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Research article

Fuzzy knowledge spaces based on β evaluation criteria

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Abstract: In KST, it is always assumed that the knowledge state represents items that an individual can solve in ideal conditions. Namely, the answers of individuals to items can be encoded as either correct or incorrect. The correct answer indicates a complete mastery of the item, but the incorrect answer may indicate a partial mastery of the item. It is reasonable to use a fuzzy knowledge state to represent the partial mastery of items instead of complete mastery. The fuzzy knowledge state of an individual is represented by a fuzzy set in $\mathcal{F}(Q)$ that the individual is capable of solving. For any fuzzy knowledge state, each item has a value that represents the level of individual mastery of the item. Fuzzy knowledge spaces and fuzzy learning spaces are generalizations of knowledge spaces and learning spaces. The generalization based on partial order is helpful to distinguish the equally informative items, which can directly induce a discriminative fuzzy knowledge structure. It is effective to use fuzzy knowledge spaces and fuzzy learning spaces to assess knowledge and guide further learning. A fuzzy knowledge space and a fuzzy learning space can be faithfully summarized by the fuzzy knowledge basis, since they are union-closed. Any fuzzy knowledge state of a fuzzy knowledge space can be generated by forming the union of some fuzzy knowledge states in the basis. A fuzzy knowledge basis is a generalization of the knowledge basis of a knowledge space.

Keywords: fuzzy knowledge state; fuzzy knowledge space; fuzzy learning space; fuzzy knowledge basis
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1. Introduction

The theory of knowledge spaces (KST) provides a mathematical framework for assessment of knowledge and advices for further learning [8, 14]. KST makes a dynamic evaluation process, where the accurate dynamic evaluation is based on individuals' responses to items and the quasi order (surmise relation) on domain Q [8, 36].

The ways to construct knowledge state of an individual or knowledge structure of a population are the core of KST. The query [13, 18, 24, 25] and ps-query [6] algorithms have been designed to build knowledge structures and knowledge states by interviewing experienced experts. As a result, based on the expert's judgment, the individual's answer to an item is only extreme cases of completely correct or incorrect. The assumption excludes the possibility of intermediate degrees of mastering an item. For instance, it does not make sense even if an individual has mastered 80% or 90% of an item, which indicates that the individual does not fully master the item.

If the item is elementary, it is reasonable that the answer can be clearly classified as "correct" (1) or "incorrect" (0). For example, for an item 3+7=?, the answer of the item is unique. It is unreasonable for some complex items to be only classified as "correct" (1) or "incorrect" (0). For example, what is the key factor affecting people's life span? The answer could be heredity, diet or wealth, etc. Therefore, it is natural to use a fuzzy set in $\mathcal{F}(Q)$ to represent a partial mastery of items.

The knowledge state of an individual is inferred from their responses to items. A fuzzy knowledge state assigns a value in $[0, 1]$ for each item q instead of $\{0, 1\}$. The degree of membership $\widetilde{K}(q)$ represents the mastery degree in item q of individual. It is natural to use fuzzy knowledge states to represent the items that individuals can partially solve. A fuzzy knowledge state can be formed according to the test of "item raw scores" by subjects, and what we get is always a fuzzy knowledge state of an individual or a fuzzy knowledge structure of a certain population. Fuzzy knowledge spaces and fuzzy learning spaces can be developed as a theoretical framework in which partial mastery of items can be handled mathematically. A fuzzy knowledge state provides a broad way for knowledge assessment and it can accurately evaluate knowledge, guide future learning and take some remedial course for individuals. Specifically, the dichotomous knowledge state K is included in the set $\{0, 1\}$ as an extreme case of a fuzzy knowledge state.

In 1997, Schrepp attempted to generalize the main KST concepts to items with more than two response alternatives [28]. In 2020, Stefanutti et al. provided the mathematical foundation for the generalization of KST to the case of more than two ordered response categories [32]. Another related study was proposed by Bartl and Belohlavek in 2011 from the perspective of complete residuated lattice in fuzzy logic [3]. They studied the generalization of KST from the linearly ordered set, the complete lattice and complete residuated lattice, respectively. Recently, Heller generalized quasi-ordinal knowledge spaces to polytomous items, where each item allows for partially ordered response values forming lattices [23, 37]. They generalized KST from a perspective of quasi-order, that is, surmise relation. However, our work starts from a partial order on Q. The generalization of KST based on partial order is helpful to distinguish the equally informative items, which can directly derive a discriminative fuzzy knowledge structure.

In KST, the knowledge state of an individual represents the subject's complete mastery of items. That means, the evaluation criteria on Q is $\beta = (1, 1, \cdots, 1)$. Rely on partial order on Q, we generalize the evaluation criteria $\beta = (1, 1, \cdots, 1)$ to the evaluation criteria $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, where $\beta_i \in [0, 1]$ for each $1 \leq i \leq n$. For a given $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, we say that the item q has mastered under β, as long as the level of proficiency in item q reaches β_q. The fuzzy learning smoothness indicates that there is only one item that can be learned at a time under β. The fuzzy learning consistency shows that knowing more does not prevent from partially learning something new, where partially learning an item q means reaching the evaluation criteria β_q. A fuzzy learning space can help to guide learning for individuals, and guide teaching for educators.

A fuzzy knowledge spaces can be faithfully summarized by a subfamily of their fuzzy knowledge states. Any fuzzy knowledge state in $(Q, \widetilde{\mathcal{K}})$ can be generated by forming the union of some fuzzy knowledge states in the fuzzy knowledge basis. The fuzzy knowledge basis is smaller than the fuzzy knowledge space, which results in a substantial economy of storage in a computer memory. In 1993, Dowling proposed an algorithm for finding the basis of a knowledge space [11]. The algorithm can be extended to a fuzzy knowledge space and it can also obtained the fuzzy knowledge basis.

The rest of paper is organized as follows. Section 2 presents some relevant background about KST and the fuzzy set theory. We introduce fuzzy knowledge states and fuzzy knowledge spaces in Sections 3 and 4. Fuzzy learning spaces are discussed in Section 5. Section 6 introduces the fuzzy basis of fuzzy knowledge space. Section 7 summarizes the major results of this paper.

2. Overview of KST and fuzzy set theory

A field of knowledge that can be parsed into a nonempty finite set of items, is denoted by Q. Sometimes, Q can be conceptualized as comprising a specified set of notions, where a notion can be identified with a problem or an equivalence class of problems. The knowledge state is the subset of Q, which an individual is capable of solving in ideal conditions, denoted by K. A collection of knowledge state is called knowledge structure, denoted by $(Q, \mathcal{K})$, where $\mathcal{K}$ contains at least $\emptyset$ and Q. Since $\bigcup \mathcal{K} = Q$, we shall sometimes simply say that $\mathcal{K}$ is the knowledge structure when the domain can be omitted without ambiguity. There are two special types of knowledge structures, namely, knowledge spaces and learning spaces. A knowledge structure $\mathcal{K}$ is called a knowledge space if it is closed under union, i.e., $K, L \in \mathcal{K}$ implies $K \cup L \in \mathcal{K}$. A knowledge space is called a quasi-ordinal space if it is additionally closed under intersection, i.e., $K, L \in \mathcal{K}$ implies $K \cap L \in \mathcal{K}$. Another special knowledge structure is called learning space if it satisfies the two following conditions [7, 14]:

[L1] *Learning smoothness*. For any two states K, L such that $K \subset L$, there exists a finite chain of states $K = K_0 \subset K_1 \subset \cdots \subset K_p = L$ such that $|K_i \backslash K_{i-1}| = 1$ for $1 \leq i \leq p$ and so $|L \backslash K| = p$. In pedagogical view: If the state K of the learner is included in some other state L, then the learner can reach state L by mastering the missing items one at a time.

[L2] *Learning consistency*. If K, L are two states satisfying $K \subset L$ and q is an item such that $K + \{q\} \in \mathcal{K}$, then $L \cup \{q\} \in \mathcal{K}$, where $K + \{q\}$ means $q \notin K$. In pedagogical view: Knowing more does not prevent from learning something new. For a detailed description of KST, please refer to Falmagne and Doignon [8, 10, 14, 15].

Let $\mathbf{I}$ be the unit closed interval $[0, 1]$ and Q be a nonempty finite set of items. A mapping from Q to $\mathbf{I}$ is said to be a *fuzzy set* on Q. Alternatively, the fuzzy power set of Q is denoted by $\mathcal{F}(Q)$. A fuzzy set $\widetilde{K} \in \mathcal{F}(Q)$ is denoted by $\{\frac{\widetilde{K}(q)}{q} \mid q \in Q\}$ and some $\frac{\widetilde{K}(q)}{q}$ are omitted if $\widetilde{K}(q) = 0$ for $q \in Q$. For any fuzzy set $\widetilde{K} \in \mathcal{F}(Q)$, $\widetilde{K}(q)$ is the membership degree of q to $\widetilde{K}$. Some operations of fuzzy sets in $\mathcal{F}(Q)$ are simply defined as follows [35, 38]:

(1). $\widetilde{K_1} = \widetilde{K_2} \Longleftrightarrow \widetilde{K}_1(q) = \widetilde{K}_2(q), \forall q \in Q$; (2). $\widetilde{K_1} \subseteq \widetilde{K_2} \Longleftrightarrow \widetilde{K}_1(q) \leq \widetilde{K}_2(q), \forall q \in Q$;
(3). $(\widetilde{K}_1 \cup \widetilde{K}_2)(q) = \widetilde{K}_1(q) \vee \widetilde{K}_2(q), \forall q \in Q$; (4). $(\widetilde{K}_1 \cap \widetilde{K}_2)(q) = \widetilde{K}_1(q) \wedge \widetilde{K}_2(q), \forall q \in Q$;
(5). $\widetilde{K}^c(q) = 1 - \widetilde{K}(q), \forall q \in Q$; (6). $\widetilde{K}_1 \backslash \widetilde{K}_2 = \widetilde{K}_1 \cap \widetilde{K}_2^c$.

In the paper, the domain Q is a finite set of items, denoted by $Q = \{q_1, q_2, \cdots, q_n\}$, and $\widetilde{\mathcal{K}}$ is a family of fuzzy knowledge states.

3. Fuzzy knowledge state

Consider the domain $U = \{a, b, c, d, e, f\}$ equipped with the knowledge structure:

$$\mathcal{K} = \{\emptyset, \{d\}, \{a, c\}, \{e, f\}, \{a, b, c\}, \{a, c, d\}, \{d, e, f\}$$

$$\{a, b, c, d\}, \{a, c, e, f\}, \{a, c, d, e, f\}, Q\}.$$

This knowledge structure contains 11 knowledge states, including $\emptyset$ and Q [14]. Scanning from left to right, we can find some learning paths from $\emptyset$ to Q. The concept of learning path in the finite case was introduced by Falmagne and Doignon [13, 16]. At first, a student knows nothing about the field, and is thus in knowledge state $\emptyset$. He/she may then gradually learn from $\emptyset$ until completely masters all items in Q.

It's necessary to analyze the relationship between items for knowledge assessment. For a knowledge structure $\mathcal{K}$, denote by $\mathcal{K}_q = \{K \mid q \in K, \forall K \in \mathcal{K}\}$ the collection of all knowledge states in $\mathcal{K}$ containing item q. Let $\mathcal{K}$ be a knowledge structure, the surmise relation "$\preceq$" on Q be defined as $r \preceq q \iff r \in \bigcap \mathcal{K}_q$ [8, 14]. In the above knowledge structure, we get $\bigcap \mathcal{K}_a = \{a, c\}, \bigcap \mathcal{K}_b = \{a, b, c\}, \bigcap \mathcal{K}_c = \{a, c\}, \bigcap \mathcal{K}_d = \{d\}, \bigcap \mathcal{K}_e = \{e, f\}$, $\bigcap \mathcal{K}_f = \{e, f\}$, which actually reveals the surmise relation of items. We can verify that the surmise relation of the above knowledge structure is $\{a \preceq c, c \preceq a, e \preceq f, f \preceq e, a \preceq b, c \preceq b\}$, see Figure 1.

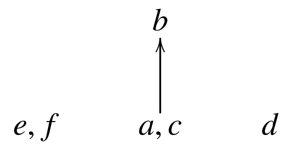

Figure 1. Hasse diagram of the surmise relation of $\mathcal{K}$.

The interpretation of the surmise relation between items is given as follows. Item a, c is simpler than b, they are the premise or foundation of b. If a subject can solve b, he must be able to solve a, c, but, if he can solve a and c, he will not necessarily be able to solve b. The relationship between items is determined when the knowledge domain is given. It is the essential feature of knowledge domain Q, and it will not change with the knowledge sates of individuals or knowledge structure of a certain population. For instance, the item "2 + 3 =?" is simpler than item "2.07 + 1.931 =?", but not other relation.

