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Research article

Iterative approximation of fixed points of generalized α_m-nonexpansive mappings in modular spaces

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Abstract: Our aim of this work is to approximate the fixed points of generalized α_m-nonexpansive mappings employing AA-iterative scheme in the structure of modular spaces. The results of fixed points for generalized α_m-nonexpansive mappings is proven in this context. Moreover, the stability of the scheme and data dependence results are given for m-contraction mappings. In order to demonstrate that the AA-iterative scheme converges faster than some other schemes for generalized α_m-nonexpansive mappings, numerical examples are shown at the end.

Keywords: fixed point; AA-Iteration; modular space; generalized α_m-nonexpansive mapping
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1. Introduction

There are several problems in nonlinear analysis that can be modeled by fixed point equations involving certain nonlinear operators. However, there are many fixed point equations that cannot be solved analytically, for instance $\cos x = x$. To overcome this problem, iterative processes provide useful tools to approximate the fixed point of nonlinear operators. For instance, the equilibrium problem, the variational inequality problem, the saddle point problem, the problem of finding the roots of polynomials, signal processing, image restoration, tomography and intensity-modulated radiation therapy are some well-known problems whose solutions are approximated with some suitable iterative processes. For more details, we refer to [8–12, 24, 31, 37, 38].

The convergence of the iterative process, known as the Picard iterative process, is used to prove the well-known Banach contraction principle [5]. This principle solves a fixed point problem for contraction mappings defined on a complete metric space, and it has become a useful tool for proving the existence and approximation of solutions to nonlinear functional equations such as differential equations, partial differential equations and integral equations. To approximate the solution of linear and nonlinear equations, as well as inclusion, several authors have used various iterative approaches. On the one hand, a study of nonexpansive mappings is crucial due to:

(1) Existence of fixed points of such mappings rely on the geometric properties of the underlying space instead of compactness properties.

(2) These mappings are an important generalization of contraction mappings.

(3) This class of mappings is used as the transition operators for certain initial value problems of differential inclusions involving accretive or dissipative operators.

(4) Different problems appearing in many real life situations involve nonexpansive mappings (see Bruck [7]).

There are several iterative algorithms of practical interest which are generated using nonexpansive mappings. For instance, the successive projection approach is utilized to address intersection problems arising in tomography and signal processing. The proximity operator of a convex function is a nonexpansive mapping that acts as projection in the context of optimization. This is employed in image processing problems like total variation denoising. Nonexpansive mappings are used to model the flow of traffic and congestion dynamics. In signal processing, nonexpansive mappings are used for signal recovery and reconstruction. This is why the study of nonexpansive mappings has attracted the attention of several mathematicians. Notice that the Banach contraction principle states that the sequence generated by Picard iterative scheme converges to a unique fixed point in the case of contraction mappings. The Picard iteration, on the other hand, may not converge to a fixed point of nonexpansive mapping. For instance, if we take the mapping $Tp = 1 - p$ on $[0, 1]$, then the Picard iteration does not converge to its fixed point (which is $\frac{1}{2}$), for any choice of $p \in [0, 1]$ other than $\frac{1}{2}$. Motivated by this fact, several authors proposed and implemented various iterative schemes to approximate fixed points of nonexpansive mappings.

Mann [25] proposed an iterative scheme to approximate the fixed point for nonexpansive mappings. The proposed scheme is given as follows: Let p_1 be an initial guess, then,

$$p_{n+1} = (1 - \eta_n)p_n + \eta_n T(p_n), \quad n \in \mathbb{Z}^+, \tag{1.1}$$

where $\{\eta_n\}$ is an appropriate sequence in $(0, 1)$ and $\mathbb{Z}^+$ represents the set of positive integers.

Later, Ishikawa [16] introduced an iterative scheme to estimate the fixed point of pseudo-contractive mapping. The sequence $\{p_n\}$ proposed by Ishikawa iterative scheme is given as: If p_1 is an initial guess, then,

$$\begin{cases} q_n = (1 - \rho_n)p_n + \rho_n T(p_n), \\ p_{n+1} = (1 - \eta_n)p_n + \eta_n T(q_n), \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.2}$$

where $\{\eta_n\}$ and $\{\rho_n\}$ are appropriate sequences in $(0, 1)$. Similarly, Noor [27], Agarwal et al. [4], Abbas and Nazir [2], Thakur et al. [35], Ullah and Arshad [36] proposed different iterative schemes for approximating the solution of nonlinear problems involving operators satisfying certain contraction conditions. Let p_1 be an initial guess then the following schemes are given as Table 1:

Table 1. Different iterative schemes.

Name	Iterative scheme
Noor	$p_{n+1} = (1-\eta_n)p_n + \eta_n T(q_n)$ $q_n = (1-\rho_n)p_n + \rho_n T(r_n)$ $r_n = (1-\sigma_n)p_n + \sigma_n T(p_n)$
Agarwal et al.	$p_{n+1} = (1-\eta_n)T(p_n) + \eta_n T(q_n)$ $q_n = (1-\rho_n)p_n + \rho_n T(p_n)$
Abbas et al.	$p_{n+1} = (1-\eta_n)T(q_n) + \eta_n T(r_n)$ $q_n = (1-\rho_n)T(p_n) + \rho_n T(r_n)$ $r_n = (1-\sigma_n)p_n + \sigma_n T(p_n)$
Thakur et al.	$p_{n+1} = (1-\eta_n)T(r_n) + \eta_n T(q_n)$ $q_n = (1-\rho_n)r_n + \rho_n T(r_n)$ $r_n = (1-\sigma_n)p_n + \sigma_n T(p_n)$
Ullah	$p_{n+1} = T(q_n)$ $q_n = T(r_n)$ $r_n = (1-\eta_n)p_n + \eta_n T(p_n)$

Where $\{\eta_n\}$, $\{\rho_n\}$ and $\{\sigma_n\}$ are appropriate sequences of parameters in $(0, 1)$.

Later, Abbas et al. [1] proposed an iterative scheme called the AA-iterative scheme that converges faster than the iterative schemes mentioned above. The sequence is defined as follows: For an initial guess p_1,

$$\begin{cases} p_{n+1} &= T(q_n), \\ q_n &= T((1-\eta_n)T(s_n) + \eta_n T(r_n)), \\ r_n &= T((1-\rho_n)s_n + \rho_n T(s_n)), \\ s_n &= (1-\sigma_n)p_n + \sigma_n T(p_n), \quad n \in \mathbb{Z}^+, \end{cases} \tag{1.3}$$

where $\{\eta_n\}$, $\{\rho_n\}$ and $\{\sigma_n\}$ are appropriate sequences in $(0, 1)$.

Most of the iterative processes for a certain class of mappings are primarily defined on Banach spaces along with some appropriate geometric structure, most frequently on uniformly convex spaces. Studying iterative processes on modular spaces is a recent trend and is attracting the attention of several researchers now. This is because modular spaces such as Orlicz spaces or Lebesgue spaces constitute a suitable framework to solve nonlinear problems arising in different branches of mathematics. Kassab and Ţurcanu [18] used the Thakur et al. iterative scheme in the structure of modular spaces. Their mapping in a modular context satisfy the condition (E) in the modular version given by García-Falset et al. [14] (see also, Khan [19] and references mentioned therein). Moreover, several authors have developed and studied different iterative schemes to solve fixed point problems and nonlinear equations (for further details, we refer to [17, 23, 33]). Furthermore, modular spaces provide an appropriate framework to model certain nonlinear problems and hence are being considered by many authors, for instance, see [13, 22, 28] and references mentioned therein.

