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Research article

Hopf bifurcation problems near double positive equilibrium points for a class of quartic Kolmogorov model

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Abstract: The Kolmogorov model is a class of significant ecological models and is initially introduced to describe the interaction between two species occupying the same ecological habitat. Limit cycle bifurcation problem is close to Hilbertis 16th problem. In this paper, we focus on investigating bifurcation of limit cycle for a class of quartic Kolmogorov model with two positive equilibrium points. Using the singular values method, we obtain the Lyapunov constants for each positive equilibrium point and investigate their limit cycle bifurcations behavior. Furthermore, based on the analysis of their Lyapunov constants' structure and Hopf bifurcation, we give the condition that each one positive equilibrium point of studied model can bifurcate 5 limit cycles, which include 3 stable limit cycles.

Keywords: limit cycles; Kolmogorov model; Poincar*é* succession function; stable cycles
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1. Introduction

The following differential autonomous systems in a planar vector field

$$\dot{x} = F(x, y), \ \dot{y} = G(x, y) \tag{1}$$

have been widely studied and a great deal of attentions have been paid to this problem in many literatures. This activity reflects the breadth of interest in Hilbert's 16th problem and the fact that the above systems are often used as mathematical models to describe real-life problems. Hilbert's 16th problem is to find the maximum number of limit cycles of system (1).

The predator-prey system, the competition system and the cooperation model are the three most basic types of systems in mathematical ecology. Theoretically, many natural predator-prey systems can be discussed and investigated by some kinds of ecological models. The qualitative properties of differential systems are often used to describe the characteristics of ecosystems, as they have been investigated in some literatures, for example, Liénard systems ([2,3,19,21]), Kolmogorov systems ([2–12, 16–20, 22–26]) and some other differential systems ([13–15]) and so on. The Kolmogorov systems (introduced by A. Kolmogorov in 1936 [12]), as a class of significant ecological models, were initially introduced to describe the interaction between two species occupying the same ecological habitat. The form Kolmogorov models are as follows:

$$\begin{cases} \frac{dx}{dt} = x f(x,y), \\ \frac{dy}{dt} = y g(x,y), \end{cases} \tag{2}$$

in which $f(x,y)$ and $g(x,y)$ are polynomials in x and y. The variables x and y are often described as the number of species in two ecological populations, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the growth rates of x and y. Hence, attention is often restricted to the behavior of orbits in the 'realistic quadrant' $\{(x,y) : x > 0, y > 0\}$. Particular significance in applications is the existence of limit cycles and the number of limit cycles that can occur near positive equilibrium points, because a limit cycle corresponds to an equilibrium state of the system. The existence and stability of limit cycles are closely related to the positive equilibrium points. Hence, many references studying Kolmogorov models pay more attention to the limit cycles problem.

On the qualitative analysis and the bifurcation of limit cycles for cubic planar Kolmogorov systems, some papers are as follows: Ref. [1] characterized the center conditions for a cubic Kolmogorov differential system and obtained the condition that the positive equilibrium point can become a fine focus of order five. Ref. [23,24] studied a class of cubic Kolmogorov systems that can bifurcate three limit cycles from the positive equilibrium point (1,1). Ref. [4] investigated the limit cycles bifurcation problem for a class of cubic Kolmogorov system and showed that this class of the Kolmogorov system could bifurcate five limit cycles including 3 stable cycles. Ref. [22] showed that a class of the cubic Kolmogorov system could bifurcate 6 limit cycles. Ref. [25] obtained the condition of integrability and non-linearizability of weak saddles for a cubic Kolmogorov model. Ref. [6] investigated limit cycles in a class of the quartic Kolmogorov model with three positive equilibrium points. Ref. [7] studied the three-Dimensional Hopf Bifurcation for a class of the cubic Kolmogorov model. Ref. [19] investigated the integrability of a class of 3-dimensional Kolmogorov system and provided the phase portraits.

In addition to cubic Kolmogorov systems, there is also a lot of literature on the bifurcation of limit cycles for some generalized Kolmogorov systems and higher order Kolmogorov systems. For example: Ref. [10,26] studied a general Kolmogorov model and obtained the condition for the existence and uniqueness of limit cycles and it classified a series of models. Ref. [8] investigated the bifurcation of limit cycles for a class of the quartic Kolmogorov model with two symmetrical positive singular points. Ref. [9] investigated the Hopf bifurcation problem about small amplitude limit cycles and the local bifurcation of critical periods for a quartic Kolmogrov system at the single positive equilibrium point (1,1) and proved that the maximum number of small amplitude limit cycles bifurcating from the equilibrium point (1,1) was 7. Ref. [20] considered the Kolmogorov system of degree 3 in R^2 and R^3, and showed it had an equilibrium point in the positive quadrant and octant, and provided sufficient conditions in order that the equilibrium point will be a Hopf point for the planar case and a zero-Hopf point for the spatial one, and studied the limit cycles bifurcating from these equilibria using averaging theory of second and first order.

As far as limit cycles of Kolmogorov models are concerned, many good results have been obtained, especially in lower degree system by analyzing a sole positive equilibrium point's state. However, results for Simultaneous limit cycles bifurcating from several different equilibrium points' is less seen, and perhaps it is difficult to investigate this kind of problem. From an ecological perspective, investigation about multiple positive equilibrium points is meaningful, the equilibrium point (a, b) represents that the ratio of the density about predator and prey is $a : b$, and so it is possible for several equilibrium points to occur in ecosystem.

In this paper, we study a class of the following quartic Kolmogorov models

$$\begin{cases} \frac{dx}{dt} = \frac{1}{6}x(y-1)(x^2 - 24ay - 6A_{10}x + 6A_{10}xy - 7y^2 + 30ay^2 \\ \quad - 6a) = P(x, y), \\ \frac{dy}{dt} = \frac{1}{3}y(3b + 3B_{10}x - x^2 + 6x^3 + 9by - 6B_{10}xy - 26x^2y \\ \quad - 2y^2 - 27by^2 + 39xy^2 + 3B_{10}xy^2 - 16y^3 + 15by^3) = Q(x, y), \end{cases} \tag{3}$$

in which a, b, A_{10} *and* $B_{10} \in \mathbb{R}$.

Clearly, model (3) has two positive equilibrium points namely, (1,1) and (2,1). We will focus on the limit cycles bifurcations of the two positive equilibrium points. By analyzing and proving carefully, we obtain that each one of the two positive equilibrium points can be a 5th-order fine focus. Furthermore, we find the condition that each positive equilibrium point can bifurcate five limit cycles, of which three limit cycles are stable. Our results are concise (especial that in the expressions of focal values) and the proof about existence of limit cycles is algebraic and symbolic.

This paper includes 4 sections. In Section 2, we introduce the method to compute focal value offered by [18]. In Section 3, we respectively compute the focal values of the two positive equilibrium points of model (3) and obtain the condition that they can be two 5th-order fine focuses. In Section 4, we discuss the bifurcation of limit cycles of model (3) and obtain that each one of the two positive equilibrium points of model (3) can have five small limit cycles; we give an example that three stable limit cycles can occur near each positive equilibrium point under a certain condition.