A direct way to determine a fuzzy knowledge state is from the perspective of surmise relation, that is a special quasi order. We can verify

$$\mathcal{K}_a = \mathcal{K}_c = \{\{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}, \{a, c, e, f\}, \{a, c, d, e, f\}, Q\}.$$

That means, items a and c are equally informative, any individual whose state contains item a has necessarily mastered item c, and vice versa. Similarly, e and f are also equally informative. Therefore, in order to avoid redundant computation, we start from a partial order on Q to extend the knowledge state. A partial order is a more strictly relation than surmise relation on Q. Theorem 2 indicates a discriminative fuzzy knowledge structure can be directly derived from a partially ordered set $(Q, \mathcal{R})$.

Definition 1. *For a partially ordered set $(Q, \mathcal{R})$, a function $\widetilde{K} : Q \to [0,1]$ is called a fuzzy knowledge state if $\widetilde{K}(q) \le \widetilde{K}(r)$ when $(r,q) \in \mathcal{R}$ for any $r, q \in Q$.*

Remark 1. *The $\widetilde{K}(r)$ is called degree of membership of fuzzy knowledge state $\widetilde{K}$ about r. Let $Q = \{a,b,c,d,e\}$ and a partial order on Q be*

$$\mathcal{R} = \{(b,c),(c,d),(a,e),(b,e),(c,e),(d,e)\}.$$

The partial order can determine whether a fuzzy set in $\mathcal{F}(Q)$ is a fuzzy knowledge state. Any fuzzy set $\widetilde{K} : Q \to [0,1]$ can be regard as a fuzzy knowledge state of $\mathcal{R}$ if it simultaneously satisfies $\widetilde{K}(c) \le \widetilde{K}(b)$, $\widetilde{K}(d) \le \widetilde{K}(c)$, $\widetilde{K}(e) \le \widetilde{K}(a)$, $\widetilde{K}(e) \le \widetilde{K}(b)$, $\widetilde{K}(e) \le \widetilde{K}(c)$, $\widetilde{K}(e) \le \widetilde{K}(d)$. It's obvious that $(q,q) \in \mathcal{R}$ for any $q \in Q$, they can be omitted from $\mathcal{R}$ without ambiguity. Let $\widetilde{K}_1 = \{\frac{0.3}{a}, \frac{0.8}{b}, \frac{0.7}{c}, \frac{0.6}{d}, \frac{0.2}{e}\}$, $\widetilde{K}_2$=$\{\frac{0.7}{a}, \frac{0.9}{b}, \frac{0.8}{c}, \frac{0.6}{d}, \frac{0.5}{e}\}$, $\widetilde{K}_3 = \{\frac{0.9}{a}, \frac{0.7}{b}, \frac{0.6}{c}, \frac{0.5}{d}, \frac{0.4}{e}\}$, the knowledge states can be regard as fuzzy knowledge states on $\mathcal{R}$, since all the fuzzy knowledge states satisfy $\widetilde{K}(q) \le \widetilde{K}(r)$ for any $(r,q) \in \mathcal{R}$. Let $\widetilde{K}_4 = \{\frac{0.3}{a}, \frac{0.2}{b}, \frac{0.8}{c}, \frac{1}{d}, \frac{0.6}{e}\}$, then $\widetilde{K}_4$ is not a fuzzy knowledge state of $\mathcal{R}$, since $\widetilde{K}_4(b) \le \widetilde{K}_4(c)$. A partial order on Q is derived by experienced experts or teacher analyzing the relationships between items. A family of fuzzy knowledge states be called a fuzzy knowledge structure when it contains fuzzy knowledge states $\{\frac{0}{q_1}, \frac{0}{q_2}, \cdots, \frac{0}{q_n}\}$ and $\{\frac{1}{q_1}, \frac{1}{q_2}, \cdots, \frac{1}{q_n}\}$.

Definition 2. *Let $(Q, \widetilde{\mathcal{K}})$ be a fuzzy knowledge structure. The set of all the items contained in the same membership degree of fuzzy knowledge states as item q is called a notion. Denoted by q^*, where $q^* = \{r \in Q \mid \widetilde{K}(q) = \widetilde{K}(r), \forall \widetilde{K} \in \widetilde{\mathcal{K}}\}$.*

It's obvious that the collection Q^* of all notions is a partition of the domain Q. The equivalence class in q^* means the same degree of membership of items in a fuzzy knowledge structure. A fuzzy knowledge structure in which each notion contains a single item is called a discriminative fuzzy knowledge structure.

Example 1. *Let $Q = \{a,b,c,d,e\}$ and a partial order on Q be*

$$\mathcal{R} = \{(b,c),(c,d),(a,e),(b,e),(c,e),(d,e)\}.$$

A fuzzy knowledge structure: $\widetilde{\mathcal{K}}$=$\{\{\frac{0}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\{\frac{0.8}{a}, \frac{0.8}{b}, \frac{0.6}{c}, \frac{0.5}{d}, \frac{0.2}{e}\}$, $\{\frac{0.7}{a}, \frac{0.7}{b}, \frac{0.4}{c}, \frac{0.3}{d}, \frac{0.3}{e}\}$, $\{\frac{0.5}{a}, \frac{0.5}{b}, \frac{0.3}{c}, \frac{0.2}{d}, \frac{0.1}{e}\}$, $\{\frac{0.9}{a}, \frac{0.9}{b}, \frac{0.8}{c}, \frac{0.7}{d}, \frac{0.6}{e}\}$, $\{\frac{0.8}{a}, \frac{0.8}{b}, \frac{0.7}{c}, \frac{0.6}{d}, \frac{0.5}{e}\}$, $\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}\}\}$ is not discriminative, since for each fuzzy knowledge state $\widetilde{K} \in \widetilde{\mathcal{K}}$, we have $\widetilde{K}(a) = \widetilde{K}(b)$. That is, items a and b are equally informative. A discriminative fuzzy knowledge structure can always be manufactured from any fuzzy knowledge structure $(Q, \widetilde{\mathcal{K}})$ by forming the notions. A discriminative fuzzy knowledge structure $\widetilde{\mathcal{K}}^*$ induced by $\widetilde{\mathcal{K}}$ on Q^* via the definition a new domain $Q^* = \{q^* \mid q \in Q\}$. The discriminative fuzzy knowledge structure $\widetilde{\mathcal{K}}^* = \{\widetilde{K}^* | \widetilde{K} \in \widetilde{\mathcal{K}}\}$, where $\widetilde{K}^* : Q^* \to [0,1]$ such that $\widetilde{K}^*(q^*) = \widetilde{K}(q)$ for any $q \in Q, q^* \in Q^*$. We have the following conclusion to determine whether a fuzzy knowledge structure is discriminative.

Theorem 1. *A fuzzy knowledge structure $(Q, \widetilde{\mathcal{K}})$ is discriminative if and only if there exists a $\widetilde{K} \in \widetilde{\mathcal{K}}$ such that $\widetilde{K}(q) \ne \widetilde{K}(r)$ when $(q,r) \notin \mathcal{R}$.*

Proof. Let $\mathcal{R}$ be the partial order of the fuzzy knowledge structure $(Q, \widetilde{\mathcal{K}})$. Each notions $q^* = \{q\}$ is singletons when $(Q, \widetilde{\mathcal{K}})$ is discriminative. Necessity: If $\widetilde{K}(q) = \widetilde{K}(r)$ for all $\widetilde{K} \in \widetilde{\mathcal{K}}$ and $(q,r) \notin \mathcal{R}$, then

$q^* = \{q, r\}$, negating the notions $q^* = \{q\}$ is singletons of a discriminative fuzzy knowledge structure. Hence, there is $\widetilde{K} \in \widetilde{\mathcal{K}}$ such that $\widetilde{K}(q) \neq \widetilde{K}(r)$. Sufficiency: If for any $(q, r) \notin \mathcal{R}$, there exists $\widetilde{K} \in \widetilde{\mathcal{K}}$ such that $\widetilde{K}(q) \neq \widetilde{K}(r)$, then we have $r \notin q^*$. Moreover we have $p \notin q^*$ for any $(p, q) \in \mathcal{R}$, since $\mathcal{R}$ is a partial order. Thus the fuzzy notion $q^* = \{q\}$ for any $q \in Q$. Hence, $(Q, \widetilde{\mathcal{K}})$ is discriminative. □

It should be noted that fuzzy knowledge structures are different from probabilistic knowledge structures $(Q, \mathcal{K}, p)$. Probabilistic knowledge structures that applied the basic local independence model to real knowledge space data were introduced and developed in KST by Falmagne and Doignon [12, 16, 19, 22, 29, 33]. It mostly considers two kinds of practical cases: the knowledge states may have different frequencies in the population of reference, and the response to an item may be an error for careless or guess the correct answer. It makes sense to introduce conditional probabilities of responses, given the knowledge states. The most difference between probabilistic knowledge structures and fuzzy knowledge structures is that the probability knowledge structure describes the probability of knowledge states, which is still completely "master" or "not master" to each item. However, a fuzzy knowledge state means partial mastery of items.

4. Fuzzy knowledge spaces

We focus on a special fuzzy knowledge structure, called fuzzy knowledge space, by assuming that any subfamily of fuzzy knowledge states of $\widetilde{\mathcal{K}}$ is closed under union. An item-state table can be used to represent a fuzzy knowledge structure, where rows represent fuzzy knowledge states and columns represent items. The closure under union is a rather reasonable property: two students interacting for a while will end up to merge their initial fuzzy knowledge states into a single one which is the union of the two fuzzy knowledge states. A fuzzy knowledge space is important for assessment knowledge and latent cognitive abilities.

Definition 3. *A fuzzy knowledge structure $(Q, \widetilde{\mathcal{K}})$ is called a fuzzy knowledge space (resp. fuzzy closure space) if it is union-closed (resp. intersection-closed), that is $\widetilde{K} \cup \widetilde{L} \in (Q, \widetilde{\mathcal{K}})$ (resp. $\widetilde{K} \cap \widetilde{L} \in (Q, \widetilde{\mathcal{K}})$) for any $\widetilde{K}, \widetilde{L} \in (Q, \widetilde{\mathcal{K}})$.*

Example 2. *Let $Q = \{a, b, c, d, e\}$ and a partial order on Q be*

$$\mathcal{R} = \{(b, c), (c, d), (a, e), (b, e), (c, e), (d, e)\}.$$

It's obvious that $\widetilde{K}_1, \widetilde{K}_2, \cdots, \widetilde{K}_{15}$ are fuzzy knowledge states on $(Q, \mathcal{R})$, see Table 1. The fuzzy knowledge structure $\widetilde{\mathcal{K}}$ in Table 1 is a fuzzy knowledge space, that is $\widetilde{K} \cup \widetilde{L} \in \widetilde{\mathcal{K}}$ for any $\widetilde{K}, \widetilde{L} \in \widetilde{\mathcal{K}}$. We now study the properties of a fuzzy knowledge structure induced by a given partial order.

Theorem 2. *All the fuzzy knowledge states induced by a partial order $\mathcal{R}$ on Q generates a union-closed and intersection-closed discriminative fuzzy knowledge structure $(Q, \widetilde{\mathcal{K}})$.*

Table 1. A fuzzy knowledge space $\widetilde{\mathcal{K}}$ on $(Q, \mathcal{R})$.

	a	b	c	d	e
$\widetilde{K}_1$	0	0	0	0	0
$\widetilde{K}_2$	0.4	0.9	0	0	0
$\widetilde{K}_3$	0	0.8	0.7	0	0
$\widetilde{K}_4$	0	0.7	0.7	0.6	0
$\widetilde{K}_5$	0.4	0.9	0.7	0	0
$\widetilde{K}_6$	0	0.8	0.7	0.6	0
$\widetilde{K}_7$	0.4	0.9	0.7	0.6	0
$\widetilde{K}_8$	0.3	0.6	0.5	0.5	0.3
$\widetilde{K}_9$	0.3	0.8	0.7	0.5	0.3
$\widetilde{K}_{10}$	0.4	0.9	0.5	0.5	0.3
$\widetilde{K}_{11}$	0.4	0.9	0.7	0.5	0.3
$\widetilde{K}_{12}$	0.4	0.9	0.7	0.6	0.3
$\widetilde{K}_{13}$	0.7	0.8	0.8	0.7	0.6
$\widetilde{K}_{14}$	0.7	0.9	0.8	0.7	0.6
$\widetilde{K}_{15}$	1	1	1	1	1

