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Research article

Global attracting sets and exponential stability of nonlinear uncertain differential equations

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Abstract: Uncertain differential equation is a type of differential equation driven by canonical Liu process. By applying some uncertain theories, the sufficient conditions of the exponential stability in mean square is obtained for nonlinear uncertain differential equations. At the same time, some new criteria ensuring the existence of the global attracting sets of considered equations are presented.

Keywords: canonical Liu process; uncertain differential equation; exponential stability; global attracting set
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1. Introduction

Uncertainty is everywhere in life, such as predicting heads or tails before flipping a coin, predicting tomorrow's stock price, predicting tomorrow's market demand, and predicting travel time. In order to deal with this kind of uncertainty phenomenon reasonably, there are two mathematical systems, one is probability theory, the other is uncertainty theory. Probability theory and uncertainty theory are completely different sets of mathematical theories. Probability theory is used to describe frequency, uncertainty theory is used to describe reliability. In other words, probability theory is based on people's perception of frequency, while uncertainty theory is based on the human belief degree of the event. In practice, for certain quantities, by any means, we can get the distribution function, and then if we believe that the distribution function is close to the true frequency of events, we should treat these quantities as random variables, otherwise, we should treat these quantities as uncertain variables, that is, if the distribution function is close to the frequency, we should use probability theory, otherwise we have to use uncertainty theory. The uncertain theory was instituted by Liu [1]. In the past period of time, other scholars in this field have also drawn some excellent conclusions (see [2–4]).

Uncertain process was proposed by Liu for modeling the evolution of uncertain phenomena.

Canoical Liu process is an uncertain process with stationary and independent normal uncertain increments. Uncertain differential equation is a type of differential equations which is driven by canonical Liu process. Subsequently, uncertain differential equations are widely used in different fields to dispose of dynamic uncertain process. For example, in finance field, Liu [5] applied it to the financial markets, then, Zhang et al. [6] evaluaed the convertible bonds within the uncertain differential equation, the option pricing problem of look back options is investigated by Zhang et al. [7], Zhang et al. [8] investigated the valuation of stock loan. In biology field, Sheng et al. [9] presented an uncertain population model with age-structure, then Zhang and Yang [10] presented some uncertain population models. But in practice, uncertain differential equation models applied in the finance field or biological field are usually nonlinear. Hence, it is of practical significance to study nonlinear differential equations.

In rencent years, stability of uncertain differential equation was a hot spot. Scholars have made fruitful achievements in the research of this theory. Yao et al. [11] have discussed some stability theorems of uncertain differential equation. Yang et al. [12] have researched stability in inverse distribution for uncertain differential equations. Then, Yao et al. [13] have discussed stability in mean for the uncertain differential equation. Almost sure stability for the uncertain differential equation had been proposed by Liu et al. [14]. Subsequently, Chen and Ning [15] have studied the pth moment exponential stability of the uncertain differential equation. Sheng and Gao [16] have studied exponential stability of the uncertain differential equation. These scholars study uncertain differential equations with single variable. On this basis, Zhang et al. [17] investigated the stability of multifactor uncertain differential equation. Furthermore, Sheng and Shi [18] studied stability in p-th moment for multifactor uncertain differential equation. Then, almost sure and pth moment stability of uncertain differential equations with time-varying delay was investigated by Wang [19]. Wang and Ning [20] studied a new stability analysis of uncertain delay differential equations.

Many researchers investigated the stability of the linear uncertain differential equation. For example, Chen and Gao [21] introduced stability analysis of linear uncertain differential equations. And in the aspect of exponential stability, Sheng and Gao [16] have studied exponential stability of the uncertain differential equation. The p th moment exponential stability of the uncertain differential equation was proposed by Chen and Ning [15].

Tao and Zhu [22] introduced some attractivity concepts of uncertain differential systems and gave sufficient and necessary conditions of attractivity for the linear uncertain differential equations. Subsequently, Tao and Zhu [23] introduced two concepts of stability and attractivity in optimistic value for dynamical systems with uncertainty and acquired a sufficient condition of stability and a sufficient and necessary condition of attractivity for linear dynamical systems with uncertainty. Furthermore, they discussed the relationship between stability in measure (in mean, in distribution) and stability in optimistic value, and the relationship between attractivity in measure (in mean, in distribution) and attractivity in optimistic value. As we can see Tao and Zhu only discussed the condition of attractivity for linear uncertain differential equations. Therefore, we will explore global attracting sets for the nonlinear uncertain differential equations.