2. A kind of method to calculate focal values

In order to use the algorithm of the singular point value to compute focal values and construct the Poincaré succession function, we need to give some properties of focal values and singular point values.

Consider the following system

$$\begin{cases} \frac{dx}{dt} = \delta x - y + \sum_{k=1}^{\infty} X_k(x, y), \\ \frac{dy}{dt} = x + \delta y + \sum_{k=1}^{\infty} Y_k(x, y), \end{cases} \tag{4}$$

where $X_k(x, y)$, $Y_k(x, y)$ are homogeneous polynomials of degree k on x, y. Under the polar coordinates $x = rcos\theta, y = rsin\theta$, system (4) takes the form

$$\frac{dr}{d\theta} = r\frac{\delta + \sum_{k=2}^{\infty} r^{k-1}\varphi_{k+1}(\theta)}{1 + \sum_{k=2}^{\infty} r^{k-1}\psi_{k+1}(\theta)}. \tag{5}$$

For sufficiently small h, let

$$d(h) = r(2\pi, h) - h\,,\ \ r = r(\theta, h) = \sum_{m=1}^{\infty} v(\theta)h^m, \tag{6}$$

be the Poincaré succession function and the solution of Eq (6) satisfies the initial value condition $r|_{\theta=0} = h$. It is evident that

$$v_1(\theta) = e^{\delta\theta} > 0, v_m(\theta) = 0, m = 2, 3, \cdots \tag{7}$$

Lemma 2.1. *For system (4) and any positive integer m, among* $v_{2m}(2\pi), v_k(2\pi)$ *and* $v_k(\pi)$*, there exists expression of the relation*

$$v_{2m}(2\pi) = \frac{1}{1 + v_1(\pi)}[\xi_m^{(0)}(v_1(2\pi) - 1) + \sum_{k=1}^{m-1} \xi_m^{(k)} v_{2k+1}(2\pi)], \tag{8}$$

where $\xi_m^{(k)}$ are all polynomials of $v_1(\pi), v_2(\pi), \cdots, v_m(\pi)$ and $v_1(2\pi), v_2(2\pi), \cdots, v_m(2\pi)$ with rational coefficients.

In addition to indicating that $v_{2m} = 0$ under the conditions $v_1(2\pi) = 1, v_{2k+1}(2\pi) = 0, k = 1, 2, \cdots, m - 1$. Lemma 2.1 plays an important role in construction of Poincaré succession function.

Definition 2.1. *For system (4), in the expression (6), if* $v_1(2\pi) = 1$*, then the origin is called the rough focus (strong focus); if* $v_1(2\pi) = 1$*, and* $v_2(2\pi) = v_3(2\pi) = \cdots = v_{2k}(2\pi) = 0, v_{2k+1}(2\pi) \neq 0$*, then the origin is called the fine focus (weak focus) of order k, and the quantity of* $v_{2k+1}(2\pi)$ *is called the kth focal values at the origin* ($k = 1, 2, \cdots$)*; if* $v_1(2\pi) = 1$*, and for any positive integer* $k, v_{2k+1}(2\pi) = 0$*, then the origin is called a center.*

By means of transformation

$$z = x + y\,i, w = x - y\,i, T = i\,t, i = \sqrt{-1}, \tag{9}$$

system $(4)|_{\delta=0}$ can be transformed into the following systems

$$\begin{cases} \frac{dz}{dT} = z + \sum\limits_{k=2}^{\infty} Z_k(z, w) = Z(z, w), \\ \frac{dw}{dT} = -w - \sum\limits_{k=2}^{\infty} W_k(z, w) = -W(z, w), \end{cases} \tag{10}$$

where z, w, T are complex variables and

$$Z_k(z, w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta, \ W_k(z, w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta.$$

It is obvious that the coefficients of system (10) satisfy conjugate condition $\forall a_{\alpha\beta} = \overline{b_{\alpha\beta}}$, we call that system $(4)|_{\delta=0}$ and (10) are concomitant.

Lemma 2.2. *(See [16]) For system (10), we can derive successively the terms of the following formal series*

$$M(z, w) = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta} z^\alpha w^\beta, \tag{11}$$

such that

$$\frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} = \sum_{m=1}^{\infty} (m + 1)\mu_m (zw)^k, \tag{12}$$

where $c_{kk} \in \mathbb{R}$, $k = 1, 2, \cdots$, and to any integer m, μ_m is determined by following recursion formulas

$$c_{0,0} = 1,$$

$$\text{when } (\alpha = \beta > 0) \text{ or } \alpha < 0, \text{ or} \beta < 0, \; c_{\alpha,\beta} = 0,$$

else

$$c_{\alpha,\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta+2} [(\alpha + 1)a_{k,j-1} - (\beta + 1)b_{j,k-1}]c_{\alpha-k+1,\beta-j+1}, \tag{13}$$

$$\mu_m = \sum_{k+j=3}^{2m+2} (a_{k,j-1} - b_{j,k-1})c_{m-k+1,m-j+1}. \tag{14}$$

Lemma 2.3. *(See[18]) For systems $(5)|_{\delta=0}$, (10) and any positive integer m, the following assertion holds*

$$v_{2m+1}(2\pi) = i\pi(\mu_m + \sum_{k=1}^{m-1} \xi_m^{(k)} \mu_k), \tag{15}$$

where $\xi_m^{(k)}(k = 1, 2, \cdots, m - 1)$ be polynomial functions of coefficients of system (10).

3. Focal values of the two positive equilibrium points of model (3)

For the linearized system of model (3), its coefficient matrix in point (x_0, y_0) is as follows:

$$A_{(x_0,y_0)} = \begin{bmatrix} \frac{\partial P(x,y)}{\partial x} & \frac{\partial P(x,y)}{\partial y} \\ \frac{\partial Q(x,y)}{\partial x} & \frac{\partial Q(x,y)}{\partial y} \end{bmatrix}_{(x_0,y_0)},$$

in which

$$\begin{aligned} \frac{\partial P(x,y)}{\partial x} &= \frac{1}{6}(y - 1)(-6a - 12A_{10}x + 3x^2 - 24ay + 12A_{10}xy - 7y^2 + 30ay^2), \\ \frac{\partial P(x,y)}{\partial y} &= \frac{1}{6}x(18a - 12A_{10}x + x^2 + 14y - 108ay + 12A_{10}xy - 21y^2 + 90ay^2), \\ \frac{\partial Q(x,y)}{\partial x} &= \frac{1}{3}y(3B_{10} - 2x + 18x^2 - 6B_{10}y - 52xy + 39y^2 + 3B_{10}y^2), \\ \frac{\partial Q(x,y)}{\partial y} &= \frac{1}{3}(3b + 3B_{10}x - x^2 + 6x^3 + 18by - 12B_{10}xy - 52x^2y - 6y^2 \\ &\quad - 81by^2 + 117xy^2 + 9B_{10}xy^2 - 64y^3 + 60by^3). \end{aligned}$$

For the two positive equilibrium points (1,1) and (2,1) of model (3), their coefficient matrixes of the linearized system of model (3) become

$$A_{(1,1)} = A_{(2,1)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, $A_{(1,1)}$ and $A_{(2,1)}$ have the same two eigenvalues $\pm i$. Hence, model (3) can be changed into the system (4) by making some appropriate transformations.

