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Research article

Multiplicity of solutions to non-local problems of Kirchhoff type involving Hardy potential

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Abstract: The aim of this paper is to establish the existence of a sequence of infinitely many small energy solutions to nonlocal problems of Kirchhoff type involving Hardy potential. To this end, we used the Dual Fountain Theorem as a key tool. In particular, we describe this multiplicity result on a class of the Kirchhoff coefficient and the nonlinear term which differ from previous related works. To the best of our belief, the present paper is the first attempt to obtain the multiplicity result for nonlocal problems of Kirchhoff type involving Hardy potential by utilizing the Dual Fountain Theorem.

Keywords: fractional p-Laplacian; Kirchhoff function; Hardy potential; weak solution; Dual Fountain Theorem
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1. Introduction

In the last few decades, an increasing deal of attention has been devoted to the study of fractional Sobolev spaces and the corresponding nonlocal equations because they can be corroborated as a model for many physical phenomena, which arose in the research of optimization, fractional quantum mechanics, the thin obstacle problem, anomalous diffusion in plasma, frames propagation, geophysical fluid dynamics, American options in finances, image process, game theory and Lévy processes; see [9, 26, 38, 52, 58].

In this paper, we are concerned with a Kirchhoff type problem driven by the nonlocal fractional p-Laplacian as follows:

$$\begin{cases} M\left([v]_{s,p}\right)\mathcal{L}v(y) + \frac{|v|^{p-2}v}{|y|^{sp}} = \lambda h(y,v) & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary $\partial\Omega$, $[v]_{s,p} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y) - v(z)|^p \mathcal{K}(y,z)\,dy\,dz$, $M \in C(\mathbb{R}^+)$ is a Kirchhoff type function, $s \in (0,1)$, $p \in (1,+\infty)$, $sp < N$

and $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition satisfying the subcritical and p-superlinear nonlinearity. Here $\mathcal{L}$ is non-local operator defined pointwise as

$$\mathcal{L}v(y) = 2\int_{\mathbb{R}^N} |v(y) - v(z)|^{p-2}(v(y) - v(z))\mathcal{K}(y,z)dz \quad \text{for all } y \in \mathbb{R}^N,$$

where $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ is a kernel function with the following properties

($\mathcal{L}1$) $m\mathcal{K} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, where $m(y,z) = \min\{|y - z|^p, 1\}$;

($\mathcal{L}2$) there exists a positive constant γ_0 such that $\mathcal{K}(y,z) \geq \gamma_0 |y-z|^{-(N+sp)}$ for almost all $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$ and $y \neq z$;

($\mathcal{L}3$) $\mathcal{K}(y,z) = \mathcal{K}(z,y)$ for all $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$.

When $\mathcal{K}(y,z) = |y-z|^{-(N+sp)}$, the operator $\mathfrak{L}$ becomes the fractional p-Laplacian operator $(-\Delta)^s_p$ defined as

$$(-\Delta)^s_p v(y) = 2\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(y)} \frac{|v(y) - v(z)|^{p-2}\,(v(y) - v(z))}{|y - z|^{N+sp}}\,dz, \quad y \in \mathbb{R}^N,$$

where $B_\varepsilon(y) := \{y \in \mathbb{R}^N : |y - z| \leq \varepsilon\}$.

Let us assume that the Kirchhoff function $M : [0,\infty) \to \mathbb{R}^+$ fulfills the conditions as follows:

($\mathcal{K}1$) $M \in C(\mathbb{R}^+)$ fulfils $\inf_{\zeta \in \mathbb{R}^+} M(\zeta) \geq m_0 > 0$, where m_0 is a constant;

($\mathcal{K}2$) there exist a constant $\vartheta \geq 1$ and a nonnegative constant K such that $\vartheta\mathcal{M}(\zeta) = \vartheta\int_0^\zeta M(\tau)d\tau \geq M(\zeta)\zeta$ and

$$\widehat{\mathcal{M}}(t\zeta) \leq \widehat{\mathcal{M}}(\zeta) + K$$

for $\zeta \geq 0$ and $t \in [0,1]$, where $\widehat{\mathcal{M}}(\zeta) = \vartheta\mathcal{M}(\zeta) - M(\zeta)\zeta$.

In order to study an extension of the classic D'Alembert's wave equation by taking the changes in the length of the strings during the vibrations into account, Kirchhoff [37] initially proposed a stationary version of the equation:

$$\rho\frac{\partial^2 v}{\partial\zeta^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial v}{\partial y}\Big| dy\Big)\frac{\partial^2 v}{\partial y^2} = 0,$$

where ρ, ρ_0, h, L and E are constants. The variational problems of Kirchhoff type have attractively interested diverse applications in physics and have been extensively investigated by many researchers in recent years; see [3, 5, 13, 15, 27, 30, 41, 43, 44, 48, 49, 54, 63, 67, 69]. Fiscella-Valdinoci [22] first gave a detailed discussion about the physical meaning underlying the fractional Kirchhoff model. In particular, by taking into account the mountain pass theorem and a truncation argument, they established the existence of nontrivial solutions to a nonlocal elliptic problem with the nondegenerate Kirchhoff terms that is an increasing and continuous function, see also [53]. This increasing condition gets rid of the case that is not monotone. In 2015, Pucci, Xiang and Zhang [54] established the existence of multiple solutions to a class of Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian when the continuous Kirchhoff function M with ($\mathcal{K}1$) satisfies the following condition:

($\mathcal{K}3$) For $0 < s < 1$, there exists $\vartheta \in [1, \frac{N}{N-sp})$ such that $\vartheta\mathcal{M}(\zeta) \geq M(\zeta)\zeta$ for any $\zeta \geq 0$.

Very recently, the existence result of a positive ground state solution for elliptic problem of Kirchhoff type with critical exponential growth has been investigated [28] when the Kirchhoff function holds the condition:

($\mathcal{K}4$) There exists $\vartheta > 1$ such that $\frac{M(\zeta)}{\zeta^{\vartheta-1}}$ is nonincreasing for $\zeta > 0$.

From this condition and a simple calculation, it is immediate that $\widehat{\mathcal{M}}(\zeta)$ is nondecreasing for all $\zeta \geq 0$ and thus we get

($\mathcal{K}5$) there exists $\vartheta > 1$ such that $\vartheta\mathcal{M}(\zeta) \geq M(\zeta)\zeta$ for any $\zeta \geq 0$.

Hence we know that the condition ($\mathcal{K}3$) is weaker than ($\mathcal{K}5$). A typical model for M satisfying ($\mathcal{K}1$) and ($\mathcal{K}3$) (or ($\mathcal{K}5$)) is given by $M(\zeta) = 1 + a\zeta^{\vartheta}$ with $a \geq 0$ for all $\zeta \geq 0$. Hence the conditions ($\mathcal{K}4$) and ($\mathcal{K}5$) include the above classical example as well as the case that is not monotone. In this light, many researchers in recent years have tended to focus on the nonlinear elliptic equations with Kirchhoff coefficient satisfying ($\mathcal{K}3$) (or ($\mathcal{K}5$)); see [4, 15, 21, 27, 54, 64–66]. However, the present paper is devoted to deriving the multiplicity result of solutions to our problem on a class of a nonlocal Kirchhoff coefficient M, which differs slightly from the above related works. For example, let us consider

$$M(\zeta) = \left(1 + \frac{\zeta^r}{\sqrt{1+\zeta^{2r}}}\right)\zeta^{r-1} + (1+\zeta)^{-\alpha}$$

with its primitive function

$$\mathcal{M}(\zeta) = \frac{1}{r}\left(\zeta^r + \sqrt{1+\zeta^{2r}} - 1\right) + \frac{1}{1-\alpha}(1+\zeta)^{1-\alpha} - \frac{1}{1-\alpha}$$

for all $\zeta \geq 0$. Then it is clear that

$$\widehat{\mathcal{M}}(\zeta) = \left(\frac{\vartheta}{r} - 1\right)\zeta^r + \left(\frac{\vartheta}{r} - \frac{\zeta^{2r}}{1+\zeta^{2r}}\right)\sqrt{1+\zeta^{2r}} + \left(\frac{\vartheta}{1-\alpha}(1+\zeta) - \zeta\right)(1+\zeta)^{-\alpha} - \frac{\vartheta}{1-\alpha} - \frac{\vartheta}{r}.$$

If $r = 2$ and $N = 4$ in ($\mathcal{K}3$), then we cannot find a constant $\vartheta \in [1, 2)$ satisfying $\widehat{\mathcal{M}}(\zeta) \geq 0$ for any $\zeta \geq 0$ by being $\lim_{\zeta\to\infty} \widehat{\mathcal{M}}(\zeta) = -\infty$. Also, if we set $r = \vartheta = 1.5$ and $1 < \alpha \leq r$, then we have $\widehat{\mathcal{M}}(\zeta)$ is not nondecreasing and $\widehat{\mathcal{M}}(\zeta) \geq 0$ for all $\zeta \geq 0$ from a direct computation. Hence this example does not satisfy the condition ($\mathcal{K}4$). This implies that

$$\widehat{\mathcal{M}}(\zeta) - \widehat{\mathcal{M}}(t\zeta) \geq 0$$

does not hold. However we can choose a positive constant K satisfying our condition ($\mathcal{K}2$).

The main reason for considering the Kirchhoff coefficient satisfying ($\mathcal{K}2$) is closely related to condition (B2) among the following conditions of the nonlinear term h:

(B1) $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist a $\rho_2 > 0$ and a function $0 \leq \rho_1 \in L^\infty(\Omega)$ such that

$$|h(y, \zeta)| \leq \rho_1(y) + \rho_2 |\zeta|^{\ell-1}$$

for all $(y, \zeta) \in \Omega \times \mathbb{R}$ where $p < \ell < p_s^*$;

(B2) there exists a constant $C > 0$ such that

$$\mathcal{H}(y,\zeta) \le \mathcal{H}(y,t) + C$$

for any $y \in \Omega$ and $0 < \zeta < t$ or $t < \zeta < 0$, where $H(y,\zeta) = \int_0^\zeta h(y,\tau)\,d\tau$ and $\mathcal{H}(y,\zeta) = h(y,\zeta)\zeta - p\vartheta H(y,\zeta)$;

(B3) there exist $C > 0$, $1 < m < p$ and a positive function $\nu \in L^\infty(\Omega)$ such that

$$\liminf_{|\zeta|\to 0} \frac{h(y,\zeta)}{\nu(y)\,|\zeta|^{m-2}\,\zeta} \ge C$$

uniformly for almost all $y \in \Omega$.

Let us consider the function

$$h(y,\zeta) = \rho(y)\left(\nu(y)\,|\zeta|^{m-2}\,\zeta + |\zeta|^{\ell-2}\,\zeta\ln(1+|\zeta|) + \frac{|\zeta|^{\ell-1}\,\zeta}{1+|\zeta|}\right)$$

with its primitive function

$$H(y,\zeta) = \rho(y)\left(\frac{\nu(y)}{m}\,|\zeta|^m + \frac{1}{\ell}\,|\zeta|^\ell\ln(1+|\zeta|)\right)$$

for all $\zeta \in \mathbb{R}$, where $p < \ell$ and $\rho \in C(\mathbb{R}^N,\mathbb{R})$ with $0 < \inf_{y\in\Omega}\rho(y) \le \sup_{y\in\Omega}\rho(y) < \infty$ and ν, m are given in (B3). Then, this example fulfills the assumptions (B1)–(B3).