In this paper, we will introduce a new definition of exponential stability of nonlinear uncertain differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \tag{1.1}$$

where C_t is a Liu process, and f and g are two nonlinear functions.

The rest of this paper is organized as follows: In Section 2, we will introduce some preliminaries about uncertainty theory. In Section 3, exponential stability of nonlinear uncertain differential equation is proposed. In Section 4, we will present for the global attracting sets for nonlinear uncertain differential equations. In Section 5, we present some examples to illustrate the advantage of our results. At last, we will make a laconic conclusion in Section 6.

2. Preliminaries

In this section, we will review some essential theories of uncertain differential equations.

Definition 2.1. *[1] Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. An uncertain measure is a function $\mathcal{M}$: $\mathcal{L} \to [0,1]$ such that:*

Axiom 2.2. *(Normality axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.*

Axiom 2.3. *(Duality axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.*

Axiom 2.4. *(Subadditivity axiom) For every countable sequence of events Λ_i, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leqslant \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{2.1}$$

A set $\Lambda \in \mathcal{L}$ is called an event. The uncertain measure $\mathcal{M}\{\Lambda\}$ indicates the degree of belief that λ will occur. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertain space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined.

Axiom 2.5. *(Product axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure on the product σ-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$ satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}, \tag{2.2}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2.6. *[9] An uncertain variable is a measureable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real number, i.e., $\{\xi \in B\}$ is an event for any Borel set B.*

Definition 2.7. *[17] The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\} \tag{2.3}$$

for any real number x.

Definition 2.8. *Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.*

Definition 2.9. *[9] Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geqslant r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leqslant r\}dr, \tag{2.4}$$

provided that at least one of the two integrals is finite.

Theorem 2.10. *[11] Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ be strictly increasing and continuous with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing and continuous with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n) \tag{2.5}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \Phi_n^{-1}(1-\alpha)). \tag{2.6}$$

Definition 2.11. *An uncertain process C_t is said to be a Liu process if:*
(i) C_0 and almost all sample paths are Lipschitz continuous;
(ii) C_t has stationary and independent increments;
(iii) Every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

Definition 2.12. *[16] Let X_t be an uncertain process and C_t be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is define as*

$$\triangle = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then, the Liu integral of X_t is defined as

$$\int_a^b X_t dC_t = \lim_{\triangle \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}), \tag{2.7}$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be Liu integrable.

Definition 2.13. *A m-dimensional process $\left\{C_t = (C_t^1, C_t^2, \cdots, C_t^m)\right\}$ is called a m-dimensional canonical process if $\left\{C_t^i\right\}$ is a one dimensional canonical process, and $\left\{C_t^1\right\}$, $\left\{C_t^2\right\}$, $\cdots$, $\left\{C_t^i\right\}$ are independent.*

Let C_t be the given m-dimensional canonical process defined on the space $(\Gamma, \mathcal{L}, \mathcal{M})$. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^d$ associated with an inner product $< \cdot, \cdot >$.

Definition 2.14. *[24] Suppose that C_t is a canonical process, and X_t is an integrable uncertain process on$[a, b]$ with respect to t. Then the inequality*

$$\Big| \int_a^b X_t(\gamma) dC_t(\gamma) \Big| \le K(\gamma) \int_a^b |X_t(\gamma)| dt$$

holds, where $K(\gamma)$ is the Lipschitz constant of the sample path $X_t(\gamma)$.

3. Exponential stability

In this section, we will explore the exponential stability in mean square of a nonlinear uncertain differential equation.