For convenience, we will respectively compute the focal values of each positive equilibrium point of model (3).

3.1. Focal values of the positive equilibrium points (1,1) of model (3)

By means of transformation

$$x = u + 1,\ y = v + 1, \tag{16}$$

model (3) takes the following form

$$\begin{cases} \frac{du}{dt} = -v - uv + \frac{1}{6}(u+1)v(2u + u^2 - 14v + 36av + 6A_{10}v + 6A_{10}uv \\ \quad - 7v^2 + 30av^2), \\ \frac{dv}{dt} = u + uv + \frac{1}{3}(v+1)(6u^3 - 9u^2 + 26uv - 26u^2v - 11v^2 + 18bv^2 \\ \quad + 3B_{10}v^2 + 39uv^2 + 3B_{10}uv^2 - 16v^3 + 15bv^3), \end{cases} \tag{17}$$

and the equilibrium point (1, 1) of model (3) becomes the origin of (17) correspondingly.

Under the transformation

$$z = u + i\,v,\ w = u - i\,v,\ T = i\,t,\ i = \sqrt{-1}, \tag{18}$$

system (17) becomes the following complex system

$$\begin{cases} \frac{dz}{dT} = z + Z_2(z,w) + Z_3(z,w) + Z_4(z,w), \\ \frac{dw}{dT} = -w - W_2(z,w) - W_3(z,w) - W_4(z,w), \end{cases} \tag{19}$$

in which

$$\begin{aligned} Z_2(z,w) = {} & \tfrac{1}{12}i(22 + 18a + 3A_{10} + 18ib + 3iB_{10})w^2 + \tfrac{1}{6}(-20 + 7i - 18ia - 3iA_{10} \\ & + 18b + 3B_{10})wz + \tfrac{1}{12}i(-36 - 4i + 18a + 3A_{10} + 18ib + 3iB_{10})z^2, \\ Z_3(z,w) = {} & \tfrac{i}{8}[-5 + 18i + (6 + 5i)a + 2A_{10} - 11b - (1 - i)B_{10}]w^3 + \tfrac{1}{24}[74 - 109i \\ & + (45 - 18i)a - 6iA_{10} + 99ib + (3 + 9i)B_{10}]w^2z - \tfrac{1}{24}[-92 - 123i \\ & + (45 + 18i)a + 6iA_{10} + 99ib - (3 - 9i)B_{10}]wz^2 + \tfrac{1}{24}[-64 + i \\ & + (15 + 18i)a + 6iA_{10} + 33ib - (3 - 3i)B_{10}]z^3, \\ Z_4(z,w) = {} & -\tfrac{1}{48}(-14 + 33i + 15a - 3iA_{10} - 15b + 3iB_{10})w^4 + \tfrac{1}{24}(29 + 45i + 15a \\ & - 30b + 3iB_{10})w^3z + \tfrac{1}{24}(-74 - 3iA_{10} + 45b)w^2z^2 - \tfrac{1}{24}(-35 + 45i \\ & + 15a + 30b + 3iB_{10})wz^3 + \tfrac{1}{16}(2 + 11i + 5a + iA_{10} + 5b + iB_{10})z^4, \\ W_2(z,w) = {} & \tfrac{1}{12}(4 + 36i - 18ia - 3iA_{10} - 18b - 3B_{10})w^2 + \tfrac{1}{6}(-20 - 7i + 18ia \\ & + 3iA_{10} + 18b + 3B_{10})wz - \tfrac{1}{12}i(22 + 18a + 3A_{10} - 18ib - 3iB_{10})z^2, \\ W_3(z,w) = {} & -\tfrac{1}{24}i[1 - 64i + (18 + 15i)a + 6A_{10} + 33b + (3 - 3i)B_{10}]w^3 + \\ & \tfrac{1}{24}[92 - 123i - (45 - 18i)a + 6iA_{10} + 99ib + (3 + 9i)B_{10}]w^2z \\ & + \tfrac{1}{24}[74 + 109i + (45 + 18i)a + 6iA_{10} - 99ib + (3 - 9i)B_{10}]wz^2 \\ & - \tfrac{1}{8}[18 - 5i + (5 + 6i)a + 2iA_{10} - 11ib + (1 - i)B_{10}]z^3, \\ W_4(z,w) = {} & \tfrac{1}{16}(2 - 11i + 5a - iA_{10} + 5b - iB_{10})w^4 - \tfrac{1}{24}(-35 - 45i + 15a + 30b \\ & - 3iB_{10})w^3z + \tfrac{1}{24}(-74 + 3iA_{10} + 45b)w^2z^2 + \tfrac{1}{24}(29 - 45i + 15a - 30b \\ & - 3iB_{10})wz^3 - \tfrac{1}{48}(-14 - 33i + 15a + 3iA_{10} - 15b - 3iB_{10})z^4. \end{aligned}$$

Obviously, system (19) belongs to the type of system (10). Then, we can compute the focal values of the origin of (19) (namely the focal values of the equilibrium point (1, 1) of model (3)) by using the method of Section 2. According to the recursion formulas (13) and (14) offered by Lemma 2.2, the following theorem holds. For convenience, here we note that $B_{10} = \lambda - 6b$.

Theorem 3.1. *The first five singular point values at the origin of system (19) are as follows:*

$\mu_1 = -\frac{1}{36}i(379 - 378a - 54A_{10} + 135b - 102\lambda + 108a\lambda + 18A_{10}\lambda)$;
$\mu_2 = \frac{1}{9720}iA_{10}f_1 + \frac{1}{29160}if_2$;
$\mu_3 = \frac{1}{41990400}ig_2 + \frac{1}{4665600}iA_{10}(9A_{10}g_3 + g_4)$; $\mu_4 = -\frac{1}{226748160000}i(486A_{10}^4g_5 + 3A_{10}g_6 + g_7 + 9A_{10}^2g_8 + 81A_{10}^3g_9 + 7290A_{10}^5g_{10} + 3280500A_{10}^6g_{11})$;
$\mu_5 = -\frac{1}{2203992115200000}i(f_3 + 3A_{10}f_4 + 27A_{10}f_5\lambda + 27A_{10}f_6\lambda^2 + 81A_{10}f_7\lambda^3 + 243A_{10}f_8\lambda^4 + 2187A_{10}f_9\lambda^5 + 98415A_{10}f_{10}\lambda^6 + 98415A_{10}f_{11}\lambda^6)$;
in which f_i, g_i, $(i \in \{1, 2, \cdots 11\})$ are the expressions about A_{10}, a, λ, their expressions can be obtained via computing by reader.

According to the relation between model (3) and system (17) and system (19), from Theorem 3.1 and Lemma 2.3, the following theorem is visible.

