://doi.org/10.3233/JIFS-16169

12. M. Atef, A. E. F. El Atik, Some extensions of covering-based multigranulation fuzzy rough sets from new perspectives, *Soft Comput.*, **25** (2021), 6633–6651. https://doi.org/10.1007/s00500-021-05659-8

13. M. Atef, A. E. F. El Atik, A. S. Nawar, Fuzzy topological structures via fuzzy graphs and their applications, *Soft Comput.*, **25** (2021), 6013–6027. https://doi.org/10.1007/s00500-021-05594-8

14. D. Chen, J. Li, R. Lin, Y. Chen, Information entropy and optimal scale combination in multi-scale covering decision systems, *IEEE Access*, **8** (2020), 182908–182917. https://doi.org/10.1109/ACCESS.2020.3029157

15. J. H. Dai, S. C. Gao, G. J. Zheng, Generalized rough set models determined by multiple neighborhoods generated from a similarity relation, *Soft Comput.*, **22** (2018), 2081–2094. https://doi.org/10.1007/s00500-017-2672-x

16. H. Dou, X. Yang, X. Song, H. Yu, W. Z. Wu, J. Yang, Decision-theoretic rough set: A multicost strategy, *Knowl-Based Syst.*, **91** (2016), 71–83. https://doi.org/10.1016/j.knosys.2015.09.011

17. A. E. F. A. El Atik, A. A. Nasef, Some topological structures of fractals and their related graphs, *Filomat*, **34** (2020), 153–165. https://doi.org/10.2298/FIL2001153A

18. A. E. F. El Atik, A. Nawar, M. Atef, Rough approximation models via graphs based on neighborhood systems, *Granul. Comput.*, **6** (2021), 1025–1035. https://doi.org/10.1007/s41066-020-00245-z

19. A. E. F. A. El Atik, A. S. Wahba, Topological approaches of graphs and their applications by neighborhood systems and rough sets, *J. Intell. Fuzzy Syst.*, **39** (2020), 6979–6992. https://doi.org/10.3233/JIFS-200126

20. M. K. El-Bably, E. A. Abo-Tabl, A topological reduction for predicting of a lung cancer disease based on generalized rough sets, *J. Intell. Fuzzy Syst.*, **41** (2021), 3045–3060. https://doi.org/10.3233/JIFS-210167

21. M. K. El-Bably, R. Abu-Gdairi, M. A. El-Gayar, Medical diagnosis for the problem of Chikungunya disease using soft rough sets, *AIMS Mathematics*, **8** (2023), 9082–9105. https://doi.org/10.3934/math.2023455

22. M. K. El-Bably, M. I. Ali, E. A. Abo-Tabl, New topological approaches to generalized soft rough approximations with medical applications, *J. Math.*, **2021** (2021), 2559495. https://doi.org/10.1155/2021/2559495
23. M. K. El-Bably, T. M. Al-Shami, Different kinds of generalized rough sets based on neighborhoods with a medical application, *Int. J. Biomath.*, **14** (2021), 2150086. https://doi.org/10.1142/S1793524521500868
24. M. K. El-Bably, A. E. F. A. El Atik, Soft β-rough sets and their application to determine COVID-19, *Turk. J. Math.*, **45** (2021), 4. https://doi.org/10.3906/mat-2008-93
25. A. El-Fattah A. El-Atik, M. E. A. El Monsef, E. I. Lashin, On finite T0 topological spaces, In: *Proceedings of the Ninth Prague Topological Symposium*, 2002, 75–90.
26. A. El-Fattah A. El-Atik, H. Z. Hassan, Some nano topological structures via ideals and graphs, *J. Egypt. Math. Soc.*, **28** (2020), 41. https://doi.org/10.1186/s42787-020-00093-5
27. M. A. El-Gayar, R. Abu-Gdairi, M. K. El-Bably, D. I. Taher, Economic decision-making using rough topological structures, *J. Math.*, **2023** (2023), 4723233. https://doi.org/10.1155/2023/4723233
28. M. A. El-Gayar, A. E. F. El Atik, Topological models of rough sets and decision making of COVID-19, *Complexity*, **2022** (2022), 2989236. https://doi.org/10.1155/2022/2989236
29. M. K. El-Bably, M. El-Sayed, Three methods to generalize Pawlak approximations via simply open concepts with economic applications, *Soft Comput.*, **26** (2022), 4685–4700. https://doi.org/10.1007/s00500-022-06816-3
30. M. K. El-Bably, K. K. Fleifel, O. A. Embaby, Topological approaches to rough approximations based on closure operators, *Granul. Comput.*, **7** (2022), 1–14. https://doi.org/10.1007/s41066-020-00247-x
31. M. El Sayed, M. A. El Safty, M. K. El-Bably, Topological approach for decision-making of COVID-19 infection via a nano-topology model, *AIMS Mathematics*, **6** (2021), 7872–7894. https://doi.org/10.3934/math.2021457
32. M. M. El-Sharkasy, Topological model for recombination of DNA and RNA, *Int. J. Biomath.*, **11** (2018), 1850097. https://doi.org/10.1142/S1793524518500973
33. H. H. Hung, Symmetric and tufted assignments of neighborhoods and metrization, *Topol. Appl.*, **155** (2008), 2137–2142. https://doi.org/10.1016/j.topol.2007.11.009
34. Z. Huang, J. Li, Discernibility measures for fuzzy β covering and their application, *IEEE T. Cybernetics*, **52** (2022), 9722–9735. https://doi.org/10.1109/TCYB.2021.3054742
35. Z. Huang, J. Li, Feature subset selection with multi-scale fuzzy granulation, *IEEE T. Artif. Intell.*, **4** (2023), 121–134. https://doi.org/10.1109/TAI.2022.3144242
36. Z. Huang, J. Li, Noise-tolerant discrimination indexes for fuzzy γ covering and feature subset selection, *IEEE T. Neural. Netw. Learn. Syst.*, (2022), 1–15. https://doi.org/10.1109/TNNLS.2022.3175922
37. Z. Huang, J. Li, Y. Qian, Noise-tolerant fuzzy-β-covering-based multigranulation rough sets and feature subset selection, *IEEE T. Fuzzy Syst.*, **30** (2022), 2721–2735. https://doi.org/10.1109/TFUZZ.2021.3093202