Definition 3.1. *[15] The nonlinear uncertain differential equation $dX_t = f(t, X_t)dt + g(t, X_t)dC_t$ is said to be exponential stable in mean square, if for any solutions X_t and Y_t with different initial values X_0 and Y_0, there exists a pair of positive constants C and δ such that*

$$E|X_t - Y_t|^2 \leq C|X_0 - Y_0|^2 exp(-\delta t), \qquad \forall t \geq 0.$$

Remark 3.2. If an uncertain differential equation is p-th moment exponential stable, then it is p-th moment stable. When $p = 2$, the uncertain differential equation is exponential stable in square mean. If an uncertain differential equation is exponentially stable, then it is stable in mean. The reverse is not true.

Proof. Let $0 < p < +\infty$. The nonlinear uncertain differential equation $dX_t = f(t, X_t)dt + g(t, X_t)dC_t$ is said to be p-th exponential stable, if for any solutions X_t and Y_t with different initial values X_0 and Y_0, there exists a pair of positive constants C and δ such that

$$E|X_t - Y_t|^p \leq C|X_0 - Y_0|^p exp(-\delta t), \qquad \forall t \geq 0.$$

Then we have

$$0 \leqslant \lim_{X_0 - Y_0 \to 0} E|X_t - Y_t|^p \leqslant \lim_{X_0 - Y_0 \to 0} C|X_0 - Y_0|^p exp(-\delta t), \qquad \forall t \geq 0.$$

Hence,

$$\lim_{X_0 - Y_0 \to 0} E|X_t - Y_t|^p = 0, \qquad \forall t \geq 0.$$

Therefore, if an uncertain differential equation is p-th moment exponential stable, then it is p-th moment stable.

When $p = 1, A = C|X_0 - Y_0|$ and $\delta = \alpha$, we can get

$$E|X_t - Y_t| \leqslant Aexp(-\alpha t), \qquad \forall t \geq 0.$$

That mean p-th exponential stability implies exponential stability.

Let $A \to 0$ as $|X_0 - Y_0| \to 0$, we have

$$\lim_{X_0 - Y_0 \to 0} E|X_t - Y_t| = 0, \qquad \forall t \geq 0.$$

Which means the uncertain differential equation is stable in mean.

When $p = 2$, the uncertain differential equation is exponential stable in square mean. □

Example 3.3. *Consider the following nonlinear uncertain differential equation:*

$$dX_t = (-X_t + {X_t}^2)dt - exp(-t)(-X_t + {X_t}^2)dC_t. \tag{3.1}$$

Suppose that X_t and Y_t are two solutions of the Eq (1.1) with different initial values X_0 and Y_0, respectively. Then we have

$$X_t = \frac{X_0 exp(-t + \int_0^t exp(-s)dC_s)}{X_0 exp(-t + \int_0^t exp(-s)dC_s) - X_0 + 1}$$

and

$$Y_t = \frac{Y_0 exp(-t + \int_0^t exp(-s)dC_s)}{Y_0 exp(-t + \int_0^t exp(-s)dC_s) - Y_0 + 1}.$$

Let

$$Z_t = exp(-t + \int_0^t exp(-s)dC_s),$$

we have

$$\begin{aligned}
|X_t - Y_t| &= \Big|\frac{X_0 Z_t}{X_0 Z_t - X_0 + 1} - \frac{Y_0 Z_t}{Y_0 Z_t - Y_0 + 1}\Big| \\
&= \Big|\frac{X_0 Z_t (Y_0 Z_t - Y_0 + 1) - Y_0 Z_t (X_0 Z_t - X_0 + 1)}{(X_0 Z_t - X_0 + 1)(Y_0 Z_t - Y_0 + 1)}\Big| \\
&= \Big|\frac{X_0 Z_t - Y_0 Z_t}{(X_0 Z_t - X_0 + 1)(Y_0 Z_t - Y_0 + 1)}\Big| \\
&\leq |(X_0 - Y_0) exp(-t + \int_0^t exp(-s)dC_s)| \\
&\leq |X_0 - Y_0| exp(-t) exp(\int_0^t exp(-s)dC_s)).
\end{aligned}$$