The study of generalized α_m-nonexpansive mappings has become a very active area of research these days and several interesting results have been obtained in this direction (for example, [15, 32]).

Motivated by the work in [18, 19], we obtained the approximation results using the AA-iterative scheme (1.3) for the mapping required to satisfy a modular counterpart of the generalized α_m-nonexpansive mappings in [30]. Our results extend, unify and generalize the corresponding results that exist in literature.

This paper is structured as follows: Section 1 contains some introductory materials needed in the sequel. In Section 2, we reviewed the definitions regarding modular spaces and their basic properties. Section 3 deals with the mappings satisfying the modular version of generalized α_m-nonexpansive mappings with an example of one of such mappings. In Section 4, we discuss the convergence of the iterative scheme defined in (1.3). The fifth section focuses on the investigation on stability and data dependence. Finally, numerical examples are presented to support the results proved herein.

2. Preliminaries

To make the section self-contained, some basic concepts of modular spaces are presented. Most of these materials are taken from [3, 20, 21, 26].

Definition 2.1. *[20] Let $\mathcal{V}$ be a vector space over $\mathbb{R}$. A mapping m: $\mathcal{V} \to [0, +\infty]$ is called modular if it satisfies the following: For any $p, q \in \mathcal{V}$,*

(1) $m(p) = 0$ if and only if $p = 0$,

(2) $m(\alpha p) = m(p)$ for $|\alpha| = 1$,

(3) $m(\alpha p + (1 - \alpha)q) \leq m(p) + m(q)$, where $\alpha \in [0, 1]$.

If for any $\alpha \in [0, 1]$ and $p, q \in \mathcal{V}$, the condition (3) is replaced with the following condition:

$$m(\alpha p + (1 - \alpha)q) \leq \alpha m(p) + (1 - \alpha)m(q),$$

then m is called a convex modular.

Throughout this paper we shall presume that m is a convex modular.

Example 2.2. Let $\mathcal{V} = \mathbb{R}$ and m: $\mathbb{R} \to [0, +\infty]$ be defined by $m(p) = p^2$. Note that all the conditions of Definition 2.1 are satisfied. Also, m is even and convex and hence is a convex modular. Clearly, m does not satisfy the triangular inequality. Indeed, if we take $p = \frac{1}{2}$ and $q = 2$, then

$$m(\frac{1}{2} + 2) = m(\frac{5}{2}) = \frac{25}{4} > \frac{17}{4} = m(\frac{1}{2}) + m(2).$$

Definition 2.3. *[26] Let m be a convex modular defined on $\mathcal{V}$. The set*

$$\mathcal{V}_m = \{p \in \mathcal{V} : \lim_{\alpha \to 0} m(\alpha p) = 0\}$$

is called a modular space with the norm $||.||_m$ defined as follows:

$$||p||_m = \inf\{\alpha > 0 : m(\frac{p}{\alpha}) \leq 1\}.$$

Definition 2.4. *[20] Assume that $\mathcal{V}$ is a vector space and m a modular function on $\mathcal{V}$. Then:*

(1) A sequence $\{p_n\} \subset \mathcal{V}_m$ is said to be m-convergent to $p \in \mathcal{V}_m$ if $\lim_{n\to+\infty} m(p_n - p) = 0$.

(2) A sequence $\{p_n\} \subset \mathcal{V}_m$ is said to be m-Cauchy if $\lim_{n,r\to+\infty} m(p_n - p_r) = 0$.

(3) We say $\mathcal{V}_m$ is m-complete if any m-Cauchy sequence in $\mathcal{V}_m$ is m-convergent.

(4) A set $C \subset \mathcal{V}_m$ is said to be m-closed if any sequence $\{p_n\} \subset C$ which is m-convergent to some point p implies that $p \in C$.

(5) A set $C \subset \mathcal{V}_m$ is said to be m-bounded if the m-diameter of C is finite.

(6) A set $\mathcal{K} \subset \mathcal{V}_m$ is said to be m-compact if any sequence $\{p_n\} \subset \mathcal{K}$ has a subsequence which is m-convergent to some $p \in \mathcal{K}$.

(7) m satisfies the Fatou property if for any $p, q, q_n \in \mathcal{V}_m$,

$$m(p - q) \leq \liminf_{n\to+\infty} m(p - q_n),$$

whenever $\{q_n\}$ m- converges to q.

Note that the Fatou property is crucial to understand the geometric characteristics of the modular in the framework of normed spaces and modular spaces.

Definition 2.5. *[20] The uniformly convex type properties of m are given as:*

(a) For $\epsilon > 0, r > 0$, define

$$\mathcal{D}_1(r, \epsilon) = \{(p, q) : p, q \in \mathcal{V}_m, m(p) \leq r, m(q) \leq r, m(p - q) \geq \epsilon r\}.$$

If $\mathcal{D}_1(r, \epsilon) \neq \emptyset$, let

$$\rho_1(r, \epsilon) = \inf\left\{1 - \frac{1}{r} m\left(\frac{p + q}{2}\right) : (p, q) \in \mathcal{D}_1(r, \epsilon)\right\}.$$

If $\mathcal{D}_1(r, \epsilon) = \emptyset$, then take $\rho_1(r, \epsilon) = 1$. We say that m fulfills the condition ($\mathcal{UUC}1$) if for every $s, \epsilon > 0$, there exists $\delta_1(s, \epsilon) > 0$, depending on s and ϵ such that

$$\rho_1(r, \epsilon) > \delta_1(s, \epsilon) > 0, \quad \text{for} \quad r > s.$$

(b) For $\epsilon > 0, r > 0$, define

$$\mathcal{D}_2(r, \epsilon) = \left\{(p, q) : p, q \in \mathcal{V}_m, m(p) \leq r, m(q) \leq r, m\left(\frac{p - q}{2}\right) \geq \epsilon r\right\}.$$

If $\mathcal{D}_2(r, \epsilon) \neq \emptyset$, let

$$\rho_2(r, \epsilon) = \inf\left\{1 - \frac{1}{r} m\left(\frac{p + q}{2}\right) : (p, q) \in \mathcal{D}_2(r, \epsilon)\right\}.$$

If $\mathcal{D}_2(r, \epsilon) = \emptyset$, then take $\rho_2(r, \epsilon) = 1$. We say that m satisfies the condition ($\mathcal{UUC}2$) if for every $s, \epsilon > 0$, there exists $\delta_2(s, \epsilon) > 0$, depending on s and ϵ such that

$$\rho_2(r, \epsilon) > \delta_2(s, \epsilon) > 0, \quad \text{for} \quad r > s.$$

Lemma 2.6. *[21] Assume that m is a convex modular which satisfies the condition ($\mathcal{UUC}1$) and let $\{\alpha_n\} \in (0, 1)$ be a sequence bounded away from 0 and 1. Suppose there exists $r > 0$ such that*

$$\limsup_{n\to+\infty} m(p_n) \leq r, \quad \limsup_{n\to+\infty} m(q_n) \leq r, \quad \limsup_{n\to+\infty} m(\alpha_n p_n + (1 - \alpha_n) q_n) = r,$$

where $\{p_n\}$ and $\{q_n\}$ are sequences in $\mathcal{V}_m$. Then, $\lim_{n\to+\infty} m(p_n - q_n) = 0$.

Definition 2.7. *[18] Let $\{p_n\}$ be a sequence in a modular space $\mathcal{V}_m$. Suppose $C \subset \mathcal{V}_m$ is nonempty. The function ϕ: $C \to [0, +\infty]$ is defined by*

$$\phi(p) = \limsup_{n\to+\infty} m(p - p_n),$$

is known as the m-type function related to the sequence $\{p_n\}$.

