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Research article

The generalized circular intuitionistic fuzzy set and its operations

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Abstract: The circular intuitionistic fuzzy set (*CIFS*) is an extension of the intuitionistic fuzzy set (*IFS*), where each element is represented as a circle in the *IFS* interpretation triangle (*IFIT*) instead of a point. The center of the circle corresponds to the coordinate formed by membership ($\mathcal{M}$) and non-membership ($\mathcal{N}$) degrees, while the radius, r, represents the imprecise area around the coordinate. However, despite enhancing the representation of *IFS*, *CIFS* remains limited to the rigid $IFIT$ space, where the sum of $\mathcal{M}$ and $\mathcal{N}$ cannot exceed one. In contrast, the generalized *IFS* (*GIFS*) allows for a more flexible *IFIT* space based on the relationship between $\mathcal{M}$ and $\mathcal{N}$ degrees. To address this limitation, we propose a generalized circular intuitionistic fuzzy set (*GCIFS*) that enables the expansion or narrowing of the *IFIT* area while retaining the characteristics of *CIFS*. Specifically, we utilize the generalized form introduced by Jamkhaneh and Nadarajah. First, we provide the formal definitions of *GCIFS* along with its relations and operations. Second, we introduce arithmetic and geometric means as basic operators for *GCIFS* and then extend them to the generalized arithmetic and geometric means. We thoroughly analyze their properties, including idempotency, inclusion, commutativity, absorption and distributivity. Third, we define and investigate some modal operators of *GCIFS* and examine their properties. To demonstrate their practical applicability, we provide some examples. In conclusion, we primarily contribute to the expansion of *CIFS* theory by providing generality concerning the relationship of imprecise membership and non-membership degrees.

Keywords: generalized circular intuitionistic fuzzy set; circular intuitionistic fuzzy set; arithmetic-geometric means; generalized arithmetic-geometric means; modal operators
Mathematics Subject Classification: 03E72, 47S40

1. Introduction

The intuitionistic fuzzy set (*IFS*) [1] was introduced by Atanassov in 1986 as an extension of the fuzzy set (*FS*) theory [2]. In *FS*, each element is characterized only by the membership degree. However, in *IFS*, each element is indicated by both membership ($\mathcal{M}$) and non-membership ($\mathcal{N}$) degrees, as well as a hesitancy degree. Additionally, various extension forms of *FS* have been proposed, including interval valued *FS* (*IVFS*) [3], type-2 *FS* [4], Hesitant *FS* [5,6] and others. These extensions aim to provide generality in representing imprecise membership degrees instead of precise membership degrees. Similarly, *IFS* has been expanded to include interval valued *IFS* (*IVIFS*) [7], type-2 *IFS* (*T2IFS*) [8] and hesitant *IFS* [9]. These extensions address problems related to imprecise membership and non-membership degrees. *IFS* has been reported to be better at presenting a higher level of complexity and uncertainty compared to *FS* due to its flexibility. Since its introduction, numerous studies have been carried out on *IFS*, especially its applications in various decision-making models (see [10–14]). Furthermore, research focusing on advancing *IFS* theoretically has also emerged, including studies on algebraic aspects of *IFS* in group theory [15], graph theory [16,17], topology [18], aggregation operators [19–21], distance, similarity and entropy measures [22–25], to mention a few.

In addition to that, another research direction on generalizing *IFS* has emerged to solve problems beyond the existing constraint of *IFS*, i.e., $\mathcal{M} + \mathcal{N} \leq 1$. The generalizations of *IFS* are normally conducted with respect to the relation between $\mathcal{M}$ and $\mathcal{N}$ degrees. One of the representations of *IFS* that has been mostly studied is the *IFS* interpretation triangle (*IFIT*). Based on this interpretation, numerous developments of generalized *IFS* (*GIFS*) have been proposed (see Table 1). Mondal and Samanta [26] were the first to propose $GIFS_{MS}$, introducing an additional condition to the existing *IFS* and allowing for cases where $\mathcal{M} + \mathcal{N} > 1$ to be considered. However, it is still limited to $\mathcal{M} + \mathcal{N} \leq 1.5$. Then, Liu [27] defined $GIFS_L$ through linear extension for interpretational surface. Hence, other cases beyond $\mathcal{M} + \mathcal{N} > 1.5$ are also established. Furthermore, this $GIFS_L$ includes $GIFS_{MS}$ as a special case. In another study, Despi et al. [28] proposed six types of *GIFS* ($GIFS1_{DOY}$–$GIFS6_{DOY}$), which extended various possible combinations between $\mathcal{M}$ and $\mathcal{N}$. All the proposed *GIFS*s provide flexibility in dealing with the possible cases of $\mathcal{M} + \mathcal{N} > 1$. Another *GIFS* has been proposed by Jamkhaneh and Nadarajah, $GIFS_{JN}$ [29] based on power and root-type of $\mathcal{M}$ and $\mathcal{N}$. They modify the relation between $\mathcal{M}$ and $\mathcal{N}$ functions to expand and narrow the *IFS* surface interpretation area under the *IFIT*. This type of $GIFS_{JN}$ covers some of the well-known extensions of *IFS* in the literature (see, [30–34]). In general, the above generalizations aim to enhance the expressive power of $\mathcal{M}$ and $\mathcal{N}$ degrees by extending the definition of *IFS* in terms of the *IFIT*.

Table 1. Comparison of some *GIFS*s.

GIFS	Condition	Relation
$GIFS_{MS}$ [26]	$\mathcal{M} \cap \mathcal{N} \leq 0.5$	$IFS \subset GIFS_{MS}$
$GIFS_{L}$ [27]	$\mathcal{M} + \mathcal{N} \leq 1 + L$ where $L \in [0, 1]$	$IFS \subset GIFS_{MS} \subset GIFS_{L}$
$GIFS_{DOY}$ [28]		
$GIFS1_{DOY}$	(1) $\mathcal{M} + \mathcal{N} \geq 1$	-
$GIFS2_{DOY}$	(2) $\mathcal{M} \leq \mathcal{N}$	-
$GIFS3_{DOY}$	(3) $\mathcal{M} \geq \mathcal{N}$	-
$GIFS4_{DOY}$	(1) and (3) or	
	(2) and $\mathcal{M} + \mathcal{N} \leq 1$	-
$GIFS5_{DOY}$	(1) and (2) or	
	(3) and $\mathcal{M} + \mathcal{N} \leq 1$	-
$GIFS6_{DOY}$	$\mathcal{M}^2 + \mathcal{N}^2 \leq 1$	-
$GIFS_{JN}$ [29]	$\mathcal{M}^{\delta} + \mathcal{N}^{\delta} \leq 1$	if $\delta = n$ then $IFS \subset GIFS_{JN}$
	where $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^{+}$	if $\delta = \frac{1}{n}$ then $GIFS_{JN} \subset IFS$

It is evident that $GIFS_{JN}$ concept is the most natural expression to overcome the problems mentioned above and covers a lot of special cases of the existing extensions of *IFS*. In its formal definition, $\mathcal{M}$ and $\mathcal{N}$ are parameterized by δ. This concept holds true in several forms, for example: if $\delta = 1$, then it reduces to *IFS*; if $\delta = 2$, then it will be *IFS* 2-type (*IFS2T*) [30] or Pythagorean *FS* (*PFS*) [33]; if $\delta = 3$, then it represents Fermatean *FS* (*FFS*) [35]; if $\delta = n$, for a positive integer n, then it represents *IFS* n-type (*IFS-nT*) or generalized orthopair *FS* [34]. Moreover, if $\delta = \frac{1}{2}$, then it will be reduced to the *IFS* root type (*IFSRT*) [32]. The existence of these generalizations has sparked numerous further studies, such as the proposal of generalized *IVIFS* [36], new operations in *GIFS* [37], defining level operators for *GIFS* [38] and determining reliability analysis based on *GIFS* two-parameter Pareto distribution [39].

In recent years, Atanassov [40] proposed another extension of *IFS* known as circular *IFS* (*CIFS*). In *CIFS*, each element is represented as a circle in the *IFIT* instead of a point. The center of the circle corresponds to the coordinate formed by ($\mathcal{M}$,$\mathcal{N}$), while the radius, r, represents the imprecise area around the coordinate. Initially, the radius takes values from the unit interval [0, 1] [40] and it has later been expanded to $[0, \sqrt{2}]$ [41] to cover the whole area of *IFIT*. Though still in the early research stage, the theory of *CIFS* has already attracted significant research attention. Several studies have begun to explore both the theoretical aspects and applications of *CIFS*. Researchers have expanded the use of *CIFS* in various domains, including introducing distance and divergence measures for *CIFS* [41–43], applying it in decision-making models [44–46] and utilizing it in present worth analysis [47]. The only distinction between *CIFS* and *IFS* resides in the radius component; when the radius equals zero, *CIFS* reverts to *IFS*.

However, as *CIFS* is a direct extension of the *IFS*, its representation is still limited to the existing *IFIT*. Considering this limitation, it becomes interesting to extend *CIFS* based on a more flexible interpretation area, which allows increasing or decreasing the interpretation of *IFIT*. Following the same idea, a generalization of *CIFS* is proposed here, specifically using the *GIFS* concept proposed by Jamkhaneh and Nadarajah [48]. Here, instead of representing $\mathcal{M}$ and $\mathcal{N}$ degrees of an element as a point, a circular region is allowed. These considerations lead us to the objectives of this study:

(1) To introduce the generalized *CIFS* (*GCIFS*) along with its corresponding relations and operations.
(2) To propose arithmetic and geometric means of *GCIFS* as the aggregation operators and extend them to generalized arithmetic mean and generalized geometric mean and verify their applicable algebraic properties.
(3) To examine some modal operators of *GCIFS* and combine them with the previously proposed main operations.

The remaining parts of this paper are summarized as follows: Section 2 provides an outline of fundamental concepts related to *IFS*, *GIFS* and *CIFS*. In Section 3, the generalized *CIFS* (*GCIFS*) is presented in a general form, along with its basic relations and operations. Section 4 introduces the arithmetic and geometric means of *GCIFS* and the generalized arithmetic mean and generalized geometric mean are defined. Section 5 examines some modal operators, which are then applied in conjunction with the arithmetic and geometric means. Finally, Section 6 presents the conclusions and suggestions derived from this paper.

2. Preliminaries

In this section, some basic definitions are given, in particular *IFS*, *GIFS* and *CIFS*. It is defined that $\mathcal{M}(x)$ represents the degree of membership and $\mathcal{N}(x)$ denotes the degree of non-membership of $x \in X$ within the unit interval, $I = [0, 1]$. Atanassov [1] defined the *IFS* as the following.

Definition 2.1. [1] An *IFS* $\mathcal{A}$ in X is defined as an object of the form $\mathcal{A} = \{\langle x, \mathcal{M}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x)\rangle | x \in X\}$, where $\mathcal{M}_{\mathcal{A}} : X \to I$ and $\mathcal{N}_{\mathcal{A}} : X \to I$ that satisfy $0 \leq \mathcal{M}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x) \leq 1$ for each $x \in X$. The collection of all *IFS*s is denoted by *IFS*(X).

Furthermore, Jamkhaneh and Nadarajah [29] proposed the generalized *IFS* by modifying the relationship between $\mathcal{M}$ and $\mathcal{N}$ functions on *IFS* and obtain the following definition.

Definition 2.2. [29] A generalized *IFS* $\mathcal{A}^*$ (denoted $GIFS_{JN}$ $\mathcal{A}^*$) in X is defined as an object of the form $\mathcal{A}^* = \{\langle x, \mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x)\rangle | x \in X\}$, where $\mathcal{M}_{\mathcal{A}^*} : X \to I$ and $\mathcal{N}_{\mathcal{A}^*} : X \to I$ that satisfy $0 \leq {\mathcal{M}_{\mathcal{A}^*}}^{\delta}(x) + {\mathcal{N}_{\mathcal{A}^*}}^{\delta}(x) \leq 1$ for each $x \in X$ with $\delta = n$ or $\frac{1}{n}$, for $n \in \mathbb{Z}^+$. The collection of all generalized *IFS*s is denoted by $GIFS_{JN}(\delta, X)$.

The interpretation area of $GIFS_{JN}$ is depicted such in Figure 1 and some special cases of it with respect to δ are shown in Table 2.

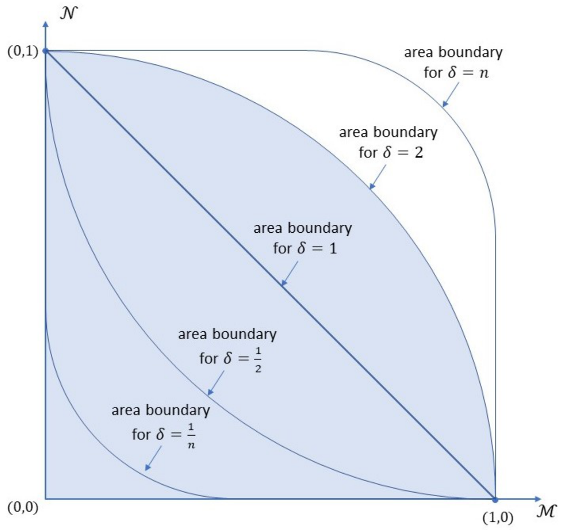

Figure 1. Geometric interpretation of $GIFS_{JN}$ for $\delta = n$ or $\delta = \frac{1}{n}$.

Table 2. Special cases of $GIFS_{JN}$.