Taking the p-th moment of $|X_t - Y_t|$, we have

$$E|X_t - Y_t|^p = |X_0 - Y_0|^p exp(-pt) E[exp(p\int_0^t exp(-s)dC_s)].$$

When $t \to \infty$, $p\int_0^t exp(-s)dC_s = p > \frac{\pi}{\sqrt{3}}$, the expected value is

$$E\Big[exp(p\int_0^t exp(-s)dC_s)\Big] \to +\infty.$$

So when $p > \frac{\pi}{\sqrt{3}}$, none of a positive constant C is satisfied

$$E|X_t - Y_t|^p \leq C|X_0 - Y_0|^p exp(-\delta t), \quad \forall t \geq 0.$$

Which means the Eq (3.1) is not p-th moment exponential stable. Because $2 > \frac{\pi}{\sqrt{3}}$, the Eq (3.1) is not exponential stable in mean square. Taking expected value of $|X_t - Y_t|$, we have

$$E|X_t - Y_t| = |X_0 - Y_0| exp(-t) E[exp(\int_0^t exp(-s)dC_s)].$$

By $\int_0^s f(t)dC_t \backsim N(0, \int_0^s |f(t)|dt)$, we can obtain

$$\int_0^t exp(-s)dC_s \backsim N(0, \int_0^s |exp(-s)|ds).$$

When

$$\int_0^t exp(-s) = 1 - exp(-t) < \frac{\pi}{\sqrt{3}}$$

for all $t \geqslant 0$, the expected value is

$$E\Big[exp(\int_0^t exp(-s)dC_s)\Big] < +\infty.$$

Existing

$$A = |X_0 - Y_0| E\Big[exp(\int_0^t exp(-s)dC_s)\Big], \alpha = 1,$$

then, the Eq (3.1) is exponential stable. Meanwhile, if $|X_0 - Y_0| \to 0$, we can get

$$\lim_{|X_0 - Y_0| \to 0} E||X_t - Y_t| = 0.$$

So the Eq (3.1) is stable in mean.

To sum up, exponential stability can not derive p-th moment exponential stability. If Eq (1.1) is not p-th moment exponential stability, then it is not exponential stable in mean square.

Theorem 3.4. *Assume the nonlinear uncertain differential equation has a unique solution for each given initial state. Then it is exponential stable in mean square if there exist a positive constant $\delta > 0$ such that*

$$< X_t - Y_t, f(t, X_t) - f(t, Y_t) > \vee < X_t - Y_t, g(t, X_t) - g(t, Y_t) > \leq -\delta |X_t - Y_t|^2. \tag{3.2}$$

Proof. Suppose that X_t and Y_t are two solutions of the Eq (1.1) with different initial values X_0 and Y_0, respectively. Then, we have

$$X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dC_s \tag{3.3}$$

and

$$Y_t = Y_0 + \int_0^t f(s, Y_s)ds + \int_0^t g(s, Y_s)dC_s. \tag{3.4}$$

By Definition 2.11, we have

$$\begin{aligned} e^{\alpha t}|X_t - Y_t|^2 &= e^{\alpha t}\Big|X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dC_s - Y_0 - \int_0^t f(s, Y_s)ds - \int_0^t g(s, Y_s)dC_s\Big|^2 \\ &= e^{\alpha t}\Big|X_0 - Y_0 + \int_0^t f(s, X_s)ds - \int_0^t f(s, Y_s)ds + \int_0^t g(s, X_s)dC_s - \int_0^t g(s, Y_s)dC_s\Big|^2 \\ &\leq |X_0 - Y_0|^2 + \int_0^t \alpha e^{\alpha s}|X_s - Y_s|^2 ds + 2\int_0^t e^{\alpha s} < X_s - Y_s, f(s, X_s) - f(s, Y_s) > ds \\ &\quad + 2\int_0^t e^{\alpha s} < X_s - Y_s, g(s, X_s) - g(s, Y_s) > dC_s \\ &\leq |X_0 - Y_0|^2 + \int_0^t \alpha e^{\alpha s}|X_s - Y_s|^2 ds - \int_0^t e^{\alpha s}\delta|X_s - Y_s|^2 ds - K(\gamma)\int_0^t e^{\alpha s}\delta|X_s - Y_s|^2 ds \\ &\leq |X_0 - Y_0|^2 + \alpha\int_0^t e^{\alpha s}|X_s - Y_s|^2 ds - \delta\int_0^t e^{\alpha s}|X_s - Y_s|^2 ds - \delta K(\gamma)\int_0^t e^{\alpha s}|X_s - Y_s|^2 ds \\ &\leq |X_0 - Y_0|^2 + (\alpha - \delta(1 + K(\gamma)))\int_0^t e^{\alpha s}|X_s - Y_s|^2 ds. \end{aligned} \tag{3.5}$$