Proof. Let $\widetilde{K}_i, \widetilde{K}_j \in \widetilde{\mathcal{K}}$ be any fuzzy knowledge states induced by the partial order $\mathcal{R}$. For any $(r, q) \in \mathcal{R}$, we have $\widetilde{K}_i(r) \geq \widetilde{K}_i(q), \widetilde{K}_j(r) \geq \widetilde{K}_j(q)$. Then

$$(\widetilde{K}_i \cup \widetilde{K}_j)(r) = \widetilde{K}_i(r) \vee \widetilde{K}_j(r) = max\{\widetilde{K}_i(r), \widetilde{K}_j(r)\}$$

$$= max\{\widetilde{K}_i(r), \widetilde{K}_j(r), \widetilde{K}_i(q), \widetilde{K}_j(q)\}$$

and $(\widetilde{K}_i \cup \widetilde{K}_j)(q) = \widetilde{K}_i(q) \vee \widetilde{K}_j(q) = max\{\widetilde{K}_i(q), \widetilde{K}_j(q)\}$. Hence, $(\widetilde{K}_i \cup \widetilde{K}_j)(r) \geq (\widetilde{K}_i \cup \widetilde{K}_j)(q)$, the union-closed is satisfied. By the similarity method, we have

$$(\widetilde{K}_i \cap \widetilde{K}_j)(r) = \widetilde{K}_i(r) \wedge \widetilde{K}_j(r) = min\{\widetilde{K}_i(r), \widetilde{K}_j(r)\}$$

$$(\widetilde{K}_i \cap \widetilde{K}_j)(q) = \widetilde{K}_i(q) \wedge \widetilde{K}_j(q) = min\{\widetilde{K}_i(q), \widetilde{K}_j(q)\}$$
$$= min\{\widetilde{K}_i(r), \widetilde{K}_j(r), \widetilde{K}_i(q), \widetilde{K}_j(q)\}.$$

Hence, $(\widetilde{K}_i \cap \widetilde{K}_j)(r) \geq (\widetilde{K}_i \cap \widetilde{K}_j)(q)$, the intersection-closed is satisfied. If $\widetilde{\mathcal{K}}$ is not discriminative, then there exist two different items r, q such that $\widetilde{K}(r) = \widetilde{K}(q)$ for any $\widetilde{K} \in \widetilde{\mathcal{K}}$. Thus $\widetilde{K}(r) \geq \widetilde{K}(q)$ and $\widetilde{K}(r) \leq \widetilde{K}(q)$ for any $\widetilde{K} \in \widetilde{\mathcal{K}}$, that means $(r, q) \in \mathcal{R}, (q, r) \in \mathcal{R}$. We get $r = q$, since $\mathcal{R}$ is a partial order relation on Q. Hence $(Q, \widetilde{\mathcal{K}})$ is discriminative. □

From Theorem 2, we say that the fuzzy knowledge structure induced by partial order relation $\mathcal{R}$ is a discriminative fuzzy quasi ordinal space on Q, which is consistent with Birkhoff theorem [4, 5]. Such a correspondence is defined by the equivalences

$$(r, q) \in \mathcal{R} \Longleftrightarrow (\forall \widetilde{K} \in \widetilde{\mathcal{K}} : \widetilde{K}(q) \leq \widetilde{K}(r)),$$

$$\widetilde{K} \in \widetilde{\mathcal{K}} \Longleftrightarrow (\forall (r, q) \in \mathcal{R} : \widetilde{K}(q) \leq \widetilde{K}(r)).$$

Theorem 3. *A discriminative fuzzy quasi ordinal space* $(Q, \widetilde{\mathcal{K}})$ *determined a partially ordered set* $(Q, \mathcal{R})$.

Proof. For any discriminative fuzzy quasi ordinal space $(Q, \widetilde{\mathcal{K}})$, we define a relation $\mathcal{R}$ on Q as

$$(r, q) \in \mathcal{R} \Longleftarrow (\forall \widetilde{K} \in \widetilde{\mathcal{K}} : \widetilde{K}(q) \leq \widetilde{K}(r)).$$

Then $\mathcal{R}$ is reflexive on Q, since $\widetilde{K}(r) \leq \widetilde{K}(r)$ holds for any $r \in Q$. That is $(r, r) \in \mathcal{R}$ for any $r \in Q$. $\mathcal{R}$ is transitive on Q, since if $\widetilde{K}(p) \leq \widetilde{K}(r)$ and $\widetilde{K}(q) \leq \widetilde{K}(p)$, we have $\widetilde{K}(q) \leq \widetilde{K}(r)$ holds for all $r, p, q \in Q$. That is $(r, p) \in \mathcal{R}$ and $(p, q) \in \mathcal{R}$, then $(r, q) \in \mathcal{R}$. $\mathcal{R}$ is antisymmetric on Q, since $(Q, \widetilde{\mathcal{K}})$ is discriminative, then $\widetilde{K}(r) \geq \widetilde{K}(p)$ and $\widetilde{K}(p) \geq \widetilde{K}(r)$ implies $r = p$ for all $r, p \in Q$. □

It should be observed that the result may not hold for any fuzzy knowledge structure that is not discriminative fuzzy quasi ordinal space, since it may induce a quasi order on Q.

5. Fuzzy learning space

Learning space is a special knowledge space, which satisfies the axioms of learning smoothness and learning consistency [14]. Cosyn and Uzun showed that the knowledge structures satisfying the two axioms are equivalent to well-graded [9, 16] knowledge spaces [7]. In learning space, the axiom of learning smoothness indicates that if the knowledge state of a learner is included in some other knowledge state, then the learner can reach the knowledge state by mastering the missing items, one at a time. There always exists an evaluation criteria for assessment knowledge and guidance future learning of an individual or population. Relying on partial order, the evaluation criteria $\beta = (1, 1, \cdots, 1)$ for ordinary learning spaces can be generalized to the evaluation criteria $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, where $\beta_i \in [0, 1]$ for each $1 \leq i \leq n$. This evaluation criteria is not necessarily fixed, it may be dynamically changed with the evaluation needs.

Definition 4. *Let* $(Q, \mathcal{R})$ *be a partially order set,* $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ *is called an evaluation criteria on* Q *if it satisfies* $\beta_j \leq \beta_i$ *when* $(q_i, q_j) \in \mathcal{R}$ *for each* $\beta_i, \beta_j \in [0, 1]$.

Remark 2. *The evaluation criteria* $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ *assigns a criteria* β_{q_i} *for item* q_i*, the criteria* β_{q_i} *simply denoted by* β_i*. The value* β_i *in* β *corresponds to the criteria for mastering of item* q_i*. The quality of response to items of individuals may be judged by points. When a subject's response normalization score for an item is not less than the value* β_i*, we think that they can solve the item under evaluation criteria* β*. The better the mastery of an item, the more points are assigned to it. Note that the ordinary knowledge state represents the subject's complete mastery of the items, where the evaluation criteria can be regard as* $\beta = (1, 1, \cdots, 1)$.

Definition 5. *A family of fuzzy knowledge states is called a* β*-fuzzy knowledge structure if it contains* β*-fuzzy empty set* $\widetilde{\varnothing}$ *and* β*-fuzzy universal set* $\widetilde{Q}$*, denoted by* $(Q, \widetilde{\mathcal{K}})$*, where* $\widetilde{\varnothing} \in \mathcal{F}(Q)$ *satisfies* $\widetilde{\varnothing}(q_i) < \beta_i$ *for any* $q_i \in Q$*,* $\widetilde{Q} \in \mathcal{F}(Q)$ *satisfies* $\widetilde{Q}(q_i) \geq \beta_i$ *for any* $q_i \in Q$.

The $\widetilde{\varnothing}$ means that the individual does not master all the items in Q under the evaluation criteria β. The $\widetilde{Q}$ indicates that the individual completely masters all items in Q under the evaluation criteria β. We main concern whether the subjects meet the evaluation criteria when the value of β is given.

Notation 1. *Let $(Q, \mathcal{R})$ be a partially ordered set and β be an evaluation criteria. For any $K \in \mathcal{F}(Q)$, $\widetilde{K}_\beta \in \mathcal{F}(Q)$ such that $\widetilde{K}_\beta(q) = \widetilde{K}(q)$ when $\widetilde{K}(q) \geq \beta_q$, otherwise $\widetilde{K}_\beta(q) = 0$ when $\widetilde{K}(q) < \beta_q$.*

Remark 3. *β_q is the criteria which assigns to item q in evaluation criteria β. For fuzzy knowledge states $\widetilde{K}$ and $\widetilde{L}$, if $\widetilde{K}_\beta = \widetilde{L}_\beta$, then the two fuzzy knowledge states can be considered to be indistinguishable under β. It can determinate a partition on $\widetilde{\mathcal{F}}(Q)$: Let $\widetilde{\mathcal{K}}$ be a family of fuzzy knowledge states and $\widetilde{K}_i, \widetilde{K}_j \in \widetilde{\mathcal{K}}$, the equivalence relation $\sim_\beta$ on $\widetilde{\mathcal{K}}$ means that $\widetilde{K}_i \sim_\beta \widetilde{K}_j \Leftrightarrow \widetilde{K}_\beta = \widetilde{L}_\beta$. Consider the partition on $\mathcal{F}(Q)$ determined by $\sim_\beta$: For any $\widetilde{K} \in \mathcal{F}(Q)$, the set $\{\widetilde{K}' : \widetilde{K}' \sim_\beta \widetilde{K}, \widetilde{K}' \in \mathcal{F}(Q)\}$ is denoted by $[\widetilde{K}]$, the set $\{[\widetilde{K}] : \widetilde{K} \in \mathcal{F}(Q)\}$ is denoted by $\mathcal{F}(Q)/\sim_\beta$.*

Theorem 4. *For any two fuzzy knowledge states $\widetilde{K}_i, \widetilde{K}_j \in \widetilde{\mathcal{K}}$, $\widetilde{K}_i \subseteq \widetilde{K}_j$ implies $(\widetilde{K}_i)_\beta \subseteq (\widetilde{K}_j)_\beta$.*

Proof. Let $\widetilde{K}_i, \widetilde{K}_j$ be two fuzzy knowledge states and evaluation criteria $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$. We obtain $\widetilde{K}_i(q) \leq \widetilde{K}_j(q)$ for any $q \in Q$, since $\widetilde{K}_i \subseteq \widetilde{K}_j$. If $\widetilde{K}_i(q_1) \geq \beta_1, \widetilde{K}_i(q_2) \geq \beta_2, \cdots, \widetilde{K}_i(q_n) \geq \beta_n$, then $\widetilde{K}_j(q_1) \geq \widetilde{K}_i(q_1) \geq \beta_1, \widetilde{K}_j(q_2) \geq \widetilde{K}_i(q_2) \geq \beta_2, \cdots, \widetilde{K}_j(q_n) \geq \widetilde{K}_i(q_n) \geq \beta_n$, where $\beta_i \in [0, 1], 1 \leq i \leq n$. Hence $(\widetilde{K}_i)_\beta \subseteq (\widetilde{K}_j)_\beta$. □

Note that the $(\widetilde{K}_i)_\beta \subseteq (\widetilde{K}_j)_\beta$ and $(\widetilde{K}_j)_\beta \subseteq (\widetilde{K}_i)_\beta$ do not imply $\widetilde{K}_i = \widetilde{K}_j$. The converse implication is false, and the example below disproves the converse.

Example 3. *Let $Q = \{a, b, c, d, e\}$ and $\beta = (0.8, 0.6, 0.5, 0.6, 0.4)$, the partial order is $\mathcal{R} = \{(b, c), (a, d), (a, e), (b, e), (c, e), (d, e)\}$. A fuzzy knowledge structure is $\widetilde{\mathcal{K}}$=$\{\{\frac{0}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}, \{\frac{0.9}{a}, \frac{0.4}{b}, \frac{0.3}{c}, \frac{0.2}{d}, \frac{0.1}{e}\}, \{\frac{0.8}{a}, \frac{0.5}{b}, \frac{0.2}{c}, \frac{0.5}{d}, \frac{0.2}{e}\}, \{\frac{0.8}{a}, \frac{0.6}{b}, \frac{0.6}{c}, \frac{0.4}{d}, \frac{0.1}{e}\}, \{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}\}\}$. It's obvious that $\{\frac{0.8}{a}, \frac{0.5}{b}, \frac{0.2}{c}, \frac{0.5}{d}, \frac{0.2}{e}\}_\beta = \{\frac{0.8}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}$ and $\{\frac{0.8}{a}, \frac{0.6}{b}, \frac{0.6}{c}, \frac{0.4}{d}, \frac{0.1}{e}\}_\beta = \{\frac{0.8}{a}, \frac{0.6}{b}, \frac{0.6}{c}, \frac{0}{d}, \frac{0}{e}\}$, that is $\{\frac{0.8}{a}, \frac{0.5}{b}, \frac{0.2}{c}, \frac{0.5}{d}, \frac{0.2}{e}\}_\beta \subseteq \{\frac{0.8}{a}, \frac{0.6}{b}, \frac{0.6}{c}, \frac{0.4}{d}, \frac{0.1}{e}\}_\beta$, but $\{\frac{0.8}{a}, \frac{0.5}{b}, \frac{0.2}{c}, \frac{0.5}{d}, \frac{0.2}{e}\} \nsubseteq \{\frac{0.8}{a}, \frac{0.6}{b}, \frac{0.6}{c}, \frac{0.4}{d}, \frac{0.1}{e}\}$. Theorem 4 and Example 3 show that the evaluation criteria for assessment knowledge of individuals is critical, and it is meaningless to discard evaluation criteria for assessment knowledge. For any fuzzy knowledge structure, we need adjust the concept of distance between two fuzzy knowledge states, rather than counting the number of items by which two states differ.*