IFS extention type	Condition	Relation
IFS [1]	$\delta = 1$	$GIFS_{JN}(1, X) = IFS$
PFS [33] or *IFS*-2*T* [5]	$\delta = 2$	$IFS \subset GIFS_{JN}(2, X)$
FFS [35]	$\delta = 3$	$IFS \subset GIFS_{JN}(2, X) \subset GIFS_{JN}(3, X)$
IFS-*nT* [31] or *q*-*ROFS* [34]	$\delta = n, n \in \mathbb{Z}^+$	$IFS \subset GIFS_{JN}(2, X) \subset GIFS_{JN}(3, X) \subset GIFS_{JN}(n, X)$
IFSRT [32]	$\delta = \frac{1}{2}$	$GIFS_{JN}(\frac{1}{2}, X) \subset IFS \subset IFSRT$

In the following, for simplicity, the notation $GIFS_{JN}$ is referred to *GIFS*. In 2020, Atanassov [40] expanded the representation of the elements in *IFS* from points to circles and introduced the concept of circular intuitionistic fuzzy set (*CIFS*).

Definition 2.3. [40] A circular *IFS* $\mathcal{A}_r$ (denoted *CIFS* $\mathcal{A}_r$) in X is defined as an object of the form $\mathcal{A}_r = \{\langle x, \mathcal{M}_{\mathcal{A}}(x), \mathcal{N}_{\mathcal{A}}(x); r\rangle | x \in X\}$, where $\mathcal{M}_{\mathcal{A}} : X \to I$ and $\mathcal{N}_{\mathcal{A}} : X \to I$ that satisfy $0 \leq \mathcal{M}_{\mathcal{A}}(x) + \mathcal{N}_{\mathcal{A}}(x) \leq 1$ for each $x \in X$ and $r \in [0, \sqrt{2}]$ is a radius of the circle around each element $x \in X$.

The collection of all *CIFSs* is denoted by $CIFS(X)$. There is clear that if $r = 0$, then $\mathcal{A}_0$ is *IFS*, but for $r > 0$ it cannot be represented by *IFS*. Let $L = \{\langle p, q\rangle | p, q \in [0, 1]$ and $p + q \leq 1\}$, then $\mathcal{A}_r$ can also be written in the form,

$$\mathcal{A}_r = \{\langle x, O_r(\mathcal{M}_{\mathcal{A}}, \mathcal{N}_{\mathcal{A}}); r\rangle | x \in X\}$$

where $O_r(\mathcal{M}_{\mathcal{A}_r}, \mathcal{N}_{\mathcal{A}_r}) = \{\langle p, q\rangle | p, q \in [0, 1]$ and $\sqrt{(\mathcal{M}_{\mathcal{A}}(x) - p)^2 + (\mathcal{N}_{\mathcal{A}}(x) - q)^2} \leq r\} \cap L$.

Remark 2.1. Based on the definition and interpretation of L, it is clear that the region is triangular with corner coordinates $(0, 0), (1, 0)$ *and* $(0, 1)$*. The region can be modified to be wider or narrower by adding powers to the relation between p and q. This is the basic form of GIFS from Jamkhaneh and Nadarajah's concept. In the next section, we will use the same concept but applied to CIFS.*

3. Generalized circular intuitionistic fuzzy set

In this section, we propose the Generalized Circular Intuitionistic Fuzzy Set (*GCIFS*) based on the concepts of $GIFS_{JN}$ and *CIFS*.

Definition 3.1. A generalized *CIFS* $\mathcal{A}_r^*$ (denoted *GCIFS* $\mathcal{A}_r^*$) in X is defined as an object of the form, $\mathcal{A}_r^* = \{\langle x, \mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x); r\rangle | x \in X\}$, where $\mathcal{M}_{\mathcal{A}^*} : X \to I$ and $\mathcal{N}_{\mathcal{A}^*} : X \to I$ denoted, respectively the degrees of membership and non-membership of x, radius $r \in [0, \sqrt{2}]$ that satisfy $0 \leq \mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \mathcal{N}^{\delta}_{\mathcal{A}^*}(x) \leq 1$ for each $x \in X$, with $\delta = n$ or $\frac{1}{n}$, for $n \in \mathbb{Z}^+$. The collection of all of the generalized *CIFS*s is denoted by $GCIFS(\delta, X)$ with the interpretation shown on Figure 2.

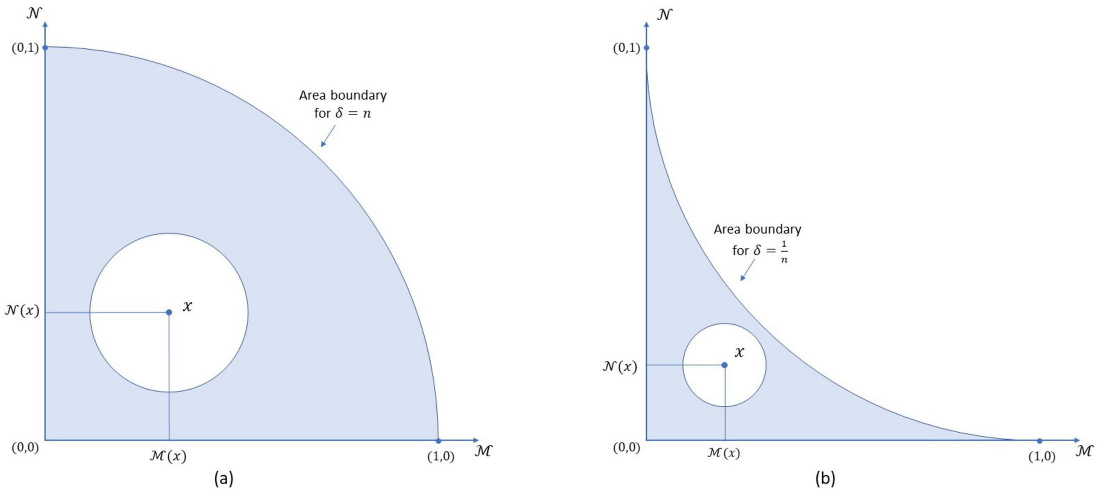

Figure 2. Geometric interpretation of *GCIFS* for (a) $\delta = n$ and (b) $\delta = \frac{1}{n}$.

Remark 3.1. It is known that for all real numbers $p, q \in [0, 1]$ and $\delta = n$ or $\frac{1}{n}$ with $n \in \mathbb{Z}^+$, the following conditions apply:

- *Let $\delta \geq 1$, if $0 \leq p+q \leq 1$ then $0 \leq p^{\delta}+q^{\delta} \leq 1$. It means if $\mathcal{A}_r^* \in CIFS(X)$ then $\mathcal{A}_r^* \in GCIFS(\delta, X)$.*
- *Let $\delta < 1$, if $0 \leq p^{\delta} + q^{\delta} \leq 1$ then $0 \leq p + q \leq 1$. It means if $\mathcal{A}_r^* \in GCIFS\ (\delta, X)$ then $\mathcal{A}_r^* \in CIFS(X)$.*

For special case, if $\delta = 1$ then $GCIFS(1, X)$=$CIFS(X)$. Fundamentally, the relations in *GCIFS* correspond to those in *CIFS* [40] and thus, they are redefined as follows.

Definition 3.2. Let $\mathcal{A}_r^*, \mathcal{B}_s^* \in GCIFS(\delta, X)$. For every $x \in X$, the relations between $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ are defined as follows:

- $\mathcal{A}_r^* \subset_\rho \mathcal{B}_s^* \Leftrightarrow (r < s)\ (\mathcal{M}_{\mathcal{A}^*}(x) = \mathcal{M}_{\mathcal{B}^*}(x)$ and $\mathcal{N}_{\mathcal{A}^*}(x) = \mathcal{N}_{\mathcal{B}^*}(x))$.
- $\mathcal{A}_r^* \subset_\nu \mathcal{B}_s^* \Leftrightarrow (r = s)$ and one of the conditions below is met,
$$\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x) \text{ and } \mathcal{N}_{\mathcal{A}^*}(x) \geq \mathcal{N}_{\mathcal{B}^*}(x),$$
$$\mathcal{M}_{\mathcal{A}^*}(x) \leq \mathcal{M}_{\mathcal{B}^*}(x) \text{ and } \mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x),$$
$$\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x) \text{ and } \mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x).$$
- $\mathcal{A}_r^* \subset \mathcal{B}_s^* \Leftrightarrow (r < s)$ and one of the conditions below is satisfied,
$$\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x) \text{ and } \mathcal{N}_{\mathcal{A}^*}(x) \geq \mathcal{N}_{\mathcal{B}^*}(x),$$
$$\mathcal{M}_{\mathcal{A}^*}(x) \leq \mathcal{M}_{\mathcal{B}^*}(x) \text{ and } \mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x),$$

$$\mathcal{M}_{\mathcal{A}^*}(x) < \mathcal{M}_{\mathcal{B}^*}(x) \text{ and } \mathcal{N}_{\mathcal{A}^*}(x) > \mathcal{N}_{\mathcal{B}^*}(x).$$

- $\mathcal{A}_r^* =_\rho \mathcal{B}_s^* \Leftrightarrow r = s$.
- $\mathcal{A}_r^* =_\nu \mathcal{B}_s^* \Leftrightarrow \mathcal{M}_{\mathcal{A}^*}(x) = \mathcal{M}_{\mathcal{B}^*}(x)$ and $\mathcal{N}_{\mathcal{A}^*}(x) = \mathcal{N}_{\mathcal{B}^*}(x)$.
- $\mathcal{A}_r^* = \mathcal{B}_s^* \Leftrightarrow (r = s)$ $(\mathcal{M}_{\mathcal{A}^*}(x) = \mathcal{M}_{\mathcal{B}^*}(x)$ and $\mathcal{N}_{\mathcal{A}^*}(x) = \mathcal{N}_{\mathcal{B}^*}(x))$.

In the previous work, Atanassov [40] defined radius operations as max and min within $[0, 1]$ domain. Here, we expand these operations to $[0, \sqrt{2}]$ and introduce four more : algebraic product, algebraic sum, arithmetic mean and geometric mean, denoted as $\otimes, \oplus, \circledast$ and $\circledcirc$, respectively. Note that this expansion of the domain covers the entire *IFS* interpretation triangle, as extreme case.

Definition 3.3. Let $r, s \in [0, \sqrt{2}]$ and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$. The operations $\otimes, \oplus, \circledast$ and $\circledcirc$ on radius are defined respectively as follows,

$$\otimes(r,s) = \frac{rs}{\sqrt{2}}, \oplus(r,s) = \left(r^\delta + s^\delta - \left(\frac{rs}{\sqrt{2}}\right)^\delta\right)^{\frac{1}{\delta}},$$

$$\circledast(r,s) = \left(\frac{r^\delta + s^\delta}{2}\right)^{\frac{1}{\delta}}, \circledcirc(r,s) = \left(\sqrt{r^\delta s^\delta}\right)^{\frac{1}{\delta}}.$$

Theorem 3.1. The operations in Definition 3.3 have the closure property.

Proof. To prove the validity of these operations, we need to demonstrate that, for $r, s \in [0, \sqrt{2}]$ and $\delta = n$ or $\frac{1}{n}$ for any $n \in \mathbb{Z}^+$, the closure property holds true for $\otimes, \oplus, \circledast, \circledcirc \in [0, \sqrt{2}]$ and within $[0, \sqrt{2}]$. Let's begin with the operation $\otimes(r, s)$. When $0 \leq r, s \leq \sqrt{2}$, it is evident that $0 \leq \frac{rs}{\sqrt{2}} \leq \frac{2}{\sqrt{2}} = \sqrt{2}$. Moving on to the operation $\oplus(r, s)$, our aim is to prove $r^\delta + s^\delta - \left(\frac{rs}{\sqrt{2}}\right)^\delta \leq \sqrt{2}^\delta$. Using the contradiction, suppose it is true for $r^\delta + s^\delta - \left(\frac{rs}{\sqrt{2}}\right)^\delta > \sqrt{2}^\delta$ such that,

$$r^\delta + s^\delta - \left(\frac{rs}{\sqrt{2}}\right)^\delta - \sqrt{2}^\delta > 0,$$

$$\sqrt{2}^\delta r^\delta + \sqrt{2}^\delta s^\delta - r^\delta s^\delta - \sqrt{2}^{2\delta} > 0,$$

$$r^\delta\left(\sqrt{2}^\delta - s^\delta\right) - \sqrt{2}^\delta\left(\sqrt{2}^\delta - s^\delta\right) < 0,$$

$$\left(r^\delta - \sqrt{2}^\delta\right)\left(\sqrt{2}^\delta - s^\delta\right) > 0.$$

For any $\delta = n$ and $\frac{1}{n}$, it is obtained $\left(r^\delta - \sqrt{2}^\delta\right)\left(\sqrt{2}^\delta - s^\delta\right) \leq 0$. Therefore, it is contradicted, hence $0 \leq \left(r^\delta + s^\delta - \left(\frac{rs}{\sqrt{2}}\right)^\delta\right)^{\frac{1}{\delta}} \leq \left(\sqrt{2}^\delta\right)^{\frac{1}{\delta}} = \sqrt{2}$. For operation $\circledast(r, s)$, since $r^\delta \leq \sqrt{2}^\delta$ and $s^\delta \leq \sqrt{2}^\delta$ then, $0 \leq \circledast(r, s) = \left(\frac{r^\delta + s^\delta}{2}\right)^{\frac{1}{\delta}} \leq \left(\frac{2\sqrt{2}^\delta}{2}\right)^{\frac{1}{\delta}} = \sqrt{2}$. Lastly, for the operation $\circledcirc(r, s)$, it follows that $0 \leq \circledcirc(r, s) = \left(\sqrt{r^\delta s^\delta}\right)^{\frac{1}{\delta}} \leq \left(\sqrt{\sqrt{2}^{2\delta}}\right)^{\frac{1}{\delta}} = \sqrt{2}$. □

The operations defined in Definition 3.3 are the operations that will take effect at *GCIFS* radius. Next, we will define the general operations that apply to *GCIFS*.