Theorem 3.2. *The simplified expressions of the first five focal values in the equilibrium* $(1, 1)$ *of model (3) (or the first five focal values at the origin of system (19)) are as follows:*

$v_3 = \frac{1}{36}\pi(379 - 378a - 54A_{10} + 135b - 102\lambda + 108a\lambda + 18A_{10}\lambda)$;
$v_5 = -\frac{1}{9720}\pi A_{10}f_1 - \frac{1}{29160}\pi f_2$;
$v_7 = -\frac{1}{41990400}\pi g_2 - \frac{1}{4665600}\pi A_{10}(9A_{10}g_3 + g_4)$;
$v_9 = \frac{1}{226748160000}\pi(486A_{10}^4g_5 + 3A_{10}g_6 + g_7 + 9A_{10}^2g_8 + 81A_{10}^3g_9 + 7290A_{10}^5g_{10} + 3280500A_{10}^6g_{11})$;
$v_{11} = \frac{1}{2203992115200000}\pi(f_3 + 3A_{10}f_4 + 27A_{10}f_5\lambda + 27A_{10}f_6\lambda^2 + 81A_{10}f_7\lambda^3 + 243A_{10}f_8\lambda^4 + 2187A_{10}f_9\lambda^5 + 98415A_{10}f_{10}\lambda^6 + 98415A_{10}f_{11}\lambda^6)$.

From Theorem 3.2, we have the following theorems.

Theorem 3.3. *The equilibrium point* $(1, 1)$ *of model (3) can be a 5th-order fine focus at most.*

Proof. According to Definition 2.1, we need to prove that there exits a group of real values about λ, A_{10}, a, b such that $v_3 = v_5 = v_7 = v_9 = 0$, $v_{11} \neq 0$.

At first, we prove $v_3 = v_5 = v_7 = v_9 = 0$ have real solutions. Let $v_3 = 0$, then we have

$$b = \frac{1}{135}(-379 + 378a + 54A_{10} + 102\lambda - 108a\lambda - 18A_{10}\lambda). \tag{20}$$

From the quality of resultant, if $f(x, y) = 0, g(x, y) = 0$ have solutions, then the resultant of $f(x, y), g(x, y)$ with respect to x or y will vanish. When computer soft Mathematica 6.0 is used, $f(x, y), g(x, y)$ with respect to x is shown as *Resultant* $[f, g, x]$. Hence, $v_3 = v_5 = v_7 = v_9 = 0$ hold if and only if Eq (20) holds and

$$\begin{cases} r_{57} = Resultant[v_5, v_7, A_{10}] = 0, \\ r_{59} = Resultant[v_5, v_9, A_{10}] = 0. \end{cases} \tag{21}$$

While Eq (21) hold if and only if

$$r_{579} = Resultant[r_{57}, r_{59}, a] = 0. \tag{22}$$

By computing, we obtain

$$r_{579} = Resultant[r_{57}, r_{59}, a] = 2401(53 - 65\lambda + 10\lambda^2)^2(4261 - 1966\lambda + 200\lambda^2)^2 g(\lambda),$$

in which $g(\lambda)$ is a 242 degrees function on λ . It can be seen that Eq (22) has some real solutions such as $\lambda = \frac{1}{20}(65 \pm \sqrt{2105})$ et al. Hence, $v_3 = v_5 = v_7 = v_9 = 0$ have real solutions.

In fact, we can find 13 groups of real number solutions such that $v_3 = v_5 = v_7 = v_9 = 0$, namely

1) $A_{10} \approx 0.4595187227963886370 74$, $a \approx 0.20757605414524699732908423106$,
$\lambda \approx -187.1523806676270387309$, $b \approx -100.90105602889687249731131061$;
2) $A_{10} \approx -51.8458491820252202 2025$, $a \approx 7.93574370354953138511508896 31$,
$b \approx 7.0067883583684724917 11$, $\lambda \approx 6.3137057412876138734129416 2$;
3) $A_{10} \approx 25.4945035125257951113 62$, $a \approx -3.0713964069508928040606126 9$,
$b \approx -1.5321798409138807741 92$, $\lambda \approx 1.7292255602055614104479639 7$;
4) $A_{10} \approx -25.0705122534245863071 22$, $a \approx 3.3378306098395340157292123 6$,
$b \approx 3.2637203827648718321 66$, $\lambda \approx 4.7291903784535522155281032 1$;
5) $A_{10} \approx -14.4490470614177171996 00$, $a \approx 3.0881596659995090262034251 0$,
$b \approx -0.6293697824769614696 00$, $\lambda \approx -3.2575458907960717440511100 0$;
6) $A_{10} \approx 8.1119241068368980224 34$, $a \approx -1.5337263884537755469521001 0$,
$b \approx 0.1264460676366500420 23$, $\lambda \approx 0.5314229438408376308210001 0$;
7) $A_{10} \approx 5.6219241678621057893 22$, $a \approx 0.0016582425852494311001200 0$,
$b \approx -0.5145360380895625710 70$, $\lambda \approx 8.5057139013267634602350896 8$;
8) $A_{10} \approx 3.0566036738431139791 551$, $a \approx -0.9894928118130848100123180 0$,
$b \approx 1.3119509650856638974 769$, $\lambda \approx 4.9730463363908633647369990 0$;
9) $A_{10} \approx 6.6897266686712602090 401$, $a \approx -0.2512928016597584784101067 1$,
$b \approx -1.7878517956323023614 398$, $\lambda \approx -14.741925190322941760808641 07$;
10) $A_{10} \approx 1.6274879010489748450 201$, $a \approx 0.1841931036627221851867221 0$,
$b \approx 0.4716932080651445815 482$, $\lambda \approx 5.3996683595764002071101167 1$;
11) $A_{10} \approx 0.1394463380540504608 910$, $a \approx -0.0883138758318200176924526 6$,
$b \approx 1.6869344282004140469 154$, $\lambda \approx 5.8020827381090768864590738 2$;
12) $A_{10} \approx -8.0555918046326485797 027$, $a \approx 0.6820327891810201830672829 0$,
$b \approx 0.8147669629127921204 300$, $\lambda \approx 3.8432146783984716372301567 8$;
13) $A_{10} \approx -7.2153236483212033683 720$, $a \approx 1.1235738011951273827388272 2$,
$b \approx 2.6297408958156117722 512$, $\lambda \approx 6.3234643726151895011243291 8$.

Next, we prove $v_{11} \neq 0$ if $v_3 = v_5 = v_7 = v_9 = 0$.

Let $r_1 = Resultant\,[v_5, v_{11}, A_{10}]$, $r_2 = Resultant\,[v_7, v_{11}, A_{10}]$, $r_3 = Resultant\,[v_9, v_{11}, A_{10}]$, and $r_{12} = Resultant\,[r_1, r_2, a]$, $r_{13} = Resultant\,[r_1, r_3, a]$.

If $v_3 = v_5 = v_7 = v_9 = v_{11} = 0$, then $r_{123} = Resultant\,[r_{12}, r_{13}, \lambda] = 0$. In fact, by computing we obtain $r_{123} = Resultant\,[r_{12}, r_{13}, \lambda] = 307659009886023883306504 0 \cdots \neq 0$.

Hence, $v_{11} \neq 0$ if $v_3 = v_5 = v_7 = v_9 = 0$, then the equilibrium point $(1, 1)$ of model (3) can be a 5th-order fine focus at most.