Theorem 5. *For any two fuzzy knowledge states $\widetilde{K}, \widetilde{L}$, $(\widetilde{K} \cup \widetilde{L})_\beta = \widetilde{K}_\beta \cup \widetilde{L}_\beta$, $(\widetilde{K} \cap \widetilde{L})_\beta = \widetilde{K}_\beta \cap \widetilde{L}_\beta$, where β is an evaluation criteria.*

Proof. According to Notation 1, $(\widetilde{K} \cup \widetilde{L})_\beta(q) = (\widetilde{K} \cup \widetilde{L})(q) = \widetilde{K}(q) \vee \widetilde{L}(q) = \widetilde{K})_\beta(q) \vee \widetilde{L}_\beta(q) = (\widetilde{K}_\beta \cup \widetilde{L}_\beta)(q)$ for any $(\widetilde{K} \cup \widetilde{L})(q) = \widetilde{K}(q) \vee \widetilde{L}(q) \geq \beta_q$, otherwise $(\widetilde{K} \cup \widetilde{L})_\beta(q) = 0$ when $(\widetilde{K} \cup \widetilde{L})(q) < \beta_q$. Hence, $(\widetilde{K} \cup \widetilde{L})_\beta = \widetilde{K}_\beta \cup \widetilde{L}_\beta$. Similarly, we can get $(\widetilde{K} \cap \widetilde{L})_\beta(q) = (\widetilde{K} \cap \widetilde{L})(q) = \widetilde{K}(q) \wedge \widetilde{L}(q) = \widetilde{K}_\beta(q) \wedge \widetilde{L}_\beta(q) = (\widetilde{K}_\beta \cap \widetilde{L}_\beta)(q)$ for any $(\widetilde{K} \cap \widetilde{L})(q) = \widetilde{K}(q) \wedge \widetilde{L}(q) \geq \beta_q$, otherwise $(\widetilde{K} \cap \widetilde{L})_\beta(q) = 0$ when $(\widetilde{K} \cap \widetilde{L})(q) < \beta_q$. Hence, $(\widetilde{K} \cap \widetilde{L})_\beta = \widetilde{K}_\beta \cap \widetilde{L}_\beta$. □

Definition 6. *Let $(Q, \widetilde{\mathcal{K}})$ be a fuzzy knowledge structure and β be an evaluation criteria. For any $\widetilde{K}_i, \widetilde{K}_j \in \widetilde{\mathcal{K}}$, the β-relative distance between $\widetilde{K}_i, \widetilde{K}_j$ is $d_\beta(\widetilde{K}_i, \widetilde{K}_j) = |(\widetilde{K}_i)_\beta \cup (\widetilde{K}_j)_\beta| - |(\widetilde{K}_i)_\beta \cap (\widetilde{K}_j)_\beta|$, where $|\widetilde{K}_\beta| = |\{q \mid \widetilde{K}_\beta(q) \neq 0\}|$ for any $\widetilde{K} \in \widetilde{\mathcal{K}}$.*

Intuitively, the β-relative distance between two fuzzy knowledge states represents the number of different items with reaching the evaluation criteria β.

Corollary 1. *Let $(Q, \widetilde{\mathcal{K}})$ be a fuzzy knowledge structure and β be an evaluation criteria. For any $\widetilde{K}_i, \widetilde{K}_j \in \widetilde{\mathcal{K}}$, the β-relative distance $d_\beta(\widetilde{K}_i, \widetilde{K}_j) = |(\widetilde{K}_i \cup \widetilde{K}_j)_\beta| - |(\widetilde{K}_i \cap \widetilde{K}_j)_\beta|$.*

Proof. It can be directly obtained from Definition 6 and Theorem 5. □

Example 4. *In Example 1, let evaluation criteria* $\beta = (0.8, 0.9, 0.7, 0.7, 0.6)$. *Take* $\widetilde{K} = \{\frac{0.4}{a}, \frac{0.9}{b}, \frac{0.7}{c}, \frac{0.6}{d}, \frac{0.3}{e}\}$ *and* $\widetilde{L} = \{\frac{0.7}{a}, \frac{0.8}{b}, \frac{0.8}{c}, \frac{0.7}{d}, \frac{0.6}{e}\}$. *Then* $\widetilde{K}_\beta = \{\frac{0}{a}, \frac{0.9}{b}, \frac{0.7}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\widetilde{L}_\beta = \{\frac{0}{a}, \frac{0}{b}, \frac{0.8}{c}, \frac{0.7}{d}, \frac{0.6}{e}\}$, $d_\beta(\widetilde{K}, \widetilde{L}) = |\widetilde{K}_\beta \cup \widetilde{L}_\beta| - |\widetilde{K}_\beta \cap \widetilde{L}_\beta| = |(\widetilde{K} \cup \widetilde{L})_\beta| - |(\widetilde{K} \cap \widetilde{L})_\beta| = 3$. *Note that the* β*-relative distance* $d_\beta(K, L)$ *is equivalent to the symmetric difference distance* $d(K, L)$ *for any ordinary knowledge states* K, L *when the evaluation criteria* $\beta = (1, 1, \cdots, 1)$ *[14].*

Notation 2. *Let* $(Q, \mathcal{R})$ *be a partially ordered set and* β *be an evaluation criteria, a fuzzy set* $\widetilde{q} \in \mathcal{F}(Q)$ *satisfy* $\widetilde{q}(x) \geq \beta_q$ *when* $x = q$, *otherwise* $\widetilde{q}(x) = 0$.

For a given evaluation criteria β, different individuals may have different degrees of membership $\widetilde{q}(x)$ when the criteria of q is reached. If $x = q$, the degree of membership $\widetilde{q}(x)$ is not fixed, but $\widetilde{q}(x) \in [\beta_q, 1]$. If $x \neq q$, the degree of membership $\widetilde{q}(x) = 0$. It's similar with the interval-valued fuzzy set on Q [17, 39], but $\widetilde{q} \in \mathcal{F}(Q)$.

Definition 7. *A* β*-fuzzy knowledge structure* $(Q, \widetilde{\mathcal{K}})$ *is called a fuzzy learning space if it satisfies the following two conditions.*

$[\widetilde{L}1]$ *fuzzy learning smoothness: For any two fuzzy knowledge states* $\widetilde{K}, \widetilde{L} \in (Q, \widetilde{\mathcal{K}})$ *such that* $\widetilde{K} \subset \widetilde{L}$, *there exists a finite chain of fuzzy knowledge states* $\widetilde{K} = \widetilde{K}_0 \subset \widetilde{K}_1 \subset \cdots \subset \widetilde{K}_p = \widetilde{L}$ *such that* $d_\beta(\widetilde{K}_i, \widetilde{K}_{i-1}) = 1$ *for* $1 \leq i \leq p$ *and* $d_\beta(\widetilde{K}, \widetilde{L}) = p$.

$[\widetilde{L}2]$ *fuzzy learning consistency: If* $\widetilde{K}$ *and* $\widetilde{L}$ *are two fuzzy knowledge states satisfying* $\widetilde{K} \subset \widetilde{L}$ *and* $\widetilde{q} \in \mathcal{F}(Q)$ *such that* $\widetilde{K} \cup \widetilde{q} \in \widetilde{\mathcal{K}}$, *then* $\widetilde{L} \cup \widetilde{q} \in \widetilde{\mathcal{K}}$.

Remark 4. *The fuzzy learning smoothness indicates that if the fuzzy knowledge state* $\widetilde{K}$ *of a learner is included in some other fuzzy knowledge state* $\widetilde{L}$, *then the learner can reach the fuzzy knowledge state* $\widetilde{L}$ *by partially mastering the missing items one at a time. Partially mastering the missing items one at a time means that the learner only needs to reach the criteria for solving the missing items. We refer to this chain as an* $\widetilde{L}1$*-chain from* $\widetilde{K}$ *to* $\widetilde{L}$. *The fuzzy learning consistency indicates that knowing more does not prevent from partially learning something new, where partially learning an item q means reaching the criteria* β_q. $[\widetilde{L}1]$ *and* $[\widetilde{L}2]$ *can be called fuzzy learning axioms under evaluation criteria* β, *since they can be regarded as generalizations of ordinary learning axioms. It's equivalent between fuzzy learning axioms and ordinary learning axioms when* $\beta = (1, 1, \cdots, 1)$. *Although learnstep number [14, 16] (Here, represent by* $d_\beta(\widetilde{K}_i, \widetilde{K}_{i-1})$*) is "1" in both axioms* $[\widetilde{L}1]$ *and* $[L1]$, *they have completely different meanings. The learnstep number "1" in* $[\widetilde{L}1]$ *indicates that there exists an item q such that only one fuzzy knowledge state reach the criteria* β_q *in the two fuzzy knowledge states. However, The learnstep number "1" in* $[L1]$ *represent there is exactly an different item q between the two knowledge states.*

Example 5. *Let* $Q = \{a, b, c, d, e\}$ *and* $\beta = (0.8, 0.6, 0.5, 0.6, 0.4)$, *the partial order is* $\mathcal{R} = \{(b, c), (a, d), (a, e), (b, e), (c, e), (d, e)\}$. *A* β*-fuzzy knowledge structure is* $\widetilde{\mathcal{K}} = \{\{\frac{0}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\{\frac{0.9}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\{\frac{0}{a}, \frac{0.6}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\{\frac{0.9}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0.7}{d}, \frac{0}{e}\}$, $\{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\{\frac{0}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0}{c}, \frac{0.7}{d}, \frac{0}{e}\}$, $\{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\{\frac{0}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0.7}{d}, \frac{0}{e}\}$, $\{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0.7}{d}, \frac{0}{e}\}$, $\{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0.7}{d}, \frac{0.5}{e}\}\}$. *We can verify that the fuzzy knowledge structure is a fuzzy learning space. For instance, take two fuzzy knowledge states* $\widetilde{K}_1 = \{\frac{0.9}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}\}$ *and* $\widetilde{K}_2 = \{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0.7}{d}, \frac{0}{e}\}$, *that is* $\widetilde{K}_1 \subset \widetilde{K}_2$. *There exists a* $\widetilde{L}1$*-chain from* $\widetilde{K}_1$ *to* $\widetilde{K}_2$, *that is* $\{\frac{0.9}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}\} \subset \{\frac{0.9}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0.7}{d}, \frac{0}{e}\} \subset \{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0}{c}, \frac{0.7}{d}, \frac{0}{e}\} \subset \{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0.7}{d}, \frac{0}{e}\}$. *It is easy to verify that* $d_\beta(\widetilde{K}_1, \widetilde{K}_2) = 3$. *Take two fuzzy knowledge states* $\widetilde{K}_1 = \{\frac{0.9}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}$ *and* $\widetilde{K}_3 = \{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0}{d}, \frac{0}{e}\}$, *that is* $\widetilde{K}_1 \subset \widetilde{K}_3$. *Take* $\widetilde{d} = \{\frac{0}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0.7}{d}, \frac{0}{e}\}$, *then* $\widetilde{K}_1 \cup \widetilde{d} \in \widetilde{\mathcal{K}}$ *such that* $\widetilde{K}_3 \cup \widetilde{d} \in \widetilde{\mathcal{K}}$. *There exist some* $\widetilde{L}1$*-chains from* $\widetilde{\varnothing}$ *to* $\widetilde{Q}$, *one of that is* $\{\frac{0}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\} \subseteq \{\frac{0.9}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}\} \subseteq \{\frac{0.9}{a}, \frac{0}{b}, \frac{0}{c}, \frac{0.7}{d}, \frac{0}{e}\} \subseteq \{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0}{c}, \frac{0.7}{d}, \frac{0}{e}\} \subseteq \{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0.7}{d}, \frac{0}{e}\} \subseteq \{\frac{0.9}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0.7}{d}, \frac{0.5}{e}\}$.

Theorem 6. *For any fuzzy learning space* $(Q, \widetilde{\mathcal{K}})$ *and* $\widetilde{K} \in \widetilde{\mathcal{K}}$, *if* $\widetilde{K} \cup \widetilde{q} \in \widetilde{\mathcal{K}}$ *then* $\widetilde{K} \cup \widetilde{p} \in \widetilde{\mathcal{K}}$ *for any* $(p, q) \in \mathcal{R}$.