A sequence $\{x_n\}$ in C is said to be a minimizing sequence of ϕ if $\lim_{n\to+\infty} \phi(x_n) = \inf_{x\in C} \phi(x)$.

Example 2.8. Consider the set of real numbers which is a modular vector space with convex modular $m(p) = p^2$. Take $C = \mathbb{Q} \subset \mathbb{R}$ and the sequence $\{p_n\} = \{\frac{1}{\sqrt{n}}\}, n \geq 1$.

The m-type function in this case is

$$\phi(p) = \limsup_{n\to+\infty} m(p - \frac{1}{\sqrt{n}}) = p^2,$$

which is clearly unbounded. The minimizing sequence $\{x_n\}$ of ϕ is given by $x_n = \frac{1}{n}, n \geq 1$.

Lemma 2.9. *[3] Suppose that $\mathcal{V}_m$ is m-complete and m satisfies the Fatou property. Let C be a nonempty convex and m-closed subset of $\mathcal{V}_m$ and ϕ: $C \to [0, +\infty]$ the m-type function related to the sequence $\{p_n\}$ in $\mathcal{V}_m$.*

Assume that $\phi_0 = \inf_{p\in C} \phi(p) < +\infty$.

(a) If m satisfies the condition ($\mathcal{UUC}1$), then all minimizing sequences of ϕ are m-convergent to the same point.
(b) If m satisfies the condition ($\mathcal{UUC}2$) and $\{x_n\}$ is a minimizing sequence of ϕ, then the sequence $\{\frac{x_n}{2}\}$ m-converges to a point which is independent of $\{x_n\}$.

Definition 2.10. *[18] Suppose that $\mathcal{V}_m$ is a modular space. We say that the modular m fulfills the Δ_2 condition if there exists a constant $J \geq 0$ such that $m(2p) \leq Jm(p)$ for any $p \in \mathcal{V}_m$. We denote the smallest such constant J by π_2.*

Note that the modular defined in Example 2.2 satisfies the Δ_2 condition with $J = 4$.

Lemma 2.11. *[6] Let $\{u_n\}$ and $\{t_n\}$ be sequences of positive real numbers that fulfill the following inequality:*

$$u_{n+1} \leq (1 - \upsilon_n)u_n + t_n,$$

where $\upsilon_n \in (0, 1)$ for all $n \in \mathbb{Z}^+$ with $\sum_{n=0}^{+\infty} \upsilon_n = +\infty$. If $\lim_{n\to+\infty} \frac{t_n}{\upsilon_n} = 0$, then $\lim_{n\to+\infty} u_n = 0$.

Lemma 2.12. *[34] Let $\{u_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers such that there exists n_0 so that for $n \geq n_0$, it satisfies the following inequality:*

$$u_{n+1} \leq (1 - \upsilon_n)u_n + \upsilon_n t_n,$$

where $\upsilon_n \in (0, 1)$ for all $n \in \mathbb{Z}^+$ with $\sum_{n=0}^{+\infty} \upsilon_n = +\infty$. Then,

$$0 \leq \limsup_{n\to+\infty} u_n \leq \limsup_{n\to+\infty} t_n.$$

3. Generalized α_m-nonexpansive mappings

Pant and Shukla [30] introduced the class of generalized α_m-nonexpansive mappings in the context of Banach spaces. Here, we adopt the definition from [30] in the framework of modular spaces.

Definition 3.1. *Suppose that C is a nonempty subset of the modular space $\mathcal{V}_m$. A mapping $T\colon C \to \mathcal{V}_m$ is called the generalized α_m-nonexpansive mappings if there exists $\alpha \in (0,1)$ such that for all $p, q \in C$,*

$$\frac{1}{2}m(p - T(q)) \leq m(p - q)$$

implies that

$$m(T(p) - T(q)) \leq \alpha m(q - T(p)) + \alpha m(p - T(q)) + (1 - 2\alpha)m(p - q).$$

Example 3.2. The modular m established in Example 2.2 presents $\mathbb{R}$ with the modular space. Take the subset of $\mathbb{R}$ that is $C = [0, +\infty)$. Define a mapping $T\colon C \to C$ as follows

$$T(p) = \begin{cases} \frac{p}{2}, & \text{if} \quad p > 2, \\ 0, & \text{if} \quad p \in [0, 2]. \end{cases}$$

Then T is a generalized α_m-nonexpansive mapping. Indeed, take $\alpha = \frac{1}{3}$.

Case (I) Let $p > 2$ and $q > 2$. Then, we have

$$\begin{aligned} \frac{1}{3}m(q - T(p)) + \frac{1}{3}m(p - T(q)) + \frac{1}{3}m(p - q) &= \frac{1}{3}(q - \frac{p}{2})^2 + \frac{1}{3}(p - \frac{q}{2})^2 + \frac{1}{3}(p - q)^2 \\ &= \frac{1}{3}[\frac{9}{4}(p^2 + q^2) - 4pq] \\ &\geq \frac{1}{3}[\frac{9}{4}(p^2 - q^2) + \frac{1}{2}pq] \\ &\geq [\frac{3}{4}(p - q)^2] \\ &\geq [\frac{1}{4}(p - q)^2] \\ &= m(T(p) - T(q)). \end{aligned}$$

Case (II) Let $p > 2$ and $q \in [0, 2]$. Then, we have

$$\begin{aligned} \frac{1}{3}m(q - T(p)) + \frac{1}{3}m(p - T(q)) + \frac{1}{3}m(p - q) &= \frac{1}{3}(q - \frac{p}{2})^2 + \frac{1}{3}(p - 0)^2 + \frac{1}{3}(p - q)^2 \\ &= \frac{1}{3}(\frac{9}{4}p^2 + 2q^2 - 3pq) \\ &\geq \frac{1}{3}\Big(\frac{6}{4}(p^2 + q^2) - \frac{12}{4}pq + \frac{1}{2}q^2 + \frac{3}{4}p^2\Big) \\ &= \frac{1}{3}\Big(\frac{3}{2}(p - q)^2 + \frac{1}{2}q^2 + \frac{3}{4}p^2\Big) \\ &\geq \frac{1}{4}p^2 = m(T(p) - T(q)). \end{aligned}$$

Case (III) Let $p \in [0,2]$ and $q \in [0,2]$. Then, we have

$$\begin{aligned}\frac{1}{3}m(q-T(p))+\frac{1}{3}m(p-T(q))+\frac{1}{3}m(p-q) &= \frac{1}{3}(q-0)^2+\frac{1}{3}(p-0)^2+\frac{1}{3}(p-q)^2\\ &\geq 0\\ &= m(T(p)-T(q)).\end{aligned}$$

So, T is a generalized α_m-nonexpansive mapping.

Throughout this paper we denote the set of fixed points of T by $\mathcal{F}(T)$.