Definition 3.4. Let $\mathcal{A}_r^*, \mathcal{B}_s^* \in GCIFS(\delta, X)$, with $r, s \in [0, \sqrt{2}]$ and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$. For every

$x \in X$, $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$ be the radius operators, the operations between $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ can be defined as follows:

- $\neg\mathcal{A}_r^* = \{\langle x, \mathcal{N}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{A}^*}(x); r\rangle | x \in X\}$.
- $\mathcal{A}_r^* \cap_\propto \mathcal{B}_s^* = \{\langle x, \min\left[\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\right], \max\left[\mathcal{N}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{B}^*}(x)\right]; \propto(r,s)\rangle | x \in X\}$.
- $\mathcal{A}_r^* \cup_\propto \mathcal{B}_s^* = \{\langle x, \max\left[\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\right], \min\left[\mathcal{N}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{B}^*}(x)\right]; \propto(r,s)\rangle | x \in X\}$.
- $\mathcal{A}_r^* +_\propto \mathcal{B}_s^* = \{\langle x, \left(\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{M}^\delta_{\mathcal{B}^*}(x) - \mathcal{M}^\delta_{\mathcal{A}^*}(x)\mathcal{M}^\delta_{\mathcal{B}^*}(x)\right)^{\frac{1}{\delta}}, \mathcal{N}_{\mathcal{A}^*}(x)\mathcal{N}_{\mathcal{B}^*}(x); \propto(r,s)\rangle | x \in X\}$.
- $\mathcal{A}_r^* \circ_\propto \mathcal{B}_s^* = \{\langle x, \mathcal{M}_{\mathcal{A}^*}(x)\mathcal{M}_{\mathcal{B}^*}(x), \left(\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \mathcal{N}^\delta_{\mathcal{B}^*}(x) - \mathcal{N}^\delta_{\mathcal{A}^*}(x)\mathcal{N}^\delta_{\mathcal{B}^*}(x)\right)^{\frac{1}{\delta}}; \propto(r,s)\rangle | x \in X\}$.

Theorem 3.2. For $\mathcal{A}_r^*, \mathcal{B}_s^* \in GCIFS$, $\varphi \in \{\cap, \cup, +, \circ\}$ and $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$, it holds that $\mathcal{A}_r^* \varphi_\propto \mathcal{B}_s^* \in GCIFS$.

Proof. The proofs for the radius have already been established in Theorem 3.1. To demonstrated this theorem, we will divide it into two types of operations: **(1)** For operations $\cap_\propto$ and $\cup_\propto$, considering the case $\mathcal{A}_r^* \cap_\propto \mathcal{B}_s^*$ where $\max\{\mathcal{N}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{B}^*}(x)\} = \mathcal{N}_{\mathcal{A}^*}(x)$, we have,

$$\begin{aligned}0 &\le \left(\mathcal{M}_{\mathcal{A}_r^* \cap_\propto \mathcal{B}_s^*}(x)\right)^\delta + \left(\mathcal{N}_{\mathcal{A}_r^* \cap_\propto \mathcal{B}_s^*}(x)\right)^\delta \\ &= \left(\min\{\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\}\right)^\delta + \left(\mathcal{N}_{\mathcal{A}^*}(x)\right)^\delta \le \left(\mathcal{M}_{\mathcal{A}^*}(x)\right)^\delta + \left(\mathcal{N}_{\mathcal{A}^*}(x)\right)^\delta \le 1.\end{aligned}$$

If $\max\{\mathcal{N}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{B}^*}(x)\} = \mathcal{N}_{\mathcal{B}^*}(x)$, then similarly to the previous proof we obtain,

$$0 \le \left(\min\{\mathcal{M}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{B}^*}(x)\}\right)^\delta + \left(\mathcal{N}_{\mathcal{A}^*}(x)\right)^\delta \le \left(\mathcal{M}_{\mathcal{B}^*}(x)\right)^\delta + \left(\mathcal{N}_{\mathcal{B}^*}(x)\right)^\delta \le 1.$$

The same approach is applied for $\mathcal{A}_r^* \cup_\propto \mathcal{B}_s^*$. Moving on to **(2)** operations $+_\propto$ and $\circ_\propto$, in the case of $\mathcal{A}_r^* +_\propto \mathcal{B}_s^*$ we have,

$$\begin{aligned}0 &\le \left(\mathcal{M}_{\mathcal{A}_r^* +_\propto \mathcal{B}_s^*}(x)\right)^\delta + \left(\mathcal{N}_{\mathcal{A}_r^* +_\propto \mathcal{B}_s^*}(x)\right)^\delta \\ &\le \mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{M}^\delta_{\mathcal{B}^*}(x) - \mathcal{M}^\delta_{\mathcal{A}^*}(x)\mathcal{M}^\delta_{\mathcal{B}^*}(x) + \left(1 - \mathcal{M}^\delta_{\mathcal{A}^*}(x)\right)\left(1 - \mathcal{M}^\delta_{\mathcal{A}^*}(x)\right) \\ &= \mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{M}^\delta_{\mathcal{B}^*}(x) - \mathcal{M}^\delta_{\mathcal{A}^*}(x)\mathcal{M}^\delta_{\mathcal{B}^*}(x) + 1 - \mathcal{M}^\delta_{\mathcal{A}^*}(x) - \mathcal{M}^\delta_{\mathcal{B}^*}(x) + \mathcal{M}^\delta_{\mathcal{A}^*}(x)\mathcal{M}^\delta_{\mathcal{B}^*}(x) = 1.\end{aligned}$$

Similarly, this holds for $\mathcal{A}_r^* \circ_\propto \mathcal{B}_s^*$. Therefore, it is proven that the operations defined in Definition 3.4 also *GCIFS*. □

4. Arithmetic and geometric mean operators for *GCIFS*

Previously, arithmetic and geometric mean operations were introduced in the context of *IFS*. These operations were subsequently extended to $GIFS_{JN}$ [48] and explored in other studies [49]. Similarly, these operations have also been proposed for *CIFS* [40]. In the following, we extend these operations, contributing to establishment of generalized operations for arithmetic and geometric means within *GCIFS*.

Definition 4.1. Let $\mathcal{A}_r^*, \mathcal{B}_s^* \in GCIFS(\delta, X)$, with $r, s \in [0, \sqrt{2}]$ and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$. For every $x \in X$ and $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$ be the radius operators, the arithmetic mean, $@_\propto$ and geometric mean, $\$_\propto$ between $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ can be defined as follows:

- $\mathcal{A}_r^*@_\propto\mathcal{B}_s^* = \{\langle x, \left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)+\mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+\mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}; \propto(r,s)\rangle | x\in X\}$.
- $\mathcal{A}_r^*\$_\propto\mathcal{B}_s^* = \{\langle x, \left(\sqrt{\mathcal{M}^\delta_{\mathcal{A}^*}(x)\mathcal{M}^\delta_{\mathcal{B}^*}(x)}\right)^{\frac{1}{\delta}}, \left(\sqrt{\mathcal{N}^\delta_{\mathcal{A}^*}(x)\mathcal{N}^\delta_{\mathcal{B}^*}(x)}\right)^{\frac{1}{\delta}}; \propto(r,s)\rangle | x\in X\}$.

Theorem 4.1. The operations in Definition 4.1 have also the closure property.

Proof. To prove these operations, we must show that for $r, s \in [0, \sqrt{2}]$ and $\delta = n$ or $\frac{1}{n}$ for any $n \in \mathbb{Z}^+$, the closure property for $@_\propto$ and $\$_\propto$ is valid. For operation $@_\propto$ we obtain,

$$\begin{aligned}0 &\le \left(\mathcal{M}_{\mathcal{A}_r^*@_\propto\mathcal{B}_s^*}(x)\right)^\delta + \left(\mathcal{N}_{\mathcal{A}_r^*@_\propto\mathcal{B}_s^*}(x)\right)^\delta \\ &= \frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)+\mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2} + \frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+\mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2} \le \frac{1}{2}+\frac{1}{2} = 1.\end{aligned}$$

Likewise for $\mathcal{A}_r^*\$_\propto\mathcal{B}_s^*$, we have,

$$\begin{aligned}0 &\le \left(\mathcal{M}_{\mathcal{A}_r^*\$_\propto\mathcal{B}_s^*}(x)\right)^\delta + (\mathcal{N}_{A_r\$_\propto B_s}(x))^\delta = \sqrt{\mathcal{M}^\delta_{\mathcal{A}^*}(x)\mathcal{M}^\delta_{\mathcal{B}^*}(x)} + \sqrt{\mathcal{N}^\delta_{\mathcal{A}^*}(x)\mathcal{N}^\delta_{\mathcal{B}^*}(x)} \\ &\le \frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)+\mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2} + \frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+\mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2} \le 1.\end{aligned}$$

It is proven that the operations defined in Definition 4.1 have the closure property. □

Example 4.1. Let $\mathcal{A}_r^* = \{\langle x_1, 0.01, 0.8; 0.02\rangle, \langle x_2, 0.2, 0.3; 0.02\rangle, \langle x_3, 0.1, 0.1; 0.02\rangle\}$ and $\mathcal{B}_s^* = \{\langle x_1, 0.71, 0.02; 0.07\rangle, \langle x_2, 0.05, 0.2; 0.07\rangle, \langle x_3, 0.32, 0.12; 0.07\rangle\}$ are two *CIFS*s. The operations $\mathcal{A}_r^*@_\circledast\mathcal{B}_s^*$ and $\mathcal{A}_r^*\$_\circledcirc\mathcal{B}_s^*$ with $\delta = \frac{1}{3}$ and $\delta = 3$ are demonstrated in Table 3.

Table 3. Results of @ and \$ on *GCIFS* with $\delta = \frac{1}{3}$ (No 1. and 2.) and $\delta = 3$ (No 3. and 4.).

No	Result
(1)	$\mathcal{A}_r^*@_\circledast\mathcal{B}_s^* = \{\langle x_1, 0.170, 0.216; 0.040\rangle, \langle x_2, 0.108, 0.247; 0.040\rangle, \langle x_3, 0.189, 0.110; 0.040\rangle\}$
(2)	$\mathcal{A}_r^*\$_\circledcirc\mathcal{B}_s^* = \{\langle x_1, 0.084, 0.126; 0.037\rangle, \langle x_2, 0.100, 0.245; 0.037\rangle, \langle x_3, 0.179, 0.110; 0.037\rangle\}$
(3)	$\mathcal{A}_r^*@_\circledast\mathcal{B}_s^* = \{\langle x_1, 0.564, 0.635; 0.056\rangle, \langle x_2, 0.160, 0.260; 0.056\rangle, \langle x_3, 0.257, 0.111; 0.056\rangle\}$
(4)	$\mathcal{A}_r^*\$_\circledcirc\mathcal{B}_s^* = \{\langle x_1, 0.084, 0.126; 0.037\rangle, \langle x_2, 0.100, 0.245; 0.037\rangle, \langle x_3, 0.179, 0.110; 0.037\rangle\}$

Remark 4.1. It can be shown that $\mathcal{A}_r^*\$_\propto\mathcal{B}_s^* = \{\langle x, \sqrt{\mathcal{M}_{\mathcal{A}^*}(x)\mathcal{M}_{\mathcal{B}^*}(x)}, \sqrt{\mathcal{N}_{\mathcal{A}^*}(x)\mathcal{N}_{\mathcal{B}^*}(x)}; \propto(r,s)\rangle | x\in X\}$. *This indicates the existence of* δ *parameter, but its significance in this operation is eliminated.*

The following discussion concerns the algebraic properties that apply to these operations. The properties are evidenced in, among others, idempotency, inclusion, commutativity, distributivity and absorption.

Theorem 4.2. *(Idempotency)* Let $\mathcal{A}_r^*$ be *GCIFS*, $\varphi \in \{@, \$\}$ and $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$, then $\mathcal{A}_r^*\varphi_\propto\mathcal{A}_r^* = \mathcal{A}_r^*$.

Proof. The proof is immediately fulfilled by using Definitions 3.3 and 4.1. □

Lemma 4.1. Let $r, s \in [0, \sqrt{2}]$ and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$, then the following expressions hold:

(1) $\otimes(r, s) < r$ or s.
(2) $\oplus(r, s) > r$ or s.

Proof. We prove this lemma by contradiction.

(1) Suppose that $\otimes(r, s) = \frac{rs}{\sqrt{2}} > r$, then,
$$\frac{rs}{\sqrt{2}} - r = \frac{r}{\sqrt{2}}(s - \sqrt{2}) > 0.$$
Note that, since $s \in [0, \sqrt{2}]$ then we have $(s - \sqrt{2}) \leq 0$. Therefore, the assumption is wrong and $\otimes(r, s) < r$. Similarly, we can prove the same way for $\otimes(r, s) < s$.

(2) Suppose that $\oplus(r, s) = \left(r^\delta + s^\delta - \left(\frac{rs}{\sqrt{2}}\right)^\delta\right)^{\frac{1}{\delta}} < r$, then,
$$s^\delta - \left(\frac{rs}{\sqrt{2}}\right)^\delta = \frac{s^\delta}{\sqrt{2}^\delta}\left(\sqrt{2}^\delta - r^\delta\right) < 0.$$
Since $s \in [0, \sqrt{2}]$ then we have $(\sqrt{2}^\delta - r^\delta) \geq 0$. Therefore $\oplus(r, s) > r$ and it applies in a similar manner to $\otimes(r, s) > s$.

The proof is now completed. □

Lemma 4.1 is used to determine the consistency of inclusion property in *GCIFS*.

Theorem 4.3. *(Inclusion)* For every two *GCIFS*s $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ with $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$, we have:

(1) If $\mathcal{A}_r^* \subseteq \mathcal{B}_s^*$, then $\mathcal{A}_r^* @_\propto \mathcal{B}_s^* \subseteq \mathcal{B}_s^*$.
(2) If $\mathcal{A}_r^* \subseteq \mathcal{B}_s^*$, then $\mathcal{A}_r^* \$_\propto \mathcal{B}_s^* \subseteq \mathcal{B}_s^*$.