Proof. Suppose that $\widetilde{K} \cup \widetilde{p} \notin \widetilde{\mathcal{K}}$ for some $(p, q) \in \mathcal{R}$. We have $\widetilde{K}(p) < \beta_p$, since $\widetilde{K} \cup \widetilde{p} \notin \widetilde{\mathcal{K}}$. Then $(\widetilde{K} \cup \widetilde{q})(q) \geq \beta_q$ imply $(\widetilde{K} \cup \widetilde{q})(p) \geq \beta_p$, since $(p, q) \in \mathcal{R}$ and $\widetilde{K} \cup \widetilde{q}$ is a fuzzy knowledge state. That is $\widetilde{K} \subset \widetilde{K} \cup \widetilde{q}$ and $(\widetilde{K} \cup \widetilde{q})(p) \geq \beta_p$, but $\widetilde{K}(p) < \beta_p$. Note that $\widetilde{K}(q) < \beta_q$, since $\widetilde{K}(p) < \beta_p$ and $(p, q) \in \mathcal{R}$. That means, there doesn't exist a $[\widetilde{L}1]$-chain from $\widetilde{K}$ to $\widetilde{K} \cup \widetilde{q}$. This is a contradiction with the fact that $(Q, \widetilde{\mathcal{K}})$ is a fuzzy learning space. Therefore, $\widetilde{K} \cup \widetilde{p} \in \widetilde{\mathcal{K}}$ for any $(p, q) \in \mathcal{R}$. □

Corollary 2. *For any ordinary learning space* $(Q, \mathcal{K})$ *and* $K \in \mathcal{K}$, *if* $K \cup \{q\} \in \mathcal{K}$ *then* $K \cup \{p\} \in \mathcal{K}$ *for any* $p \preceq q$, *where "*$\preceq$*" is the surmise relation of* $(Q, \mathcal{K})$.

Proof. Suppose that $K \cup \{p\} \notin \mathcal{K}$ for some $p \preceq q$. We have $p \notin K$, since $K \cup \{p\} \notin \mathcal{K}$. Then $q \in K \cup \{q\}$ imply $p \in K \cup \{q\}$, since $p \preceq q$. That is $K \subset K \cup \{q\}$ and $p \in K \cup \{q\}$, but $p \notin K$. Note that $q \notin K$, since $p \notin K$ and $p \preceq q$. Then there doesn't exist a $[L1]$-chain from K to $K \subset K \cup \{q\}$. This is a contradiction with the fact that $(Q, \mathcal{K})$ is a learning space. Therefore, $K \cup \{p\} \in \mathcal{K}$ for any $p \preceq q$. □

Theorem 6 and Corollary 2 show that an individual always learns from the simple to the complex items that he has not yet mastered.

Definition 8. *A family of fuzzy knowledge states* $\widetilde{\mathcal{K}}$ *is called well-graded if for any two distinct fuzzy knowledge states* $\widetilde{K}, \widetilde{L} \in \widetilde{\mathcal{K}}$, *there is a finite sequence of fuzzy knowledge states* $\widetilde{K} = \widetilde{K}_0, \widetilde{K}_1, \cdots, \widetilde{K}_p = \widetilde{L}$ *such that* $d_\beta(\widetilde{K}_i, \widetilde{K}_{i-1}) = 1$ *for* $1 \leq i \leq p$ *and* $d_\beta(\widetilde{K}, \widetilde{L}) = p$.

Theorem 7. *For any* β*-fuzzy knowledge structure* $(Q, \widetilde{\mathcal{K}})$*,the following three conditions are equivalent:*

(1) $(Q, \widetilde{\mathcal{K}})$ *is a fuzzy learning space;*
(2) $(Q, \widetilde{\mathcal{K}})$ *is union-closed and there is some fuzzy set* $\widetilde{q} \in \mathcal{F}(Q)$ *such that* $\widetilde{K} \backslash \widetilde{q} \in (Q, \widetilde{\mathcal{K}})$ *for any nonempty fuzzy knowledge state* $\widetilde{K}$*;*
(3) $(Q, \widetilde{\mathcal{K}})$ *is well-graded fuzzy knowledge space.*

Proof. (1) ⇒ (2). Suppose that $(Q, \widetilde{\mathcal{K}})$ is a fuzzy learning space. For any fuzzy knowledge state $\widetilde{K}$, there is an $\widetilde{L}1$-chain from $\widetilde{\varnothing}$ to $\widetilde{K}$, so $\widetilde{K} \backslash \widetilde{q} \in (Q, \widetilde{\mathcal{K}})$ for some $\widetilde{q} \in \mathcal{F}(Q)$. Turning to union closure, we take any two fuzzy states $\widetilde{K}$, $\widetilde{L} \in \widetilde{\mathcal{K}}$, and suppose that neither of them is $\widetilde{\varnothing}$ or a subset of the other (otherwise union-closure holds trivially). Since $\widetilde{\varnothing} \subset \widetilde{L}$, axiom $[\widetilde{L}1]$ implies the existence of $\widetilde{L}1$-chain $\widetilde{\varnothing} \subset \widetilde{q}_1 \subset \cdots \subset \cup_{i=1}^{m} \widetilde{q}_m = \widetilde{L}$, where each $\cup_{i=1}^{j} \widetilde{q}_j \in \widetilde{\mathcal{K}}$ $(j = 1 \cdots m)$. Since $\widetilde{\varnothing} \subset \widetilde{K}$ $(\widetilde{\varnothing}, \widetilde{K} \in \widetilde{\mathcal{K}})$, and $\widetilde{\varnothing} \cup \widetilde{q}_1 = \widetilde{q}_1 \in \widetilde{\mathcal{K}}$, axiom $[\widetilde{L}2]$ implies that $\widetilde{K} \cup \widetilde{q}_1 \in \widetilde{\mathcal{K}}$. Repeatedly applying of axiom $[\widetilde{L}2]$, since $\widetilde{q}_1 \subseteq (\widetilde{K} \cup \widetilde{q}_1)$ and $\widetilde{q}_1 \cup \widetilde{q}_2 \in \widetilde{\mathcal{K}}$ be included in $\widetilde{L}1$-chain. Therefore we have $(\widetilde{K} \cup \widetilde{q}_1) \cup \widetilde{q}_2 = \widetilde{K} \cup (\widetilde{q}_1 \cup \widetilde{q}_2) \in \widetilde{\mathcal{K}}$. Applying the mathematical induction yields $\widetilde{K} \cup \widetilde{L} \in \widetilde{\mathcal{K}}$.

(2) ⇒ (3). Only need verify $(Q, \widetilde{\mathcal{K}})$ is well-graded. Take any two fuzzy knowledge states $\widetilde{K}, \widetilde{L}$ with $\widetilde{K} \subset \widetilde{L}$ (with possibly $\widetilde{K} = \widetilde{\varnothing}$). Repeatedly applying of condition (2) to state $\widetilde{L}$, we get a sequence of fuzzy knowledge states $\widetilde{L}_0 = \widetilde{L}, \widetilde{L}_1, \cdots, \widetilde{L}_k = \widetilde{\varnothing}$ such that $\widetilde{q}_{i-1} \subseteq \widetilde{L}_{i-1}$ and $\widetilde{L}_i = \widetilde{L}_{i-1} \backslash \widetilde{q}_{i-1}$ for i=1, 2, $\cdots$, k. Let j be the largest index such that $\widetilde{q}_j \nsubseteq \widetilde{K}$. We obtain $\widetilde{K} \subset \widetilde{K} \cup \widetilde{q}_j = \widetilde{K} \cup \widetilde{L}_j \subseteq \widetilde{L}$.

Replacing $\widetilde{K}$ with $\widetilde{K} \cup \widetilde{q}_j$ and using the induction we see that the condition of the fuzzy well-graded of $(Q, \widetilde{\mathcal{K}})$ is satisfied.

(3) $\Rightarrow$ (1). Axiom $\widetilde{L}1$ results from the fuzzy wellgradedness condition. Suppose that $\widetilde{K} \subset \widetilde{L}$ for two fuzzy knowledge states and $\widetilde{K} \cup \widetilde{q}$ is also a fuzzy knowledge state. By union-closure, the set $(\widetilde{K} \cup \widetilde{q}) \cup \widetilde{L} = \widetilde{L} \cup \widetilde{q}$ is also a fuzzy knowledge state. So, $\widetilde{L}2$ holds. □

How to use a simple method to determine that a fuzzy knowledge structure is a fuzzy learning space? Now, Theorems 8 and 10 can make a simple decision.

Theorem 8. *A fuzzy knowledge structure $(Q, \widetilde{\mathcal{K}})$ is a fuzzy learning space if and only if $(Q, \widetilde{\mathcal{K}})$ satisfies the following two conditions:*

(1) For any nonempty fuzzy knowledge state $\widetilde{K}$, there exists $\widetilde{q} \in \mathcal{F}(Q)$ such that $\widetilde{K} \setminus \widetilde{q} \in \widetilde{\mathcal{K}}$;
(2) For any fuzzy knowledge state $\widetilde{K}$ and $\widetilde{q}, \widetilde{r} \in \mathcal{F}(Q)$, if $\widetilde{K} \cup \widetilde{q} \in \widetilde{\mathcal{K}}, \widetilde{K} \cup \widetilde{r} \in \widetilde{\mathcal{K}}$, then $\widetilde{K} \cup \{\widetilde{q} \cup \widetilde{r}\} \in \widetilde{\mathcal{K}}$.

Proof. Necessity. Since $(Q, \widetilde{\mathcal{K}})$ is a fuzzy learning space, the condition (1) is a direct consequence of axiom $\widetilde{L}1$. We only need to notice $\widetilde{K} \cup (\widetilde{q} \cup \widetilde{r}) = (\widetilde{K} \cup \widetilde{q}) \cup (\widetilde{K} \cup \widetilde{r})$. Applying the assumed closure under union of a fuzzy learning space which was proved in Theorem 7, we have $\widetilde{K} \cup (\widetilde{q} \cup \widetilde{r}) \in (Q, \widetilde{\mathcal{K}})$.

Sufficiency. To prove fuzzy learning smoothness, we consider two fuzzy knowledge states $\widetilde{K}$ and $\widetilde{L}$ such that $\widetilde{K} \subset \widetilde{L}$. According to condition (1), there exist two sequences $\widetilde{K}_0 = \widetilde{\varnothing}, \widetilde{K}_1, \cdots, \widetilde{K}_n = \widetilde{K}$ and $\widetilde{L}_0 = \widetilde{\varnothing}, \widetilde{L}_1, \cdots, \widetilde{L}_m = \widetilde{L}$ of fuzzy knowledge states such that $d_\beta(\widetilde{K}_i, \widetilde{K}_{i-1}) = 1$ for $1 \le i \le n$ and $d_\beta(\widetilde{L}_j, \widetilde{L}_{j-1}) = 1$ for $1 \le j \le m$. By applying condition (2) repeatedly, we derive $\widetilde{K}_0 \cup \widetilde{L}_0, \widetilde{K}_1 \cup \widetilde{L}_0, \cdots, \widetilde{K}_n \cup \widetilde{L}_0 \in \widetilde{\mathcal{K}}$, next $\widetilde{K}_0 \cup \widetilde{L}_1, \widetilde{K}_1 \cup \widetilde{L}_1, \cdots, \widetilde{K}_n \cup \widetilde{L}_1 \in \widetilde{\mathcal{K}}$, etc., and $\widetilde{K}_0 \cup \widetilde{L}_{m-1}, \widetilde{K}_1 \cup \widetilde{L}_{m-1}, \cdots, \widetilde{K}_n \cup \widetilde{L}_{m-1} \in \widetilde{\mathcal{K}}$, finally $\widetilde{K}_0 \cup \widetilde{L}_m, \widetilde{K}_1 \cup \widetilde{L}_m, \cdots, \widetilde{K}_n \cup \widetilde{L}_m \in \widetilde{\mathcal{K}}$. Thus $\widetilde{K} \cup \widetilde{L}_0 = \widetilde{K}, \widetilde{K} \cup \widetilde{L}_1, \cdots, \widetilde{K} \cup \widetilde{L}_{m-1}, \widetilde{L}$ are all in $\widetilde{\mathcal{K}}$, where $\widetilde{K}_n = \widetilde{K}, \widetilde{L}_m = \widetilde{L}$, and $\widetilde{K} \subset \widetilde{L}$. After deleting of repetitions, we obtain the desired sequence from $\widetilde{K}$ to $\widetilde{L}$. To prove fuzzy learning consistency, if there exists $\widetilde{q} \in \mathcal{F}(Q)$ such that $\widetilde{K} \cup \widetilde{q} \in \widetilde{\mathcal{K}}$. We have clarified the existence of a sequence $\widetilde{K} \cup \widetilde{L}_0 = \widetilde{K}, \widetilde{K} \cup \widetilde{L}_1, \cdots, \widetilde{K} \cup \widetilde{L}_{m-1}, \widetilde{L}$ of fuzzy knowledge states such that $d_\beta(\widetilde{K} \cup \widetilde{L}_i, \widetilde{K} \cup \widetilde{L}_{i-1}) = 1$ for $1 \le i \le m$. Applying condition (2) repeatedly, we obtain $(\widetilde{K} \cup \widetilde{L}_1) \in \widetilde{\mathcal{K}}, \widetilde{K} \cup \widetilde{q} \in \widetilde{\mathcal{K}}$, that is $(\widetilde{K} \cup \widetilde{L}_1) \cup \widetilde{q} \in \widetilde{\mathcal{K}}$. By the mathematical induction yields $(\widetilde{K} \cup \widetilde{L}_2) \cup \widetilde{q} \in \widetilde{\mathcal{K}}, \cdots, \widetilde{L} \cup \widetilde{q} \in \widetilde{\mathcal{K}}$, where the last one is fuzzy learning consistency as expected. □