In particular, the condition (B2) is firstly considered by Miyagaki-Souto [50] in the case of $p \equiv 2$. Under this condition, the authors established the existence of a nontrivial solution for the superlinear problems. Inspired by this work, Li-Yang [40] proved the existence of at least one nontrivial weak solution to the following elliptic Dirichlet problem

$$\begin{cases} -\Delta_p v = \lambda h(y,v) & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

see also [46,47]. Some researchers have tried to generalize the results of Miyagaki-Souto. For example, Wei-Su [61] obtained the existence of infinitely many weak solutions to the fractional Laplacian problem, and Choudhuri [13] carried out an investigation of the existence of infinitely many solutions to a fractional p-Kirchhoff-type problem involving a superlinear term and a singular nonlinearity. The existence of a nontrivial solution for the $p(x)$-Laplacian Dirichlet problems can be found in Ge [24]. Following basic ideas of Li-Yang [40], Chung-Toan [14] established the existence and multiplicity results to a class of nonlinear and nonhomogeneous problems in an Orlicz-Sobolev spaces setting.

In order to illustrate such existence results to the superlinear p-Laplacian problems, the Carathéodory function $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ fulfills the conditions (B1), (B2) and

(h) $H(y,\zeta) = o(|\zeta|^p)$ as $\zeta \to 0$ uniformly for all $y \in \Omega$.

However, even if we proceed the analogous ways from others' research [14, 24, 40, 46, 47, 50], the same existence results cannot be obtained because of the presence of a nonlocal Kirchhoff coefficient

M. More precisely, under assumptions (B2) and (h) we cannot guarantee the compactness condition of the Palais-Smale type for an energy functional corresponding to (1.1) when conditions ($\mathcal{K}1$) and ($\mathcal{K}5$) hold. Especially, to ensure this compactness condition of an energy functional corresponding to problems of elliptic type with the nonlinear term satisfying (B2), the fact that $\widehat{\mathcal{M}}(\zeta)$ is nondecreasing for all $\zeta \geq 0$ is essential. Because of this reason, when ($\mathcal{K}5$) holds, many researchers have considered some conditions of the nonlinear term which is different from (B2); see [4, 15, 21, 27, 54, 55, 64–66]. From this perspective, one of the novelties of the present paper is to establish the existence of infinitely many small energy solutions to (1.1) without the monotonicity of $\widehat{\mathcal{M}}$ and without assuming the condition (h) which is crucial to verify the compactness condition of Palais-Smale type and ensure assumptions in the Dual Fountain Theorem. These arguments are motivated by the recent work [35].

On the other hand, stationary problems involving singular nonlinearities arise in the context of chemical catalyst kinetics and chemical heterogeneous catalysts in the theory of heat conduction in electrically conducting materials, and in the study of relativistic matter in magnetic fluid; see [16, 17, 51]. Moreover, motivated by this large interest, singular problems have been investigated more in the recent years; see [13, 19, 31–33, 42, 45, 69]. For a very recent study on the existence of solutions to nonlocal singular problems with variable exponents, we refer to Aberqi and Ouaziz [1]. In the local setting ($s = 1$), Ferrara-Bisci [19] studied the existence of at least one nontrivial weak solution of the following Dirichlet boundary value problem

$$\begin{cases} -\Delta_p v = \lambda \frac{|v|^{p-2}v}{|y|^p} + \mu h(y, v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu > 0$ and $\lambda \geq 0$ are two real parameters, $1 < p < N$ and $h : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying a suitable subcritical growth condition. The main tool is a refinement of the variational principle of Ricceri [56]. Inspired by this work and by employing three critical points theorem [57], Khodabakhshi-Hadjian [33] obtained the existences of three weak solutions of the following problem:

$$\begin{cases} -\Delta_p v + \frac{|v|^{p-2}v}{|y|^p} = \lambda f(y, v) + \mu h(y, v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where f and h are Carathódory functions; see [42] for double phase problems. For $\mu = 0$ in (1.2), Liu-Zhao [45] investigated the existence of triple solutions to problem (1.2) with Dirichlet-Neumann boundary conditions. Also the multiplicity results of solutions to (1.2) with $\mu = 0$ have been provided by the works of Khodabakhshi-Aminpour-Afrouzi-Hadjian [31] and Khodabakhshi-Afrouzi-Hadjian [32]. The main tools for obtaining these existence results of multiple solutions are various critical point theorems of either Ricceri's type in [56, 57] or Bonanno's type in [7, 8]. Another new aspect of this paper is to consider a different approach from [19,31–33,42,45] to derive the multiplicity result to the nonlocal elliptic problems involving the Hardy potential. This approach is inspired by Chen-Thin [12] and Fiscella [20]. The authors in [12] obtained the multiplicity result of solutions to the nonlocal $p_1\&\cdots\&p_m$ fractional Laplacian problems of Kirchhoff type with the Hardy potential when the Kirchhoff coefficients satisfied ($\mathcal{K}3$) and a condition on h differed from (B2). In [20], the existence of multiple solutions to fractional p-Laplacian equation of Schrödinger-Kirchhoff-Hardy type in $\mathbb{R}^N$ has been investigated when $M(\zeta) = a + b\zeta^{\vartheta}$ ($a > 0$, $b \geq 0$), which can be regarded as a special case of ($\mathcal{K}3$). The main tool for obtaining such multiplicity results is the Fountain Theorem. In order to apply this theorem, the compactness condition of the Palais-Smale type for an energy functional is essential. However, as mentioned before, we cannot ensure this condition from the analogous way as in [12, 20] if we assume the conditions ($\mathcal{K}3$) and (B1)–(B3).

To this end, by taking advantage of the Dual Fountain Theorem as the key tool, we illustrate the existence of a sequence of infinitely many small energy solutions on a class of the Kirchhoff coefficient M and the nonlinear term h, which are different from the previous related works [4, 12, 15, 20, 21, 27, 54, 55, 64, 65]. To the best of our belief, although the basic idea of our proof for obtaining this existence result of multiple solutions comes from recent studies [35, 36], this paper is the first effort to establish the existence of a sequence of small energy solutions to nonlocal problems of Kirchhoff type with Hardy potential, using the Dual Fountain Theorem as a primary tool.

The outline of this paper is as follows. We present some necessary preliminary knowledge of function spaces for the present paper. Next, we provide the variational framework related to problem (1.1), and then we illustrate the existence result of infinitely many nontrivial small energy solutions under suitable assumptions.

2. Preliminaries

In this section, we shortly present some useful definitions and fundamental properties of the fractional Sobolev spaces that will be used in the present paper. Let $0 < s < 1 < p < +\infty$ be real numbers and p_s^* is the fractional critical Sobolev exponent, that is

$$p_s^* := \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \geq N. \end{cases}$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. We define the fractional Sobolev space $W^{s,p}(\Omega)$ as follows:

$$W^{s,p}(\Omega) := \left\{ v \in L^p(\Omega) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y) - v(z)|^p}{|y - z|^{N+ps}} \, dydz < +\infty \right\},$$

endowed with the norm

$$|v|_{W^{s,p}(\Omega)} := \left(|v|_{L^p(\Omega)}^p + |v|_{W^{s,p}(\mathbb{R}^N)}^p \right)^{\frac{1}{p}},$$

where

$$|v|_{L^p(\Omega)}^p := \int_\Omega |v(y)|^p \, dy \qquad \text{and} \qquad |v|_{W^{s,p}(\mathbb{R}^N)}^p := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(y) - v(z)|^p}{|y - z|^{N+ps}} \, dydz.$$

Then $W^{s,p}(\Omega)$ is a separable and reflexive Banach space. The space $C_0^\infty(\Omega)$ is dense in $W^{s,p}(\Omega)$, that is $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$ ([2, 52]).

Lemma 2.1. *Let $s \in (0, 1)$ and $p \in (1, +\infty)$, then the following continuous embeddings hold [52]:*

$$\begin{aligned} &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for all } q \in [1, p_s^*], && \text{if } sp < N; \\ &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \text{for every } q \in [1, \infty), && \text{if } sp = N; \\ &W^{s,p}(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) && \text{for all } \lambda < s - N/p, && \text{if } sp > N. \end{aligned}$$

In particular, the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, p_s^*)$.

We define the fractional Sobolev space $W^{s,p}_{\mathcal{K}}(\mathbb{R}^N)$ as follows:

$$W^{s,p}_{\mathcal{K}}(\mathbb{R}^N) := \left\{ v \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dydz < +\infty \right\},$$

where $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,+\infty)$ is a kernel function with the properties $(\mathcal{L}1)$–$(\mathcal{L}3)$. By the condition $(\mathcal{L}1)$, the function

$$(y,z) \mapsto (v(y)-v(z))\mathcal{K}^{\frac{1}{p}}(y,z) \in L^p(\mathbb{R}^N)$$

for all $v \in C_0^\infty(\mathbb{R}^N)$. We consider the problem (1.1) in the closed linear subspace defined by

$$X := \left\{ v \in W^{s,p}_{\mathcal{K}}(\mathbb{R}^N) : v(y) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\}$$

with respect to the norm

$$\|v\|_X := \left(\|v\|^p_{L^p(\Omega)} + |v|^p_X \right)^{\frac{1}{p}},$$

where

$$|v|^p_X := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dydz.$$

The following useful Lemmas 2.2 and 2.3 can be found in [65].

Lemma 2.2. *Let $0 < s < 1 < p < +\infty$ with $ps < N$, and let $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ be a kernel function satisfying the conditions $(\mathcal{L}1)$–$(\mathcal{L}3)$. If $v \in X$, then $v \in W^{s,p}(\Omega)$. Moreover, we have*

$$\|v\|_{W^{s,p}(\Omega)} \le \max\{1, \gamma_0^{-\frac{1}{p}}\}\|v\|_X,$$

where γ_0 is given in $(\mathcal{L}2)$.

From Lemmas 2.1 and 2.2, we can obtain the following assertion immediately.

Lemma 2.3. *Let $0 < s < 1 < p < +\infty$ with $ps < N$, and let $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ satisfy the conditions $(\mathcal{L}1)$–$(\mathcal{L}3)$. Then there exists a positive constant $C_0 = C_0(N,p,s)$ such that for any $v \in X$ and $1 \le q \le p_s^*$,*

$$\begin{aligned} \|v\|^p_{L^q(\Omega)} &\le C_0 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(y)-v(z)|^p}{|y-z|^{N+ps}} dydz \\ &\le \frac{C_0}{\gamma_0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dydz, \end{aligned}$$

where γ_0 is given in $(\mathcal{L}2)$. Consequently, the space X is continuously embedded in $L^q(\Omega)$ for any $q \in [1, p_s^*]$. In addition, the embedding

$$X \hookrightarrow L^q(\Omega)$$

is compact for $q \in (1, p_s^*)$.

The following assertion is the fractional Hardy inequality given in research by Frank-Seiringer [23].

Lemma 2.4. *Let $N \geq 1,\ 0 < s < 1 \leq p$ and let $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ satisfy the conditions $(\mathcal{L}1)$–$(\mathcal{L}3)$. Then for all $v \in X$, in case $sp < N$, and for all $v \in X \backslash \{0\}$, in case $sp > N$,*

$$\begin{aligned}\int_\Omega \frac{|v(y)|^p}{|y|^{sp}} dy &\leq c_H \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(y)-v(z)|^p}{|y-z|^{N+sp}} dydz,\\ &\leq \frac{c_H}{\gamma_0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p\, \mathcal{K}(y,z)dydz,\end{aligned}$$

where $c_H := c_H(N,s,p)$ is a positive constant.

Throughout this paper, the kernel function $\mathcal{K} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ ensures the assumptions $(\mathcal{L}1)$–$(\mathcal{L}3)$. Also, let $0 < s < 1 < p < +\infty$ with $ps < N$, and let the Kirchhoff function M satisfy the conditions $(\mathcal{K}1)$ and $(\mathcal{K}2)$. Moreover, $\langle \cdot, \cdot \rangle$ denotes the pairing of X and its dual X^*.

3. Variational setting and main result

In this section, the existence result of multiple small energy solutions to (1.1) is provided by taking into account the Dual Fountain Theorem under appropriate assumptions. Before going to our main result, we introduce the variational setting corresponding to the problem (1.1).