38. Z. Li, T. Xie, Q. Li, Topological structure of generalized rough sets, *Comput. Math. Appl.* **63** (2021), 1066–1071. https://doi.org/10.1016/j.camwa.2011.12.011

39. T. Y. Lin, Neighborhood systems and approximation in relational databases and knowledge bases, In: *Proceedings of the Fourth International Symposium on Methodologies of Intelligent Systems*, 1988.

40. T. Y. Lin, Granular computing on binary relations I: Data mining and neighborhood systems, In: *Rough set representations and belief functions, rough sets in knowledge discovery 1*, Heidelberg: Physica -Verlag, 1998, 107–140.

41. S. Liang, X. Yang, X. Chen, J. Li, Stable attribute reduction for neighborhood rough set, *Filomat*, **32** (2018), 1809–1815. https://doi.org/10.2298/FIL1805809L

42. H. Lu, A. M. Khalil, W. Alharbi, M. A. El-Gayar, A new type of generalized picture fuzzy soft set and its application in decision making, *J. Intell. Fuzzy Syst.*, **40** (2021), 12459–12475. https://doi.org/10.3233/JIFS-201706

43. S. Nada, A. E. F. El Atik, M. Atef, New types of topological structures via graphs, *Math. Method. Appl. Sci.*, **41** (2018), 5801–5810. https://doi.org/10.1002/mma.4726

44. A. S. Nawar, A. A. El Atik, A model of a human heart via graph nano topological spaces, *Int. J. Biomath.*, **12** (2019), 1950006. https://doi.org/10.1142/S1793524519500062

45. A. S. Nawar, M. A. El-Gayar, M. K. El-Bably, R. A. Hosny, $\theta\beta$-ideal approximation spaces and their applications, *AIMS Mathematics*, **7** (2022), 2479–2497. https://doi.org/10.3934/math.2022139

46. Z. Pawlak, Rough sets, *Int. J. Inform. Comput. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956

47. Z. Pawlak, *Rough sets: Theoretical aspects of reasoning about data*, Dordrecht: Springer, 1991. https://doi.org/10.1007/978-94-011-3534-4

48. K. Qin, J. Yang, Z. Pei, Generalized rough sets based on reflexive and transitive relations, *Inform. Sci.*, **178** (2008), 4138–4141. https://doi.org/10.1016/j.ins.2008.07.002

49. M. Shokry, R. E. Aly, Topological properties on graph vs medical application in human heart, *Int. J. Appl. Math.*, **15** (2013), 1103–1108.