Definition 9. *A fuzzy knowledge structure $(Q, \widetilde{\mathcal{K}})$ is β-relative discriminative if the set $q^* = \{r \in Q \mid \widetilde{K}(q) \ge \beta_q \Leftrightarrow \widetilde{K}(r) \ge \beta_r, \forall \widetilde{K} \in \widetilde{\mathcal{K}}\}$ contains a single item for any $q \in Q$.*

Theorem 9. *Let $\widetilde{\mathcal{K}}$ be a fuzzy knowledge structure and $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ satisfies $\beta_i = \beta_j$ for each $1 \le i, j \le n$. Then $\widetilde{\mathcal{K}}$ is a discriminative fuzzy knowledge structure if and only if $\widetilde{\mathcal{K}}$ is a β-relative discriminative fuzzy knowledge structure.*

Proof. Sufficiency: Suppose that $\widetilde{\mathcal{K}}$ is not a discriminative fuzzy knowledge structure, then there exist $q, r \in Q$, $q \ne r$, such that $\widetilde{K}(q) = \widetilde{K}(r)$ for all $\widetilde{K} \in \widetilde{\mathcal{K}}$. Hence $\widetilde{K}(q) \ge \beta_q \Leftrightarrow \widetilde{K}(r) \ge \beta_r$, since $\beta_q = \beta_r$. That means $\widetilde{\mathcal{K}}$ is not β-relative discriminative fuzzy knowledge structure. Necessity: Suppose $\widetilde{\mathcal{K}}$ is not a β-relative discriminative fuzzy knowledge structure, then there exist $q, r \in Q$, $q \ne r$ such that $\widetilde{K}(q) \ge \beta_q \Leftrightarrow \widetilde{K}(r) \ge \beta_r$ for any $\widetilde{K} \in \widetilde{\mathcal{K}}$. Then $\widetilde{K}(q) = \widetilde{K}(r)$ for any $\widetilde{K} \in \widetilde{\mathcal{K}}$, since $\beta_q = \beta_r$ in β. That means $\widetilde{\mathcal{K}}$ is not a discriminative fuzzy knowledge structure. □

Note that a β-relative discriminative fuzzy knowledge structure is equivalent to a discriminative knowledge structure when the $\beta = (1, 1, \cdots, 1)$. The Example 6 below shows that the Theorem 9 may not hold if the evaluation criteria β is not satisfy $\beta_i = \beta_j$ for all $1 \le i, j \le n$.

Example 6. *Let $Q = \{a, b, c\}$ and a partial order $\mathcal{R} = \{(a, b), (b, c)\}$. A fuzzy knowledge structure is $\widetilde{\mathcal{K}} = \{\{\frac{0}{a}, \frac{0}{b}, \frac{0}{c}\}, \{\frac{0.9}{a}, \frac{0.8}{b}, \frac{0.3}{c}\}, \{\frac{0.8}{a}, \frac{0.7}{b}, \frac{0.2}{c}\}, \{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\}\}$, it's a discriminative fuzzy knowledge structure since there are some fuzzy knowledge states $\widetilde{K} \in \widetilde{\mathcal{K}}$ such that $\widetilde{K}(a) \neq \widetilde{K}(b)$ and $\widetilde{K}(b) \neq \widetilde{K}(c)$, $\widetilde{K}(a) \neq \widetilde{K}(c)$. However, this fuzzy knowledge structure is not β-relative discriminative when $\beta = (0.7, 0.6, 0.5)$, since $\widetilde{K}(a) \geq 0.7 \Leftrightarrow \widetilde{K}(b) \geq 0.6$ for any $\widetilde{K} \in \widetilde{\mathcal{K}}$. Considering the fuzzy knowledge structure $\widetilde{\mathcal{K}} = \{\{\frac{0}{a}, \frac{0}{b}, \frac{0}{c}\}, \{\frac{0.6}{a}, \frac{0.6}{b}, \frac{0.5}{c}\}, \{\frac{0.8}{a}, \frac{0.8}{b}, \frac{0.3}{c}\}, \{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\}\}$. Take $\widetilde{K} = \{\frac{0.6}{a}, \frac{0.6}{b}, \frac{0.5}{c}\}$, then $\widetilde{K}(a) < 0.7$, but $\widetilde{K}(b) \geq 0.6$. Take $\widetilde{K} = \{\frac{0.8}{a}, \frac{0.8}{b}, \frac{0.3}{c}\}$, then $\widetilde{K}(a) \geq 0.7$, but $\widetilde{K}(c) < 0.3$. Take $\widetilde{K} = \{\frac{0.6}{a}, \frac{0.6}{b}, \frac{0.5}{c}\}$, then $\widetilde{K}(b) \geq 0.6$, but $\widetilde{K}(c) < 0.3$. That means, the fuzzy knowledge structure is β-relative discriminative when $\beta = (0.7, 0.6, 0.5)$. However, the fuzzy knowledge structure is not a discriminative fuzzy knowledge structure. Therefore, when a fuzzy knowledge structure is not β-relative discriminative, it is not a fuzzy learning space. We have the following conclusion.*

Theorem 10. *Any fuzzy learning space is β-relative discriminative.*

Proof. Let $(Q, \widetilde{\mathcal{K}})$ be a fuzzy learning space. Suppose it is not β-relative discriminative, then there exist two items $q_i, q_j \in Q$ such that $\widetilde{K}(q_i) \geq \beta_i \Leftrightarrow \widetilde{K}(q_j) \geq \beta_j$ for every $\widetilde{K} \in \widetilde{\mathcal{K}}$. Since $\widetilde{\varnothing} \subset \widetilde{Q}$, by fuzzy learning smoothness axiom $\widetilde{L}1$, there exists a $\widetilde{L}1$-chain $\widetilde{\varnothing} \subset \widetilde{K}_1 \subset \widetilde{K}_2 \subset \cdots \subset \widetilde{K}_n \subset \widetilde{Q}$. Then, we derive $\widetilde{\varnothing} \subset \{\widetilde{q}_i \cup \widetilde{q}_j\} = \widetilde{K}_1$ from Theorem 7 condition (2) by induction. Finally, we get either $\widetilde{\varnothing} \subset \{\widetilde{q}_i\} \subset \{\widetilde{q}_i \cup \widetilde{q}_j\}$ or $\widetilde{\varnothing} \subset \{\widetilde{q}_j\} \subset \{\widetilde{q}_i \cup \widetilde{q}_j\}$ which contradicts with $\widetilde{K}(q_i) \geq \beta_i \Leftrightarrow \widetilde{K}(q_j) \geq \beta_j$. So, the $\widetilde{\mathcal{K}}$ is β-relative discriminative. □

Theorem 10 can be regard as one of the judgment of a fuzzy learning space, since if a fuzzy knowledge structure is not β-relative discriminative, then it isn't a fuzzy learning space.

6. Fuzzy knowledge basis

A knowledge space can be faithfully summarized by a subfamily of their knowledge states. For a fuzzy knowledge space, any fuzzy knowledge state of the space can be generated by forming the union of some states in the fuzzy knowledge basis. Fuzzy knowledge basis plays an important role in knowledge assessment and guidance learning.

Definition 10. *The fuzzy knowledge basis of a fuzzy knowledge space $(Q, \widetilde{\mathcal{K}})$ is a minimal subfamily $\widetilde{\mathcal{B}}$ of $\widetilde{\mathcal{K}}$ spanning $\widetilde{\mathcal{K}}$, where the span of $\widetilde{\mathcal{B}}$ is the family $\widetilde{\mathcal{K}}$ containing any union of arbitrary subfamily of $\widetilde{\mathcal{B}}$.*

The fuzzy knowledge basis of $(Q, \widetilde{\mathcal{K}})$ is denoted as $\widetilde{\mathcal{B}} = \{\widetilde{B}_1, \widetilde{B}_2, \cdots, \widetilde{B}_m\}$. Any fuzzy knowledge state $\widetilde{K} \in \widetilde{\mathcal{K}}$ satisfies $\widetilde{K} = \cup\widetilde{\mathcal{B}}'$ for some $\widetilde{\mathcal{B}}' \subseteq \widetilde{\mathcal{B}}$. The minimal subfamily $\widetilde{\mathcal{B}}$ is with respect to set inclusion.

Example 7. *In Example 2, the fuzzy knowledge structure is a fuzzy knowledge space, and we can find $\widetilde{K}_5 = \widetilde{K}_2 \cup \widetilde{K}_3$, $\widetilde{K}_6 = \widetilde{K}_3 \cup \widetilde{K}_4$, $\widetilde{K}_7 = \widetilde{K}_2 \cup \widetilde{K}_4$, $\widetilde{K}_9 = \widetilde{K}_3 \cup \widetilde{K}_8$, $\widetilde{K}_{10} = \widetilde{K}_2 \cup \widetilde{K}_8$, $\widetilde{K}_{11} = \widetilde{K}_2 \cup \widetilde{K}_3 \cup \widetilde{K}_8$, $\widetilde{K}_{12} = \widetilde{K}_4 \cup \widetilde{K}_9 \cup \widetilde{K}_7 \cup \widetilde{K}_8$, $\widetilde{K}_{14} = \widetilde{K}_2 \cup \widetilde{K}_{13}$. The fuzzy knowledge basis is $\widetilde{\mathcal{B}} = \{\widetilde{K}_2, \widetilde{K}_3, \widetilde{K}_4, \widetilde{K}_8, \widetilde{K}_{13}, \widetilde{K}_{15}\}$, see Table 2.*

Definition 11. *Let $(Q, \widetilde{\mathcal{K}})$ be a fuzzy knowledge space. A fuzzy knowledge state $\widetilde{K}$ is an atom at q if and only if $\widetilde{K}' \nsubseteq \widetilde{K}$ for any $\widetilde{K}' \in \widetilde{\mathcal{F}}^q_{K'}$, where $\widetilde{\mathcal{F}}^q_K = \{\widetilde{K}' \mid \widetilde{K}'(q) = \widetilde{K}(q), \widetilde{K} \neq \widetilde{K}', \widetilde{K}' \in \widetilde{\mathcal{K}}\}$. A fuzzy knowledge state $\widetilde{K} \in \widetilde{\mathcal{K}}$ is called an atom if it is an atom at q for some $q \in Q$.*

Table 2. The fuzzy knowledge basis of $(Q, \widetilde{\mathcal{K}})$.

	a	b	c	d	e
$\widetilde{K}_2$	0.4	0.9	0	0	0
$\widetilde{K}_3$	0	0.8	0.7	0	0
$\widetilde{K}_4$	0	0.7	0.7	0.6	0
$\widetilde{K}_8$	0.3	0.6	0.5	0.5	0.3
$\widetilde{K}_{13}$	0.7	0.8	0.8	0.7	0.6
$\widetilde{K}_{15}$	1	1	1	1	1