3.2. Focal values of the positive equilibrium points (2,1) of model (3)

Next, we compute the focal values of the positive equilibrium point (2,1) of model (3). By means of transformation

$$x = \widetilde{u} + 2,\ y = \widetilde{v} + 1, \tag{23}$$

model (3) takes the following form:

$$\begin{cases} \frac{d\widetilde{u}}{dt} = -\widetilde{v} - \frac{\widetilde{uv}}{2} + \frac{\widetilde{v}}{6}(\widetilde{u} + 2)(4\widetilde{u} + \widetilde{u}^2 - 14\widetilde{v} + 36a\widetilde{v} + 12A_{10}\widetilde{v} \\ \quad + 6A_{10}\widetilde{uv} - 7\widetilde{v}^2 + 30a\widetilde{v}^2), \\ \frac{d\widetilde{v}}{dt} = \widetilde{u} + \widetilde{uv} + \frac{1}{3}(\widetilde{v} + 1)(6\widetilde{u}^3 + 9\widetilde{u}^2 - 26\widetilde{uv} - 26\widetilde{u}^2\widetilde{v} + 28\widetilde{v}^2 \\ \quad + 18b\widetilde{v}^2 + 6B_{10}\widetilde{v}^2 + 39\widetilde{uv}^2 + 3B_{10}\widetilde{uv}^2 - 16\widetilde{v}^3 + 15b\widetilde{v}^3), \end{cases} \tag{24}$$

and the equilibrium point (2, 1) of model (3) becomes the origin of system (24) correspondingly. Under the transformation

$$\widetilde{z} = \widetilde{u} + i\widetilde{v},\ \widetilde{w} = \widetilde{u} - i\widetilde{v},\ T = it,\ i = \sqrt{-1}, \tag{25}$$

system (24) becomes its concomitant complex system, i.e.,

$$\begin{cases} \frac{d\widetilde{z}}{dT} = \widetilde{z} + Z_2(\widetilde{z}, \widetilde{w}) + Z_3(\widetilde{z}, \widetilde{w}) + Z_4(\widetilde{z}, \widetilde{w}), \\ \frac{d\widetilde{w}}{dT} = -\widetilde{w} - W_2(\widetilde{z}, \widetilde{w}) - W_3(\widetilde{z}, \widetilde{w}) - W_4(\widetilde{z}, \widetilde{w}), \end{cases} \tag{26}$$

in which

$$\begin{aligned} Z_2(\widetilde{z}, \widetilde{w}) = {} & \tfrac{1}{24}i(-74 + 33i + 72a + 24A_{10} + 36ib + 12iB_{10})\widetilde{w}^2 + \tfrac{1}{6}(37 + 14i \\ & - 36ia - 12iA_{10} + 18b + 6B_{10})\widetilde{wz} + \tfrac{1}{24}i(18 + 43i + 72a + 24A_{10} + 36ib \\ & + 12iB_{10})\widetilde{z}^2, \\ Z_3(\widetilde{z}, \widetilde{w}) = {} & \tfrac{1}{8}i[-12 - i + (6 + 10i)a + 4A_{10} - 11b - (2 - i)B_{10})\widetilde{w}^3 + \tfrac{1}{24}[13 + 26i \\ & + (90 - 18i)a - 12iA_{10} + 99ib + (3 + 18i)B_{10}]\widetilde{w}^2\widetilde{z} - \tfrac{1}{24}[-49 + 12i + \\ & (90 + 18i)a + 12iA_{10} + 99ib - (3 - 18i)B_{10}]\widetilde{wz}^2 + \tfrac{1}{24}[-17 + 22i + \\ & (30 + 18i)a + 12iA_{10} + 33ib - (3 - 6i)B_{10}]\widetilde{z}^3, \\ Z_4(\widetilde{z}, \widetilde{w}) = {} & \tfrac{1}{48}(14 - 33i - 15a + 3iA_{10} + 15b - 3iB_{10})\widetilde{w}^4 + \tfrac{1}{24}(29 + 45i + 15a \\ & - 30b + 3iB_{10})\widetilde{w}^3\widetilde{z} + \tfrac{1}{24}(-74 - 3iA_{10} + 45b)\widetilde{w}^2\widetilde{z}^2 - \tfrac{1}{24}(-35 + 45i + \\ & 15a + 30b + 3iB_{10})\widetilde{wz}^3 + \tfrac{1}{16}(2 + 11i + 5a + iA_{10} + 5b + iB_{10})\widetilde{z}^4, \\ W_2(\widetilde{z}, \widetilde{w}) = {} & -\tfrac{1}{24}i(18 - 43i + 72a + 24A_{10} - 36ib - 12iB_{10})\widetilde{w}^2 + \tfrac{1}{6}(37 - 14i \\ & + 36ia + 12iA_{10} + 18b + 6B_{10})\widetilde{wz} - \tfrac{1}{24}i(-74 - 33i + 72a + 24A_{10} \\ & - 36ib - 12iB_{10})\widetilde{z}^2, \\ W_3(\widetilde{z}, \widetilde{w}) = {} & -\tfrac{1}{24}i[22 - 17i + (18 + 30i)a + 12A_{10} + 33b + (6 - 3i)B_{10}]\widetilde{w}^3 \\ & + \tfrac{1}{24}[49 + 12i - (90 - 18i)a + 12iA_{10} + 99ib + (3 + 18i)B_{10}]\widetilde{w}^2\widetilde{z} \\ & + \tfrac{1}{24}[13 - 26i + (90 + 18i)a + 12iA_{10} - 99ib + (3 - 18i)B_{10}]\widetilde{wz}^2 \\ & - \tfrac{1}{8}[-1 - 12i + (10 + 6i)a + 4iA_{10} - 11ib + (1 - 2i)B_{10}]\widetilde{z}^3, \\ W_4(\widetilde{z}, \widetilde{w}) = {} & \tfrac{1}{16}(2 - 11i + 5a - iA_{10} + 5b - iB_{10})\widetilde{w}^4 - \tfrac{1}{24}(-35 - 45i + 15a + 30b \\ & - 3iB_{10})\widetilde{w}^3\widetilde{z} + \tfrac{1}{24}(-74 + 3iA_{10} + 45b)\widetilde{w}^2\widetilde{z}^2 + \tfrac{1}{24}(29 - 45i + 15a \\ & - 30b - 3iB_{10})\widetilde{wz}^3 - \tfrac{1}{48}(-14 - 33i + 15a + 3iA_{10} - 15b - 3iB_{10})\widetilde{z}^4. \end{aligned}$$

Obviously, system (26) belongs to the class of system (10). Next, we begin to compute the focal values of the origin of system (24) (namely the focal values of the equilibrium point (2, 1) of model (3)). According to the recursion formulas of Lemma 2.2, we have the following result. For convenience, we note that $B_{10} = \lambda_1 - 3b - \frac{17}{6}$.