Definition 3.1. *We say that $v \in X$ is a weak solution of* (1.1) *if*

$$\begin{aligned}M\left([v]_{s,p}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^{p-2}(v(y)-v(z))(\omega(y)-\omega(z))\mathcal{K}(y,z)\,dydz\\ + \int_\Omega \frac{|v(y)|^{p-2}}{|y|^{sp}} v\omega\,dy = \lambda \int_\Omega h(y,v)\omega\,dy\end{aligned}$$

for any $\omega \in X$, where

$$[v]_{s,p} := \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^p \mathcal{K}(y,z)\,dy\,dz.$$

Let us define the functional $A : X \to \mathbb{R}$ by

$$A(v) = \mathcal{M}\left([v]_{s,p}\right) + \frac{1}{p}\int_\Omega \frac{|v(y)|^p}{|y|^{sp}}\,dy.$$

Thus, it is not difficult to prove that A is well defined on X, and we get the following result if we follow the lines of the proof of [54, Lemma 2].

Lemma 3.2. *The functional $A : X \to \mathbb{R}$ is of class $C^1(X,\mathbb{R})$ and its Fréchet derivative is*

$$\begin{aligned}\langle A'(v),\omega\rangle =& M([v]_{s,p})\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^{p-2}\,(v(y)-v(z))(\omega(y)-\omega(z))\mathcal{K}(y,z)dydz\\ &+ \int_\Omega \frac{|v(y)|^{p-2}}{|y|^{sp}} v\omega dy\end{aligned} \tag{3.1}$$

for any $v, \omega \in X$.

Proof. It is easy to show that A has Fréchet derivative in X and (3.1) holds for any $v, \omega \in X$. Let $\{w_n\}_{n\in\mathbb{N}} \subset X$ be a sequence satisfying $w_n \to w$ strongly in X as $n \to \infty$. Without loss of generality, we suppose that $w_n \to w$ a.e. in $\mathbb{R}^N$. Then the sequence

$$\left\{|w_n(y) - w_n(z)|^{p-2}(w_n(y) - w_n(z))\mathcal{K}(y,z)^{\frac{1}{p'}}\right\}_{n\in\mathbb{N}}$$

is bounded in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$, as well as a.e. in $\mathbb{R}^N \times \mathbb{R}^N$

$$\begin{aligned}&\mathcal{A}_n(y,z) := |w_n(y) - w_n(z)|^{p-2}(w_n(y) - w_n(z))\mathcal{K}(y,z)^{\frac{1}{p'}}\\ &\longrightarrow \mathcal{A}(y,z) := |w(y) - w(z)|^{p-2}(w(y) - w(z))\mathcal{K}(y,z)^{\frac{1}{p'}} \quad \text{as } n \to \infty.\end{aligned}$$

Thus, by virtue of the Brezis-Lieb Lemma [6], one has

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\mathcal{A}_n(y,z) - \mathcal{A}(y,z)|^{p'}\,dy\,dz\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(|w_n(y) - w_n(z)|^p\mathcal{K}(y,z) - |w(y) - w(z)|^p\mathcal{K}(y,z)\right)\,dy\,dz.\end{aligned} \tag{3.2}$$

The fact that $\{w_n\}_{n\in\mathbb{N}}$ converges strongly to w in X as $n \to \infty$ yields that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(|w_n(y) - w_n(z)|^p\mathcal{K}(y,z) - |w(y) - w(z)|^p\mathcal{K}(y,z)\right)\,dy\,dz = 0.$$

Owing to (3.2), we infer that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\mathcal{A}_n(y,z) - \mathcal{A}(y,z)|^{p'}\,dy\,dz = 0. \tag{3.3}$$

Furthermore, by the continuity of M, we deduce

$$\lim_{n\to\infty} M\left([w_n]_{s,p}\right) = M\left([w]_{s,p}\right). \tag{3.4}$$

On the other hand, the sequence

$$\left\{\frac{|w_n(y)|^{p-2}w_n(y)}{|y|^{\frac{sp}{p'}}}\right\}_{n\in\mathbb{N}}$$

is bounded in $L^{p'}(\Omega)$, as well as a.e. in Ω

$$\widetilde{\mathcal{A}}_n(y,z) := \frac{|w_n(y)|^{p-2}w_n(y)}{|y|^{\frac{sp}{p'}}} \longrightarrow \widetilde{\mathcal{A}}(y,z) := \frac{|w(y)|^{p-2}w(y)}{|y|^{\frac{sp}{p'}}} \quad \text{as } n \to \infty.$$

Thus, we have

$$\lim_{n\to\infty}\int_\Omega \left|\widetilde{\mathcal{A}}_n(y,z) - \widetilde{\mathcal{A}}(y,z)\right|^{p'}\,dy = \lim_{n\to\infty}\int_\Omega\left(\frac{|w_n(y)|^p}{|y|^{sp}} - \frac{|w(y)|^p}{|y|^{sp}}\right)dy. \tag{3.5}$$

The fact that $\{w_n\}_{n\in\mathbb{N}}$ converges strongly to w in X as $n \to \infty$ and Lemma 2.4 yield that

$$\lim_{n\to\infty}\int_\Omega\left(\frac{|w_n(y)|^p}{|y|^{sp}} - \frac{|w(y)|^p}{|y|^{sp}}\right)dy = \lim_{n\to\infty}\int_\Omega \frac{|w_n(y) - w(y)|^p}{|y|^{sp}}dy$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty} \frac{c_H}{\gamma_0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |(w_n-w)(y)-(w_n-w)(z)|^p\,\mathcal{K}(y,z)dydz\\
&\leq \lim_{n\to\infty} \frac{c_H}{\gamma_0} \|w_n-w\|_X^p\\
&= 0. \qquad (3.6)
\end{aligned}
$$

In accordance with (3.5) and (3.6), it follows that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|\widetilde{\mathcal{A}}_n(y,z)-\widetilde{\mathcal{A}}(y,z)\right|^{p'}\,dy\,dz = 0. \qquad (3.7)
$$

Combining (3.3), (3.4) and (3.7) with the Hölder inequality, we arrive

$$
\|A'(w_n)-A'(w)\|_{X^*} = \sup_{w\in X, \|w\|_X=1} |\langle A'(w_n)-A'(w),\omega\rangle| \longrightarrow 0
$$

as $n\to\infty$. Therefore, we assert $A\in C^1(X,\mathbb{R})$. □

Let the functional $B_\lambda : X\to\mathbb{R}$ be defined by

$$
B_\lambda(v) = \lambda\int_\Omega H(y,v)\,dy.
$$

Then it is obvious that $B_\lambda\in C^1(X,\mathbb{R})$ and its Fréchet derivative is

$$
\langle B'_\lambda(v),\omega\rangle = \lambda\int_\Omega h(y,v)\omega\,dy
$$

for any $v,\omega\in X$.

Next, the functional $I_\lambda : X\to\mathbb{R}$ is defined by

$$
I_\lambda(v) = A(v) - B_\lambda(v). \qquad (3.8)
$$

Afterward, the functional $I_\lambda\in C^1(X,\mathbb{R})$, and its Fréchet derivative is

$$
\begin{aligned}
\langle I'_\lambda(v),\omega\rangle =& M\left([v]_{s,p}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y)-v(z)|^{p-2}(v(y)-v(z))(\omega(y)-\omega(z))\mathcal{K}(y,z)\,dy\,dz\\
&+\int_\Omega \frac{|v(y)|^{p-2}}{|y|^{sp}}v\omega\,dy - \lambda\int_\Omega h(y,v)\omega\,dy
\end{aligned}
$$

for any $v,\omega\in X$.

The following definition was initially provided by Cerami [11].

Definition 3.3. *We say that I_λ satisfies the Cerami condition at level c ($(C)_c$-condition for short) in X, if any $(C)_c$-sequence $\{v_n\}_{n\in\mathbb{N}}\subset X$, i.e., $I_\lambda(v_n)\to c$ and $\|I'_\lambda(v_n)\|_{X^*}(1+\|v_n\|_X)\to 0$ as $n\to\infty$, has a convergent subsequence in X.*

From now on, we present the useful preliminary consequences, which play an indispensable role to deriving our main result.

Lemma 3.4. *Suppose that* (B1) *holds, then B_λ and B'_λ are weakly strongly continuous on X for any $\lambda > 0$.*

Proof. Let us assume that $v_n \rightharpoonup v$ in X as $n \to \infty$. We begin by proving prove that B_λ is weakly strongly continuous on X, and then it is easy to check that $v_n \to v$ in $L^\kappa(\Omega)$ where $1 < \kappa < p_s^*$ by Lemma 2.3. By the convergence principle, there exist a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, in X and a function $u \in L^p(\Omega)$ such that $v_n(y) \to v(y)$ for almost all $y \in \Omega$ as $n \to \infty$ and $|v_n(y)| \le u(y)$ for all $n \in \mathbb{N}$ and for almost all $y \in \Omega$. Therefore, taking (B1) into account, we deduce

$$\begin{aligned}\int_\Omega |H(y, v_n) - H(y, v)|\, dy &\le \int_\Omega \rho_1(y)|v_n(y)| + \rho_2|v_n(y)|^\ell + \rho_1(y)|v(y)| + \rho_2|v(y)|^\ell\, dy \\ &\le \int_\Omega \rho_1(y)|u(y)| + \rho_2|u(y)|^\ell + \rho_1(y)|v(y)| + \rho_2|v(y)|^\ell\, dy\end{aligned}$$

and thus the integral at the left-hand side is dominated by an integrable function. Since h is the Carathéodory function, we have that $H(y, v_n) \to H(y, v)$ as $n \to \infty$ for almost all $y \in \mathbb{R}^N$ by (B1). By the Lebesgue dominated convergence theorem, we have

$$\int_\Omega H(y, v_n)\, dy \to \int_\Omega H(y, v)\, dy$$

as $n \to \infty$. This implies that B_λ is weakly strongly continuous on X.

Next we prove that B'_λ is weakly strongly continuous on X. Note that

$$\sup_{\|\varphi\|_X \le 1} \left|\langle B'_\lambda(v_n) - B'_\lambda(v), \varphi\rangle\right| = \sup_{\|\varphi\|_X \le 1} \left|\int_{\mathbb{R}^N} (h(y, v_n) - h(y, v))\varphi\, dy\right|$$

for any $\varphi \in X$. Since $1 < p < p_s^*$, the compact embedding

$$X \hookrightarrow\hookrightarrow L^p(\Omega) \quad \text{implies} \quad v_n \to v \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad n \to \infty.$$

This together with the continuity of the Nemytskii operator with h and acting from $L^p(\Omega)$ into $L^{\ell'}(\Omega)$ yields that the right side of the equality (3.9) tends to 0 as $n \to \infty$. This implies that B'_λ is weakly strongly continuous in X. Therefore, the proof is completed. □

Lemma 3.5. *Suppose that* (B1) *and* (B2) *hold. Furthermore, we assume that*

(B4) $\lim_{|s|\to\infty} \frac{H(y,s)}{|s|^{\vartheta p}} = \infty$ *uniformly for almost all $y \in \mathbb{R}^N$.*

Then, the functional I_λ satisfies the $(C)_c$-condition for any $\lambda > 0$.