Example 8. *An atom at item q is a minimal fuzzy knowledge state of all fuzzy knowledge states with the same degree of membership at q, and an atom at item q must satisfy* $\widetilde{K}(q) \neq 0$. *By convention,* $\widetilde{\varnothing}$ *is the union of the empty subfamily of the basis. Thus, since the basis is minimal,* $\widetilde{\varnothing}$ *never belongs to a basis. In Example 2, the atoms are* $\{\widetilde{K}_2, \widetilde{K}_3, \widetilde{K}_4, \widetilde{K}_8, \widetilde{K}_{13}, \widetilde{K}_{15}\}$, *where* $\widetilde{K}_2 = \{\frac{0.4}{a}, \frac{0.9}{b}, \frac{0}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\widetilde{K}_3 = \{\frac{0}{a}, \frac{0.8}{b}, \frac{0.7}{c}, \frac{0}{d}, \frac{0}{e}\}$, $\widetilde{K}_4 = \{\frac{0}{a}, \frac{0.7}{b}, \frac{0.7}{c}, \frac{0.6}{d}, \frac{0}{e}\}$, $\widetilde{K}_8 = \{\frac{0.3}{a}, \frac{0.6}{b}, \frac{0.5}{c}, \frac{0.5}{d}, \frac{0.3}{e}\}$, $\widetilde{K}_{13} = \{\frac{0.7}{a}, \frac{0.8}{b}, \frac{0.8}{c}, \frac{0.7}{d}, \frac{0.6}{e}\}$, $\widetilde{K}_{15} = \{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}\}$. *For the fuzzy knowledge state* $\widetilde{K}_{13}$, *we get* $\widetilde{\mathcal{F}}^a_{\widetilde{K}_{13}} = \{\widetilde{K}_{14}\}$ *and* $\widetilde{K}_{14} \nsubseteq \widetilde{K}_{13}$. *Thus,* $\widetilde{K}_{13}$ *is an atom at a. We get* $\widetilde{\mathcal{F}}^b_{\widetilde{K}_{13}} = \{\widetilde{K}_3, \widetilde{K}_6, \widetilde{K}_9\}$ *and* $\widetilde{K}_3 \subseteq \widetilde{K}_{13}, \widetilde{K}_6 \subseteq \widetilde{K}_{13}, \widetilde{K}_9 \subseteq \widetilde{K}_{13}$. *Thus,* $\widetilde{K}_{13}$ *is not an atom at b. We get* $\widetilde{\mathcal{F}}^c_{\widetilde{K}_{13}} = \{\widetilde{K}_{14}\}$ *and* $\widetilde{K}_{14} \nsubseteq \widetilde{K}_{13}$. *Thus,* $\widetilde{K}_{13}$ *is an atom at c. We get* $\widetilde{\mathcal{F}}^d_{\widetilde{K}_{13}} = \{\widetilde{K}_{14}\}$ *and* $\widetilde{K}_{14} \nsubseteq \widetilde{K}_{13}$. *Thus,* $\widetilde{K}_{13}$ *is an atom at d. We get* $\widetilde{\mathcal{F}}^e_{\widetilde{K}_{13}} = \{\widetilde{K}_{14}\}$ *and* $\widetilde{K}_{14} \nsubseteq \widetilde{K}_{13}$. *Thus,* $\widetilde{K}_{13}$ *is an atom at e. Similarly, we can verify* $\widetilde{K}_2$ *is an atom at a and b,* $\widetilde{K}_3$ *is an atom at b and c,* $\widetilde{K}_4$ *is an atom at b, c and d,* $\widetilde{K}_8$ *is an atom at a, b, c, d and e,* $\widetilde{K}_{15}$ *is an atom at a, b, c, d and e.*

Theorem 11. *Let* $(Q, \widetilde{\mathcal{K}})$ *be a fuzzy knowledge space. A fuzzy knowledge state* $\widetilde{K}$ *is an atom at q if and only if* $\widetilde{K}(q) \neq \bigcup \widetilde{\mathcal{F}}_{\widetilde{K}}(q)$ *for some* $q \in Q$, *where* $\widetilde{\mathcal{F}}_{\widetilde{K}} = \{\widetilde{K}' \mid \widetilde{K}' \subset \widetilde{K}, \widetilde{K}' \in \widetilde{\mathcal{K}}\}$.

Proof. Necessity: Suppose $\widetilde{K} = \bigcup \widetilde{\mathcal{F}}_{\widetilde{K}}$. Then for every $q \in Q$, there exists a $\widetilde{K}' \in \widetilde{\mathcal{F}}_{\widetilde{K}}$ such that $\widetilde{K}(q) = \widetilde{K}'(q)$. That is, there exists $\widetilde{K}' \in \widetilde{\mathcal{F}}^q_{\widetilde{K}}$ such that $\widetilde{K}' \subseteq \widetilde{K}$ for any $q \in Q$, where $\widetilde{\mathcal{F}}^q_{\widetilde{K}} = \{\widetilde{K}' \mid \widetilde{K}'(q) = \widetilde{K}(q), \widetilde{K} \neq \widetilde{K}', \widetilde{K}' \in \widetilde{\mathcal{K}}\}$. Hence, $\widetilde{K}$ is not an atom at item q. Sufficiency: Suppose $\widetilde{K}$ is not an atom at q. According to Definition 11, there exists a $\widetilde{K}' \in \widetilde{\mathcal{F}}^q_{\widetilde{K}}$ such that $\widetilde{K}' \subseteq \widetilde{K}$, where $\widetilde{\mathcal{F}}^q_{\widetilde{K}} = \{\widetilde{K}' \mid \widetilde{K}'(q) = \widetilde{K}(q), \widetilde{K} \neq \widetilde{K}', \widetilde{K}' \in \widetilde{\mathcal{K}}\}$. That means $\widetilde{K}' \in \widetilde{\mathcal{F}}_{\widetilde{K}} = \{\widetilde{K}' \mid \widetilde{K}' \subset \widetilde{K}, \widetilde{K}' \in \widetilde{\mathcal{K}}\}$. Since $\widetilde{K}' \in \widetilde{\mathcal{F}}_{\widetilde{K}}$ and $\widetilde{K}(q) = \widetilde{K}'(q)$, then we have $\widetilde{K}(q) = \bigcup \widetilde{\mathcal{F}}_{\widetilde{K}}(q)$ for some $q \in Q$. We have a contradiction with the fact that $\widetilde{K}(q) \neq \bigcup \widetilde{\mathcal{F}}(q)$. □

Theorem 12. *The fuzzy knowledge basis of a fuzzy knowledge space* $(Q, \widetilde{\mathcal{K}})$ *is formed by the collection of all the atoms.*

Proof. Let $\widetilde{\mathcal{B}}$ be the fuzzy knowledge basis of $(Q, \widetilde{\mathcal{K}})$, and let $\widetilde{\mathcal{A}}$ be the family of all the atoms of $(Q, \widetilde{\mathcal{K}})$. We have to show that $\widetilde{\mathcal{A}} = \widetilde{\mathcal{B}}$. If some $\widetilde{K} \in \widetilde{\mathcal{B}}$ is not an atom. According to Theorem 11, we get $\widetilde{K} = \bigcup \widetilde{\mathcal{F}}$, where $\widetilde{\mathcal{F}}_{\widetilde{K}} = \{\widetilde{K}' \mid \widetilde{K}' \subset \widetilde{K}, \widetilde{K}' \in \widetilde{\mathcal{K}}\}$. Furthermore, $\widetilde{K}' = \bigcup \widetilde{\mathcal{B}}'$ for any $\widetilde{K}' \in \widetilde{\mathcal{F}}$, since $\widetilde{K}' \in \widetilde{\mathcal{K}}$. That means $\widetilde{K}$ is an union of some fuzzy knowledge states in $\widetilde{\mathcal{B}}$, which is a contradiction. Thus, $\widetilde{K}$ is an atom. Eventually, we have $\widetilde{\mathcal{B}} \subseteq \widetilde{\mathcal{A}}$. Conversely, take any $\widetilde{K} \in \widetilde{\mathcal{A}}$. Suppose $\widetilde{K}$ is an atom at q. Then, $\widetilde{K}(q) \neq \bigcup \widetilde{\mathcal{F}}(q)$, where $\widetilde{\mathcal{F}}_{\widetilde{K}} = \{\widetilde{K}' \mid \widetilde{K}' \subset \widetilde{K}\}$. That means, there doesn't exist a family of fuzzy knowledge states $\widetilde{\mathcal{F}}$ such that $\widetilde{K} = \bigcup \widetilde{\mathcal{F}}$. Therefore $\widetilde{K} \in \widetilde{\mathcal{B}}$, then $\widetilde{\mathcal{A}} \subseteq \widetilde{\mathcal{B}}$. Eventually, $\widetilde{\mathcal{A}} = \widetilde{\mathcal{B}}$. □

We give two algorithms for finding the fuzzy knowledge basis of a fuzzy knowledge space which are based on Theorem 12 and Definition 11 respectively.

Step 1. List the fuzzy knowledge states $\widetilde{K}_1, \widetilde{K}_2, \cdots, \widetilde{K}_m$ in such a way that $\widetilde{K}_i \subset \widetilde{K}_j$ implies $i < j$ for $i, j \in \{1, 2, \cdots, m\}$. Thus, list the fuzzy knowledge states according to the order of their nondecreasing size, and arbitrarily for states of the incomparable fuzzy knowledge states. Form an $m \times n$ array $T = (T_{ij})$ with the rows and columns representing the fuzzy knowledge states and items, respectively. The rows are indexed from 1 to m and the columns from 1 to n.

Step 2. Let $\widetilde{K}_1 = \widetilde{\varnothing}$ and take $\widetilde{K}_1$ out of $\widetilde{\mathcal{B}}$, compute $\bigcup_{j=1}^{i-1} \widetilde{K}_j$. If $\widetilde{K}_i = \bigcup_{j=1}^{i-1} \widetilde{K}_j$ holds, then take $\widetilde{K}_i$ out of $\widetilde{\mathcal{B}}$.

Step 3. If a fuzzy state $\widetilde{K}_i$ satisfies $\widetilde{K}_i \neq \bigcup_{j=1}^{i-1} \widetilde{K}_j$, then put $\widetilde{K}_i$ in $\widetilde{\mathcal{B}}$.

Step 4. Repeat Steps 2 and 3. Eventually we get the fuzzy knowledge basis $\widetilde{\mathcal{B}}$.

The process of finding fuzzy knowledge basis is to remove union-reducible fuzzy knowledge states. The above algorithm is similar to the Dowling proposal in 1993 [11] and that of Rusch and Wille proposed in 1996 [27]. We extend it to the fuzzy knowledge spaces. According to Theorem 12, we can see the process of finding the fuzzy knowledge basis is to find all the atoms. Another algorithm for finding the fuzzy knowledge basis of a fuzzy knowledge space $(Q, \widetilde{\mathcal{K}})$ is as follows.

Step 1. For any fuzzy knowledge state $\widetilde{K}$, compute $\widetilde{\mathcal{F}}^q_{\widetilde{K}}$ for any item $q \in Q$, where $\widetilde{\mathcal{F}}^q_{\widetilde{K}} = \{\widetilde{K}' \mid \widetilde{K}'(q) = \widetilde{K}(q), \widetilde{K} \neq \widetilde{K}'\}$.

Step 2. If $\widetilde{K}' \not\subseteq \widetilde{K}$ for any $\widetilde{K}' \in \widetilde{\mathcal{F}}_q$, put $\widetilde{K}$ in $\widetilde{\mathcal{B}}$. Otherwise, take $\widetilde{K}$ out of $\widetilde{\mathcal{B}}$.

Step 3. Repeat Steps 1 and 2 for any fuzzy knowledge state in $(Q, \widetilde{\mathcal{K}})$. Eventually we get the fuzzy knowledge basis $\widetilde{\mathcal{B}}$.

7. Empirical application

In this section, we take an example to illustrate the process of constructing fuzzy knowledge structures and β-fuzzy knowledge structures from partial order $\mathcal{R}$ on Q, where the knowledge domain Q is "Addition of fractions".

For the "Addition of fractions", Stefanutti et al. constructed a dichotomous knowledge structure through 11 items [31]. However, there are equally informative items in the work of Stefanutti et al.. Therefore, we redesign some items, and then select 3 items for the domain. These items are listed in Table 3.

Table 3. The items of the knowledge domain "Addition of fractions".

	Items
q_1	$\frac{1}{12} + \frac{5}{12}$
q_2	$\frac{4}{5} + \frac{7}{10}$
q_3	$\frac{5}{6} + \frac{1}{15}$

For the item $q_1 : \frac{1}{12} + \frac{5}{12}$, the process of solving q_1 is as follows:

q_1 :	$\frac{1}{12} + \frac{5}{12}$	
	$= \frac{6}{12}$	Proficiency 0.6 in item q_1
	$= \frac{1}{2}$	Proficiency 1 in item q_1

For the item $q_2 : \frac{4}{5} + \frac{7}{10}$, the process of solving q_2 is as follows:

$q_2 :$	$\frac{4}{5} + \frac{7}{10}$	
	$= \frac{8}{10} + \frac{7}{10}$	Proficiency 0.6 in item q_2
	$= \frac{15}{10} = \frac{3}{2}$	Proficiency 1 in item q_2

For the item $q_3 : \frac{5}{6} + \frac{1}{15}$, the process of solving q_3 is as follows:

$q_3 :$	$\frac{5}{6} + \frac{1}{15}$	
	$= \frac{25}{30} + \frac{2}{30}$	Proficiency 0.6 item q_3
	$= \frac{27}{30} = \frac{9}{10}$	Proficiency 1 item q_3

Based on the process of solving items, we assume that the partial order on Q is $\mathcal{R} = \{(q_1, q_2), (q_1, q_3)\}$. The dichotomous knowledge structure $\mathcal{K}$ is then obtained by applying

$$K \in \mathcal{K} \Leftrightarrow (\forall (p, q) \in \mathcal{R} : q \in K \Rightarrow p \in K).$$

Eventually, $\mathcal{K} = \{\emptyset, \{q_1\}, \{q_1, q_2\}, \{q_1, q_3\}, Q\}$, see Table 4. Moreover, the Hasse diagram of $\mathcal{K}$ is Figure 2.

Table 4. Constructing the dichotomous knowledge space $(Q, \mathcal{K})$ by $\mathcal{R}$.