50. W. Sierpinski, *General topology: (Mathematical expositions No. 7)*, University of Toronto press, 1952. Available from: `https://utorontopress.com/9781487584894/general-topology/`

51. A. Skowron, J. Stepaniuk, Tolerance approximation spaces, *Fund. Inform.*, **27** (1996), 245–253. https://doi.org/10.3233/FI-1996-272311

52. A. Tan, S. Shi, W. Z. Wu, J. Li, W. Pedrycz, Granularity and entropy of intuitionistic fuzzy information and their applications, *IEEE T. Cybernetics*, **52** (2022), 192–204. https://doi.org/10.1109/TCYB.2020.2973379

53. W. Z. Wu, W. X. Zhang, Neighborhood operator systems and approximations, *Inform. Sci.*, **144** (2002), 201–217. https://doi.org/10.1016/S0020-0255(02)00180-9

54. Y. Y. Yao, Generalized rough set models, In: *Rough sets in knowledge discovery 1*, Heidelberg: Physica Verlag, 1998, 286–318.

55. Y. Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, *Inform. Sci.*, **111** (1998), 239–259. https://doi.org/10.1016/S0020-0255(98)10006-3

56. Y. Y. Yao, A Comparative study of fuzzy sets and rough sets, *Inform. Sci.*, **109** (1998), 227–242. https://doi.org/10.1016/S0020-0255(98)10023-3
57. Y. Y. Yao, Granular computing using neighborhood systems, In: *Advances in soft computing*, London: Springer, 1999, 539–553. https://doi.org/10.1007/978-1-4471-0819-1_40
58. Y. Y. Yao, Three-way decisions with probabilistic rough sets, *Inform. Sci.*, **180** (2010), 341–353. https://doi.org/10.1016/j.ins.2009.09.021
59. Y. Y. Yao, Three-way decision and granular computing, *Int. J. Approx. Reason.*, **103**(2018), 107–123. https://doi.org/10.1016/j.ijar.2018.09.005
60. Z. Yu, X. Bai, Z. Yun, A study of rough sets based on 1-neighborhood systems, *Inform. Sci.*, **248** (2013), 103–113. https://doi.org/10.1016/j.ins.2013.06.031
61. C. Zhang, J. Ding, J. Zhan, A. K. Sangaiah, D. Li, Fuzzy intelligence learning based on bounded rationality in IoMT systems: A case study in Parkinson's disease, *IEEE T. Comput. Soc. Syst.*, **10** (2023), 1607–1621. https://doi.org/10.1109/TCSS.2022.3221933
62. C. Zhang, D. Li, J. Liang, Hesitant fuzzy linguistic rough set over two universes model and its applications, *Int. J. Mach. Learn. Cybern.*, **9** (2018), 577–588. https://doi.org/10.1007/s13042-016-0541-z
63. C. Zhang, D. Li, J. Liang, Multi-granularity three-way decisions with adjustable hesitant fuzzy linguistic multigranulation decision-theoretic rough sets over two universes, *Inform. Sci.*, **507** (2020), 665–683. https://doi.org/10.1016/j.ins.2019.01.033
64. P. Zhang, T. Li, C. Luo, G. Wang, AMG-DTRS: Adaptive multi-granulation decision theoretic rough sets, *Int. J. Approx. Reason.*, **140** (2022), 7–30. https://doi.org/10.1016/j.ijar.2021.09.017
65. C. Zhang, D. Li, R. Ren, Pythagorean fuzzy multigranulation rough set over two universes and its applications in merger and acquisition, *Int. J. Intell. Syst.*, **31** (2016), 921–943. https://doi.org/10.1002/int.21811
66. C. Zhang, X. Li, A. K. Sangaiah, W. Li, B. Wang, F. Cao, et al., Collaborative fuzzy linguistic learning to low-resource and robust decision system based on bounded rationality, *ACM Transactions on Asian and Low-Resource Language Information Processing*, 2023. https://doi.org/10.1145/3592605
67. C. Zhang, D. Li, Y. Yan, A dual hesitant fuzzy multigranulation rough set over two-universe model for medical diagnoses, *Comput. Math. Meth. Medicine*, **2015** (2015), 292710. https://doi.org/10.1155/2015/292710
68. N. Zhong, Y. Yao, M. Ohshima, Peculiarity oriented multidatabase mining, *IEEE T. Knowl. Data Eng.*, **15** (2003), 952–960. https://doi.org/10.1109/TKDE.2003.1209011
69. W. Zhu, F. Y. Wang, Covering based granular computing for conflict analysis, In: *Intelligence and security informatics*, Berlin, Heidelberg: Springer, (2006), 566–571. https://doi.org/10.1007/11760146_58

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