Proof. For any $c \in \mathbb{R}$, let $\{v_n\}_{n\in\mathbb{N}}$ be a $(C)_c$-sequence in X, *i.e.,*

$$I_\lambda(v_n) \to c \text{ and } \langle I'_\lambda(v_n), v_n\rangle = o(1) \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

We first prove that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in X. To this end, arguing by contradiction, it is assumed that $\|v_n\|_X > 1$ and $\|v_n\|_X \to \infty$ as $n \to \infty$, and a sequence $\{\omega_n\}_{n\in\mathbb{N}}$ is defined by $\omega_n = v_n/\|v_n\|_X$. Then, up to a subsequence, still denoted by $\{\omega_n\}_{n\in\mathbb{N}}$, we know $\omega_n \rightharpoonup \omega$ in X as $n \to \infty$, and due to Lemma 2.3, one has

$$\omega_n(y) \to \omega(y) \text{ a.e. in } \Omega, \quad \text{and} \quad \omega_n \to \omega \text{ in } L^\kappa(\Omega) \tag{3.10}$$

as $n \to \infty$, where $\kappa \in [1, p_s^*)$. Due to ($\mathcal{K}1$) and ($\mathcal{K}2$), we have that

$$
\begin{aligned}
I_\lambda(v_n) &= \mathcal{M}\Big([v_n]_{s,p}\Big) + \frac{1}{p}\int_\Omega \frac{|v_n(y)|^p}{|y|^{sp}}\,dy - \lambda\int_\Omega H(y,v_n)\,dy \\
&\geq \frac{m_0}{\vartheta p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)\,dydz + \frac{1}{p}\int_\Omega \frac{|v_n(y)|^p}{|y|^{sp}}\,dy - \lambda\int_\Omega H(y,v_n)dy \\
&\geq \frac{m_0}{\vartheta p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)dydz - \lambda\int_\Omega H(y,v_n)dy \\
&\geq \frac{m_0}{2\vartheta p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)dydz + \frac{m_0\gamma_0}{2\vartheta p C_0}|v_n|^p_{L^p(\Omega)} - \lambda\int_\Omega H(y,v_n)dy \\
&\geq \frac{m_0\min\{C_0,\gamma_0\}}{2\vartheta p C_0}|v_n|_X^p - \lambda\int_\Omega H(y,v_n)\,dy,
\end{aligned}
\tag{3.11}
$$

where γ_0 and C_0 are positive constants given in Lemma 2.3. Since $|v_n|_X \to \infty$ as $n \to \infty$, we assert by (3.11) that

$$
\lambda\int_\Omega H(y,v_n)\,dy \geq \frac{m_0\min\{C_0,\gamma_0\}}{2\vartheta p C_0}|v_n|_X^p - I_\lambda(v_n) \to \infty \quad \text{as} \quad n\to\infty. \tag{3.12}
$$

Observe that $\mathcal{M}(\zeta) \leq \mathcal{M}(1)\left(1+\zeta^\vartheta\right)$ for all $\zeta \in \mathbb{R}$ because if $0 \leq \zeta < 1$, then $\mathcal{M}(\zeta) = \int_0^\zeta M(t)\,dt \leq \mathcal{M}(1)$, and if $\zeta > 1$, then $\mathcal{M}(\zeta) \leq \mathcal{M}(1)\zeta^\vartheta$. This together with Lemma 2.4 yields that

$$
\begin{aligned}
I_\lambda(v_n) &= \mathcal{M}\Big([v_n]_{s,p}\Big) + \frac{1}{p}\int_\Omega \frac{|v_n|^p}{|y|^{sp}}dy - \lambda\int_\Omega H(y,v_n)dy \\
&\leq \mathcal{M}(1)\left(1+[v_n]_{s,p}^\vartheta\right) + \frac{c_H}{p\gamma_0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v_n(y)-v_n(z)|^p\mathcal{K}(y,z)dydz - \lambda\int_\Omega H(y,v_n)dy \\
&\leq \mathcal{M}(1)\left(1+[v_n]_{s,p}\right)^\vartheta + \frac{c_H}{\gamma_0}[v_n]_{s,p}^p - \lambda\int_\Omega H(y,v_n)dy \\
&\leq \mathcal{M}(1)\left(1+|v_n|_X^p\right)^\vartheta + \frac{c_H}{\gamma_0}|v_n|_X^p - \lambda\int_\Omega H(y,v_n)dy \\
&\leq 2^\vartheta\mathcal{M}(1)|v_n|_X^{p\vartheta} + \frac{c_H}{\gamma_0}|v_n|_X^{p\vartheta} - \lambda\int_\Omega H(y,v_n)dy \\
&\leq \left(\frac{2^\vartheta\mathcal{M}(1)\gamma_0 + c_H}{\gamma_0}\right)|v_n|_X^{p\vartheta} - \lambda\int_\Omega H(y,v_n)dy.
\end{aligned}
\tag{3.13}
$$

Then we obtain by the relation (3.13) that

$$
\frac{2^\vartheta\mathcal{M}(1)\gamma_0 + c_H}{\gamma_0} \geq \frac{1}{|v_n|_X^{\vartheta p}}\left(\lambda\int_\Omega H(y,v_n)\,dy + I_\lambda(v_n)\right). \tag{3.14}
$$

From (B4), we can choose $\zeta_0 > 1$ such that $H(y,\zeta) > |\zeta|^{\vartheta p}$ for all $y \in \Omega$ and $|\zeta| > \zeta_0$. Using (B1), there exists a positive constant $\mathcal{K}$ such that $|H(y,\zeta)| \leq \mathcal{K}$ for all $(y,\zeta) \in \Omega\times[-\zeta_0,\zeta_0]$. Hence there exists a real number $\mathcal{K}_0$ such that $H(y,\zeta) \geq \mathcal{K}_0$ for all $(y,\zeta) \in \Omega\times\mathbb{R}$, and thus

$$
\frac{H(y,v_n)-\mathcal{K}_0}{|v_n|_X^{\vartheta p}} \geq 0, \tag{3.15}
$$

for all $y \in \Omega$ and for all $n \in \mathbb{N}$. Set $\Delta_1 = \{y \in \Omega : w(y) \neq 0\}$ and suppose that $\mathrm{meas}(\Delta_1) \neq 0$. By the convergence (3.10), we infer that $|v_n(y)| = |w_n(y)| \, \|v_n\|_X \to \infty$ as $n \to \infty$, for all $y \in \Delta_1$. Furthermore, owing to (B4), one has

$$\lim_{n\to\infty} \frac{H(y, v_n)}{\|v_n\|_X^{\vartheta p}} = \lim_{n\to\infty} \frac{H(y, v_n)}{|v_n|^{\vartheta p}} |w_n|^{\vartheta p} = \infty \tag{3.16}$$

for all $y \in \Delta_1$. According to (3.12)–(3.16) and Fatou's Lemma, we deduce that

$$\begin{aligned}
\frac{2^{\vartheta}\mathcal{M}(1)\gamma_0 + c_H}{\gamma_0} &= \liminf_{n\to\infty} \frac{\lambda\left(2^{\vartheta}\mathcal{M}(1)\gamma_0 + c_H\right)\int_\Omega H(y, v_n)\,dy}{\lambda\gamma_0 \int_\Omega H(y, v_n)\,dy + I_\lambda(v_n)} \\
&\geq \lambda \liminf_{n\to\infty} \int_\Omega \frac{H(y, v_n)}{\|v_n\|_X^{\vartheta p}}\,dy \\
&= \lambda \liminf_{n\to\infty} \int_\Omega \frac{H(y, v_n)}{\|v_n\|_X^{\vartheta p}}\,dy - \lambda \limsup_{n\to\infty} \int_\Omega \frac{\mathcal{K}_0}{\|v_n\|_X^{\vartheta p}}\,dy \\
&\geq \lambda \liminf_{n\to\infty} \int_{\Delta_1} \frac{H(y, v_n) - \mathcal{K}_0}{\|v_n\|_X^{\vartheta p}}\,dy \\
&\geq \lambda \int_{\Delta_1} \liminf_{n\to\infty} \frac{H(y, v_n) - \mathcal{K}_0}{\|v_n\|_X^{\vartheta p}}\,dy \\
&= \lambda \int_{\Delta_1} \liminf_{n\to\infty} \frac{H(y, v_n)}{|v_n|^{\vartheta p}} |w_n|^{\vartheta p}\,dy - \lambda \int_{\Delta_1} \limsup_{n\to\infty} \frac{\mathcal{K}_0}{\|v_n\|_X^{\vartheta p}}\,dy = \infty,
\end{aligned}$$

which is a contradiction. Hence we have that $\mathrm{meas}(\Delta_1) = 0$ and $w(y) = 0$ for almost all $y \in \Omega$. As $I_\lambda(\tau v_n)$ is continuous in $\tau \in [0, 1]$, for each $n \in \mathbb{N}$, there exists $\tau_n \in [0, 1]$, such that

$$I_\lambda(\tau_n v_n) := \max_{\tau\in[0,1]} I_\lambda(\tau v_n).$$

Let $\{a_k\}_{k\in\mathbb{N}}$ be a positive sequence of real numbers satisfying $\lim_{k\to\infty} a_k = \infty$ and $a_k > 1$ for any k. Then, it is immediate that $\|a_k\omega_n\|_X = a_k > 1$ for any k and n. Let k be fixed. Because $\omega_n \to 0$ strongly in $L^{\ell}(\Omega)$ as $n \to \infty$, it follows from the continuity of the Nemytskii operator that $H(y, a_k\omega_n) \to 0$ in $L^1(\Omega)$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \int_\Omega H(y, a_k\omega_n)\,dy = 0. \tag{3.17}$$

Because $\|v_n\|_X \to \infty$ as $n \to \infty$, we have $\|v_n\|_X > a_k$ for sufficiently large n. Thus, by (3.17), we have

$$\begin{aligned}
&I_\lambda(\tau_n v_n) \geq I_\lambda\left(\frac{a_k}{\|v_n\|_X} v_n\right) = I_\lambda(a_k\omega_n) \\
&= \mathcal{M}\left([a_k\omega_n]_{s,p}\right) + \frac{1}{p}\int_\Omega \frac{|a_k\omega_n|^p}{|y|^{sp}}\,dy - \lambda \int_\Omega H(y, a_k\omega_n)\,dy \\
&\geq \frac{m_0}{\vartheta p} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |a_k\omega_n(y) - a_k\omega_n(z)|^p \mathcal{K}(y,z)\,dy\,dz - \lambda \int_\Omega H(y, a_k\omega_n)\,dy \\
&\geq \frac{m_0}{2\vartheta p} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |a_k\omega_n(y) - a_k\omega_n(z)|^p \mathcal{K}(y,z)\,dy\,dz + \frac{m_0\gamma_0}{2\vartheta C_0}\|a_k\omega_n\|_{L^p(\Omega)}^p - \lambda \int_\Omega H(y, a_k\omega_n)\,dy \\
&\geq \frac{m_0 \min\{C_0, \gamma_0\}}{2\vartheta p C_0} \|a_k\omega_n\|_X^p - \lambda C_1
\end{aligned}$$

for sufficiently large n and a positive constant C_1. From this, letting n and k tend to infinity, we have

$$\lim_{n\to\infty} I_\lambda(\tau_n v_n) = \infty. \tag{3.18}$$

Because $I_\lambda(0) = 0$ and $I_\lambda(v_n) \to c$ as $n \to \infty$, it proves that $\tau_n \in (0,1)$ and $\left\langle I'_\lambda(\tau_n v_n), \tau_n v_n \right\rangle = 0$. Therefore, by ($\mathcal{K}$2), (B2) and (3.9), we have