Theorem 3.4. *The first five singular point values at the origin of system (24) are as follows:*

$\widetilde{\mu}_1 = -\frac{1}{12}i(312a + 110A_{10} + 45b + 8\lambda_1 + 144a\lambda_1 + 48A_{10}\lambda_1);$
$\widetilde{\mu}_2 = \frac{1}{4860}i(h_1 + \lambda_1 h_2 + 108\lambda_1^2 h_3);$
$\widetilde{\mu}_3 = \frac{1}{27993600}i(-15240960\lambda_1^4 h_4 - 1440\lambda_1^3 h_5 + 12\lambda_1^2 h_6 + \lambda_1 h_7 + h_8);$
$\widetilde{\mu}_4 = -\frac{1}{604661760000}i(-10484051097600\lambda_1^6 h_9 - 207360\lambda_1^5 h_{10} - 41472\lambda_1^4 h_{11} + 192\lambda_1^3 h_{12} + 4\lambda_1^2 h_{13} - \lambda_1 h_{14} + h_{15});$
$\widetilde{\mu}_5 = -\frac{1}{7836416409600000}i(6216291333365760000\lambda_1^8 h_{16} + 149299200\lambda_1^7 h_{17} + 1244160 \times \lambda_1^6 h_{18} - 6912\lambda_1^5 h_{19} - 1728\lambda_1^4 h_{20} + 48\lambda_1^3 h_{21} + 4\lambda_1^2 h_{22} + \lambda_1 h_{23} + h_{24});$

in which h_i, $(i \in \{1, 2, \cdots 24\})$ are the expressions about A_{10}, a.

Considering the relation between model (3) and system (24) and system (26), from Theorem 3.4 and Lemma 2.3, the following theorem is visible.

Theorem 3.5. *The first five focal values of the equilibrium point* $(2, 1)$ *of model (3) (or the first five focal values at the origin of system (24)) are as follows:*

$\widetilde{v}_3 = \frac{1}{12}\pi(312a + 110A_{10} + 45b + 8\lambda_1 + 144a\lambda_1 + 48A_{10}\lambda_1);$
$\widetilde{v}_5 = -\frac{1}{4860}\pi(h_1 + \lambda_1 h_2 + 108\lambda_1^2 h_3);$
$\widetilde{v}_7 = -\frac{1}{27993600}\pi(-15240960\lambda_1^4 h_4 - 1440\lambda_1^3 h_5 + 12\lambda_1^2 h_6 + \lambda_1 h_7 + h_8);$
$\widetilde{v}_9 = \frac{1}{604661760000}\pi(-10484051097600\lambda_1^6 h_9 - 207360\lambda_1^5 h_{10} - 41472\lambda_1^4 h_{11} + 192\lambda_1^3 h_{12} + 4\lambda_1^2 h_{13} - \lambda_1 h_{14} + h_{15});$
$\widetilde{v}_{11} = \frac{1}{7836416409600000}\pi(6216291333365760000\lambda_1^8 h_{16} + 149299200\lambda_1^7 h_{17} + 1244160 \times \lambda_1^6 h_{18} - 6912\lambda_1^5 h_{19} - 1728\lambda_1^4 h_{20} + 48\lambda_1^3 h_{21} + 4\lambda_1^2 h_{22} + \lambda_1 h_{23} + h_{24}).$

From Theorem 3.5, we have

Theorem 3.6. *The equilibrium point* $(2, 1)$ *of model (3) can be a 5th-order fine focus at most.*

Proof. According to Definition 2.1, we need to prove that there exits a group of real values about λ_1, A_{10}, a, b such that $\widetilde{v}_3 = \widetilde{v}_5 = \widetilde{v}_7 = \widetilde{v}_9 = 0$, $\widetilde{v}_{11} \neq 0$.

At first, we prove that $\widetilde{v}_3 = \widetilde{v}_5 = \widetilde{v}_7 = \widetilde{v}_9 = 0$ have real number solutions.

Let $\widetilde{v}_3 = 0$, we have

$$b = -\frac{2}{45}(156a + 55A_{10} + 4\lambda_1 + 72a\lambda_1 + 24A_{10}\lambda_1). \tag{27}$$

By using computer soft Mathematica 6.0 to compute, $\widetilde{v}_3 = \widetilde{v}_5 = \widetilde{v}_7 = \widetilde{v}_9 = 0$ hold if and only if Eq (27) holds and

$$\begin{cases} \widetilde{r}_{57} = Resultant\ [\widetilde{v}_5, \widetilde{v}_7, A_{10}] = 0, \\ \widetilde{r}_{59} = Resultant\ [\widetilde{v}_5, \widetilde{v}_9, A_{10}] = 0. \end{cases} \tag{28}$$

While Eq (28) hold if and only if

$$\widetilde{r}_{579} = Resultant[\widetilde{r}_{57}, \widetilde{r}_{59}, a] = 0. \tag{29}$$

By computing, we obtain

$$\widetilde{r}_{579} = Resultant\ [r_{57}, r_{59}, a] = (3107 + 3210\lambda_1 + 720\lambda_1^2)^2(13964 + 10293\lambda_1 + 1800\lambda_1^2)^2\widetilde{g}(\lambda_1),$$

in which $\widetilde{g}(\lambda)$ is a 242 degrees function on λ_1. It can be seen that Eq (29) has some real solutions such as $\lambda_1 = \frac{1}{240}(-535 \pm 9\sqrt{465})$ et al. Hence, $\widetilde{v}_3 = \widetilde{v}_5 = \widetilde{v}_7 = \widetilde{v}_9 = 0$ have real number solutions.

In fact, we can find 15 groups of real number solutions such that $\widetilde{v}_3 = \widetilde{v}_5 = \widetilde{v}_7 = \widetilde{v}_9 = 0$, namely

1) $A_{10} \approx -4643.3018425501515263, \quad a \approx 1550.8973397671423233,$
$b \approx 1036.3362485482110284, \quad \lambda_1 \approx -43.0579290316570603;$

2) $A_{10} \approx 0.9959258074047267, \quad a \approx -0.0496867911910842,$
$b \approx -215.6922502839121132, \quad \lambda_1 \approx 197.5784671350668527;$

3) $A_{10} \approx 17.6221151647949041, \quad a \approx -5.8674840306887030,$
$b \approx -5.1961268584644113, \quad \lambda_1 \approx 14.0933015765297504;$

4) $A_{10} \approx -22.4133844512194665, \quad a \approx 7.6232290779662400,$
$b \approx -1.4651057037980100, \quad \lambda_1 \approx 5.1151041647768187;$

5) $A_{10} \approx 6.0154299290528854, \quad a \approx -3.1637820182235810,$
$b \approx -7.3080323729222197, \quad \lambda_1 \approx -4.1189107630353050;$

6) $A_{10} \approx -5.3540843597677715, \quad a \approx 2.210452369910683,$
$b \approx 1.9959489233153501, \quad \lambda_1 \approx -2.748983659910321;$

7) $A_{10} \approx -3.2879140223160507, \quad a \approx 1.636028779039390,$
$b \approx 1.4395307705732038, \quad \lambda_1 \approx -2.489840572873740;$

8) $A_{10} \approx -4.1024413004886361, \quad a \approx 2.121698633718532,$
$b \approx -1.3456036840775022, \quad \lambda_1 \approx -1.287647593274435;$

9) $A_{10} \approx -2.7095465588712933, \quad a \approx 1.607999835283884,$
$b \approx 0.1502206861914896, \quad \lambda_1 \approx -1.921623601914863;$

10) $A_{10} \approx 0.0859310460513803, \quad a \approx 0.219273746634790,$
$b \approx 1.1721730119425250, \quad \lambda_1 \approx -2.988862274797442;$

11) $A_{10} \approx 0.3976086548746663, \quad a \approx 0.119165192985334,$
$b \approx 1.0985939724503711, \quad \lambda_1 \approx -2.946168239121486;$

12) $A_{10} \approx 0.3522522540516692, \quad a \approx 0.044767143636023,$
$b \approx 0.9813497899428434, \quad \lambda_1 \approx -3.089686791404887;$

13) $A_{10} \approx 0.5942698666771841, \quad a \approx 0.294891555364813,$
$b \approx 1.1887051809847815, \quad \lambda_1 \approx -2.669570213107159;$

14) $A_{10} \approx 1.0727163914454935, \quad a \approx 0.107785966220006,$
$b \approx 1.1631832585231690, \quad \lambda_1 \approx -2.719197614164823;$

15) $A_{10} \approx 1.5505080546244919, \quad a \approx -1.101693440557772,$
$b \approx -2.0337942792540713, \quad \lambda_1 \approx -3.472776931203230.$

Next, we prove $\widetilde{v}_{11} \neq 0$ if $\widetilde{v}_3 = \widetilde{v}_5 = \widetilde{v}_7 = \widetilde{v}_9 = 0$.