4. Convergence analysis

Theorem 4.1. *Let C be a nonempty m-closed convex subset of a modular space $\mathcal{V}_m$ and $T\colon C \to C$ be a generalized α_m-nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. Choose any $p_1 \in C$ and any $p^* \in \mathcal{F}(T)$. If $\{p_n\}$ is the sequence defined by the AA-iterative scheme* (1.3)*, and $m(p_j - p^*) < +\infty$ for some $j \geq 1$, then $\lim_{n\to+\infty} m(p_n - p^*)$ exists for all $p^* \in \mathcal{F}(T)$.*

Proof. Let $p^* \in \mathcal{F}(T)$. Since T is a generalized α_m-nonexpansive mapping it satisfies the condition of m-nonexpansive given in (Definition 4.1 in [3]). Using the iterative scheme (1.3) and the convexity of m, we have

$$\begin{aligned}m(s_n-p^*) &= m((1-\sigma_n)p_n+\sigma_n T(p_n)-p^*)\\ &\leq (1-\sigma_n)m(p_n-p^*)+\sigma_n m(T(p_n)-p^*).\end{aligned} \tag{4.1}$$

Recall that T is a generalized α_m-nonexpansive mapping with $T(p^*) = p^*$, hence, we have

$$\begin{aligned}m(T(p_n)-p^*) &\leq \alpha m(p^*-T(p_n))+\alpha m(p_n-T(p^*))+(1-2\alpha)m(p_n-p^*)\\ &\leq \alpha[m(p^*-T(p^*))+m(T(p_n)-T(p^*))]+\alpha m(p_n-T(a^*))\\ &\quad +(1-2\alpha)m(p_n-p^*)\\ &\leq m(p_n-p^*).\end{aligned} \tag{4.2}$$

Using (4.2) in (4.1), we obtain

$$m(s_n-p^*) \leq (1-\sigma_n)m(p_n-p^*)+\sigma_n m(p_n-p^*) = m(p_n-p^*). \tag{4.3}$$

If

$$t_n = (1-\rho_n)s_n+\rho_n T(s_n),$$

then,

$$m(r_n-p^*) = m(T(t_n)-p^*). \tag{4.4}$$

Now, since $T(p^*) = p^*$,

$$\begin{aligned}m(T(t_n)-T(p^*)) &\leq \alpha m(p^*-T(t_n))+\alpha m(t_n-T(p^*))+(1-2\alpha)m(t_n-p^*)\\ &\leq m(t_n-p^*).\end{aligned} \tag{4.5}$$

Also,

$$m(t_n - p^*) = m((1-\rho_n)s_n + \rho_n T(s_n) - p^*)$$
$$\leq (1-\rho_n)m(s_n - p^*) - \rho_n m(T(s_n) - p^*), \tag{4.6}$$

and again using $T(p^*) = p^*$,

$$m(T(s_n) - p^*) \leq \alpha m(p^* - s_n) + \alpha m(s_n - T(p^*)) + (1-2\alpha)m(s_n - p^*)$$
$$\leq m(s_n - p^*). \tag{4.7}$$

Putting (4.7) and (4.3) in (4.6), we get

$$m(t_n - p^*) \leq m(p_n - p^*). \tag{4.8}$$

Using (4.8) and (4.5), we have

$$m(T(t_n) - p^*) \leq m(p_n - p^*). \tag{4.9}$$

By (4.9) and (4.4) we get

$$m(r_n - p^*) \leq m(p_n - p^*). \tag{4.10}$$

Now, taking

$$u_n = (1-\eta_n)T(s_n) + \eta_n T(r_n),$$

so, $T(u_n) = q_n$. Then,

$$m(q_n - p^*) \leq m(T(u_n) - p^*) \leq m(u_n - p^*)$$
$$\leq \alpha m(p^* - T(u_n)) + \alpha m(u_n - T(p^*)) + (1-2\alpha)m(u_n - p^*)$$
$$\leq \alpha m(T(u_n) - (p^*)) + (1-\alpha)m(u_n - p^*)$$
$$\leq m(u_n - p^*). \tag{4.11}$$

Also,

$$m(u_n - p^*) = m((1-\eta_n)T(s_n) + \eta_n T(r_n) - p^*)$$
$$\leq (1-\eta_n)m(T(s_n) - p^*) + \eta_n m(T(r_n) - p^*) \tag{4.12}$$
$$\leq (1-\eta_n)m(T(s_n) - p^*) + \eta_n m(T(r_n) - p^*)$$

and

$$m(T(r_n) - p^*) \leq \alpha m(p^* - T(r_n)) + \alpha m((r_n) - T(p^*)) + (1-2\alpha)m(r_n - p^*)$$
$$\leq \alpha m(T(r_n) - (p^*)) + (1-\alpha)m(r_n - p^*)$$
$$\leq m(r_n - p^*). \tag{4.13}$$

Using (4.13) and (4.7) in (4.12), we obtain

$$m(u_n - p^*) \leq m(p_n - p^*). \tag{4.14}$$

Putting (4.14) in (4.11), we get

$$m(q_n - p^*) \leq m(p_n - p^*). \tag{4.15}$$

Now,

$$m(p_{n+1} - p^*) = m(T(q_n) - p^*) \tag{4.16}$$

and

$$\begin{aligned} m(T(q_n) - p^*) &\leq \alpha m(p^* - T(q_n)) + \alpha m(q_n - T(p^*)) + (1 - 2\alpha) m(q_n - p^*) \\ &\leq \alpha m(T(q_n) - T(p^*)) + (1 - \alpha) m(q_n - p^*) \\ &\leq m(q_n - p^*). \end{aligned} \tag{4.17}$$

Putting (4.17) in (4.16), we have

$$m(p_{n+1} - p^*) \leq m(q_n - p^*). \tag{4.18}$$

From (4.18) and (4.15), we obtain

$$m(p_{n+1} - p^*) \leq m(p_n - p^*). \tag{4.19}$$

This shows that $\{m(p_n - p^*)\}$ is decreasing and bounded below, hence, $\lim_{n\to+\infty} m(p_n - p^*)$ exists. □

Lemma 4.2. *Let C be a nonempty subset of a modular space $\mathcal{V}_m$ and $T\colon C \to C$ be a generalized α_m-nonexpansive mapping. Assume that there exists a bounded sequence $\{p_n\} \subset C$ such that*

$$\lim_{n\to+\infty} m(p_n - T(p_n)) = 0$$

and let ϕ be the m-type function defined by the sequence $\{p_n\}$. Then T leaves the minimizing sequence invariant, i.e., if $\{x_n\}$ is a minimizing sequence for ϕ, then, so is $\{T x_n\}$.

Proof. Assume that $\{p_n\} \subset C$ is such that

$$\lim_{n\to+\infty} m(p_n - T(p_n)) = 0.$$

For any $p \in C$, we have

$$\begin{aligned} m(p_n - T(p)) &\leq m(p_n - T(p_n)) + m(T(p_n) - T(p)) \\ &\leq m(p_n - p), \end{aligned} \tag{4.20}$$

which implies that

$$\phi(T(p)) = \limsup_{n\to+\infty} m(p_n - T(p)) \leq \limsup_{n\to+\infty} m(p_n - p) = \phi(p). \tag{4.21}$$

Now, assume that $\{x_n\}$ is a minimizing sequence. Using (4.21), we get

$$\inf_{p\in C} \phi(p) \leq \lim_{n\to+\infty} \phi(T(x_n)) \leq \lim_{n\to+\infty} \phi(x_n) = \inf_{p\in C} \phi(p). \tag{4.22}$$

This implies that

$$\lim_{n\to+\infty} \phi(T(x_n)) = \inf_{p\in C} \phi(p).$$

So, $\{T(x_n)\}$ is a minimizing sequence for ϕ. □

Lemma 4.3. *Assume that C is a nonempty m-closed and convex subset of a m-complete modular space $\mathcal{V}_m$ and m is ($\mathcal{UUC}1$) which fulfill the Δ_2 condition and the Fatou property. Assume that the m-type function ϕ: $C \to [0, +\infty]$ is defined by a sequence $\{p_n\}$ in $\mathcal{V}_m$ and that*

$$\phi_0 = \inf_{p \in C} \phi(p) < +\infty.$$

Let $\{x_n\}$ and $\{y_n\}$ be two minimizing sequences for ϕ. Then,

(i) *each convex combination of $\{x_n\}$ and $\{y_n\}$ is a minimizing sequence for ϕ,*
(ii) $\lim_{n\to+\infty} m(x_n - y_n) = 0$.