$$\begin{aligned}
I_\lambda(\tau_n v_n) &= I_\lambda(\tau_n v_n) - \frac{1}{p\vartheta}\langle I'_\lambda(\tau_n v_n), \tau_n v_n\rangle \\
&= \mathcal{M}\Big([\tau_n v_n]_{s,p}\Big) - \frac{1}{p\vartheta} M\left([\tau_n v_n]_{s,p}\right)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\tau_n v_n(y) - \tau_n v_n(z)|^p \mathcal{K}(y,z)\,dy\,dz\right) \\
&\quad + \frac{1}{p}\int_\Omega \frac{|\tau_n v_n|^p}{|y|^{sp}}dy - \frac{1}{p\vartheta}\int_\Omega \frac{|\tau_n v_n|^p}{|y|^{sp}}dy \\
&\quad + \lambda\int_\Omega \left(\frac{1}{p\vartheta} h(y,\tau_n v_n)\tau_n v_n - H(y,\tau_n v_n)\right) dy \\
&\le \mathcal{M}\Big([\tau_n v_n]_{s,p}\Big) - \frac{1}{\vartheta} M\left([\tau_n v_n]_{s,p}\right)[\tau_n v_n]_{s,p} + \frac{\vartheta - 1}{p\vartheta}\int_\Omega \frac{|\tau_n v_n|^p}{|y|^{sp}}dy \\
&\quad + \frac{\lambda}{p\vartheta}\int_\Omega \left(h(y,\tau_n v_n)\tau_n v_n - p\vartheta H(y,\tau_n v_n)\right) dy \\
&= \frac{1}{\vartheta}\widehat{\mathcal{M}}\Big([\tau_n v_n]_{s,p}\Big) + \frac{\vartheta - 1}{p\vartheta}\int_\Omega \frac{|\tau_n v_n|^p}{|y|^{sp}}dy + \frac{\lambda}{p\vartheta}\int_\Omega \mathcal{H}(y,\tau_n v_n)\,dy \\
&\le \frac{1}{\vartheta}\widehat{\mathcal{M}}\Big([v_n]_{s,p}\Big) + \frac{\vartheta - 1}{p\vartheta}\int_\Omega \frac{|v_n|^p}{|y|^{sp}}dy\,dy + \frac{\lambda}{p\vartheta}\int_\Omega \mathcal{H}(y,v_n)\,dy + K + C \\
&\le \mathcal{M}\Big([v_n]_{s,p}\Big) + \frac{1}{p}\int_\Omega \frac{|v_n|^p}{|y|^{sp}}\,dy - \lambda\int_\Omega H(y,v_n)\,dy \\
&\quad - \frac{1}{p\vartheta} M\left([v_n]_{s,p}\right)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v_n(y) - v_n(z)|^p \mathcal{K}(y,z)\,dy\,dz\right) \\
&\quad - \frac{1}{p\vartheta}\int_\Omega \frac{|v_n|^p}{|y|^{sp}}\,dy + \frac{\lambda}{p\vartheta}\int_\Omega h(y,v_n)v_n\,dy + K + C \\
&= I_\lambda(v_n) - \frac{1}{p\vartheta}\langle I'_\lambda(v_n), v_n\rangle + K + C \to c + K + C
\end{aligned}$$

as $n \to \infty$, which contradicts to (3.18) and so $\{v_n\}_{n\in\mathbb{N}}$ is bounded in X.

Passing to the limit, if necessary, to a subsequence by Lemma 2.3, we have

$$v_n \rightharpoonup v \text{ in } X, \quad v_n(y) \to v(y) \text{ a.e. in } \Omega \quad \text{and} \quad v_n \to v \text{ in } L^\kappa(\Omega) \text{ as } n \to \infty, \tag{3.19}$$

where $\kappa \in [1, p_s^*)$. To prove that $\{v_n\}_{n\in\mathbb{N}}$ converges strongly to v in X as $n \to \infty$, let $\varphi \in X$ be fixed and let $\tilde{\Phi}_\varphi$ denote the linear functional on X with

$$\tilde{\Phi}_\varphi(w) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(y) - \varphi(z)|^{p-2}(\varphi(y) - \varphi(z))(w(y) - w(z))\mathcal{K}(y,z)\,dy\,dz$$

for all $w \in X$. By the Hölder inequality, $\tilde{\Phi}_\varphi$ is also continuous, as

$$\left|\tilde{\Phi}_\varphi(w)\right| \le \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(y) - \varphi(z)|^p \mathcal{K}(y,z)\,dy\,dz\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w(y) - w(z)|^p \mathcal{K}(y,z)\,dy\,dz\Big)^{\frac{1}{p}}$$

$$\leq \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(y)-\varphi(z)|^p \mathcal{K}(y,z)\,dy\,dz\Big)^{\frac{p-1}{p}} \|w\|_X \tag{3.20}$$

for any $w \in X$. Hence, relation (3.20) yields

$$\lim_{n\to\infty}\Big[M\left([v_n]_{s,p}\right) - M\left([v]_{s,p}\right)\Big]\tilde{\Phi}_z(v_n - v) = 0, \tag{3.21}$$

because the sequence $\left\{M\left([v_n]_{s,p}\right) - M\left([v]_{s,p}\right)\right\}_{n\in\mathbb{N}}$ is bounded in Ω. Using (B1) and (3.19), it follows that

$$\begin{aligned}\int_\Omega |(h(y,v_n)-h(y,v))(v_n-v)|\,dy &\leq \int_\Omega \rho_2(|v_n|^{\ell-1}+|v|^{\ell-1})\,|v_n-v|\,dy\\ &\leq \rho_2\Big(\|v_n\|_{L^\ell(\Omega)}^{\ell-1}+\|v\|_{L^\ell(\Omega)}^{\ell-1}\Big)\|v_n-v\|_{L^\ell(\Omega)}.\end{aligned}$$

Then, due to (3.19), one has

$$\lim_{n\to\infty}\int_\Omega (h(y,v_n)-h(y,v))(v_n-v)\,dy = 0. \tag{3.22}$$

Because $v_n \rightharpoonup v$ in X and $I_\lambda'(v_n) \to 0$ in X^* as $n \to \infty$, we have

$$\langle I_\lambda'(v_n) - I_\lambda'(v), v_n - v\rangle \to 0 \text{ as } n \to \infty. \tag{3.23}$$

We then infer that

$$\begin{aligned}&\langle I_\lambda'(v_n) - I_\lambda'(v), v_n - v\rangle\\ &= M\left([v_n]_{s,p}\right)\tilde{\Phi}_{v_n}(v_n-v) - M\left([v]_{s,p}\right)\tilde{\Phi}_v(v_n-v)\\ &\quad+\int_\Omega \frac{|v_n|^{p-2}\,v_n(v_n-v)}{|y|^{sp}}\,dy - \int_\Omega \frac{|v|^{p-2}\,v(v_n-v)}{|y|^{sp}}\,dy\\ &\quad-\lambda\int_\Omega h(y,v_n)(v_n-v)\,dy + \lambda\int_\Omega h(y,v)(v_n-v)\,dy\\ &= M\left([v_n]_{s,p}\right)\left(\tilde{\Phi}_{v_n}(v_n-v) - \tilde{\Phi}_v(v_n-v)\right)\\ &\quad+\left(M\left([v_n]_{s,p}\right) - M\left([v]_{s,p}\right)\right)\tilde{\Phi}_v(v_n-v)\\ &\quad+\int_\Omega \frac{(|v_n|^{p-2}\,v_n - |v|^{p-2}\,v)(v_n-v)}{|y|^{sp}}\,dy - \lambda\int_\Omega (h(y,v_n)-h(y,v))(v_n-v)\,dy.\end{aligned}$$

This together with (3.21)–(3.23) yields

$$\lim_{n\to\infty}\Big(M\left([v_n]_{s,p}\right)[\tilde{\Phi}_{v_n}(v_n-v) - \tilde{\Phi}_v(v_n-v)] + \int_\Omega \frac{(|v_n|^{p-2}\,v_n - |v|^{p-2}\,v)(v_n-v)}{|y|^{sp}}\,dy\Big) = 0.$$

By convexity and ($\mathcal{K}1$), we have in particular

$$M\left([v_n]_{s,p}\right)[\tilde{\Phi}_{v_n}(v_n-v) - \tilde{\Phi}_v(v_n-v)] \geq 0 \tag{3.24}$$

and

$$\int_{\Omega} \frac{(|v_n|^{p-2}\, v_n - |v|^{p-2}\, v)(v_n - v)}{|y|^{sp}}\, dy \geq 0, \tag{3.25}$$

and then it follows that

$$\lim_{n\to\infty}\Big(\tilde{\Phi}_{v_n}(v_n - v) - \tilde{\Phi}_{v}(v_n - v)\Big) = 0 \tag{3.26}$$

and

$$\lim_{n\to\infty} \int_{\Omega} \frac{(|v_n|^{p-2}\, v_n - |v|^{p-2}\, v)(v_n - v)}{|y|^{sp}}\, dy = 0.$$

Note that there are the well-known useful inequalities

$$|\xi - \eta|^p \leq \begin{cases} C_2(|\xi|^{p-2}\, \xi - |\eta|^{p-2}\, \eta)(\xi - \eta) & \text{if } p \geq 2, \\ C_3\Big((|\xi|^{p-2}\, \xi - |\eta|^{p-2}\, \eta)(\xi - \eta)\Big)^{\frac{p}{2}} & \\ \qquad\qquad \times(|\xi|^p + |\eta|^p)^{\frac{2-p}{2}} & \text{if } 1 < p < 2 \text{ and } (\xi, \eta) \neq (0, 0), \end{cases} \tag{3.27}$$

for all $\xi, \eta \in \mathbb{R}^N$, where C_2 and C_3 are positive constants; see [59].

It is now assumed that $p \geq 2$. Then, by (3.27),

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |(v_n - v)(y) - (v_n - v)(z)|^p \mathcal{K}(y,z)\, dy\, dz \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v_n(y) - v_n(z) - v(y) + v(z)|^p \mathcal{K}(y,z)\, dy\, dz \\ &\leq C_2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(|v_n(y) - v_n(z)|^{p-2}(v_n(y) - v_n(z)) - |v(y) - v(z)|^{p-2}(v(y) - v(z))\Big) \\ &\qquad\qquad \times (v_n(y) - v_n(z) - v(y) + v(z))\mathcal{K}(y,z)\, dy\, dz \\ &= C_2\left(\tilde{\Phi}_{v_n}(v_n - v) - \tilde{\Phi}_{v}(v_n - v)\right). \end{aligned} \tag{3.28}$$

On the other hand, we consider the case $1 < p < 2$. As $\{v_n\}_{n\in\mathbb{N}}$ is bounded in X, there exist positive constants K_0 and K_1 such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v_n(y) - v_n(z)|^p \mathcal{K}(y,z)\, dy\, dz \leq K_0 \text{ and } \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y) - v(z)|^p \mathcal{K}(y,z)\, dy\, dz \leq K_1 \tag{3.29}$$

for all $n \in \mathbb{N}$. By (3.27), (3.29) and the Hölder inequality, we have for n large enough

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |(v_n - v)(y) - (v_n - v)(z)|^p \mathcal{K}(y,z)\, dy\, dz \\ &\leq C_3 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(\Big(|v_n(y) - v_n(z)|^{p-2}(v_n(y) - v_n(z)) - |v(y) - v(z)|^{p-2}(v(y) - v(z))\Big) \\ &\quad \times (v_n(y) - v_n(z) - v(y) + v(z))\Big)^{\frac{p}{2}} (|v_n(y) - v_n(z)|^p + |v(y) - v(z)|^p)^{\frac{2-p}{2}} \mathcal{K}(y,z)\, dy\, dz \\ &\leq C_3\left(\tilde{\Phi}_{v_n}(v_n - v) - \tilde{\Phi}_{v}(v_n - v)\right)^{\frac{p}{2}} \\ &\quad \times \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v_n(y) - v_n(z)|^p \mathcal{K}(y,z)\, dy\, dz + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(y) - v(z)|^p \mathcal{K}(y,z)\, dy\, dz\right)^{\frac{2-p}{2}} \end{aligned}$$

$$\leq C_3(K_0+K_1)^{\frac{2-p}{p}}\left(\tilde{\Phi}_{v_n}(v_n-v)-\tilde{\Phi}_v(v_n-v)\right)^{\frac{p}{2}}. \tag{3.30}$$

From (3.26), (3.28) and (3.30), we obtain that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|(v_n-v)(y)-(v_n-v)(z)|^p\mathcal{K}(y,z)\,dy\,dz\to 0 \text{ as } n\to\infty.$$

Since $v_n \to v$ in $L^p(\Omega)$ as $n\to\infty$ by (3.19), we get $\|v_n-v\|_X\to 0$ as $n\to\infty$. Hence, I_λ satisfies the $(C)_c$-condition. This completes the proof. □

Let $\mathfrak{X}$ be a reflexive and separable Banach space. Then it is known [18,68] that there are $\{e_n\}_{n\in\mathbb{N}}\subseteq\mathfrak{X}$ and $\{f_n^*\}_{n\in\mathbb{N}}\subseteq\mathfrak{X}^*$ such that

$$\mathfrak{X}=\overline{\operatorname{span}\{e_n:n=1,2,\cdots\}},\quad \mathfrak{X}^*=\overline{\operatorname{span}\{f_n^*:n=1,2,\cdots\}},$$

and

$$\left\langle f_j^*,e_i\right\rangle=\begin{cases}1 & \text{if } j=i\\ 0 & \text{if } j\neq i.\end{cases}$$

Let us denote $\mathfrak{X}_n=\operatorname{span}\{e_n\}$, $\mathfrak{F}_k=\bigoplus_{n=1}^k\mathfrak{X}_n$, and $\mathfrak{G}_k=\overline{\bigoplus_{n=k}^\infty\mathfrak{X}_n}$.