Proof. Let $\mathcal{A}_r^* \subseteq \mathcal{B}_s^*$ such that $(\forall x \in X)(r \leq s)$ and assume that $\mathcal{M}_{\mathcal{A}^*}(x) \leq \mathcal{M}_{\mathcal{B}^*}(x)$ and $\mathcal{N}_{\mathcal{A}^*}(x) \geq \mathcal{N}_{\mathcal{B}^*}(x)$. Thus for operation $\mathcal{A}_r^* @_\propto \mathcal{B}_s^*$, we can show that,

$$\left(\frac{\mathcal{M}_{\mathcal{A}^*}^\delta(x) + \mathcal{M}_{\mathcal{B}^*}^\delta(x)}{2}\right)^{\frac{1}{\delta}} \leq \left(\frac{\mathcal{M}_{\mathcal{B}^*}^\delta(x) + \mathcal{M}_{\mathcal{B}^*}^\delta(x)}{2}\right)^{\frac{1}{\delta}} = \mathcal{M}_{\mathcal{B}^*}(x).$$

Analogously,

$$\left(\frac{\mathcal{N}_{\mathcal{A}^*}^\delta(x) + \mathcal{N}_{\mathcal{B}^*}^\delta(x)}{2}\right)^{\frac{1}{\delta}} \geq \left(\frac{\mathcal{N}_{\mathcal{B}^*}^\delta(x) + \mathcal{N}_{\mathcal{B}^*}^\delta(x)}{2}\right)^{\frac{1}{\delta}} = \mathcal{N}_{\mathcal{B}^*}(x).$$

This condition is promptly satisfied for the radius operations with each $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$, as per Definition 3.3 and Theorem 3.1. Hence, it is proven. Likewise, we can demonstrate the same for $\mathcal{A}_r^* \$_\propto \mathcal{B}_s^* \subseteq \mathcal{B}_s^*$. □

Theorem 4.4. *(Commutativity)* For every two *GCIFS*s $\mathcal{A}_r^*$ and $\mathcal{B}_s^*, \varphi \in \{@, \$\}$ and $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$, we have $\mathcal{A}_r^* \varphi_\propto \mathcal{B}_s^* = \mathcal{B}_s^* \varphi_\propto \mathcal{A}_r^*$.

Proof. Based on Definition 4.1 and Theorem 3.1, for $r, s \in [0, \sqrt{2}]$ it is clear that $\propto (r, s) = \propto (s, r)$; in other words, it is commutative for radius. Now we will prove the $\mathcal{M}$ and $\mathcal{N}$ parts for $\varphi \in \{@, \$\}$. We start from $\mathcal{A}_r^*@_\propto\mathcal{B}_s^*$ and thus we obtain,

$$\begin{aligned}
\mathcal{A}_r^*@_\propto\mathcal{B}_s^* &= \langle x, \left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)+\mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+\mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}; \propto(r,s)\rangle \\
&= \langle x, \left(\frac{\mathcal{M}^\delta_{\mathcal{B}^*}(x)+\mathcal{M}^\delta_{\mathcal{A}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}^\delta_{\mathcal{B}^*}(x)+\mathcal{N}^\delta_{\mathcal{A}^*}(x)}{2}\right)^{\frac{1}{\delta}}; \propto(r,s)\rangle \\
&= \mathcal{B}_s^*@_\propto\mathcal{A}_r^*.
\end{aligned}$$

Whereas for $\mathcal{A}_r^*\$_\propto\mathcal{B}_s^*$ we get,

$$\begin{aligned}
\mathcal{A}_r^*\$_\propto\mathcal{B}_s^* &= \langle x, \left(\sqrt{\mathcal{M}^\delta_{\mathcal{A}^*}(x).\mathcal{M}^\delta_{\mathcal{B}^*}(x)}\right)^{\frac{1}{\delta}}, \left(\sqrt{\mathcal{N}^\delta_{\mathcal{A}^*}(x).\mathcal{N}^\delta_{\mathcal{B}^*}(x)}\right)^{\frac{1}{\delta}}; \propto(r,s)\rangle \\
&= \langle x, \left(\sqrt{\mathcal{M}^\delta_{\mathcal{B}^*}(x).\mathcal{M}^\delta_{\mathcal{A}^*}(x)}\right)^{\frac{1}{\delta}}, \left(\sqrt{\mathcal{N}^\delta_{\mathcal{B}^*}(x).\mathcal{N}^\delta_{\mathcal{A}^*}(x)}\right)^{\frac{1}{\delta}}; \propto(r,s)\rangle \\
&= \mathcal{B}_s^*\$_\propto\mathcal{A}_r^*.
\end{aligned}$$

The proof is now completed. □

Theorem 4.5. *(Distributivity)* For every two *GCIFS*s $\mathcal{A}_r^*$ and $\mathcal{B}_s^*, \varphi \in \{@, \$\}$ and $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$, then the following relations apply:

(1) $\mathcal{A}_r^*\varphi_\propto(\mathcal{B}_s^* \cap_{\min/\max} \mathcal{C}_t^*) = (\mathcal{A}_r^*\varphi_\propto\mathcal{B}_s^*) \cap_{\min/\max} (\mathcal{A}_r^*\varphi_\propto\mathcal{C}_t^*)$.
(2) $\mathcal{A}_r^*\varphi_\circledast(\mathcal{B}_s^* \cap_\circledast \mathcal{C}_t^*) = (\mathcal{A}_r^*\varphi_\circledast\mathcal{B}_s^*) \cap_\circledast (\mathcal{A}_r^*\varphi_\circledast\mathcal{C}_t^*)$.
(3) $\mathcal{A}_r^*\varphi_\circledcirc(\mathcal{B}_s^* \cap_\circledcirc \mathcal{C}_t^*) = (\mathcal{A}_r^*\varphi_\circledcirc\mathcal{B}_s^*) \cap_\circledcirc (\mathcal{A}_r^*\varphi_\circledcirc\mathcal{C}_t^*)$.
(4) $\mathcal{A}_r^*\varphi_\propto(\mathcal{B}_s^* \cup_{\min/\max} \mathcal{C}_t^*) = (\mathcal{A}_r^*\varphi_\propto\mathcal{B}_s^*) \cup_{\min/\max} (\mathcal{A}_r^*\varphi_\propto\mathcal{C}_t^*)$.
(5) $\mathcal{A}_r^*\varphi_\circledast(\mathcal{B}_s^* \cup_\circledast \mathcal{C}_t^*) = (\mathcal{A}_r^*\varphi_\circledast\mathcal{B}_s^*) \cup_\circledast (\mathcal{A}_r^*\varphi_\circledast\mathcal{C}_t^*)$.
(6) $\mathcal{A}_r^*\varphi_\circledcirc(\mathcal{B}_s^* \cup_\circledcirc \mathcal{C}_t^*) = (\mathcal{A}_r^*\varphi_\circledcirc\mathcal{B}_s^*) \cup_\circledcirc (\mathcal{A}_r^*\varphi_\circledcirc\mathcal{C}_t^*)$.
(7) $\mathcal{A}_r^*@_\propto(\mathcal{B}_s^* \cap_{\min/\max} \mathcal{C}_t^*) = (\mathcal{A}_r^*@_\propto\mathcal{B}_s^*) \cap_{\min/\max} (\mathcal{A}_r^*@_\propto\mathcal{C}_t^*)$.
(8) $\mathcal{A}_r^*@_\circledast(\mathcal{B}_s^*@_\circledast\mathcal{C}_t^*) = (\mathcal{A}_r^*@_\circledast\mathcal{B}_s^*)@_\circledast(\mathcal{A}_r^*@_\circledast\mathcal{C}_t^*)$.
(9) $\mathcal{A}_r^*@_\circledcirc(\mathcal{B}_s^*@_\circledcirc\mathcal{C}_t^*) = (\mathcal{A}_r^*@_\circledcirc\mathcal{B}_s^*)@_\circledcirc(\mathcal{A}_r^*@_\circledcirc\mathcal{C}_t^*)$.
(10) $\mathcal{A}_r^*\$_\propto(\mathcal{B}_s^* \cap_{\min/\max} \mathcal{C}_t^*) = (\mathcal{A}_r^*\$_\propto\mathcal{B}_s^*) \cap_{\min/\max} (\mathcal{A}_r^*\$_\propto\mathcal{C}_t^*)$.
(11) $\mathcal{A}_r^*\$_\circledast(\mathcal{B}_s^*\$_\circledast\mathcal{C}_t^*) = (\mathcal{A}_r^*\$_\circledast\mathcal{B}_s^*)\$_\circledast(\mathcal{A}_r^*\$_\circledast\mathcal{C}_t^*)$.
(12) $\mathcal{A}_r^*\$_\circledcirc(\mathcal{B}_s^*\$_\circledcirc\mathcal{C}_t^*) = (\mathcal{A}_r^*\$_\circledcirc\mathcal{B}_s^*)\$_\circledcirc(\mathcal{A}_r^*\$_\circledcirc\mathcal{C}_t^*)$.

Proof. The proofs are provided for parts (1), (4), (8) and (12), and it can be shown analogously for the remaining parts with certain operator assumptions. For any two *GCIFS*s $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ with $r, s \in [0, \sqrt{2}]$ and $\delta = n$ or $\frac{1}{n}$ where $n \in \mathbb{Z}^+$ then we can demonstrate the following results.

(1) Assume that $\varphi = @$ and $\propto = \max$, so it is obtained as follows,

$$\begin{aligned}
&\mathcal{A}_r^*@_{\max}(\mathcal{B}_s^* \cap_{\min} \mathcal{C}_t^*) = \mathcal{A}_r^*@_{\max}\{\langle x, \min\{\mathcal{M}_{\mathcal{B}^*}(x), \mathcal{M}_{\mathcal{C}^*}(x)\}, \max\{\mathcal{N}_{\mathcal{B}^*}(x), \mathcal{N}_{\mathcal{C}^*}(x)\}; \min\{s,t\}\rangle\} \\
&= \{\langle x, \left[\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x) + (\min\{\mathcal{M}_{\mathcal{B}^*}(x), \mathcal{M}_{\mathcal{C}^*}(x)\})^\delta}{2}\right]^{\frac{1}{\delta}}, \left[\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x) + (\max\{\mathcal{N}_{\mathcal{B}^*}(x), \mathcal{N}_{\mathcal{C}^*}(x)\})^\delta}{2}\right]^{\frac{1}{\delta}}; \\
&\quad \max\{r, \min\{s,t\}\}\rangle\} \\
&= \{\langle x, \min\left[\left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)+\mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)+\mathcal{M}^\delta_{\mathcal{C}^*}(x)}{2}\right)^{\frac{1}{\delta}}\right], \max\left[\left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+\mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+\mathcal{N}^\delta_{\mathcal{C}^*}(x)}{2}\right)^{\frac{1}{\delta}}\right];
\end{aligned}$$

$\min\left[\max\{r,s\},\max\{r,t\}\right]\rangle\}$
$= (\mathcal{A}_r^* @_{\max} \mathcal{B}_s^*) \cap_{\min} (\mathcal{A}_r^* @_{\max} C_t^*)$.

(4) Assume that $\varphi = @$ and $\propto = \otimes$, then it can be derived as follows,

$$\mathcal{A}_r^* @_{\otimes} (\mathcal{B}_s^* \cup_{\max} C_t^*) = \mathcal{A}_r^* @_{\otimes} \{\langle x, \max\{\mathcal{M}_{\mathcal{B}^*}(x), \mathcal{M}_{C^*}(x)\}, \min\{\mathcal{N}_{\mathcal{B}^*}(x), \mathcal{N}_{C^*}(x)\}; \max\{s,t\}\rangle\}$$
$$= \{\langle x, \left[\frac{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x) + (\max\{\mathcal{M}_{\mathcal{B}^*}(x), \mathcal{M}_{C^*}(x)\})^{\delta}}{2}\right]^{\frac{1}{\delta}}, \left[\frac{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x) + (\min\{\mathcal{N}_{\mathcal{B}^*}(x), \mathcal{N}_{C^*}(x)\})^{\delta}}{2}\right]^{\frac{1}{\delta}};$$
$$\frac{r.\max\{s,t\}}{\sqrt{2}}\rangle\}$$
$$= \{\langle x, \max\left[\left(\frac{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x)+\mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x)+\mathcal{M}_{C^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}\right], \min\left[\left(\frac{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x)+\mathcal{N}_{\mathcal{B}^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x)+\mathcal{N}_{C^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}\right];$$
$$\max\left[\frac{rs}{\sqrt{2}}, \frac{rt}{\sqrt{2}}\right]\rangle\}$$
$$= (\mathcal{A}_r^* @_{\otimes} \mathcal{B}_s^*) \cup_{\max} (\mathcal{A}_r^* @_{\otimes} C_t^*).$$