Let
$\widetilde{r}_1 = Resultant\ [\widetilde{v}_5, \widetilde{v}_{11}, A_{10}]$, $\widetilde{r}_2 = Resultant\ [\widetilde{v}_7, \widetilde{v}_{11}, A_{10}]$, $\widetilde{r}_3 = Resultant\ [\widetilde{v}_9, \widetilde{v}_{11}, A_{10}]$, and $\widetilde{r}_{12} = Resultant\ [\widetilde{r}_1, \widetilde{r}_2, a]$, $\widetilde{r}_{13} = Resultant\ [\widetilde{r}_1, \widetilde{r}_3, a]$.

If $\widetilde{v}_3 = \widetilde{v}_5 = \widetilde{v}_7 = \widetilde{v}_9 = 0$, then $\widetilde{r}_{123} = Resultant\ [\widetilde{r}_{12}, \widetilde{r}_{13}, \lambda_1] = 0$. By computing, we obtain: $\widetilde{r}_{123} = Resultant\ [\widetilde{r}_{12}, \widetilde{r}_{13}, \lambda_1] = 5465588712932574981288128547O922 \cdots \neq 0$.

Hence, $\widetilde{v}_{11} \neq 0$ if $\widetilde{v}_3 = \widetilde{v}_5 = \widetilde{v}_7 = \widetilde{v}_9 = 0$, then the equilibrium point $(2, 1)$ of model (3) can be a 5th-order fine focus at most. Proof end.

4. The bifurcations of limit cycles of model (3)

After finding focal values of two positive equilibrium points of model (3), we will consider the limit cycle bifurcation near $(1, 1)$ and $(2, 1)$ of perturbed model (3).

Lemma 4.1. *Let J_1 be the Jacobin of the function group (v_3, v_5, v_7, v_9) with respect to the variables (a, λ, A_{10}, b), if the equilibrium $(1, 1)$ of model (3) is a 5th-order fine focus, then $J_1 \neq 0$.*

Proof. Suppose that $J_1 = 0$, next we deduce a contradictory result. The Jacobin of the function group (v_3, v_5, v_7, v_9) with respect to the variables (a, λ, A_{10}, b) has the following form

$$J_1 = \begin{vmatrix} \frac{\partial v_3}{\partial a} & \frac{\partial v_3}{\partial A_{10}} & \frac{\partial v_3}{\partial \lambda} & \frac{\partial v_3}{\partial b} \\ \frac{\partial v_5}{\partial a} & \frac{\partial v_5}{\partial A_{10}} & \frac{\partial v_5}{\partial \lambda} & \frac{\partial v_5}{\partial b} \\ \frac{\partial v_7}{\partial a} & \frac{\partial v_7}{\partial A_{10}} & \frac{\partial v_7}{\partial \lambda} & \frac{\partial v_7}{\partial b} \\ \frac{\partial v_9}{\partial a} & \frac{\partial v_9}{\partial A_{10}} & \frac{\partial v_9}{\partial \lambda} & \frac{\partial v_9}{\partial b} \end{vmatrix}.$$

Obviously, J_1 is a function with respect to the variables (a, λ, A_{10}, b). Because the equilibrium point $(1, 1)$ of model (3) is a 5th-order fine focus, then $v_3 = v_5 = v_7 = v_9 = 0$. Suppose that $J_1 = 0$, then the resultant of $J_1, v_i, (i \in \{3, 5, 7, 9\})$ with respect to the variable b will become 0.

Let $R_1 = Resultant\ [J_1, v_3, b]$, $R_2 = Resultant\ [J_1, v_5, b]$, $R_3 = Resultant\ [J_1, v_7, b]$, $R_4 = Resultant\ [J_1, v_9, b]$, then $R_i = 0,\ i \in \{1, 2, 3, 4\}$.

While $R_i = 0,\ i \in \{1, 2, 3, 4\}$ will deduce that $R_{12} = Resultant\ [R_1, R_2, A_{10}] = 0$, $R_{13} = Resultant\ [R_1, R_3, A_{10}] = 0$, $R_{14} = Resultant\ [R_1, R_4, A_{10}] = 0$. Similarly, $R_{12} = R_{13} = R_{14} = 0$ deduce that $R_{23} = Resultant\ [R_{12}, R_{13}, a] = 0$ and $R_{24} = Resultant\ [R_{12}, R_{14}, a] = 0$. In the same way, $R_{23} = R_{24} = 0$ deduce that $R_{34} = Resultant\ [R_{23}, R_{24}, \lambda] = 0$. In fact, with help of computer, we obtain that

$R_{34} = 24231418616323066317496248857998624904907753623104 17 \cdots \neq 0$.

$R_{34} \neq 0$ pushes $J_1 \neq 0$. Proof end.

Similarly, we can obtain the following lemma.

Lemma 4.2. *Let J_2 be the Jacobin of the function group $(\widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_7, \widetilde{v}_9)$ with respect to the variables $(a, \lambda_1, A_{10}, b)$, if the equilibrium point $(2, 1)$ of model (3) is a 5th-order fine focus, then $J_2 \neq 0$.*

Theorem 4.1. *Suppose that $(1, 1)$ is a 5-th order fine focus of system (3), then by small perturbations of the parameter group (a, λ, A_{10}, b), the point $(1, 1)$ of perturbed model (3) can bifurcate at least 5 small amplitude limit cycles.*

Proof. From lemma 4.1, $J_1 \neq 0$, while $(1, 1)$ is a 5-th order fine focus of system (3), according to the theory of reference [14], the conclusion of Theorem 4.1 holds. Proof end.

Similarly, we have the following theorem.

Theorem 4.2. *Suppose that $(2, 1)$ is a 5-th fine focus of system (3), then by small perturbations of the parameter group $(a, \lambda_1, A_{10}, b)$, the point $(2, 1)$ of perturbed model (3) can bifurcate at least 5 small amplitude limit cycles.*

Next we will give a case that $(1, 1)$ of the model (3) can bifurcate 5 limit cycles of which 3 limit cycles are stable.