Definition 3.6. Suppose that $(\mathfrak{X},\|\cdot\|)$ is a real separable and reflexive Banach space. We say that $\mathcal{F}\in C^1(\mathfrak{X},\mathbb{R})$ satisfies the $(C)_c^*$-condition (with respect to $\mathfrak{F}_k$) if any sequence $\{w_k\}_{k\in\mathbb{N}}\subset\mathfrak{X}$ for which $w_k\in\mathfrak{F}_k$ for any $k\in\mathbb{N}$,

$$\mathcal{F}(w_k)\to c \quad\text{and}\quad \|(\mathcal{F}|_{\mathfrak{F}_k})'(w_k)\|_{\mathfrak{X}^*}(1+\|w_k\|)\to 0 \text{ as } k\to\infty,$$

possesses a subsequence converging to a critical point of $\mathcal{F}$.

Proposition 3.7. (Dual Fountain Theorem [29]) *Suppose that $(\mathfrak{X},\|\cdot\|)$ is a Banach space and $\mathcal{F}\in C^1(\mathfrak{X},\mathbb{R})$ is an even functional. If there is $k_0>0$ so that, for each $k\geq k_0$, there exist $\beta_k>\alpha_k>0$ such that*

(D1) $\inf\{\mathcal{F}(v):\|v\|_{\mathfrak{X}}=\beta_k, v\in\mathfrak{G}_k\}\geq 0$*;*

(D2) $b_k:=\max\{\mathcal{F}(v):\|v\|_{\mathfrak{X}}=\alpha_k, v\in\mathfrak{F}_k\}<0$*;*

(D3) $c_k:=\inf\{\mathcal{F}(v):\|v\|_{\mathfrak{X}}\leq\beta_k, v\in\mathfrak{G}_k\}\to 0$ *as* $k\to\infty$*;*

(D4) $\mathcal{F}$ *fulfills the* $(C)_c^*$*-condition for every* $c\in[c_{k_0},0)$*,*

then $\mathcal{F}$ admits a sequence of negative critical values $c_n<0$ satisfying $c_n\to 0$ as $n\to\infty$.

Lemma 3.8. *Assume that* (B1)–(B4) *hold, then I_λ satisfies the $(C)_c^*$-condition.*

Proof. Let $c\in\mathbb{R}$ and let the sequence $\{v_n\}_{n\in\mathbb{N}}$ in X be such that $v_n\in\mathfrak{F}_n$, for any $n\in\mathbb{N}$,

$$I_\lambda(v_n)\to c \quad\text{and}\quad \|(I_\lambda|_{\mathfrak{F}_n})'(v_n)\|_{X^*}(1+\|v_n\|)\to 0 \text{ as } n\to\infty.$$

Therefore, we get $c = I_\lambda(v_n) + o_n(1)$ and $\left\langle I'_\lambda(v_n), v_n \right\rangle = o_n(1)$, where $o_n(1) \to 0$ as $n \to \infty$. Repeating the argument from the proof of Lemma 3.5, we derive the boundedness of $\{v_n\}_{n\in\mathbb{N}}$ in X.

Let $\{v_n\}_{n\in\mathbb{N}}$ be any sequence in X such that $v_n \rightharpoonup v$ in X as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle A'(v_n) - A'(v), v_n - v\rangle \leq 0.$$

Then, we know by using the notation in Lemma 3.5 that

$$\limsup_{n\to\infty} \left\{ M([v_n]_{s,p}) \left(\tilde{\Phi}_{v_n}(v_n - v) - \tilde{\Phi}_v(v_n - v) \right) + \int_\Omega \frac{(|v_n|^{p-2} v_n - |v|^{p-2} v)(v_n - v)}{|y|^{sp}} \, dy \right\} \leq 0.$$

By (3.24) and (3.25) we have

$$\lim_{n\to\infty} \langle A'(v_n) - A'(v), v_n - v\rangle = 0.$$

Therefore, using (3.19), (3.28) and (3.30), we get $v_n \to v$ in X as $n \to \infty$, i.e., A' is mapping of type (S_+).

According to Lemma 3.4, B' is a compact operator on X. Since X is a reflexive Banach space, the idea of the rest of the proof is essentially the same as those in Lemma 3.12 [29]. □

Theorem 3.9. *Suppose that* (B1)–(B4) *hold. If $h(y, -\zeta) = -h(y, \zeta)$ holds for all $(y, \zeta) \in \Omega \times \mathbb{R}$, then the problem* (1.1) *has a sequence of nontrivial solutions $\{v_n\}_{n\in\mathbb{N}}$ in X such that $I_\lambda(v_n) \to 0$ as $n \to \infty$ for all $\lambda > 0$.*

Proof. With the aid of the oddness of h and Lemma 3.8, we derive that the functional I_λ is even and the $(C)^*_c$-condition is ensured for every $c \in \mathbb{R}$. Thus we will show that conditions (D1)–(D3) in Proposition 3.7 are verified.

(D1): For the sake of convenience, we denote

$$\zeta_{1,k} = \sup_{\|v\|_X=1, v\in\mathfrak{G}_k} \|v\|_{L^p(\Omega)} \quad \text{and} \quad \zeta_{2,k} = \sup_{\|v\|_X=1, v\in\mathfrak{G}_k} \|v\|_{L^\ell(\Omega)}.$$

Then, it is immediate to ensure that $\zeta_{1,k} \to 0$ and $\zeta_{2,k} \to 0$ as $k \to \infty$ (see [29]). Let us denote $\zeta_k = \max\{\zeta_{1,k}, \zeta_{2,k}\}$. From $(\mathcal{K}1)$, $(\mathcal{K}2)$, (B1), the definition of ζ_k and the analogous argument as in (3.11), it follows that

$$\begin{aligned} I_\lambda(v) &= \mathcal{M}\left([v]_{s,p}\right) + \frac{1}{p}\int_\Omega \frac{|v|^p}{|y|^{sp}}\, dy - \lambda \int_\Omega H(y, v)\, dy \\ &\geq \frac{m_0 \min\{C_0, \gamma_0\}}{2\vartheta p C_0} \|v\|_X^p - \lambda \int_\Omega H(y, v)\, dy \\ &\geq \frac{m_0 \min\{C_0, \gamma_0\}}{2\vartheta p C_0} \|v\|_X^p - \lambda \|\rho_1\|_{L^{p'}(\Omega)} \|v\|_{L^p(\Omega)} - \frac{\lambda\rho_2}{\ell} \|v\|_{L^\ell(\Omega)}^\ell \\ &\geq \frac{m_0 \min\{C_0, \gamma_0\}}{2\vartheta p C_0} \|v\|_X^p - \lambda \|\rho_1\|_{L^{p'}(\Omega)} \zeta_k \|v\|_X - \frac{\lambda\rho_2}{\ell} \zeta_k^\ell \|v\|_X^{2\ell} \end{aligned}$$

for k large enough and $\|v\|_X \geq 1$, where C_0 and γ_0 are given in Lemma 2.3. Choose

$$\beta_k = \left(\frac{4\vartheta p C_0 \lambda \rho_2}{m_0 \min\{C_0, \gamma_0\}\, \ell} \zeta_k^\ell \right)^{\frac{1}{p-2\ell}}. \tag{3.31}$$

Let $v \in \mathfrak{G}_k$ with $\|v\|_X = \beta_k > 1$ for k large enough. Then, there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
I_\lambda(v) &\geq \frac{m_0 \min\{C_0, \gamma_0\}}{2\vartheta p C_0} \|v\|_X^p - \lambda \|\rho_1\|_{L^{p'}(\Omega)} \zeta_k \|v\|_X - \frac{\lambda \rho_2}{\ell} \zeta_k^\ell \|v\|_X^{2\ell} \\
&\geq \frac{m_0 \min\{C_0, \gamma_0\}}{4\vartheta p C_0} \beta_k^p - \lambda \zeta_k^{\frac{p-\ell}{p-2\ell}} \|\rho_1\|_{L^{p'}(\Omega)} \left(\frac{4\vartheta p \lambda \rho_2}{m_0 \min\{C_0, \gamma_0\}\, \ell} \right)^{\frac{1}{p-2\ell}} \\
&\geq 0
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, because $\lim_{k\to\infty} \beta_k = \infty$. Therefore,

$$\inf\{I_\lambda(v) : v \in \mathfrak{G}_k, \|v\|_X = \beta_k\} \geq 0.$$

(D2): Since $\mathfrak{F}_k$ is finite dimensional, all the norms are equivalent. Then, we can choose positive constants $\varsigma_{1,k}$ and $\varsigma_{2,k}$ such that

$$\varsigma_{1,k} \|v\|_X \leq \|v\|_{L^m(\nu,\Omega)} \text{ and } \|v\|_{L^\ell(\Omega)} \leq \varsigma_{2,k} \|v\|_X$$

for any $v \in \mathfrak{F}_k$. Let $v \in \mathfrak{F}_k$ with $\|v\|_X \leq 1$. In accordance with (B1) and (B3), there are $C_1, C_2 > 0$ such that

$$H(y, \zeta) \geq C_1 \nu(y) |\zeta|^m - C_2 |\zeta|^\ell$$

for almost all $(y, \zeta) \in \Omega \times \mathbb{R}$. Observe that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{p} |v(y) - v(z)|^p \mathcal{K}(y, z)\, dy\, dz \leq C_3$$

for a positive constant C_3. Then we have

$$\begin{aligned}
I_\lambda(v) &= \mathcal{M}\Big([v]_{s,p}\Big) + \frac{1}{p} \int_\Omega \frac{|v|^p}{|y|^{sp}} dy - \lambda \int_\Omega H(y, v)\, dy \\
&\leq \left(\sup_{0 \leq \xi \leq C_3} M(\xi) \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{p} |v(y) - v(z)|^p \mathcal{K}(y, z)\, dy\, dz + \frac{c_H}{p\, \gamma_0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v(y) - v(z)|^p\, \mathcal{K}(y, z) dy dz \\
&\quad - \lambda C_1 \int_\Omega \nu |v|^m dy + \lambda C_2 \int_\Omega |v|^\ell\, dy \\
&\leq \frac{1}{p} \left(\sup_{0 \leq \xi \leq C_3} M(\xi) + \frac{c_H}{\gamma_0} \right) \|v\|_X^p - \lambda C_1 \int_\Omega \nu |v|^m dy + \lambda C_2 \int_\Omega |v|^\ell\, dy \\
&\leq C_4 \|v\|_X^p - \lambda C_1 \|v\|_{L^m(\nu,\Omega)}^m + \lambda C_2 \|v\|_{L^\ell(\Omega)}^\ell \\
&\leq C_4 \|v\|_X^p - \lambda C_1 \varsigma_{1,k}^m \|v\|_X^m + \lambda C_2 \varsigma_{2,k}^\ell \|v\|_X^\ell,
\end{aligned}$$

where $C_4 = \frac{1}{p} \left(\sup_{0 \leq \xi \leq C_3} M(\xi) + \frac{c_H}{\gamma_0} \right)$. Let $g(x) = C_4 x^p - \lambda C_1 \varsigma_{1,k}^m x^m + \lambda C_2 \varsigma_{2,k}^\ell x^\ell$. Since $m < p < \ell$, we infer $g(x) < 0$ for all $x \in (0, x_0)$ for sufficiently small $x_0 \in (0, 1)$. Hence, $I_\lambda(v) < 0$ for all $v \in \mathfrak{F}_k$ with $\|v\|_X = x_0$. Choosing $\alpha_k = x_0$ for all $k \in \mathbb{N}$, one has

$$b_k := \max\{I_\lambda(v) : v \in \mathfrak{F}_k, \|v\|_X = \alpha_k\} < 0.$$

If necessary, we can replace k_0 with a large value, so that $\beta_k > \alpha_k > 0$ for all $k \geq k_0$.