(8) $\mathcal{A}_r^* @_{\circledast} (\mathcal{B}_s^* @_{\circledast} C_t^*) = \mathcal{A}_r^* @_{\circledast} \{\langle x, \left(\frac{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x) + \mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x) + \mathcal{N}_{\mathcal{B}^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}; \left(\frac{s^{\delta}+t^{\delta}}{2}\right)^{\frac{1}{\delta}}\rangle\}$

$$= \{\langle x, \left(\frac{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x) + \frac{\mathcal{M}_{\mathcal{B}^*}^{\delta}(x)+\mathcal{M}_{C^*}^{\delta}(x)}{2}}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x) + \frac{\mathcal{N}_{\mathcal{B}^*}^{\delta}(x)+\mathcal{N}_{C^*}^{\delta}(x)}{2}}{2}\right)^{\frac{1}{\delta}}; \left(\frac{r^{\delta} + \frac{s^{\delta}+t^{\delta}}{2}}{2}\right)^{\frac{1}{\delta}}\rangle\}$$
$$= \{\langle x, \left(\frac{\frac{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x)+\mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}{2} + \frac{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x)+\mathcal{M}_{C^*}^{\delta}(x)}{2}}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\frac{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x)+\mathcal{N}_{\mathcal{B}^*}^{\delta}(x)}{2} + \frac{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x)+\mathcal{N}_{C^*}^{\delta}(x)}{2}}{2}\right)^{\frac{1}{\delta}}; \left(\frac{\frac{r^{\delta}+s^{\delta}}{2} + \frac{r^{\delta}+t^{\delta}}{2}}{2}\right)^{\frac{1}{\delta}}\rangle\}$$
$$= (\mathcal{A}_r^* @_{\circledast} \mathcal{B}_s^*) @_{\circledast} (\mathcal{A}_r^* @_{\circledast} C_t^*).$$

(12) $\mathcal{A}_r^* \$_{\circledcirc} (\mathcal{B}_s^* \$_{\circledcirc} C_t^*) = \mathcal{A}_r^* \$ \{\langle \sqrt{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x)\mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}^{\frac{1}{\delta}}, \sqrt{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x)\mathcal{N}_{\mathcal{B}^*}^{\delta}(x)}^{\frac{1}{\delta}}; \sqrt{s^{\delta}t^{\delta}}^{\frac{1}{\delta}}\rangle\}$

$$= \{\langle x, \left(\sqrt{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x)\sqrt{\mathcal{M}_{\mathcal{B}^*}^{\delta}(x).\mathcal{M}_{C^*}^{\delta}(x)}}\right)^{\frac{1}{\delta}}, \left(\sqrt{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x)\sqrt{\mathcal{N}_{\mathcal{B}^*}^{\delta}(x).\mathcal{N}_{C^*}^{\delta}(x)}}\right)^{\frac{1}{\delta}}; \left(\sqrt{r^{\delta}\sqrt{s^{\delta}.t^{\delta}}}\right)^{\frac{1}{\delta}}\rangle\}$$
$$= \{\langle x, \left(\sqrt{\sqrt{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x).\mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}.\sqrt{\mathcal{M}_{\mathcal{A}^*}^{\delta}(x).\mathcal{M}_{C^*}^{\delta}(x)}}\right)^{\frac{1}{\delta}}, \left(\sqrt{\sqrt{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x).\mathcal{N}_{\mathcal{B}^*}^{\delta}(x)}.\sqrt{\mathcal{N}_{\mathcal{A}^*}^{\delta}(x).\mathcal{N}_{C^*}^{\delta}(x)}}\right)^{\frac{1}{\delta}};$$
$$\left(\sqrt{\sqrt{r^{\delta}.s^{\delta}}.\sqrt{r^{\delta}.t^{\delta}}}\right)^{\frac{1}{\delta}}\rangle\}$$
$$= (\mathcal{A}_r^* \$_{\circledcirc} \mathcal{B}_s^*) \$_{\circledcirc} (\mathcal{A}_r^* \$_{\circledcirc} C_t^*).$$

The proof is now completed. □

Lemma 4.2. Let $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ are *GCIFS*s and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$, then the following relations hold:

(1) $\mathcal{A}_r^* \subseteq \mathcal{A}_r^* \cup_{\max} \mathcal{B}_s^* \subseteq_{\rho} \mathcal{A}_r^* \cup_{\oplus} \mathcal{B}_s^*$.
(2) $\mathcal{A}_r^* \subseteq \mathcal{A}_r^* +_{\max} \mathcal{B}_s^* \subseteq_{\rho} \mathcal{A}_r^* +_{\oplus} \mathcal{B}_s^*$.
(3) $\mathcal{A}_r^* \cap_{\otimes} \mathcal{B}_s^* \subseteq_{\rho} \mathcal{A}_r^* \cap_{\min} \mathcal{B}_s^* \subseteq \mathcal{A}_r^*$.
(4) $\mathcal{A}_r^* \circ_{\otimes} \mathcal{B}_s^* \subseteq_{\rho} \mathcal{A}_r^* \circ_{\min} \mathcal{B}_s^* \subseteq \mathcal{A}_r^*$.

Proof. The validity of this lemma follows from Lemma 4.1. Given $r, s \in [0, \sqrt{2}]$ and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$, then the following properties apply,

$$r \leq \max\{r,s\} \leq r^{\delta} + s^{\delta} - \left(\frac{rs}{\sqrt{2}}\right)^{\delta}.$$

Analogously,

$$\frac{rs}{\sqrt{2}} \leq \min\{r, s\} \leq r.$$

The proof is now completed. □

Theorem 4.6. *(Absorption)* For every two *GCIFS*s $\mathcal{A}_r^*$ and $\mathcal{B}_s^*, \varphi' \in \{\cup, +\}, \varphi'' \in \{\cap, \circ\}$ and $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$, then the following relations hold:

(1) $\mathcal{A}_r^* @_{\propto} (\mathcal{A}_r^* \varphi'_{\max} \mathcal{B}_s^*) \subseteq \mathcal{A}_r^* \varphi'_{\max} \mathcal{B}_s^*$; $\mathcal{A}_r^* @_{\propto} (\mathcal{A}_r^* \varphi'_{\oplus} \mathcal{B}_s^*) \subseteq \mathcal{A}_r^* \varphi'_{\oplus} \mathcal{B}_s^*$.
(2) $(\mathcal{A}_r^* \varphi''_{\min} \mathcal{B}_s^*) @_{\propto} \mathcal{A}_r^* \subseteq \mathcal{A}_r^*$; $(\mathcal{A}_r^* \varphi''_{\otimes} \mathcal{B}_s^*) @_{\propto} \mathcal{A}_r^* \subseteq \mathcal{A}_r^*$.

Proof. The proof of this theorem can be demonstrated by utilizing Lemma 4.2 and inclusion law (Theorem 4.3). □

Furthermore, based on the previous studies [29, 48], we aim to develop general aggregation operators for aggregating multiple *GCIFS*s. Specifically, we will explore operations involving generalized arithmetic mean, @ and generalized geometric mean, \$ on a family of *GCIFS*s denoted as $\mathcal{A}_{r_i}^* = \{\langle x, \mathcal{M}_{\mathcal{A}_i^*}(x), \mathcal{N}_{\mathcal{A}_i^*}(x); r_i \rangle | x \in X\}, i = \{1, 2, 3, \cdots, k\}$.

Definition 4.2. Let $\mathcal{A}_{r_i}^*$ be a family of *GCIFS*s with $i = \{1, 2, 3, \cdots, k\}$ and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$. The generalized arithmetic mean and generalized geometric mean are defined as follows:

(1) $@_{\circledast}{}_{i=1}^{k}(\mathcal{A}_{r_i}^*) = \{\langle x, \left(\frac{\Sigma_{i=1}^k \mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\Sigma_{i=1}^k \mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^k r_i^{\delta}}{k}\right)^{\frac{1}{\delta}} \rangle | x \in X\}$.

(2) $\$_{\circledcirc}{}_{i=1}^{k}(\mathcal{A}_{r_i}^*) = \{\langle x, \left(\sqrt[k]{\Pi_{i=1}^k \mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}\right)^{\frac{1}{\delta}}, \left(\sqrt[k]{\Pi_{i=1}^k \mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)}\right)^{\frac{1}{\delta}}; \left(\sqrt[k]{\Pi_{i=1}^k r_i^{\delta}}\right)^{\frac{1}{\delta}} \rangle | x \in X\}$.

Theorem 4.7. The generalized arithmetic and geometric means exhibit the closure property.

Proof. For operation $@_{\circledast}$, it is proven that $@_{\circledast}{}_{i=1}^{k}(\mathcal{A}_{r_i}^*) \in GCIFS$ since,

$$\begin{aligned} 0 &\leq \frac{\Sigma_{i=1}^k \mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{k} + \frac{\Sigma_{i=1}^k \mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)}{k} \\ &= \frac{\mathcal{M}_{\mathcal{A}_1^*}^{\delta}(x) + \mathcal{M}_{\mathcal{A}_2^*}^{\delta}(x) + \cdots + \mathcal{M}_{\mathcal{A}_k^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_1^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_2^*}^{\delta}(x) + \cdots + \mathcal{N}_{\mathcal{A}_k^*}^{\delta}(x)}{k} \\ &= \frac{[\mathcal{M}_{\mathcal{A}_1^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_1^*}^{\delta}(x)] + [\mathcal{M}_{\mathcal{A}_2^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_2^*}^{\delta}(x)] + \cdots + [\mathcal{M}_{\mathcal{A}_k^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_k^*}^{\delta}(x)]}{k} \leq \frac{k}{k} = 1. \end{aligned}$$

Likewise for $\$_{\circledcirc}$,

$$\begin{aligned} 0 &\leq \sqrt[k]{\Pi_{i=1}^k \mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)} + \sqrt[k]{\Pi_{i=1}^k \mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)} \\ &= \sqrt[k]{\mathcal{M}_{\mathcal{A}_1^*}^{\delta}(x) \times \mathcal{M}_{\mathcal{A}_2^*}^{\delta}(x) \times \cdots \times \mathcal{M}_{\mathcal{A}_k^*}^{\delta}(x)} + \sqrt[k]{\mathcal{N}_{\mathcal{A}_1^*}^{\delta}(x) \times \mathcal{N}_{\mathcal{A}_2^*}^{\delta}(x) \times \cdots \times \mathcal{N}_{\mathcal{A}_k^*}^{\delta}(x)} \\ &\leq \frac{\mathcal{M}_{\mathcal{A}_1^*}^{\delta}(x) + \mathcal{M}_{\mathcal{A}_2^*}^{\delta}(x) + \cdots + \mathcal{M}_{\mathcal{A}_k^*}^{\delta}(x)}{k} + \frac{\mathcal{N}_{\mathcal{A}_1^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_2^*}^{\delta}(x) + \cdots + \mathcal{N}_{\mathcal{A}_k^*}^{\delta}(x)}{k} \\ &= \frac{[\mathcal{M}_{\mathcal{A}_1^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_1^*}^{\delta}(x)] + [\mathcal{M}_{\mathcal{A}_2^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_2^*}^{\delta}(x)] + \cdots + [\mathcal{M}_{\mathcal{A}_k^*}^{\delta}(x) + \mathcal{N}_{\mathcal{A}_k^*}^{\delta}(x)]}{k} \leq \frac{k}{k} = 1. \end{aligned}$$

Hence, it is proven that the generalized arithmetic and geometric means have the closure property. □

Example 4.2. Next, we will illustrate some examples of the generalized arithmetic and geometric means of the *GCIFS*s. Suppose that $\mathcal{A}^*_{r_1}, \cdots, \mathcal{A}^*_{r_5} \in GCIFS\{3, X\}$ for $x_1, x_2, x_3 \in X$, given as follows:

$$\begin{aligned}
\mathcal{A}^*_{r_1} &= \{\langle x_1, 0.32, 0.43; 0.2\rangle, \langle x_2, 0.23, 0.18; 0.2\rangle, \langle x_3, 0.42, 0.77; 0.2\rangle\},\\
\mathcal{A}^*_{r_2} &= \{\langle x_1, 0.25, 0.30; 0.08\rangle, \langle x_2, 0.76, 0.54; 0.08\rangle, \langle x_3, 0.28, 0.16; 0.08\rangle\},\\
\mathcal{A}^*_{r_3} &= \{\langle x_1, 0.64, 0.55; 0.32\rangle, \langle x_2, 0.45, 0.12; 0.32\rangle, \langle x_3, 0.33, 0.83; 0.32\rangle\},\\
\mathcal{A}^*_{r_4} &= \{\langle x_1, 0.31, 0.59; 0.1\rangle, \langle x_2, 0.86, 0.48; 0.1\rangle, \langle x_3, 0.86, 0.40; 0.1\rangle\},\\
\mathcal{A}^*_{r_5} &= \{\langle x_1, 0.16, 0.77; 0.25\rangle, \langle x_2, 0.24, 0.47; 0.25\rangle, \langle x_3, 0.31, 0.65; 0.25\rangle\}.
\end{aligned}$$

For $k = 5$, the operations $@_{\circledast_{i=1}^{k}}(\mathcal{A}^*_{r_i})$ and $\$_{\circledcirc_{i=1}^{k}}(\mathcal{A}^*_{r_i})$ of these *GCIFS*s are,

- $$\begin{aligned}@_{\circledast_{i=1}^{5}}(\mathcal{A}^*_{r_i}) &= \{\langle x, \left(\frac{\Sigma_{i=1}^{5}\mathcal{M}^3_{\mathcal{A}^*_i}(x)}{5}\right)^{\frac{1}{3}}, \left(\frac{\Sigma_{i=1}^{5}\mathcal{N}^3_{\mathcal{A}^*_i}(x)}{5}\right)^{\frac{1}{3}}; \left(\frac{\Sigma_{i=1}^{5} r_i^3}{5}\right)^{\frac{1}{3}}\rangle | x \in \{x_1, x_2, x_3\}\}.\\ &= \{\langle x_1, 0.410, 0.572; 0.226\rangle, \langle x_2, 0.620, 0.423; 0.226\rangle, \langle x_3, 0.542, 0.650; 0.226\rangle\}.\end{aligned}$$