Obviously, v_i, $(i = 3, 5, 7, 9, 11)$ are functions about (a, λ, A_{10}, b). The course of Theorem 3.3 shows there exists a group solutions (a, λ, A_{10}, b)=$(\tilde{a}, \tilde{\lambda}, \tilde{A_{10}}, \tilde{b})$ such that $v_i = 0,\ i \in \{3, 5, 7, 9\},\ v_{11} \neq 0$. Hence, we may as well let

$$v_3 = \epsilon_1,\ v_5 = \epsilon_2,\ v_7 = \epsilon_3,\ v_9 = \epsilon_4, \tag{30}$$

in which $\epsilon_i,\ i \in \{1, 2, 3, 4\}$ are a group of arbitrary given real small parameters.

According to existence theorem of implicit function and the result of Lemma 4.1, Eq (30) has a group of solutions as follows:

$$\begin{aligned} a &= a(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4),\\ \lambda &= \lambda(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4),\\ A_{10} &= A_{10}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4),\\ b &= b(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4). \end{aligned} \tag{31}$$

From (30) and (31), clearly the following theorem holds.

Theorem 4.3. *Suppose that* (a, λ, A_{10}, b) *disturb by way of (31), then* $v_3 = \epsilon_1,\ v_5 = \epsilon_2,\ v_7 = \epsilon_3,\ v_9 = \epsilon_4$, *in which* $\epsilon_i,\ i \in \{1, 2, 3, 4\}$ *are a group of arbitrary given real small parameters.*

Next, we let $\epsilon_i,\ i \in \{1, 2, 3, 4\}$ be some special values, we have the following theorem.

Theorem 4.4. *Suppose that the coefficients of model (3) disturb via* $v_3 = 21076c_5\pi\epsilon^8 + o(\epsilon^9)$, $v_5 = -7645c_5\pi\epsilon^6 + o(\epsilon^7)$, $v_7 = 1023c_5\pi\epsilon^4 + o(\epsilon^5)$, $v_9 = -55c_5\pi\epsilon^2 + o(\epsilon^3)$, $c_5 = v_{11}$, *then the point (1,1) of model (3) can bifurcate 5 small limit cycles which are near to circles* $(x-1)^2 + (y-1)^2 = k^2\epsilon^2, k = 1, 2, 3, 4, 5$ *in which 3 limit cycles can be stable.*

Proof. Suppose that $v_3 = 21076c_5\pi\epsilon^8 + o(\epsilon^9)$, $v_5 = -7645c_5\pi\epsilon^6 + o(\epsilon^7)$, $v_7 = 1023c_5\pi\epsilon^4 + o(\epsilon^5)$, $v_9 = -55c_5\pi\epsilon^2 + o(\epsilon^3)$, $c_5 = v_{11}$, then we have

$$\begin{aligned} &v_1(2\pi, \epsilon, \delta) = e^{2\pi\delta} = 1 + c_0\pi\epsilon^{10} + o(\epsilon^{11}),\\ &v_3(2\pi, \epsilon, \delta) = c_1\pi\epsilon^8 + o(\epsilon^9),\\ &v_5(2\pi, \epsilon, \delta) = c_2\pi\epsilon^6 + o(\epsilon^7),\\ &v_7(2\pi, \epsilon, \delta) = c_3\pi\epsilon^4 + o(\epsilon^5),\\ &v_9(2\pi, \epsilon, \delta) = c_4\pi\epsilon^2 + o(\epsilon^3),\\ &v_{11}(2\pi, \epsilon, \delta) = c_5 + o(\epsilon), \end{aligned}$$

in which
$c_5 = v_{11}|_{\epsilon=0}$, and $c_0 = -14400c_5, c_1 = 21076c_5, c_2 = -7645c_5, c_3 = 1023c_5, c_4 = -55c_5$.

At this time, according to (6), Poincaré succession function for the point (1,1) of model (3) is as follows:

$$\begin{aligned} d(\epsilon h) &= r(2\pi, \epsilon h) - \epsilon h\\ &= (v_1(2\pi, \epsilon, \delta) - 1)\epsilon h + v_2(2\pi, \epsilon, \delta)(\epsilon h)^2 + v_3(2\pi, \epsilon, \delta)(\epsilon h)^3 + \ldots\\ &\quad + v_{11}(2\pi, \epsilon, \delta)(\epsilon h)^{11} + \ldots\\ &= \pi\epsilon^{11}h[g(h) + \epsilon h G(h, \epsilon)], \end{aligned}$$

in which

$$\begin{aligned} g(h) &= c_0 + c_1h^2 + c_2h^4 + c_3h^6 + c_4h^8 + j_0h^{10}\\ &= c_6(h^2 - 1)(h^2 - 4)(h^2 - 9)(h^2 - 16)(h^2 - 25)\\ &= -14400c_5 + 21076c_5h^2 - 7645c_5h^4 + 1023c_5h^6 - 55c_5h^8 + c_5h^{10}, \end{aligned}$$

and $G(h, \epsilon)$ is analytic at $(0, 0)$.

Obviously, $g(h) = 0$ has 5 simple positive zero points $1, 2, 3, 4, 5$. From implicit function theorem, the number of positive zero points of equation $d(\epsilon h) = 0$ is equal to one of $g(h) = 0$, and these positive zero points are close to $1, 2, 3, 4, 5$ when $0 < |\epsilon| \ll 1$. The above analysis shows there are 5 small limit cycles in a small enough neighborhood of $(1, 1)$ of model (3), which are near to circles $(x - 1)^2 + (y - 1)^2 = k^2\epsilon^2, k = 1, 2, 3, 4, 5$.

Obviously, if $v_{11} < 0$, then the point (1,1) of model (3) can bifurcate 3 stable cycles which are near circles $(x - 1)^2 + (y - 1)^2 = k^2\epsilon^2, k = 1, 3, 5$. While by analyzing the 13 groups of solutions showed in the proof course of Theorem 3.3, this kind of solutions exist such as the second group of solution. Hence, the result of Theorem 4.4 holds. Proof end.

Remark: We can also find the solution such that $v_3 = v_5 = v_7 = v_9 = 0$, $v_{11} > 0$ among the 13 groups of solutions showed in the proof course of Theorem 3.3. Hence, the point (1,1) of model (3) can also bifurcate 2 stable cycles which are near circles $(x - 1)^2 + (y - 1)^2 = k^2\epsilon^2, k = 2, 4$.

Similarly, we can obtain the following theorem.

Theorem 4.5. *The point (2,1) of model (3) can bifurcate 3 stable limit cycles under certain condition.*

5. Conclusions

The work of this paper focuses on investigating the limit cycle bifurcation of a class of the quartic Kolmogorov model, which is an interesting and significant ecological model both in theory and applications. We used the singular values method to compute focal value. First, we give the relation between focal value and singular point value at the origin, which is necessary for us to investigate bifurcations of limit cycles. Sceond, we give the singular point values' recursive formulas and respectively compute the focal values of the two positive equilibrium points of model (3) and obtain the condition that they can be two 5th-order fine focuses. Next, we discuss the bifurcation of limit cycle of model (3) and obtain that each one of the two positive equilibrium points of model (3) can have five small limit cycles. Then, we show a case that each positive equilibrium point can bifurcate 3 stable limit cycles at most by making use of algebraic and symbolic proof.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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