Proof. (i) Let $z_n = \lambda x_n + (1-\lambda)y_n, \lambda \in (0, 1),\ n \geq 1$. Then for any $p \in C$, we get

$$m(z_n - p) \leq \lambda m(x_n - p) + (1 - \lambda)m(y_n - p),$$

which gives

$$\limsup_{r\to+\infty} m(z_n - p_r) \leq \lambda \limsup_{r\to+\infty} m(x_n - p_r) + (1 - \lambda) \limsup_{r\to+\infty} m(y_n - p_r), \quad n \geq 1.$$

That is,

$$\phi(z_n) \leq \lambda\phi(x_n) + (1 - \lambda)\phi(y_n).$$

Taking limit and using the fact that $\{x_n\}$ and $\{y_n\}$ are minimizing sequences, we obtain

$$\phi_0 = \inf_{p\in C} \phi(p) \leq \lim_{n\to+\infty} \phi(z_n) \leq \lambda\phi_0 + (1 - \lambda)\phi_0 = \phi_0$$

as required.

(ii) Note that for $\lambda = \frac{1}{2}$, $z_n = \frac{1}{2}x_n + \frac{1}{2}y_n$, $n \geq 1$, we have

$$x_n - y_n = 2(z_n - y_n),$$

using (i), $\{z_n\}$ is a minimizing sequence and by using Lemma 2.9, each minimizing sequence m-converge to the same point, say r. Thus,

$$m(z_n - y_n) = m\left(\frac{x_n - y_n}{2}\right) \leq \frac{1}{2}m(x_n - r) + \frac{1}{2}m(y_n - r).$$

We get $\lim_{n\to+\infty} m(z_n - y_n) = 0$. Now, using Δ_2 condition we have

$$m(x_n - y_n) \leq \frac{\pi_2}{2}[m(z_n - y_n)].$$

By taking limit as $n \to +\infty$ we get the required result. □

Theorem 4.4. *Let C be a nonempty m-closed, m-bounded and convex subset of a m-complete modular space $\mathcal{V}_m$. Suppose that m satisfies the condition ($\mathcal{UUC}1$), the Δ_2 condition and the Fatou property. Let T: $C \to C$ be a generalized α_m-nonexpansive mapping and $\{p_n\}$ be a sequence given in the AA-iterative scheme* (1.3). *Then $\mathcal{F}(T) \neq \emptyset$ if and only if*

$$\lim_{n\to+\infty} m(p_n - T(p_n)) = 0$$

and $\{p_n\}$ is bounded.

Proof. Suppose $\mathcal{F}(T) \neq \emptyset$ and $p^* \in \mathcal{F}(T)$. By the above Theorem 4.1, $\lim_{n\to+\infty} m(p_n - p^*)$ exists and $\{p_n\}$ is bounded. Put

$$\lim_{n\to+\infty} m(p_n - p^*) = k. \tag{4.23}$$

From (4.3), (4.10) and (4.15), we have

$$\limsup_{n\to+\infty} m(s_n - p^*) \leq \limsup_{n\to+\infty} m(p_n - p^*) = k, \tag{4.24}$$

$$\limsup_{n\to+\infty} m(r_n - p^*) \leq \limsup_{n\to+\infty} m(p_n - p^*) = k, \tag{4.25}$$

$$\limsup_{n\to+\infty} m(q_n - p^*) \leq \limsup_{n\to+\infty} m(a_n - p^*) = k. \tag{4.26}$$

It follows from (4.2) that

$$\limsup_{n\to+\infty} m(T(p_n) - p^*) \leq k. \tag{4.27}$$

Thus,

$$\begin{aligned} m(p_{n+1} - p^*) &= m(T(q_n) - T(p^*)) \\ &\leq m(q_n - p^*). \end{aligned} \tag{4.28}$$

By taking lim inf as $n \to +\infty$, we get

$$k \leq \liminf_{n\to+\infty} m(q_n - p^*). \tag{4.29}$$

From (4.26) and (4.29), we have

$$\lim_{n\to+\infty} m(q_n - p^*) = k. \tag{4.30}$$

Now, from (4.28), we obtain that

$$\begin{aligned} m(p_{n+1} - p^*) &\leq m(q_n - p^*) \\ &\leq m(T(r_n) - p^*) \\ &\leq m(r_n - p^*), \end{aligned} \tag{4.31}$$

which on taking lim inf as $n \to +\infty$ gives that

$$k \leq \liminf_{n\to+\infty} m(r_n - p^*). \tag{4.32}$$

By (4.25) and (4.32), we get

$$\lim_{n\to+\infty} m(r_n - p^*) = k.$$

From (4.31), we have

$$\begin{aligned} m(p_{n+1} - p^*) &\leq m(r_n - p^*) \\ &\leq m(T(r_n) - p^*) \\ &\leq m(s_n - p^*). \end{aligned} \tag{4.33}$$

On taking lim inf as $n \to +\infty$, we obtain that

$$k \leq \liminf_{n\to+\infty} m(s_n - p^*). \tag{4.34}$$

Thus, from (4.3) and (4.34), we get

$$\lim_{n\to+\infty} m(s_n - p^*) = k.$$

Also,

$$\begin{aligned} k &= \lim_{n\to+\infty} m(s_n - p^*) \\ &= \lim_{n\to+\infty} m((1-\sigma_n)p_n + \sigma_n T(p_n)) - p^*) \\ &\leq \lim_{n\to+\infty} (1-\sigma_n)m(p_n - p^*) + \sigma_n m(T(p_n) - p^*) \\ &\leq \lim_{n\to+\infty} (1-\sigma_n)m(p_n - p^*) + \sigma_n m(p_n - p^*) \\ &\leq \lim_{n\to+\infty} m(p_n - p^*) \\ &\leq k. \end{aligned}$$

Hence,

$$\lim_{n\to+\infty} m((1-\sigma_n)(p_n - p^*) + \sigma_n(T(p_n) - p^*)) = k. \tag{4.35}$$

From (4.23), (4.27), (4.35) and Lemma 2.6, we get

$$\lim_{n\to+\infty} m(p_n - T(p_n)) = 0.$$

Conversely, assume that $\{p_n\}$ is a bounded sequence and

$$\lim_{n\to+\infty} m(p_n - T(p_n)) = 0.$$

Let ϕ: $C \to [0, +\infty]$ be the m-type function generated by $\{p_n\}$ and suppose that $\{x_n\}$ is a minimizing sequence for ϕ converging to a point $r \in C$. Using Lemmas 4.2 and 4.3, we get

$$\lim_{n\to+\infty} m(x_n - T(x_n)) = 0.$$

On the other hand using (4.20) as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} m(x_n - T(r)) = 0.$$

This gives that $\{p_n\}$ converges to $T(r)$. As we know, limit is always unique, we get $T(r) = r$. □

5. Stability and data dependence

Ostrowski [29] established the concept of stability for a fixed point iterative technique. The analogue of the Ostrowski definition in the framework of modular spaces is given as follows:

Definition 5.1. *Assume that C is a nonempty subset of a modular space $\mathcal{V}_m$ and $\{\varrho_n\}$ is an approximate sequence of $\{p_n\}$ in C. Then, the iterative process $p_{n+1} = \hbar(T, p_n)$ for a function $\hbar$, converging to a fixed point p^* of $T\colon C \to C$ is called T-stable or stable w.r.t. T if the condition that*

$$\lim_{n\to+\infty} \ell_n = 0$$

is equivalent to the condition that $\{\varrho_n\}$ is m-convergent to p^*. Here $\{\ell_n\}$ is defined as

$$\ell_n = m(\varrho_{n+1} - \hbar(T, \varrho_n)), \text{ for all } n \in \mathbb{Z}^+.$$