- $$\begin{aligned}\$_{\circledcirc_{i=1}^{5}}(\mathcal{A}^*_{r_i}) &= \{\langle x, \left(\sqrt[5]{\Pi_{i=1}^{5}\mathcal{M}^3_{\mathcal{A}^*_i}(x)}\right)^{\frac{1}{3}}, \left(\sqrt[5]{\Pi_{i=1}^{5}\mathcal{N}^3_{\mathcal{A}^*_i}(x)}\right)^{\frac{1}{3}}; \left(\sqrt[5]{\Pi_{i=1}^{5} r_i^3}\right)^{\frac{1}{3}}\rangle | x \in \{x_1, x_2, x_3\}\}.\\ &= \{\langle x_1, 0.303, 0.503; 0.167\rangle, \langle x_2, 0.439, 0.305; 0.167\rangle, \langle x_3, 0.401, 0.484; 0.167\rangle\}.\end{aligned}$$

If we change $\delta = \frac{1}{3}$, then it can be proved that $\mathcal{A}^*_{r_1}, \cdots, \mathcal{A}^*_{r_5} \notin GCIFS\{\frac{1}{3}, X\}$. Suppose that $\mathcal{B}^*_{s_1}, \cdots, \mathcal{B}^*_{s_4} \in GCIFS\{\frac{1}{3}, X\}$ for $x_1, x_2, x_3 \in X$ as follows:

$$\begin{aligned}
\mathcal{B}^*_{s_1} &= \{\langle x_1, 0.11, 0.02; 0.02\rangle, \langle x_2, 0.20, 0.07; 0.02\rangle, \langle x_3, 0.02, 0.22; 0.02\rangle\},\\
\mathcal{B}^*_{s_2} &= \{\langle x_1, 0.05, 0.15; 0.30\rangle, \langle x_2, 0.01, 0.30; 0.30\rangle, \langle x_3, 0.08, 0.16; 0.30\rangle\},\\
\mathcal{B}^*_{s_3} &= \{\langle x_1, 0.09, 0.04; 0.17\rangle, \langle x_2, 0.32, 0.02; 0.17\rangle, \langle x_3, 0.33, 0.02; 0.17\rangle\},\\
\mathcal{B}^*_{s_4} &= \{\langle x_1, 0.12, 0.13; 0.32\rangle, \langle x_2, 0.03, 0.25; 0.32\rangle, \langle x_3, 0.24, 0.05; 0.32\rangle\},
\end{aligned}$$

then for $k = 4$, the operations $@_{\circledast_{i=1}^{k}}(\mathcal{B}^*_{r_i})$ and $\$_{\circledcirc_{i=1}^{k}}(\mathcal{B}^*_{r_i})$ are,

- $$\begin{aligned}@_{\circledast_{i=1}^{4}}(\mathcal{B}^*_{s_i}) &= \{\langle x, \left(\frac{\Sigma_{i=1}^{4}\mathcal{M}^{\frac{1}{3}}_{\mathcal{B}^*_i}(x)}{4}\right)^{3}, \left(\frac{\Sigma_{i=1}^{4}\mathcal{N}^{\frac{1}{3}}_{\mathcal{B}^*_i}(x)}{4}\right)^{3}; \left(\frac{\Sigma_{i=1}^{4} s_i^{\frac{1}{3}}}{4}\right)^{3}\rangle | x \in \{x_1, x_2, x_3\}\}.\\ &= \{\langle x_1, 0.089, 0.070; 0.162\rangle, \langle x_2, 0.090, 0.122; 0.162\rangle, \langle x_3, 0.128, 0.089; 0.162\rangle\}.\end{aligned}$$

- $$\begin{aligned}\$_{\circledcirc_{i=1}^{4}}(\mathcal{B}^*_{s_i}) &= \{\langle x, \left(\sqrt[4]{\Pi_{i=1}^{4}\mathcal{M}^{\frac{1}{3}}_{\mathcal{B}^*_i}(x)}\right)^{3}, \left(\sqrt[4]{\Pi_{i=1}^{4}\mathcal{N}^{\frac{1}{3}}_{\mathcal{B}^*_i}(x)}\right)^{3}; \left(\sqrt[4]{\Pi_{i=1}^{4} s_i^{\frac{1}{3}}}\right)^{3}\rangle | x \in \{x_1, x_2, x_3\}\}.\\ &= \{\langle x_1, 0.088, 0.063; 0.134\rangle, \langle x_2, 0.066, 0.101; 0.134\rangle, \langle x_3, 0.106, 0.077; 0.134\rangle\}.\end{aligned}$$

Remark 4.2. For any $n \in \mathbb{Z}^+$ and $n \neq 1$, if $\mathcal{A}^*_r \in GCIFS(\frac{1}{n}, X)$ then $\mathcal{A}^*_r \in GCIFS(n, X)$. But it does not hold otherwise, if $\mathcal{A}^*_r \in GCIFS(n, X)$ then $\mathcal{A}^*_r$ is not necessarily GCIFS($\frac{1}{n}$, X). For the generalized geometric mean, it can be shown that $\$_{\circledcirc_{i=1}^{k}}(\mathcal{A}^*_{r_i}) = \{\langle x, \left(\sqrt[k]{\Pi_{i=1}^{k}\mathcal{M}_{\mathcal{A}^*_i}(x)}\right), \left(\sqrt[k]{\Pi_{i=1}^{k}\mathcal{N}_{\mathcal{A}^*_i}(x)}\right); \left(\sqrt[k]{\Pi_{i=1}^{k} r_i}\right)\rangle | x \in X\}$. This is the general form of the geometric mean operation in Remark 4.1.

5. Some modal operators for *GCIFS*

In this section, some other modal operators and their corresponding properties are defined for *GCIFS* over the universal set X. Atanassov [40] previously defined the notions of "necessity" and "possibility" and introduced modal operators in *CIFS*. Other studies have also defined modal operators, such as type-2 modal operators, which apply to *IFS* [50]. Therefore, in the following, the type-2 modal operator in *GCIFS* is propossed along with its corresponding properties.

Definition 5.1. For any *GCIFS* $\mathcal{A}_r^*$, $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$ and real number $\lambda, \gamma \in [0,1]$ for $\lambda + \gamma \leq 1$. Let $x \in X$, modal operator type-2 over *GCIFS* are defined as follows:

(1) $\boxplus(\mathcal{A}_r^*) = \{\langle x, \left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+1}{2}\right)^{\frac{1}{\delta}}; r\rangle\}$.

(2) $\boxtimes(\mathcal{A}_r^*) = \{\langle x, \left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)+1}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)}{2}\right)^{\frac{1}{\delta}}; r\rangle\}$.

(3) $\boxplus_\lambda(\mathcal{A}_r^*) = \{\langle x, \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x), \left(\lambda\mathcal{N}^\delta_{\mathcal{A}^*}(x) + (1-\lambda)\right)^{\frac{1}{\delta}}; r\rangle\}$.

(4) $\boxtimes_\lambda(\mathcal{A}_r^*) = \{\langle x, \left(\lambda\mathcal{M}^\delta_{\mathcal{A}^*}(x) + (1-\lambda)\right)^{\frac{1}{\delta}}, \lambda^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{A}^*}(x); r\rangle\}$.

(5) $\boxplus_{\lambda,\gamma}(\mathcal{A}_r^*) = \{\langle x, \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x), \left(\lambda\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \gamma\right)^{\frac{1}{\delta}}; r\rangle\}$.

(6) $\boxtimes_{\lambda,\gamma}(\mathcal{A}_r^*) = \{\langle x, \left(\lambda\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \gamma\right)^{\frac{1}{\delta}}, \lambda^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{A}^*}(x); r\rangle\}$.

(7) $\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*) = \{\langle x, \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x), \left(\gamma\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\}$ for any $\gamma \in [0,1]$ and $\max(\lambda,\gamma) + \eta \leq 1$.

(8) $\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*) = \{\langle x, \left(\lambda\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{A}^*}(x); r\rangle\}$ for any $\gamma \in [0,1]$ and $\max(\lambda,\gamma) + \eta \leq 1$.

It must be confirmed that some modal operators type-2 specified in Definition 5.1 are also *GCIFS*.

Theorem 5.1. The operations defined in Definition 5.1 for *GCIFS* are also *GCIFS*.

Proof. For $\mathcal{A}_r^* \in$ *GCIFS* such that $\mathcal{A}_r^* = \{\langle x, \mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x); r\rangle | x \in X\}$, $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$ and $\lambda, \gamma \in [0,1]$ for $\lambda + \gamma \leq 1$, then for each $x \in X$ we have,

(1) Since $0 \leq \mathcal{M}_{\mathcal{A}^*}(x), \mathcal{N}_{\mathcal{A}^*}(x) \leq 1$ and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$ then,
$$\mathcal{M}^\delta_{\boxplus(\mathcal{A}_r^*)}(x) + \mathcal{N}^\delta_{\boxplus(\mathcal{A}_r^*)}(x) = \left[\left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)}{2}\right)^{\frac{1}{\delta}}\right]^\delta + \left[\left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+1}{2}\right)^{\frac{1}{\delta}}\right]^\delta = \frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x)}{2} + \left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x)+1}{2}\right) \leq 1.$$

(2) The operator $\boxtimes(\mathcal{A}_r^*)$ is proved analogously.

(3) For any real number $\lambda \in [0,1]$ and *GCIFS* $\mathcal{A}_r^*$, we have $0 \leq \mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{N}^\delta_{\mathcal{A}^*}(x) \leq 1$. Since $\mathcal{M}_{\boxplus_\lambda(\mathcal{A}_r^*)}(x) = \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x)$ and $\mathcal{N}_{\boxplus_\lambda(\mathcal{A}_r^*)}(x) = (\lambda\mathcal{N}^\delta_{\mathcal{A}^*}(x) + (1-\lambda))^{\frac{1}{\delta}}$ then,
$$\begin{aligned}\mathcal{M}^\delta_{\boxplus_\lambda(\mathcal{A}_r^*)}(x) + \mathcal{N}^\delta_{\boxplus_\lambda(\mathcal{A}_r^*)}(x) &= \left[\lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}_r^*}(x)\right]^\delta + \left[\left(\lambda\mathcal{N}^\delta_{\mathcal{A}^*}(x) + (1-\lambda)\right)^{\frac{1}{\delta}}\right]^\delta \\ &= \lambda\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \lambda\mathcal{N}^\delta_{\mathcal{A}^*}(x) + (1-\lambda) \\ &= \lambda(\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{N}^\delta_{\mathcal{A}^*}(x)) + (1-\lambda) \leq 1.\end{aligned}$$

(4) Can be proved in a manner analogous to (3).

(5) For any real number $\lambda, \gamma \in [0,1]$ and *GCIFS* $\mathcal{A}_r^*$, we have $0 \leq \mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{N}^\delta_{\mathcal{A}^*}(x) \leq 1$. Since $\mathcal{M}_{\boxplus_{\lambda,\gamma}(\mathcal{A}_r^*)}(x) = \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x)$ and $\mathcal{N}_{\boxplus_{\lambda,\gamma}(\mathcal{A}_r^*)}(x) = (\lambda\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \gamma)^{\frac{1}{\delta}}$ then,

$$\mathcal{M}^{\delta}_{\boxplus_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) + \mathcal{N}^{\delta}_{\boxplus_{\lambda,\gamma}(\mathcal{A}^*_r)}(x) = \left[\lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x)\right]^{\delta} + \left[\left(\lambda\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \gamma\right)^{\frac{1}{\delta}}\right]^{\delta}$$
$$= \lambda\mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \lambda\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \gamma$$
$$= \lambda(\mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \mathcal{N}^{\delta}_{\mathcal{A}^*}(x)) + \gamma \leq 1.$$

(6) Analogous to (5).

(7) Let $\eta \in [0,1]$ and $\max(\lambda,\gamma) + \eta \leq 1$. Since $\mathcal{M}_{\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)}(x) = \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x)$ and $\mathcal{N}_{\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)}(x) = (\gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta)^{\frac{1}{\delta}}$ then,

$$\mathcal{M}^{\delta}_{\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)}(x) + \mathcal{N}^{\delta}_{\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)}(x) = \left[\lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x)\right]^{\delta} + \left[\left(\gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}\right]^{\delta} = \lambda\mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta.$$

If $\max(\lambda,\gamma) = \lambda$ then $\lambda\mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta \leq \gamma(\mathcal{M}^{\delta}_{\mathcal{A}^*} + \mathcal{N}^{\delta}_{\mathcal{A}^*}) + \eta \leq 1$. So are, if $\max(\lambda,\gamma) = \gamma$ then $\lambda\mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta \leq 1$.

(8) Similar to (7).

So it is proven that the modal operators type-2 defined in Definition 5.1 are also *GCIFS*. □

There are special cases for modal operators type-2:(1) if $\lambda = 0.5$ then $\boxplus_{\lambda}(\mathcal{A}^*_r) = \boxplus(\mathcal{A}^*_r)$; and (2) if $\gamma = 1 - \lambda$ then $\boxplus_{\lambda,\gamma}(\mathcal{A}^*_r) = \boxplus_{\lambda}(\mathcal{A}^*_r)$, if $\gamma = \lambda = 0.5$ then $\boxplus_{\lambda,\gamma}(\mathcal{A}^*_r) = \boxplus(\mathcal{A}^*_r)$. Moreover, (3) if $\gamma = \lambda$ then $\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r) = \boxplus_{\lambda,\eta}(\mathcal{A}^*_r)$, if $\gamma = 1 - \lambda$ and $\eta = 1 - \gamma$ then $\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r) = \boxplus_{\lambda}(\mathcal{A}^*_r)$ and if $\gamma = \lambda = \eta = 0.5$ then $\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r) = \boxplus(\mathcal{A}^*_r)$. This condition also applies to operator "$\boxtimes$".