K \ $\mathcal{R}$	(q_1, q_2)	(q_1, q_3)
$\emptyset$	√	√
$\{q_1\}$	√	√
$\{q_2\}$	×	√
$\{q_3\}$	√	×
$\{q_1, q_2\}$	√	√
$\{q_1, q_3\}$	√	√
$\{q_2, q_3\}$	×	×
Q	√	√

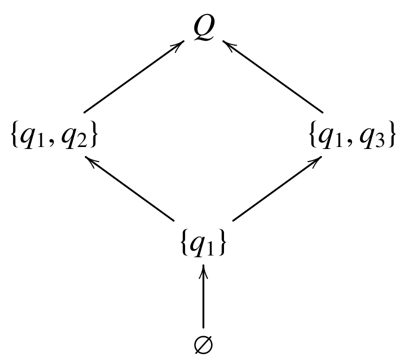

Figure 2. Hasse diagram of the dichotomous knowledge space $\mathcal{K}$.

Based on the partial order $\mathcal{R} = \{(q_1, q_2), (q_1, q_3)\}$ on Q, the fuzzy knowledge structure $\widetilde{\mathcal{K}}$ is then obtained by applying

$$\widetilde{K} \in \widetilde{\mathcal{K}} \Leftrightarrow (\forall (p, q) \in \mathcal{R} : \widetilde{K}(q) \leq \widetilde{K}(p)).$$

Eventually, the fuzzy knowledge structure induced by the partial order $\mathcal{R} = \{(q_1, q_2), (q_1, q_3)\}$ is $\widetilde{\mathcal{K}}=\{\widetilde{\emptyset}, \{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}, \{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}, \{\frac{0.6}{q_1}, \frac{0.6}{q_2}, \frac{0}{q_3}\}, \{\frac{0.6}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\}, \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}, \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}, \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{1}{q_3}\}, \{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0}{q_3}\}, \{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\}, \{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{1}{q_3}\}, \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0}{q_3}\}, \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}, \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}\}$, see Table 5. Moreover, the Hasse diagram of the fuzzy knowledge space $\widetilde{\mathcal{K}}$ is Figure 3.

Table 5. Constructing the fuzzy knowledge space $(Q, \widetilde{\mathcal{K}})$ by $\mathcal{R}$.

K \ $\mathcal{R}$	(q_1, q_2)	(q_1, q_3)	K \ $\mathcal{R}$	(q_1, q_2)	(q_1, q_3)
$\widetilde{\emptyset}$	$\surd$	$\surd$	$\{\frac{0.6}{q_1}, \frac{0.6}{q_2}, \frac{1}{q_3}\}$	$\surd$	$\times$
$\{\frac{0}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$	$\surd$	$\times$	$\{\frac{0.6}{q_1}, \frac{1}{q_2}, \frac{0}{q_3}\}$	$\times$	$\surd$
$\{\frac{0}{q_1}, \frac{0}{q_2}, \frac{1}{q_3}\}$	$\surd$	$\times$	$\{\frac{0.6}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}$	$\times$	$\surd$
$\{\frac{0}{q_1}, \frac{0.6}{q_2}, \frac{0}{q_3}\}$	$\times$	$\surd$	$\{\frac{0.6}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$	$\times$	$\times$
$\{\frac{0}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\}$	$\times$	$\times$	$\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0}{q_1}, \frac{0.6}{q_2}, \frac{1}{q_3}\}$	$\times$	$\times$	$\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0}{q_1}, \frac{1}{q_2}, \frac{0}{q_3}\}$	$\times$	$\surd$	$\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{1}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}$	$\times$	$\times$	$\{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$	$\times$	$\times$	$\{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}$	$\surd$	$\surd$	$\{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{1}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$	$\surd$	$\surd$	$\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{1}{q_3}\}$	$\surd$	$\times$	$\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0.6}{q_1}, \frac{0.6}{q_2}, \frac{0}{q_3}\}$	$\surd$	$\surd$	$\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$	$\surd$	$\surd$
$\{\frac{0.6}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\}$	$\surd$	$\surd$			

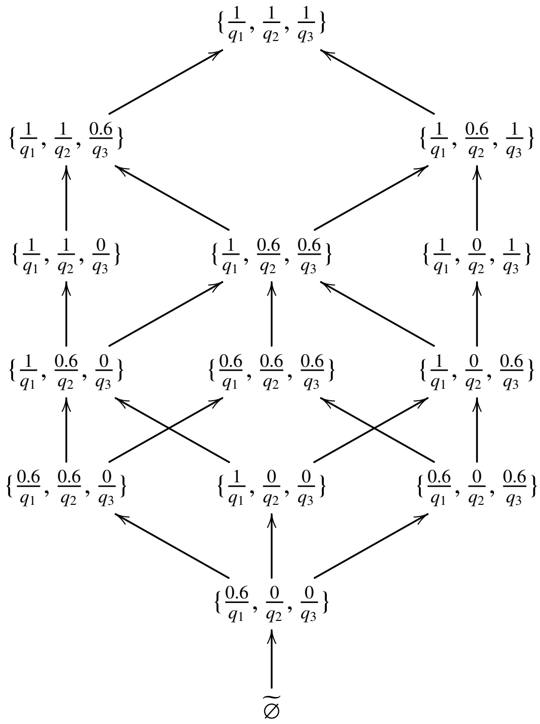

Figure 3. Hasse diagram of the fuzzy knowledge space $\widetilde{\mathcal{K}}$.

Rely on partial order $\mathcal{R} = \{(q_1, q_2), (q_1, q_3)\}$ on Q, the evaluation criteria is $\beta = (1, 1, 1)$ for the dichotomous knowledge spaces. In the example of "Addition of fractions", if $\beta = (1, 0.8, 0.5)$, then the β-fuzzy knowledge structure is $\widetilde{\mathcal{K}}_\beta = \{\widetilde{\varnothing}, \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}, \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}, \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{1}{q_3}\}, \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0}{q_3}\}, \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}, \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}\}$. Moreover, the Hasse diagram of the β-fuzzy knowledge structure $\widetilde{\mathcal{K}}_\beta$ is Figure 4.

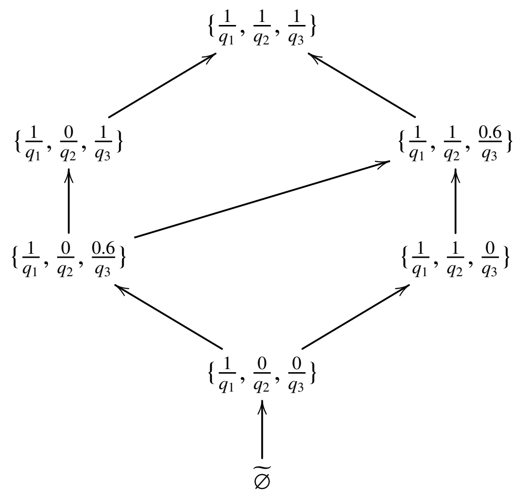

Figure 4. Hasse diagram of the β-fuzzy knowledge structure $\widetilde{\mathcal{K}}_\beta$.

In dichotomous KST, individuals need to completely solve some items at each step to reach larger knowledge states. Dichotomous learning paths exclude the possibility of intermediate degrees of mastering some items. In Figure 2, there are only two learning paths, that is $\varnothing \to \{q_1\} \to \{q_1, q_2\} \to Q$ and $\varnothing \to \{q_1\} \to \{q_1, q_3\} \to Q$.

However, for fuzzy knowledge structures, individuals only need to partially master some items at each step to reach larger fuzzy knowledge states. There are more learning paths in fuzzy knowledge structures than in dichotomous knowledge structures. Individuals and educators can choose anyone of these learning paths from $\widetilde{\varnothing}$ to $\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$. Those learning paths in fuzzy knowledge structures represent individuals' partial solving of items. There are several learning paths in Figures 3 and 4. We choose one of learning paths in Figure 3 to show the learning process for "Addition of fractions" and the learning paths in Figure 4 is similar. Learning path in $\widetilde{\mathcal{K}}$: $\widetilde{\varnothing} \to \{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\} \to \{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$

Step 1. $\widetilde{\varnothing} \to \{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}$. Solve the item q_1 to the proficiency 0.6, then the fuzzy knowledge state reaches $\{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}$;

Step 2. $\{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\} \to \{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$. Solve the item q_3 to the proficiency 0.6, then the fuzzy knowledge state reaches $\{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$;

Step 3. $\{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$. Solve the item q_1 to the proficiency 1, then the fuzzy knowledge state reaches $\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$;

Step 4. $\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\}$. Solve the item q_2 to the proficiency 0.6, then the fuzzy knowledge state reaches $\{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\}$;

Step 5. $\{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}$. Solve the item q_2 to the proficiency 1, then the fuzzy knowledge state reaches $\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}$;

Step 6. $\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$. Solve the item q_3 to the proficiency 1, then the fuzzy knowledge state reaches $\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$.

Similarly, we choose one of learning paths in Figure 4 to explain the learning process for "Addition of fractions". Learning path in $\widetilde{\mathcal{K}_\beta}$: $\widetilde{\varnothing} \to \{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\} \to \{\frac{0.6}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{0.6}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$

Step 1. $\widetilde{\varnothing} \to \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}$. Solve the item q_1 to the proficiency 1, then the fuzzy knowledge state reaches $\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\}$;

Step 2. $\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0}{q_3}\} \to \{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$. Solve the item q_3 to the proficiency 0.6, then the fuzzy knowledge state reaches $\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\}$;

Step 3. $\{\frac{1}{q_1}, \frac{0}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}$. Solve the item q_2 to the proficiency 1, then the fuzzy knowledge state reaches $\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\}$;

Step 4. $\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{0.6}{q_3}\} \to \{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$. Solve the item q_3 to the proficiency 1, then the fuzzy knowledge state reaches $\{\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}\}$;

8. Conclusions

In this paper, we proposed an approach to generalize knowledge space and learning space for assessment knowledge. We have explored various properties of fuzzy knowledge space and fuzzy learning space. The knowledge state of an individual represents the items fully mastered in ordinary knowledge space theory. The knowledge state of an individual in KST is inferred from their responses to a set of items. A strict restriction in this approach to knowledge assessment leads to the lack of a description for partial mastery of items. A fuzzy knowledge state can be used to express the individual's partial solving to items. A partial order in Definition 1 can avoid some equally informative items. Theorem 2 indicates a discriminative fuzzy knowledge structure can be directly obtained from a partially ordered set $(Q, \mathcal{R})$. Theorems 2 and 3 indicate that there is a one-to-one correspondence between the collection of all discriminative fuzzy quasi-ordinal space on $\mathcal{F}(Q)$ and the collection of all partial orders on Q.

A fuzzy knowledge space is a union-closed fuzzy knowledge structure. The closure under union is a rather reasonable property: Consider the case of two students engaged in extensive interactions for a while, and one of the students will end up to merge their initial fuzzy knowledge states into a single one which is the union of the two fuzzy knowledge states. Obviously, there is no certainty that this will happen. However, requiring the existence of a fuzzy knowledge state in the structure to cover this case is reasonable. A fuzzy learning space indicates that only one item can be learned at a time under the evaluation criteria β. Theorem 6 and Corollary 2 indicate that if an individual learns an item under β, then the individual can solve those items that are "predecessors" of the item. This indicates that an individual always learns from the simple to the complex items. Theorem 7 proves that a discriminative fuzzy learning space is equivalent to a well-graded fuzzy knowledge space. The β-relative discriminative in Definition 9 indicates that two do not simultaneously reach or not reach the criteria in a β-relative discriminative fuzzy knowledge structure. Theorem 9 indicates that a discriminative fuzzy knowledge structure is equivalent to a β-relative discriminative fuzzy knowledge structure when each value in evaluation criteria β is the same. Theorem 10 states that if a fuzzy knowledge structure is not β-relative discriminative, then it is not a fuzzy learning space.

The fuzzy knowledge basis of fuzzy knowledge space $(Q, \widetilde{\mathcal{K}})$ is a minimal subfamily of fuzzy knowledge states, which spans $(Q, \widetilde{\mathcal{K}})$. Definition 11 and Theorem 11 show that if a fuzzy knowledge state $\widetilde{K}$ is an atom at q, then $\widetilde{K}$ is related to the degree of membership $\widetilde{K}(q)$. Theorem 12 proves that the fuzzy knowledge basis is formed by the collection of all the atoms. The algorithm for finding the fuzzy knowledge basis is to find all atoms of $\mathcal{K}$. The process of finding the fuzzy knowledge basis is to remove union-reducible fuzzy knowledge states in $\mathcal{K}$. On the link between cognitive diagnostic models and knowledge space theory, Heller established the link between CDM (cognitive diagnostic model) and KST in 2015 [21]. The competence-based extension of KST was greatly developed for the assessment of knowledge and learning [1, 2, 20, 26, 30, 31, 34]. Skills identifying latent abilities can be used to interpret individuals' fuzzy knowledge states. Future work could be devoted to established the link between fuzzy knowledge structures and skill functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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