(D3): Because $\mathfrak{F}_k \cap \mathfrak{G}_k \neq \phi$ and $0 < \alpha_k < \beta_k$, we have $c_k \leq b_k < 0$ for all $k \geq k_0$. For any $v \in \mathfrak{G}_k$ with $\|v\|_X = 1$ and $0 < \delta < \beta_k$, one has

$$\begin{aligned} I_\lambda(\delta v) &= \mathcal{M}\Big([\delta v]_{s,p}\Big) + \frac{1}{p}\int_\Omega \frac{|\delta v|^p}{|y|^{sp}}dy - \lambda\int_\Omega H(y,\delta v)\,dy \\ &\geq \frac{m_0 \min\{C_0,\gamma_0\}}{2\vartheta p C_0}\|\delta v\|_X^p - \lambda\|\rho_1\|_{L^{p'}(\Omega)}\|\delta v\|_{L^p(\Omega)} - \frac{\lambda\rho_2}{\ell}\|\delta v\|_{L^\ell(\Omega)}^\ell \\ &\geq -\lambda\|\rho_1\|_{L^p(\Omega)}\beta_k\zeta_k - \frac{\lambda\rho_2}{\ell}\beta_k^\ell\zeta_k^\ell \end{aligned}$$

for large enough k. Hence, the definition of β_k implies that

$$\begin{aligned} 0 > c_k &\geq -\lambda\|\rho_1\|_{L^{p'}(\Omega)}\beta_k\zeta_k - \frac{\lambda\rho_2}{\ell}\beta_k^\ell\zeta_k^\ell \\ &= -\lambda\|\rho_1\|_{L^{p'}(\Omega)}\left(\frac{4\vartheta p C_0\lambda\rho_2}{m_0\min\{C_0,\gamma_0\}\,\ell}\right)^{\frac{1}{p-2\ell}}\zeta_k^{\frac{p-\ell}{p-2\ell}} \\ &\quad - \frac{\lambda\rho_2}{\ell}\left(\frac{4\vartheta p C_0\lambda\rho_2}{m_0\min\{C_0,\gamma_0\}\,\ell}\right)^{\frac{\ell}{p-2\ell}}\zeta_k^{\frac{\ell(p-\ell)}{p-2\ell}}. \end{aligned} \tag{3.32}$$

Because $p < \ell$ and $\zeta_k \to 0$ as $k \to \infty$, we derive that $\lim_{k\to\infty} c_k = 0$.

Consequently, all conditions of the Dual Fountain Theorem in Proposition 3.7 hold, and we arrive that problem (1.1) possesses a sequence of nontrivial solutions $\{v_n\}_{n\in\mathbb{N}}$ in X satisfying $I_\lambda(v_n) \to 0$ as $n \to \infty$ for all $\lambda > 0$. □

Finally, in terms of applying the Dual Fountain Theorem, we illustrate the differences between the present paper and the previous related studies [5,25,29,39,44,60,62,67].

Remark 3.10. *In order to apply the Dual Fountain Theorem, numerous researchers [5,44,60,62,67] considered the existence of two sequences* $0 < \alpha_k < \beta_k \to 0$ *as* $k \to \infty$. *However, our approach is different from the above sources. This is based on the papers [10,25,29,39]. In view of these papers, the conditions* (B4) *and* (h) *play an important role in proving assumptions of the Dual Fountain Theorem. Under these two conditions, researchers established the existence of two sequences* $0 < \alpha_k < \beta_k$ *large enough [10,25,29,34,39]. Regrettably, by utilizing the analogous argument [25,29], we cannot show the property (D3) in Theorem 3.9. More precisely, if we change* β_k *in* (3.31) *into*

$$\hat{\beta}_k = \left(\frac{4\vartheta p C_0\lambda\rho_2}{m_0\min\{C_0,\gamma_0\}\,\ell}\zeta_k^\ell\right)^{\frac{1}{p-\ell}},$$

then in the estimate (3.32)*,*

$$\hat{\beta}_k\zeta_k = \left(\frac{4\vartheta p C_0\lambda\rho_2}{m_0\min\{C_0,\gamma_0\}\,\ell}\right)^{\frac{\ell}{p-\ell}}\zeta_k^{\frac{p}{p-\ell}} \to \infty \text{ as } k \to \infty$$

and thus we cannot obtain the property (D3) in $\hat{\beta}_k$. *However, researchers [10,34,39] overcame this difficulty from a new setting for* β_k *as in* (3.31). *Although the basic idea for proving the conditions (D1)–(D3) in the Dual Fountain Theorem is similar to the technique above, we derive these conditions without assuming* (B4) *and* (h) *in the present paper. For this reason, the proof of Theorem 3.9 is slightly different from that of the previous related works.*

4. Conclusions

In the present paper, on a class of the Kirchhoff coefficient M and the nonlinear term h which differ from the previous related works, we give the existence result of multiple small energy solutions to nonlocal problems of Kirchhoff type involving Hardy potential. One of the novelties of the present paper is to provide our main result when we do not assume the monotonicity of $\widehat{\mathcal{M}}$ in ($\mathcal{K}2$), and the condition (h), which are crucial to prove the compactness condition of Palais-Smale type and ensuring all assumptions in the Dual Fountain Theorem. The second novelty is to consider a different approach from other works [19, 31–33, 42, 45] to derive the multiplicity result by using the Dual Fountain Theorem instead of various critical point theorems as mentioned in the introduction.

Additionally, a new research direction is the study of Kirchhoff-Schrödinger type problems:

$$M\left([v]_{s,p}\right)\mathcal{L}v(y) + \mathcal{V}(y)|v|^{p-2}w = \mu\frac{|v|^{p-2}v}{|y|^p} + \lambda h(y,v) \quad \text{in} \quad \mathbb{R}^N,$$

where $1 < p < p_s^*$, $\mu \in (-\infty, \mu^*)$ for a positive constant μ^* and $\mathcal{V} : \mathbb{R}^N \to (0,\infty)$ is a potential function with

(V) $\mathcal{V} \in L^1_{loc}(\mathbb{R}^N)$, $ess\,inf_{y\in\mathbb{R}^N}\mathcal{V}(y) > 0$, and $\lim_{|y|\to\infty}\mathcal{V}(y) = +\infty$.

Let us consider the condition

($f2$) There is a positive constant $\theta \geq 1$ such that

$$\theta\mathcal{H}(y,\zeta) \geq \mathcal{H}(y,t\zeta)$$

for $(y,\zeta) \in \mathbb{R}^N \times \mathbb{R}$ and $t \in [0,1]$, where $\mathcal{H}(y,\zeta) = h(y,\zeta)\zeta - p\vartheta H(y,\zeta)$.

When $\mu \neq 0$, the classical variational approach is not applicable to our treatment according to the presence of the term $\mu|v|^{p-2}v|y|^{-p}$. The reason is that the Hardy inequality only guarantees that the embedding of the fractional Sobolev space $W_0^{s,p}(\Omega)$ into the Lebesgue space $L^p(\Omega)$ with weight $|y|^{-p}$, also denoted by $L^p(\Omega, |y|^{-p})$ is continuous, but not compact. Hence, this situation with $\mu \neq 0$ should be much more delicate than this paper because of the lack of compactness.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. A. Aberqi, A. Ouaziz, Morse's theory and local linking for a fractional $(p_1(x,\cdot), p_2(x,\cdot))$: Laplacian problems on compact manifolds, *J. Pseudo-Differ. Oper. Appl.*, **41** (2023). https://doi.org/10.1007/s11868-023-00535-5
2. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, 2 Eds., Academic Press, New York-London, 2003.
3. D. Arcoya, J. Carmona, P. J. Martínez-Aparicio, Multiplicity of solutions for an elliptic Kirchhoff equation, *Milan J. Math.*, **90** (2022), 679–689. https://doi.org/10.1007/s00032-022-00365-y
4. G. Autuori, A. Fiscella, P. Pucci, Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity, *Nonlinear Anal.*, **125** (2015), 699–714. https://doi.org/10.1016/j.na.2015.06.014
5. R. Ayazoglu, S. Akbulut, E. Akkoyunlu, Existence and multiplicity of solutions for $p(.)$-Kirchhoff-type equations, *Turkish J. Math.*, **46** (2022). https://doi.org/10.55730/1300-0098.3164
6. B. Barrios, E. Colorado, A. De Pablo, U. Sanchez, On some critical problems for the fractional Laplacian operator, *J. Differ. Equ.*, **252** (2012), 6133–6162. https://doi.org/10.1016/j.jde.2012.02.023
7. G. Bonanno, Some remarks on a three critical points theorem, *Nonlinear Anal.*, **54** (2003), 651–665. https://doi.org/10.1016/S0362-546X(03)00092-0
8. G. Bonanno, S. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, **89** (2010), 1–10. https://doi.org/10.1080/00036810903397438
9. L. Caffarelli, Non-local equations, drifts and games, *Nonlinear Partial Differ. Equ. Abel Symp.*, **7** (2012), 37–52. https://doi.org/10.1007/978-3-642-25361-4
10. J. Cen, S. J. Kim, Y. H. Kim, S. Zeng, Multiplicity results of solutions to the double phase anisotropic variational problems involving variable exponent, *Adv. Differential Equ.*, **28** (2023), 467–504. https://doi.org/10.57262/ade028-0506-467
11. G. Cerami, An existence criterion for the critical points on unbounded manifolds, *Istit. Lombardo Accad. Sci. Lett. Rend. A*, **112** (1978), 332–336.
12. W. Chen, N. V. Thin, Existence of solutions to Kirchhoff type equations involving the nonlocal $p_1\&\cdots\&p_m$ fractional Laplacian with critical Sobolev-Hardy exponent, *Complex Var. Elliptic Equ.*, **67** (2022), 1931–1975. https://doi.org/10.1080/17476933.2021.1913129
13. D. Choudhuri, Existence and Hölder regularity of infinitely many solutions to a p Kirchhoff type problem involving a singular nonlinearity without the Ambrosetti-Rabinowitz (AR) condition, *Z. Angew. Math. Phys.*, **72** (2021). https://doi.org/10.48550/arXiv.2006.00953
14. N. T. Chung, H. Q. Toan, On a nonlinear and non-homogeneous problem without (A-R) type condition in Orlicz-Sobolev spaces, *Appl. Math. Comput.*, **219** (2013), 7820–7829. https://doi.org/10.1016/j.amc.2013.02.011
15. G. W. Dai, R. F. Hao, Existence of solutions of a $p(x)$-Kirchhoff-type equation, *J. Math. Anal. Appl.*, **359** (2009), 275–284. https://doi.org/10.1016/j.jmaa.2009.05.031

16. J. I. Diaz, Nonlinear partial differential equations and free boundaries, *Elliptic Equ. Res. Notes Math.*, **106** (1985).