Theorem 5.2. For any *GCIFS* $\mathcal{A}^*_r$ and every $\lambda, \gamma, \eta \in [0,1]$ we obtain:

(1) $\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r) \subseteq_v \boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)$.
(2) $\neg \boxplus_{\lambda,\gamma,\eta} (\neg\mathcal{A}^*_r) = \boxtimes_{\gamma,\lambda,\eta}(\mathcal{A}^*_r)$ and $\neg \boxtimes_{\lambda,\gamma,\eta} (\neg\mathcal{A}^*_r) = \boxplus_{\gamma,\lambda,\eta}(\mathcal{A}^*_r)$.
(3) $\boxplus_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)) \subseteq_v \boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r) \Leftrightarrow \gamma + \eta = 1$.
(4) $\boxtimes_{\lambda,\gamma,\eta}(\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)) \supseteq_v \boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}^*_r) \Leftrightarrow \lambda + \eta = 1$.
(5) $\boxplus_{\lambda,\gamma,\eta}(\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)) = \boxtimes_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)) \Leftrightarrow \lambda = \gamma$ or $\eta = 0$.

Proof. The proof of this theorem will be provided as follows:

(1) For the Definition 5.1 and $\lambda, \gamma, \eta \in [0,1]$ where $\max(\lambda,\gamma) + \eta \leq 1$ we have,
$\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r) = \{\langle x, \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x), \left(\gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\}$ and
$\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}^*_r) = \{\langle x, \left(\lambda\mathcal{M}^{\delta}(x) + \eta\right)^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{A}^*}(x); r\rangle\}$.
Obviously, the following expressions hold $\lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x) \leq \left(\lambda\mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}$ and $\left(\gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}} \geq \gamma^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{A}^*}(x)$, thus concluding the proof.

(2)
$$\neg \boxplus_{\lambda,\gamma,\eta} (\neg\mathcal{A}^*_r) = \neg \boxplus_{\lambda,\gamma,\eta} \{\langle x, \mathcal{N}_{\mathcal{A}^*}(x), \mathcal{M}_{\mathcal{A}^*}(x); r\rangle\}$$
$$= \neg\{\langle x, \lambda^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{A}^*}(x), \left(\gamma\mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\}$$
$$= \{\langle x, \left(\gamma\mathcal{M}^{\delta}_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}, \lambda^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{A}^*}(x); r\rangle\} = \boxtimes_{\gamma,\lambda,\eta}(\mathcal{A}^*_r).$$
Similarly with $\neg \boxtimes_{\lambda,\gamma,\eta} (\neg\mathcal{A}^*_r) = \boxplus_{\gamma,\lambda,\eta}(\mathcal{A}^*_r)$.

(3) ($\Rightarrow$)
$$\boxplus_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_r)) = \boxplus_{\lambda,\gamma,\eta}\{\langle x, \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x), \left(\gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\}$$
$$= \{\langle x, \lambda^{\frac{1}{\delta}}\left(\lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x)\right), \left(\gamma(\gamma\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \eta) + \eta\right)^{\frac{1}{\delta}}; r\rangle\}$$
$$= \{\langle x, \lambda^{\frac{2}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x), \left(\gamma^2\mathcal{N}^{\delta}_{\mathcal{A}^*}(x) + \gamma\eta + \eta\right)^{\frac{1}{\delta}}; r\rangle\}.$$

Should be noted that $\boxplus_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)) \subseteq_\nu \boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)$, therefore for non-membership we obtain,

$$\begin{gathered}\left(\gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \gamma\eta + \eta\right)^{\frac{1}{\delta}} \geq \left(\gamma \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}} \\ \gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \gamma\eta + \eta \geq \gamma \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \eta \\ \gamma \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \eta \geq \mathcal{N}_{\mathcal{A}^*}^\delta(x) \\ \eta \geq \mathcal{N}_{\mathcal{A}^*}^\delta(x)(1-\gamma).\end{gathered}$$

This is true if $1-\gamma = \eta$, so that $\gamma + \eta = 1$.
($\Leftarrow$) Let $\lambda, \gamma, \eta \in [0,1]$, then:

$$\boxplus_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)) = \{\langle x, \lambda^{\frac{2}{\delta}} \mathcal{M}_{\mathcal{A}^*}(x), \left(\gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \gamma\eta + \eta\right)^{\frac{1}{\delta}}; r\rangle\}.$$

If we have $\gamma + \eta = 1$, then it can be proved that $\boxplus_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)) \subseteq_\nu \boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)$ as follows:
(membership degree) $\lambda^{\frac{2}{\delta}} \mathcal{M}_{\mathcal{A}^*}(x) - \lambda^{\frac{1}{\delta}} \mathcal{M}_{\mathcal{A}^*}(x) = \lambda^{\frac{1}{\delta}} \mathcal{M}_{\mathcal{A}^*} x \left(\lambda^{\frac{1}{\delta}} - 1\right) \leq 0$,
(non-membership degree) $\gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \gamma(1-\gamma) + (1-\gamma) = \gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) - \gamma^2 + 1 \geq \gamma \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \eta$.
So it is clear that $\boxplus_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)) \subseteq_\nu \boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*) \Leftrightarrow \gamma + \eta = 1$.

(4) Similarly with (3).

(5) ($\Rightarrow$)
$$\begin{aligned}\boxplus_{\lambda,\gamma,\eta}\left(\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right) &= \boxplus_{\lambda,\gamma,\eta}\{\langle x, \left(\lambda \mathcal{M}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}} \mathcal{N}_{\mathcal{A}^*}(x); r\rangle\} \\ &= \{\langle x, \lambda^{\frac{1}{\delta}} \left(\lambda \mathcal{M}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}}, \left(\gamma(\gamma \mathcal{N}_{\mathcal{A}^*}^\delta(x)) + \eta\right)^{\frac{1}{\delta}}; r\}\rangle \\ &= \{\langle x, \left(\lambda^2 \mathcal{M}_{\mathcal{A}^*}^\delta(x) + \lambda\eta\right)^{\frac{1}{\delta}}, \left(\gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\},\end{aligned}$$
$$\begin{aligned}\boxtimes_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)) &= \boxtimes_{\lambda,\gamma,\eta}\{\langle x, \lambda^{\frac{1}{\delta}} \mathcal{M}_{\mathcal{A}^*}(x), \left(\gamma \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\} \\ &= \{\langle x, \left(\lambda(\lambda \mathcal{M}_{\mathcal{A}^*}^\delta(x)) + \eta\right)^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}} \left(\gamma \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\} \\ &= \{\langle x, \left(\lambda^2 \mathcal{M}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}}, \left(\gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \gamma\eta\right)^{\frac{1}{\delta}}; r\rangle\}.\end{aligned}$$
Let $\boxplus_{\lambda,\gamma,\eta}\left(\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right) = \boxtimes_{\lambda,\gamma,\eta}\left(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right)$, then in terms of membership value, $\eta(\lambda - 1) = 0$ and for non-membership value $\eta(\gamma - 1) = 0$. Hence, it is evident that $\lambda = \gamma$ or $\eta = 0$.
($\Leftarrow$) Let $\lambda, \gamma, \eta \in [0,1]$, based on Definition 5.1 equation $\boxplus_{\lambda,\gamma,\eta}\left(\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right)$ is,

$$\boxplus_{\lambda,\gamma,\eta}\left(\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right) = \{\langle x, \left(\lambda^2 \mathcal{M}_{\mathcal{A}^*}^\delta(x) + \lambda\eta\right)^{\frac{1}{\delta}}, \left(\gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\}.$$

Whereas for $\boxtimes_{\lambda,\gamma,\eta}\left(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right)$ we attain,

$$\boxtimes_{\lambda,\gamma,\eta}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)) = \{\langle x, \left(\lambda^2 \mathcal{M}_{\mathcal{A}^*}^\delta(x) + \eta\right)^{\frac{1}{\delta}}, \left(\gamma^2 \mathcal{N}_{\mathcal{A}^*}^\delta(x) + \gamma\eta\right)^{\frac{1}{\delta}}; r\rangle\}.$$

Since $\eta = 0$, this makes $\boxplus_{\lambda,\gamma,\eta}\left(\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right)$ and $\boxtimes_{\lambda,\gamma,\eta}\left(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right)$ are equal. Hence, we can conclude that $\boxplus_{\lambda,\gamma,\eta}\left(\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right) = \boxtimes_{\lambda,\gamma,\eta}\left(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*)\right)$.

The proof is now completed. □

Next, we will examine the relationship between the modal operators type-2 and the arithmetic and geometric means that have been defined previously.

Theorem 5.3. For every two *GCIFS*s $\mathcal{A}_r^*$ and $\mathcal{B}_s^*$ and $\propto \in \{\min, \max, \otimes, \oplus, \circledast, \circledcirc\}$ then, the following expressions hold true:

(1) $\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^* @_\propto \mathcal{B}_s^*) = \boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*) @_\propto \boxplus_{\lambda,\gamma,\eta} (\mathcal{B}_s^*)$.
(2) $\boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^* @_\propto \mathcal{B}_s^*) = \boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*) @_\propto \boxtimes_{\lambda,\gamma,\eta} (\mathcal{B}_s^*)$.

Proof. Using Definitions 5.1 and 4.1, for every $x \in X$ we have,

$$\begin{aligned}
(1)\ \boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^* @_\propto \mathcal{B}_s^*) &= \boxplus_{\lambda,\gamma,\eta}\{\langle x, \left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}; \propto (r,s)\rangle\} \\
&= \{\langle x, \lambda^{\frac{1}{\delta}} \left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\gamma\left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2} + \eta\right)\right)^{\frac{1}{\delta}}; \propto (r,s)\rangle\} \\
&= \{\langle x, \left(\frac{\lambda\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \lambda\mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{[\gamma\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \eta] + [\gamma\mathcal{N}^\delta_{\mathcal{B}^*}(x) + \eta]}{2}\right)^{\frac{1}{\delta}}; \propto (r,s)\rangle\} \\
&= \{\langle x, \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}^*}(x), \left(\gamma\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}; r\rangle\} @_\propto \{\langle x, \lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{B}^*}(x), \left(\gamma\mathcal{N}^\delta_{\mathcal{B}^*}(x) + \eta\right)^{\frac{1}{\delta}}; s\rangle\} \\
&= \boxplus_{\lambda,\gamma,\eta}(\mathcal{A}_r^*) @_\propto \boxplus_{\lambda,\gamma,\eta} (\mathcal{B}_s^*). \\
(2)\ \boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^* @_\propto \mathcal{B}_s^*) &= \boxtimes_{\lambda,\gamma,\eta}\{\langle x, \left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}; \propto (r,s)\rangle\} \\
&= \{\langle x, \left(\lambda\left(\frac{\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \mathcal{M}^\delta_{\mathcal{B}^*}(x)}{2} + \eta\right)\right)^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}}\left(\frac{\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}; \propto (r,s)\rangle\} \\
&= \{\langle x, \left(\frac{[\lambda\mathcal{M}^\lambda_{\mathcal{A}^*}(x) + \eta] + [\lambda\mathcal{M}^\delta_{\mathcal{B}^*}(x) + \eta]}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\gamma\mathcal{N}^\delta_{\mathcal{A}^*}(x) + \gamma\mathcal{N}^\delta_{\mathcal{B}^*}(x)}{2}\right)^{\frac{1}{\delta}}; \propto (r,s)\rangle\} \\
&= \{\langle x, \left(\lambda\mathcal{M}^\delta_{\mathcal{A}^*}(x) + \eta\right)^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{A}^*}(x); r\rangle\} @_\propto \{\langle x, \left(\lambda\mathcal{M}^\delta_{\mathcal{B}^*}(x) + \eta\right)^{\frac{1}{\delta}}, \gamma^{\frac{1}{\delta}}\mathcal{N}_{\mathcal{B}^*}(x); s\rangle\} \\
&= \boxtimes_{\lambda,\gamma,\eta}(\mathcal{A}_r^*) @_\propto \boxtimes_{\lambda,\gamma,\eta} (\mathcal{B}_s^*).
\end{aligned}$$

□

Based on the Definition 4.1, the following is a generalization of the properties that apply to the generalized arithmetic and geometric means of *GCIFS*.

Theorem 5.4. Given a family of *GCIFS*s $\mathcal{A}_{r_i}$ for $i = 1, 2, 3, \cdots, k$ and real number $\lambda, \gamma, \eta \in [0, 1]$ then the following expressions hold:

(1) $\left(@_{\circledast}{}_{i=1}^{k}(\mathcal{A}^*_{r_i})\right) @_\circledast \mathcal{B}^*_s = @_{\circledast}{}_{i=1}^{k}(\mathcal{A}^*_{r_i} @_\circledast \mathcal{B}^*_s)$ for any $\mathcal{B}^*_s \in GCIFS$.
(2) $\boxplus\left(@_{\circledast}{}_{i=1}^{k}(\mathcal{A}^*_{r_i})\right) = @_{\circledast}{}_{i=1}^{k}(\boxplus(\mathcal{A}^*_{r_i}))$.
(3) $\boxplus_\lambda\left(@_{\circledast}{}_{i=1}^{k}(\mathcal{A}^*_{r_i})\right) = @_{\circledast}{}_{i=1}^{k}(\boxplus_\lambda(\mathcal{A}^*_{r_i}))$
(4) $\boxplus_{\lambda,\gamma}\left(@_{\circledast}{}_{i=1}^{k}(\mathcal{A}^*_{r_i})\right) = @_{\circledast}{}_{i=1}^{k}(\boxplus_{\lambda,\gamma}(\mathcal{A}^*_{r_i}))$.
(5) $\boxplus_{\lambda,\gamma,\eta}\left(@_{\circledast}{}_{i=1}^{k}(\mathcal{A}^*_{r_i})\right) = @_{\circledast}{}_{i=1}^{k}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_{r_i}))$.