Theorem 5.2. *Assume that C is a nonempty m-closed and convex subset of a m-complete modular space $\mathcal{V}_m$ and $T\colon C \to C$ is a m-contraction mapping with contraction constant c. Then, the iterative scheme defined in* (1.3) *is T-stable.*

Proof. Let $\{\varrho_n\}$ be an approximate sequence of $\{p_n\}$ in C. The sequence defined by the iteration (1.3) is:

$$p_{n+1} = \hbar(T, p_n)$$

and

$$\ell_n = m(\varrho_{n+1} - \hbar(T, \varrho_n)), \quad n \in \mathbb{N}.$$

We show that $\lim_{n\to+\infty} \ell_n = 0$ if and only if

$$\lim_{n\to+\infty} m(\varrho_n - p^*) = 0.$$

If $\lim_{n\to+\infty} \ell_n = 0$, then, using the convexity modular m, Δ_2 condition, it follows from (1.3) that

$$\begin{aligned} m(\varrho_{n+1} - p^*) &\le m(\varrho_{n+1} - \hbar(T, \varrho_n)) + m(\hbar(T, \varrho_n) - p^*) \\ &= \ell_n + m(\varrho_{n+1} - p^*). \end{aligned} \tag{5.1}$$

Also,

$$m(\varrho_{n+1} - p^*) \le \ell_n + c^3[1 - (1 - c)(\eta_n + \sigma_n - \eta_n\sigma_n)]m(\varrho_n - p^*).$$

Let

$$\alpha_n = m(\varrho_n - p^*) \text{ and } \beta_n = (1 - c)(\eta_n + \sigma_n - \eta_n\sigma_n),$$

then,

$$\alpha_{n+1} \le c^3(1 - \beta_n)\alpha_n + \ell_n.$$

Since, $\lim_{n\to+\infty} \ell_n = 0$, we get

$$\frac{\ell_n}{\beta_n} \to 0 \text{ as } n \to +\infty.$$

Hence, by Lemma 2.11, we have

$$\lim_{n\to+\infty} m(\varrho_n - p^*) = 0.$$

On the other hand, if

$$\lim_{n\to+\infty} m(\varrho_n - p^*) = 0,$$

then, we have

$$\begin{aligned}
\ell_n &= m(\varrho_{n+1} - \hbar(T, \varrho_n)) \\
&\leq m(\varrho_{n+1} - p^*) + m(\hbar(T, \varrho_n) - p^*) \\
&\leq m(\varrho_{n+1} - p^*) + c^3[1 - (1-c)(\eta_n + \sigma_n - \eta_n\sigma_n)]m(\varrho_n - p^*).
\end{aligned} \tag{5.2}$$

This implies that $\lim_{n\to+\infty} \ell_n = 0$. Hence, the iterative scheme (1.3) is T-stable. □

Definition 5.3. *Assume that $T, \hat{T}\colon C \to C$ are two mappings. Then $\hat{T}$ is known as an approximate mapping of T, if there exists $\epsilon > 0$ such that, for all $p \in C$, we have*

$$m(T(p) - \hat{T}(p)) \leq \epsilon.$$

Theorem 5.4. *Assume that $\hat{T}$ is an approximate operator of a m-contraction T with maximum acceptable error ϵ. Let $\{p_n\}$ be an iterative sequence generated by* (1.3) *and define an iterative scheme $\hat{p}_n$ as follows:*

$$\begin{cases}
\hat{p}_{n+1} &= \hat{T}(\hat{q}_n), \\
\hat{q}_n &= \hat{T}((1-\eta_n)\hat{T}(\hat{s}_n) + \eta_n\hat{T}(\hat{r}_n)), \\
\hat{r}_n &= \hat{T}((1-\rho_n)\hat{s}_n + \rho_n\hat{T}(\hat{s}_n)), \\
\hat{s}_n &= (1-\sigma_n)\hat{p}_n + \sigma_n\hat{T}(\hat{p}_n),
\end{cases} \tag{5.3}$$

with real sequences $\{\eta_n\}$, $\{\rho_n\}$ and $\{\sigma_n\}$ in $(0,1)$ satisfying $\eta_n\rho_n\sigma_n \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. If $T(p^*) = p^*$ and $\hat{T}(\hat{p^*}) = \hat{p^*}$, such that

$$\lim_{n\to+\infty} m(\hat{p}_n - \hat{p^*}) = 0,$$

then,

$$m(p^* - \hat{p^*}) \leq \frac{7\pi_2^2\epsilon}{4 - 2\pi_n c}.$$

Proof. By convexity and the Δ_2 property

$$\begin{aligned}
m(s_n - \hat{s}_n) &= m((1-\sigma_n)p_n + \sigma_n T(p_n) - (1-\sigma_n)\hat{p}_n - \sigma_n\hat{T}(\hat{p}_n)) \\
&\leq (1-\sigma_n)m(p_n - \hat{p}_n) + \sigma_n m(T(p_n) - \hat{T}(\hat{p}_n)) \\
&\leq (1-\sigma_n)m(p_n - \hat{p}_n) + \sigma_n\frac{\pi_2}{2}\Big(m(T(p_n) - T(\hat{p}_n)) + m(T(\hat{p}_n) - \hat{T}(\hat{p}_n))\Big) \\
&\leq \Big(1 - \sigma_n + \sigma_n\frac{\pi_2}{2}c\Big)m(p_n - \hat{p}_n) + \sigma_n\frac{\pi_2}{2}\epsilon.
\end{aligned} \tag{5.4}$$

Now, let

$$t_n = (1-\rho_n)s_n + \rho_n T(s_n).$$

So,

$$\begin{aligned}
m(r_n - \hat{r}_n) &= m(T(t_n) - \hat{T}(\hat{t}_n)) \\
&\leq \frac{\pi_2}{2}\Big(m(T(t_n) - T(\hat{t}_n)) + m(T(\hat{t}_n) - \hat{T}(\hat{t}_n))\Big) \\
&\leq c\frac{\pi_2}{2}m(t_n - \hat{t}_n) + \frac{\pi_2}{2}\epsilon.
\end{aligned} \tag{5.5}$$

Using similar arguments as in (5.4) we get

$$m(t_n - \hat{t}_n) \leq \Big(1 - \rho_n + \rho_n \frac{\pi_2}{2} c\Big) m(s_n - \hat{s}_n) + \rho_n \frac{\pi_2}{2} \epsilon. \tag{5.6}$$

Using (5.4) and (5.6) in (5.5) we get

$$m(r_n - \hat{r}_n) \leq c\frac{\pi_2}{2}\Big(1 - \rho_n + \rho_n \frac{\pi_2}{2} c\Big)\Big[\Big(1 - \sigma_n + \sigma_n \frac{\pi_2}{2} c\Big) m(p_n - \hat{p}_n) + \sigma_n \frac{\pi_2}{2}\epsilon\Big] + \frac{\pi_2}{2}\epsilon.$$

Now, suppose that

$$v_n = (1 - \eta_n)T(s_n) + \eta_n T(r_n),$$

then,

$$\begin{aligned} m(q_n - \hat{q}_n) &= m(T(v_n) - \hat{T}(\hat{v}_n)) \\ &\leq c\frac{\pi_2}{2} m(v_n - \hat{v}_n) + \frac{\pi_2}{2}\epsilon. \end{aligned} \tag{5.7}$$