17. J. I. Diaz, J. M. Morel, L. Oswald, An elliptic equation with singular nonlinearity, *Commun. Part. Diff. Eq.*, **12** (1987), 1333–1344. https://doi.org/10.1080/03605308708820531

18. M. Fabian, P. Habala, P. Hajék, V. Montesinos, V. Zizler, *Banach space theory: The basis for linear and nonlinear analysis*, Springer, New York, 2011.

19. M. Ferrara, G. M. Bisci, Existence results for elliptic problems with Hardy potential, *Bull. Sci. Math.*, **138** (2014), 846–859. https://doi.org/10.1016/j.bulsci.2014.02.002

20. A. Fiscella, Schrödinger-Kirchhoff-Hardy p-fractional equations without the Ambrosetti-Rabinowitz condition, *Discrete Cont. Dyn.-S*, **13** (2020), 1993–2007. https://doi.org/10.3934/dcdss.2020154

21. A. Fiscella, G. Marino, A. Pinamonti, S. Verzellesi, Multiple solutions for nonlinear boundary value problems of Kirchhoff type on a double phase setting, *Rev. Mat. Complut.*, 2023, 1–32. https://doi.org/10.1007/s13163-022-00453-y

22. A. Fiscella, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.*, **94** (2014), 156–170. https://doi.org/10.1016/j.na.2013.08.011

23. R. L. Frank, R. Seiringer, Non-linear ground state representations and sharp Hardy inequalities, *J. Funct. Anal.*, **255** (2008), 3407–3430. https://doi.org/10.1016/j.jfa.2008.05.015

24. B. Ge, On the superlinear problems involving the $p(x)$-Laplacian and a non-local term without AR-condition, *Nonlinear Anal.*, **102** (2014), 133–143. https://doi.org/10.1016/j.na.2014.02.004

25. B. Ge, D. J. Lv, J. F. Lu, Multiple solutions for a class of double phase problem without the Ambrosetti-Rabinowitz conditions, *Nonlinear Anal.*, **188** (2019), 294–315. https://doi.org/10.1016/j.na.2019.06.007

26. G. Gilboa, S. Osher, Nonlocal operators with applications to image processing, *Multiscale Model. Simul.*, **7** (2008), 1005–1028. https://doi.org/10.1137/070698592

27. S. Gupta, G. Dwivedi, Kirchhoff type elliptic equations with double criticality in Musielak-Sobolev spaces, *Math. Method. Appl. Sci.*, **46** (2023), 8463–8477. https://doi.org/10.1002/mma.8991

28. T. Huang, S. Deng, Existence of ground state solutions for Kirchhoff type problem without the Ambrosetti-Rabinowitz condition, *Appl. Math. Lett.*, **113** (2021), 106866. https://doi.org/10.1016/j.aml.2020.106866

29. E. J. Hurtado, O. H. Miyagaki, R. S. Rodrigues, Existence and multiplicity of solutions for a class of elliptic equations without Ambrosetti-Rabinowitz type conditions, *J. Dyn. Differ. Equ.*, **30** (2018), 405–432. https://doi.org/10.1007/s10884-016-9542-6

30. F. Júlio, S. Corrêa, G. Figueiredo, On an elliptic equation of p-Kirchhoff type via variational methods, *Bull. Aust. Math. Soc.*, **74** (2006), 263–277. https://doi.org/10.1017/S000497270003570X

31. M. Khodabakhshi, A. M. Aminpour, G. A. Afrouzi, A. Hadjian, Existence of two weak solutions for some singular elliptic problems, *RACSAM*, **110** (2016), 385–393. https://doi.org/10.1007/s13398-015-0239-1

32. M. Khodabakhshi, G. A. Afrouzi, A. Hadjian, Existence of infinitely many weak solutions for some singular elliptic problems, *Complex Var. Elliptic Equ.*, **63** (2018), 1570–1580. https://doi.org/10.1080/17476933.2017.1397137

33. M. Khodabakhshi, A. Hadjian, Existence of three weak solutions for some singular elliptic problems, *Complex Var. Elliptic Equ.*, **63** (2018), 68–75. https://doi.org/10.1080/17476933.2017.1282949

34. J. M. Kim, Y. H. Kim, Multiple solutions to the double phase problems involving concave-convex nonlinearities, *AIMS Math.*, **8** (2023), 5060–5079. https://doi.org/10.3934/math.2023254

35. I. H. Kim, Y. H. Kim, Infinitely many small energy solutions to nonlinear Kirchhoff-Schrödinger equations with the p-Laplacian, submitted.

36. I. H. Kim, Y. H. Kim, K. Park, Multiple solutions to a non-local problem of Schrödinger-Kirchhoff type in $\mathbb{R}^N$, *Fractal Fract.*, **7** (2023), 627. https://doi.org/10.3390/fractalfract7080627

37. G. R. Kirchhoff, *Vorlesungen über mathematische physik, mechanik*, Teubner, Leipzig, 1876.

38. N. Laskin, Fractional quantum mechanics and Levy path integrals, *Phys. Lett. A*, **268** (2000), 298–305. https://doi.org/10.1016/S0375-9601(00)00201-2

39. J. Lee, J. M. Kim, Y. H. Kim, A. Scapellato, On multiple solutions to a non-local fractional $p(\cdot)$-Laplacian problem with concave-convex nonlinearities, *Adv. Cont. Discr. Mod.*, **2022** (2022), 14. https://doi.org/10.1186/s13662-022-03689-6

40. G. Li, C. Yang, The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition, *Nonlinear Anal.*, **72** (2010), 4602–4613. https://doi.org/10.1016/j.na.2010.02.037

41. L. Li, X. Zhong, Infinitely many small solutions for the Kirchhoff equation with local sublinear nonlinearities, *J. Math. Anal. Appl.*, **435** (2016), 955–967. https://doi.org/10.1016/j.jmaa.2015.10.075

42. C. B. Lian, B. L. Zhang, B. Ge, Multiple solutions for double phase problems with Hardy type potential, *Mathematics*, **9** (2021), 376. https://doi.org/10.3390/math9040376

43. J. L. Lions, On some questions in boundary value problems of mathematical physics, *North-Holland Math. Stud.*, **30** (1978), 284–346. https://doi.org/10.1016/S0304-0208(08)70870-3

44. D. C. Liu, On a p-Kirchhoff-type equation via fountain theorem and dual fountain theorem, *Nonlinear Anal.*, **72** (2010), 302–308. https://doi.org/10.1016/j.na.2009.06.052

45. J. Liu, Z. Zhao, Existence of triple solutions for elliptic equations driven by p-Laplacian-like operators with Hardy potential under Dirichlet-Neumann boundary conditions, *Bound Value Probl.*, **2023** (2023). https://doi.org/10.1186/s13661-023-01692-8

46. S. B. Liu, On superlinear problems without Ambrosetti and Rabinowitz condition, *Nonlinear Anal.*, **73** (2010), 788–795. https://doi.org/10.1016/j.na.2010.04.016

47. S. B. Liu, S. J. Li, Infinitely many solutions for a superlinear elliptic equation, *Acta Math. Sinica (Chin. Ser.)*, **46** (2003), 625–630. https://doi.org/10.12386/A2003sxxb0084

48. D. Lu, A note on Kirchhoff-type equations with Hartree-type nonlinearities, *Nonlinear Anal.*, **99** (2014), 35–48. https://doi.org/10.1016/j.na.2013.12.022

49. D. Lu, Existence and multiplicity results for perturbed Kirchhoff-type Schrödinger systems in $\mathbb{R}^3$, *Comput. Math. Appl.*, **68** (2014), 1180–1193. https://doi.org/10.1016/j.camwa.2014.08.020

50. O. H. Miyagaki, M. A. S. Souto, Superlinear problems without Ambrosetti and Rabinowitz growth condition, *J. Diff. Equ.*, **245** (2008), 3628–3638. https://doi.org/10.1016/j.jde.2008.02.035

51. A. Nachman, A. Callegari, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.*, **38** (1980), 275–281. https://doi.org/10.1137/0138024

52. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573. https://doi.org/10.48550/arXiv.1104.4345

53. P. Pucci, S. Saldi, Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators, *Rev. Mat. Iberoam.*, **32** (2016), 1–22. https://doi.org/10.4171/RMI/879

54. P. Pucci, M. Q. Xiang, B. L. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in $\mathbb{R}^N$, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 2785–2806. https://doi.org/10.1007/s00526-015-0883-5

55. P. Pucci, M. Q. Xiang, B. L. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.*, **5** (2016), 27–55. https://doi.org/10.1515/anona-2015-0102

56. B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.*, **113** (2000), 401–410. https://doi.org/10.1016/S0377-0427(99)00269-1

57. B. Ricceri, A further three critical points theorem, *Nonlinear Anal.*, **71** (2009), 4151–4157. https://doi.org/10.1016/j.na.2009.02.074

58. R. Servadei, E. Valdinoci, Mountain Pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, **389** (2012), 887–898. https://doi.org/10.1016/j.jmaa.2011.12.032

59. J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, *Journées d'Analyse Non Linéaire*, **665** (1978), 205–227. https://doi.org/10.1007/BFb0061807

60. K. Teng, Multiple solutions for a class of fractional Schrödinger equations in $\mathbb{R}^N$, *Nonlinear Anal.-Real*, **21** (2015), 76–86. https://doi.org/10.1016/j.nonrwa.2014.06.008

61. Y. Wei, X. Su, Multiplicity of solutions for non-local elliptic equations driven by the fractional Laplacian, *Calc. Var. Partial Differ. Equ.*, **52** (2015), 95–124. https://doi.org/10.1007/s00526-013-0706-5

62. M. Willem, *Minimax theorems*, Birkhauser, Basel, 1996.

63. Q. Wu, X. P. Wu, C. L. Tang, Existence of positive solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *Qual. Theor. Dyn. Syst.*, **21** (2022), 1–16. https://doi.org/10.1007/s12346-022-00696-6

64. M. Q. Xiang, B. L. Zhang, X. Y. Guo, Infinitely many solutions for a fractional Kirchhoff type problem via fountain theorem, *Nonlinear Anal.*, **120** (2015), 299–313. https://doi.org/10.1016/j.na.2015.03.015

65. M. Q. Xiang, B. L. Zhang, M. Ferrara, Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian, *J. Math. Anal. Appl.*, **424** (2015), 1021–1041. https://doi.org/10.1016/j.jmaa.2014.11.055

66. M. Q. Xiang, B. L. Zhang, M. Ferrara, Multiplicity results for the nonhomogeneous fractional p-Kirchhoff equations with concave-convex nonlinearities, *Proc. R. Soc. A*, **471** (2015), 20150034. https://doi.org/10.1098/rspa.2015.0034

67. L. Yang, T. An, Infinitely many solutions for fractional p-Kirchhoff equations, *Mediterr. J. Math.*, **15** (2018), 80. https://doi.org/10.1007/s00009-018-1124-x

68. Y. Zhou, J. Wang, L. Zhang, *Basic theory of fractional differential equations*, 2 Eds., World Scientific Publishing Co. Pte. Ltd., Singapore, 2017.

69. J. Zuo, D. Choudhuri, D. D. Repovs, Multiplicity and boundedness of solutions for critical variable-order Kirchhoff type problems involving variable singular exponent, *J. Math. Anal. Appl.*, **514** (2022), 1–18. https://doi.org/10.48550/arXiv.2204.10635

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