Proof. Let $\mathcal{A}^*_{r_i}$ be a family of *GCIFS*s and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$, then based on Definitions 4.1 and 5.1 the proof of this theorem will be provided as follows:

(1) Let $\mathcal{B}^*_s \in GCIFS$, then it applies,

$$\left(@_{\circledast}{}_{i=1}^{k}(\mathcal{A}^*_{r_i})\right) @_\circledast \mathcal{B}^*_s = \{\langle x, \left(\frac{\Sigma_{i=1}^k \mathcal{M}^\delta_{\mathcal{A}^*_i}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\Sigma_{i=1}^k \mathcal{N}^\delta_{\mathcal{A}^*_i}(x)}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^k r_i^\delta}{k}\right)^{\frac{1}{\delta}}\rangle\} @_\circledast \mathcal{B}^*_s$$

$$= \{\langle x, \left(\frac{\frac{\Sigma_{i=1}^{k}\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{k} + \mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\frac{\Sigma_{i=1}^{k}\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)}{k} + \mathcal{N}_{\mathcal{B}^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}; \left(\frac{\frac{\Sigma_{i=1}^{k}r_i^{\delta}}{k} + s^{\delta}}{2}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\frac{\mathcal{M}_{\mathcal{A}_1^*}^{\delta}(x) + \cdots + \mathcal{M}_{\mathcal{A}_k^*}^{\delta}(x) + k\cdot\mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}{2k}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}_{\mathcal{A}_1^*}^{\delta}(x) + \cdots + \mathcal{N}_{\mathcal{A}_k^*}^{\delta}(x) + k\cdot\mathcal{N}_{B}^{\delta}(x)}{2k}\right)^{\frac{1}{\delta}};$$

$$\left(\frac{r_1^{\delta} + \cdots + r_k^{\delta} + k\cdot s^{\delta}}{2k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\frac{\Sigma_{i=1}^{k}\frac{\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)+\mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}{2}}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\Sigma_{i=1}^{k}\frac{\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)+\mathcal{N}_{\mathcal{B}^*}^{\delta}(x)}{2}}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k}\frac{r_i^{\delta}+s^{\delta}}{2}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= @_{\circledast}{}_{i=1}^{k}\{\langle x, \left(\frac{\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x) + \mathcal{M}_{\mathcal{B}^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x) + \mathcal{N}_{\mathcal{B}^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}; \left(\frac{r_i^{\delta} + s^{\delta}}{2}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= @_{\circledast}{}_{i=1}^{k}(\mathcal{A}_{r_i}^* @_{\circledast} \mathcal{B}_s^*).$$

(2) $$\boxplus(@_{\circledast}{}_{i=1}^{k}(\mathcal{A}_{r_i}^*)) = \boxplus\{\langle x, \left(\frac{\Sigma_{i=1}^{k}\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\Sigma_{i=1}^{k}\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k}r_i^{\delta}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\frac{\frac{\Sigma_{i=1}^{k}\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{k}}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\frac{\Sigma_{i=1}^{k}\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)}{k} + 1}{2}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k}r_i^{\delta}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\frac{\frac{\Sigma_{i=1}^{k}\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{2}}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\frac{\Sigma_{i=1}^{k}[\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)+1]}{2}}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k}r_i^{\delta}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= @_{\circledast}{}_{i=1}^{k}\{\langle x, \left(\frac{\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{2}\right)^{\frac{1}{\delta}}, \left(\frac{\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x) + 1}{2}\right)^{\frac{1}{\delta}}; r_i\rangle\}$$

$$= @_{\circledast}{}_{i=1}^{k}(\boxplus(\mathcal{A}_{r_i}^*)).$$

(3) $$\boxplus_{\lambda}(@_{\circledast}{}_{i=1}^{k}(\mathcal{A}_{r_i}^*)) = \boxplus_{\lambda}\{\langle x, \left(\frac{\Sigma_{i=1}^{k}\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\Sigma_{i=1}^{k}\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k}r_i^{\delta}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\frac{\lambda\Sigma_{i=1}^{k}\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\lambda\Sigma_{i=1}^{k}\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x)}{k} + (1-\lambda)\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k}r_i^{\delta}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\frac{\lambda\Sigma_{i=1}^{k}\mathcal{M}_{\mathcal{A}_i^*}^{\delta}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\Sigma_{i=1}^{k}[\lambda\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x) + (1-\lambda)]}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k}r_i^{\delta}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= @_{\circledast}{}_{i=1}^{k}\{\langle x, \left(\lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}_i^*}(x)\right), \left(\lambda\mathcal{N}_{\mathcal{A}_i^*}^{\delta}(x) + (1-\lambda)\right)^{\frac{1}{\delta}}; r_i\rangle\}$$

$$= @_{\circledast}{}_{i=1}^{k}(\boxplus_{\lambda}(\mathcal{A}_{r_i}^*)).$$

(4) Analogously we can prove (4) by replacing $1-\lambda=\gamma$.

$$(5)\ \boxplus_{\lambda,\gamma,\eta}(@_{\circledast_{i=1}^{k}}(\mathcal{A}_{r_i})) = \boxplus_{\lambda,\gamma,\eta}\{\langle x, \left(\frac{\Sigma_{i=1}^{k}\mathcal{M}^{\delta}_{\mathcal{A}_i^*}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\Sigma_{i=1}^{k}\mathcal{N}^{\delta}_{\mathcal{A}_i^*}(x)}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k} r_i^{\delta}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\frac{\lambda\Sigma_{i=1}^{k}\mathcal{M}^{\delta}_{\mathcal{A}_i^*}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\gamma\Sigma_{i=1}^{k}\mathcal{N}^{\delta}_{\mathcal{A}_i^*}(x)}{k}+\eta\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k} r_i^{\delta}}{n}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\frac{\lambda\Sigma_{i=1}^{k}\mathcal{M}^{\delta}_{\mathcal{A}_i^*}(x)}{k}\right)^{\frac{1}{\delta}}, \left(\frac{\Sigma_{i=1}^{k}[\gamma\mathcal{N}^{\delta}_{\mathcal{A}_i^*}(x)+\eta]}{k}\right)^{\frac{1}{\delta}}; \left(\frac{\Sigma_{i=1}^{k} r_i^{\delta}}{k}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= @_{\circledast_{i=1}^{k}}\{\langle x, \left(\lambda^{\frac{1}{\delta}}\mathcal{M}_{\mathcal{A}_i^*}(x)\right), \left(\gamma\mathcal{N}^{\delta}_{\mathcal{A}_i^*}(x)+\eta\right)^{\frac{1}{\delta}}; r_i\rangle\}$$

$$= @_{\circledast_{i=1}^{k}}(\boxplus_{\lambda,\gamma,\eta}(\mathcal{A}^*_{r_i})).$$

The proof is now completed. □

Theorem 5.5. Given a family of *GCIFS*s $\mathcal{A}^*_{r_i}$ for $i = 1, 2, 3, \cdots, k$ and real number $\lambda, \gamma \in [0, 1]$ then we have:

$$\left(\$_{\circledcirc_{i=1}^{k}}(\mathcal{A}^*_{r_i})\right)\$_{\circledcirc}\mathcal{B}^*_s = \$_{\circledcirc_{i=1}^{k}}(\mathcal{A}^*_{r_i}\$_{\circledcirc}\mathcal{B}^*_s) \text{ for any } \mathcal{B}^*_s \in GCIFS.$$

Proof. Let $\mathcal{A}^*_{r_i}$ be a family of *GCIFS*s and $\delta = n$ or $\frac{1}{n}$ for $n \in \mathbb{Z}^+$. Based on Definitions 4.1 and 5.1 the following result is obtained. Let $\mathcal{B}^*_s \in GCIFS$, then it applies,

$$\left(\$_{\circledcirc_{i=1}^{k}}(\mathcal{A}^*_{r_i})\right)\$_{\circledcirc}\mathcal{B}^*_s = \{\langle x, \left(\sqrt[k]{\Pi_{i=1}^{k}\mathcal{M}^{\delta}_{\mathcal{A}_i^*}(x)}\right)^{\frac{1}{\delta}}, \left(\sqrt[k]{\Pi_{i=1}^{k}\mathcal{N}^{\delta}_{\mathcal{A}_i^*}(x)}\right)^{\frac{1}{\delta}}; \left(\sqrt[k]{\Pi_{i=1}^{k} r_i^{\delta}}\right)^{\frac{1}{\delta}}\rangle\}\$_{\circledcirc}\mathcal{B}^*_s$$

$$= \{\langle x, \left(\sqrt{\sqrt[k]{\Pi_{i=1}^{k}\mathcal{M}^{\delta}_{\mathcal{A}_i^*}(x)}\times\mathcal{M}^{\delta}_{\mathcal{B}^*}(x)}\right)^{\frac{1}{\delta}}, \left(\sqrt{\sqrt[k]{\Pi_{i=1}^{k}\mathcal{N}^{\delta}_{\mathcal{A}_{r_i}^*}(x)}\times\mathcal{N}^{\delta}_{\mathcal{B}^*}(x)}\right)^{\frac{1}{\delta}}; \left(\sqrt{\sqrt[k]{\Pi_{i=1}^{k} r_i^{\delta}}\times s^{\delta}}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\sqrt{\sqrt[k]{\mathcal{M}^{\delta}_{\mathcal{A}_1^*}(x)\times\cdots\times\mathcal{M}^{\delta}_{\mathcal{A}_k^*}(x)}\times\mathcal{M}^{\delta}_{\mathcal{B}^*}(x)}\right)^{\frac{1}{\delta}}, \left(\sqrt{\sqrt[k]{\mathcal{N}^{\delta}_{\mathcal{A}_1^*}(x)\times\cdots\times\mathcal{N}^{\delta}_{\mathcal{A}_k^*}(x)}\times\mathcal{N}^{\delta}_{\mathcal{B}^*}(x)}\right)^{\frac{1}{\delta}}; \left(\sqrt{\sqrt[k]{r_1^{\delta}\times\cdots\times r_k^{\delta}}\times s^{\delta}}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\sqrt{\sqrt[k]{[\mathcal{M}^{\delta}_{\mathcal{A}_1^*}(x)\times\mathcal{M}^{\delta}_{\mathcal{B}^*}(x)]\times\cdots\times[\mathcal{M}^{\delta}_{\mathcal{A}_k^*}(x)\times\mathcal{M}^{\delta}_{\mathcal{B}^*}(x)]}}\right)^{\frac{1}{\delta}}, \left(\sqrt{\sqrt[k]{[\mathcal{N}^{\delta}_{\mathcal{A}_1^*}(x)\times\mathcal{N}^{\delta}_{\mathcal{B}^*}(x)]\times\cdots\times[\mathcal{N}^{\delta}_{\mathcal{A}_k^*}(x)\times\mathcal{N}^{\delta}_{\mathcal{B}^*}(x)]}}\right)^{\frac{1}{\delta}}; \left(\sqrt{\sqrt[k]{[r_1^{\delta}\times s^{\delta}]\times\cdots\times[r_k^{\delta}\times s^{\delta}]}}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= \{\langle x, \left(\sqrt[k]{\sqrt{[\mathcal{M}^{\delta}_{\mathcal{A}_1^*}(x)\times\mathcal{M}^{\delta}_{\mathcal{B}^*}(x)]\times\cdots\times[\mathcal{M}^{\delta}_{\mathcal{A}_k^*}(x)\times\mathcal{M}^{\delta}_{\mathcal{B}^*}(x)]}}\right)^{\frac{1}{\delta}}, \left(\sqrt[k]{\sqrt{[\mathcal{N}^{\delta}_{\mathcal{A}_1^*}(x)\times\mathcal{N}^{\delta}_{\mathcal{B}^*}(x)]\times\cdots\times[\mathcal{N}^{\delta}_{\mathcal{A}_k^*}(x)\times\mathcal{N}^{\delta}_{\mathcal{B}^*}(x)]}}\right)^{\frac{1}{\delta}}; \left(\sqrt[k]{\sqrt{[r_1^{\delta}\times s^{\delta}]\times\cdots\times[r_k^{\delta}\times s^{\delta}]}}\right)^{\frac{1}{\delta}}\rangle\}$$

$$= @_{\circledast_{i=1}^{k}}\{\langle x, \sqrt{\mathcal{M}^{\delta}_{\mathcal{A}_{r_i}^*}(x)\mathcal{M}^{\delta}_{\mathcal{B}^*}(x)}^{\frac{1}{\delta}}, \sqrt{\mathcal{N}^{\delta}_{\mathcal{A}_{r_i}^*}(x)\mathcal{N}^{\delta}_{\mathcal{B}^*}(x)}^{\frac{1}{\delta}}; \sqrt{r_i^{\delta}s^{\delta}}^{\frac{1}{\delta}}\rangle\}$$

$$= \$_{\circledcirc_{i=1}^{k}}(\mathcal{A}^*_{r_i}\$_{\circledcirc}\mathcal{B}^*_s).$$

The proof is now completed. □

6. Conclusions

This study significantly enriches and deepens the existing *CIFS* theory by introducing *GCIFS* as an extension of *CIFS*. We define the basic operations and relations of *GCIFS*, along with their algebraic properties. Furthermore, we examine two operations, the arithmetic mean and geometric mean, on *GCIFS*, demonstrating their desirable properties through theoretical proofs, including idempotency, inclusion, commutativity, distributivity and absorption. Additionally, we introduce modal operators applicable to *GCIFS* and apply them to arithmetic and geometric means. In the final section, we develop aggregation operations, namely the generalized arithmetic and geometric means, extending the capabilities of these two operators. These properties are further applied to the modal operators in context of *GCIFS*.

However, it is essential to note that we do not fully explore several aspects of *GCIFS*. For instance, distance and similarity measurements, entropy, aggregation functions and other components require additional investigation for practical use in decision-making models. Furthermore, from a theoretical perspective, a deeper exploration is needed to understand the specific operating characteristics and relations of *GCIFS*. Future research should be to prioritize these areas to fully unlock the potential of *GCIFS* across various applications.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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