Following arguments similar to those given above, we get

$$m(v_n - \hat{v}_n) \leq \frac{\pi_2}{2} c\Big[1 - \eta_n\rho_n \frac{\pi_2}{2} c(1 - \frac{\pi_2}{2} c)\Big] m(p_n - \hat{p}_n) + (\sigma_n\eta_n + \sigma_n + \eta_n)(\frac{\pi_2}{2})^2 c\epsilon + \frac{\pi_2}{2}\epsilon. \tag{5.8}$$

Therefore (5.7) becomes

$$\begin{aligned} &\leq (\frac{\pi_2}{2})^2 c^2\Big[1 - \eta_n\rho_n \frac{\pi_2}{2} c(1 - \frac{\pi_2}{2} c)\Big] m(p_n - \hat{p}_n) + (\sigma_n\eta_n + \sigma_n + \eta_n)(\frac{\pi_2}{2})^3 c^2\epsilon \\ &\quad + (\frac{\pi_2}{2})^2 c\epsilon + \frac{\pi_2}{2}\epsilon. \end{aligned} \tag{5.9}$$

Now, using similar arguments as in (5.4), we have

$$\begin{aligned} m(p_{n+1} - \hat{p}_{n+1}) &= m(T(q_n) - \hat{T}(\hat{q}_n)) \\ &\leq (\frac{\pi_2}{2})^3 c^3\Big[1 - \eta_n\rho_n \frac{\pi_2}{2} c(1 - \frac{\pi_2}{2} c)\Big] m(p_n - \hat{p}_n) + (\sigma_n\eta_n + \sigma_n + \eta_n)(\frac{\pi_2}{2})^4 c^3\epsilon \\ &\quad + (\frac{\pi_2}{2})^3 c^2\epsilon + (\frac{\pi_2}{2})^2 c\epsilon + \frac{\pi_2}{2}\epsilon \\ &\leq \Big[1 - \eta_n\rho_n \frac{\pi_2}{2} c(1 - \frac{\pi_2}{2} c)\Big] m(p_n - \hat{p}_n) + 7\frac{\pi_2}{2}\eta_n\rho_n\epsilon. \end{aligned} \tag{5.10}$$

Taking

$$u_n = m(p_n - \hat{p}_n), \;\; \upsilon_n = \eta_n\rho_n \frac{\pi_2}{2} c(1 - \frac{\pi_2}{2} c)$$

and

$$t_n = \frac{7\pi_2}{2 - \pi_2 c}\epsilon$$

in Lemma 2.12, we get

$$\limsup_{n\to+\infty} \frac{7\pi_2}{2 - \pi_2 c}\epsilon \geq \limsup_{n\to+\infty} m(p_n - \hat{p}_n) \geq 0. \tag{5.11}$$

Also,

$$m(p^* - \hat{p}^*) \leq \frac{\pi_2}{2} m(p_n - \hat{p}_n) + [\frac{\pi_2}{2}]^2 \Big(m(p_n - p^*) + m(\hat{p}_n - \hat{p}^*) \Big).$$

Taking $\lim\limits_{n\to+\infty}$ and using the inequality (5.11) we obtain

$$m(p^* - \hat{p}^*) \leq \frac{7\pi_2^2 \epsilon}{4 - 2\pi_2 c}.$$

□

6. Numerical experiments

In this section, we present the numerical experiment for the applicability of our results. We used the MATLAB version R2018a for all of the numerical calculations. We compare the iterative scheme (1.3) with the existing methods for generalized α_m- nonexpansive mapping and used the Example 3.2 to implement our results. Moreover, we take different initial guesses and parameters for comparison.

Example 6.1. Let T be the generalized α_m-nonexpansive mapping defined in Example 3.2. We now discuss a numerical experiment to substantiate the convergence of iteration (1.3). Take an initial value $p_1 = 150$ and take different sequences of parameters, i.e., those we used for Figure 1,

$$\eta_n = \frac{2n^2 + n}{n^4 + 4n + 7}, \quad \rho_n = \frac{n - 1}{n^2 + n + 7}, \quad \sigma_n = \frac{2n - 1}{n^2 + n + 5},$$

for Figure 2,

$$\eta_n = \frac{n^2 + n}{n^2 + 3n + 7}, \quad \rho_n = \frac{n + 1}{n^2 + n + 1}, \quad \sigma_n = \frac{n^2 + 1}{n^2 + n + 3},$$

for Figure 3,

$$\eta_n = \frac{2n^2 + n + 1}{n^3 + 4n^2 - 1}, \quad \rho_n = \frac{(n - 1)^3}{n^5 + 2n + 1}, \quad \sigma_n = \frac{2n^4 - 1}{n^5 + n^2 + 3}$$

and for Figure 4,

$$\eta_n = \frac{2n^3 + n - 1}{4n^3 + 4n^2 + 2}, \quad \rho_n = \frac{(n - 1)^2}{n^3 + 5n^2 + 5}, \quad \sigma_n = \frac{2n^2 + n - 2}{4n^3 + n^2 + n + 7}.$$

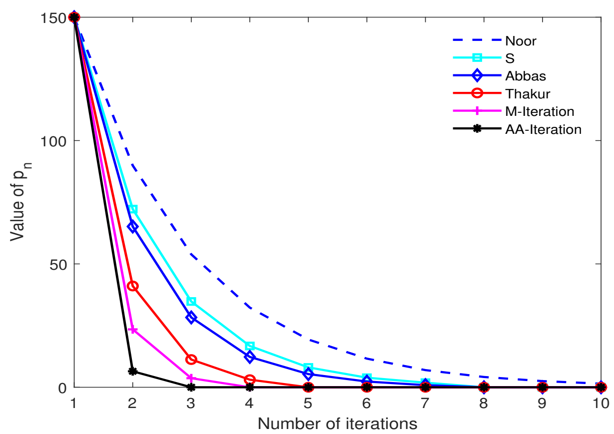

Figure 1. $\eta_n = \frac{2n^2+n}{n^4+4n+7}$, $\rho_n = \frac{n-1}{n^2+n+7}$ and $\sigma_n = \frac{2n-1}{n^2+n+5}$.

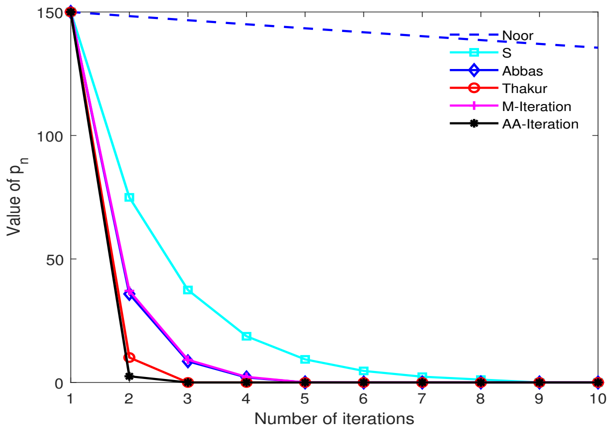

Figure 2. $\eta_n = \frac{n^2+n}{n^2+3n+7}$, $\rho_n = \frac{n+1}{n^2+n+1}$ and $\sigma_n = \frac{n^2+1}{n^2+n+3}$.

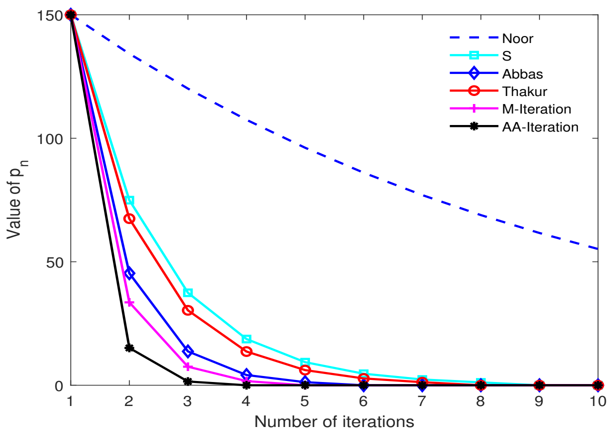

Figure 3. $\eta_n = \frac{2n^2+n+1}{n^3+4n^2-1}$, $\rho_n = \frac{(n-1)^3}{n^5+2n+1}$ and $\sigma_n = \frac{2n^4-1}{n^5+n^2+3}$.

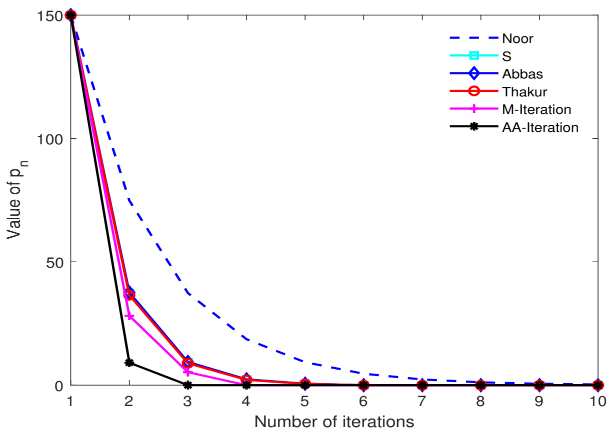

Figure 4. $\eta_n = \frac{2n^3+n-1}{4n^3+4n^2+2}$, $\rho_n = \frac{(n-1)^2}{n^3+5n^2+5}$ and $\sigma_n = \frac{2n^2+n-2}{4n^3+n^2+n+7}$.

Note that by the graphs, the convergence of iterative schemes (1.3) converges faster than the other schemes for generalized α_m-nonexpansive mapping for all choices of the used parameters. On the other hand, the other iterative schemes change their convergence behaviors by changing the parameters. Moreover, we have considered different initial guesses and observed that the whole space becomes the basin of attraction for the scheme considered herein. Thus, the iterative scheme we used is superior in this framework of study.

Example 6.2. Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows:

$$T(p, q) = (\frac{p}{2}, \frac{q}{2}).$$

Clearly, the mapping T is a generalized α_m-nonexpansive mapping with $\alpha = \frac{1}{4}$ and $m = \|.\|_2$. We illustrate the convergence behavior of the different iterative schemes along with the iteration (1.3) and take different initial values as shown in Figures 5–8 for the convergence and comparison.

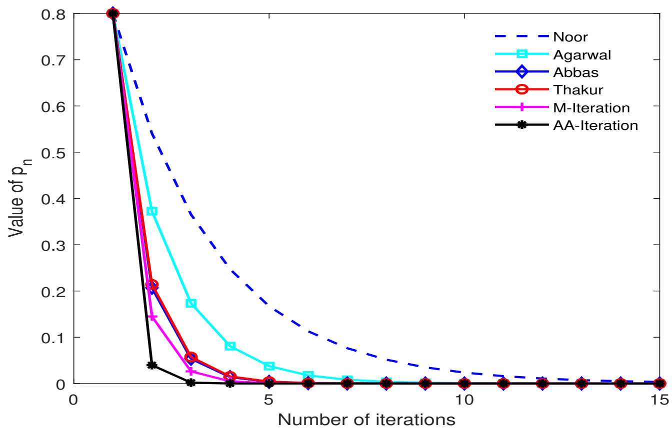

Figure 5. For $(p_1, q_1) = (0.15, 0.65)$.

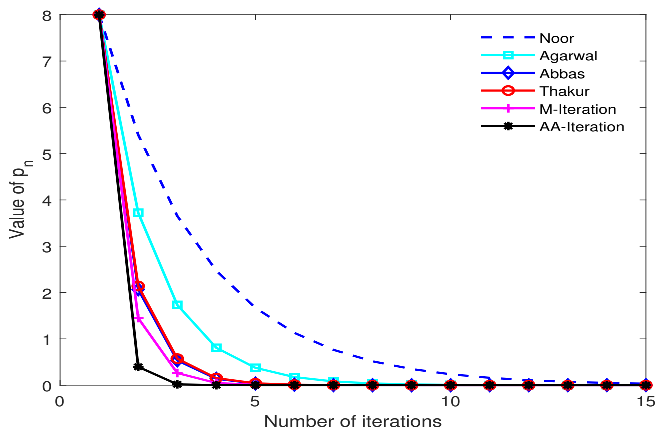

Figure 6. For $(p_1, q_1) = (3, 5)$.

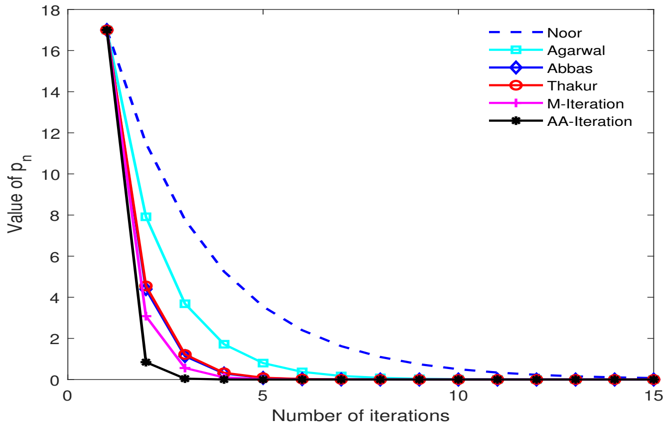

Figure 7. For $(p_1, q_1) = (10, 7)$.

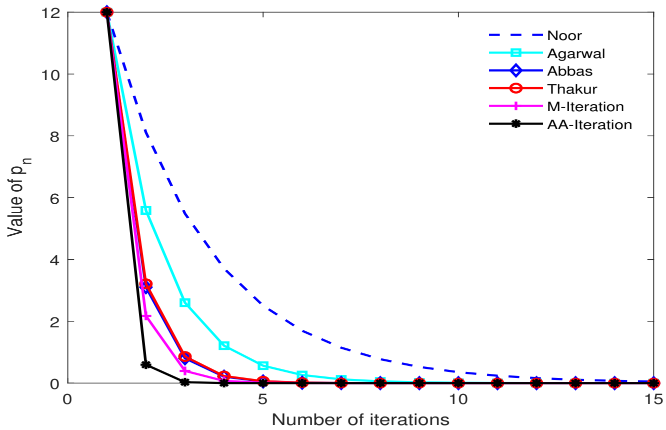

Figure 8. For $(p_1, q_1) = (6, 6)$.

7. Conclusions

Our aim of this work is to study the AA-iterative scheme proposed by Abbas et al. [1] to approximate the fixed points of generalized α_m-nonexpansive mappings in the structure of modular spaces. We proposed adequate requirements regarding the convergence of the iterative scheme to approximate the solution of a fixed point equation involving α_m-nonexpansive mappings in the framework of uniformly convex type modular spaces. Numerical examples are given and demonstrate that the AA-iterative method converges faster than certain known schemes for generalized α_m-nonexpansive mappings in the context of modular spaces. For future direction, one may apply the AA-iterative algorithm in image processing problems involving α_m-nonexpansive mappings such as image denoising and reconstruction. One may also enhance the convergence speed and preserve the image features as compared to existing methods.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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