Taking the expectation on the both sides of the above inequality, according to Gronwall inequality by Theorem 2.4 in [25], then we obtain

$$\begin{aligned} e^{\alpha t}E|X_t - Y_t|^2 &\le |X_0 - Y_0|^2 + (\alpha - \delta(1 + K(\gamma))) \int_0^t e^{\alpha s}E|X_s - Y_s|^2 ds \\ &\le |X_0 - Y_0|^2 exp(\int_0^t (\alpha - \delta(1 + K(\gamma))))ds \\ &\le |X_0 - Y_0|^2 exp(\alpha - \delta(1 + K(\gamma)))t. \end{aligned} \tag{3.6}$$

Then, we have

$$\begin{aligned} E|X_t - Y_t|^2 &\le |X_0 - Y_0|^2 exp(\alpha - \delta(1 + K(\gamma)))t - \alpha t \\ &\le |X_0 - Y_0|^2 exp(-\delta(1 + K(\gamma)t)) \\ &\le |X_0 - Y_0|^2 exp(-\delta t). \end{aligned} \tag{3.7}$$

Hence, the nonlinear uncertain differential equations is exponential stable in mean square. The proof is completed. □

4. Global attracting sets

In this section, We will introduce the global attracting sets for the nonlinear uncertain differential equation to (1.1).

Theorem 4.1. *Assume the nonlinear uncertain differential equation has a unique solution for each given initial state. Then, it is attractive if there exist a positive real number ϵ and η is a strictly positive real number. Assume that for all $t > 0$, we have*

$$< X_s, f(t, X_t)) > \vee < X_t, g(t, X_t)) > \le -\eta|X_t|^2 + \epsilon. \tag{4.1}$$

Then

$$E|X_t|^2 \le |X_0|^2 exp(-\eta t) + \epsilon(1 + K(\gamma)). \tag{4.2}$$

Hence, the set

$$S := \left\{ x \in R \middle| E|X|^2 \le \epsilon(1 + K(\gamma)) \right\}$$

is a global attracting set of the nonlinear uncertain differential equation.

Proof. Suppose that X_t is solution of the Eq (1.1) with the initial value X_0. Then, we have

$$X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dC_s. \tag{4.3}$$

By Definition 2.11, we have

$$
\begin{aligned}
e^{\alpha t}|X_t|^2 &= |X_0|^2 + \int_0^t \alpha e^{\alpha s}|X_s|^2 ds + 2\int_0^t e^{\alpha s} < X_s, f(s,X_s) > ds + 2\int_0^t e^{\alpha s} < X_s, g(s,X_s) > dC_s \\
&\le |X_0|^2 + \int_0^t \alpha e^{\alpha s}|X_s|^2 ds + \int_0^t e^{\alpha s}(-\eta|X_s|^2 + \epsilon)ds + K(\gamma)\int_0^t e^{\alpha s}(-\eta|X_s|^2 + \epsilon)ds \\
&\le |X_0|^2 + \alpha\int_0^t e^{\alpha s}|X_s|^2 ds - \eta\int_0^t e^{\alpha s}|X_s|^2 ds + \int_0^t \epsilon e^{\alpha s} - \eta K(\gamma)\int_0^t e^{\alpha s}|X_s|^2 ds + K(\gamma)\int_0^t \epsilon e^{\alpha s} \qquad (4.4)\\
&\le |X_0|^2 + (\alpha - \eta(1 + K(\gamma)))\int_0^t e^{\alpha s}|X_s|^2 ds + \epsilon(1 + K(\gamma))\int_0^t e^{\alpha s} \\
&\le |X_0|^2 + (\alpha - \eta(1 + K(\gamma)))\int_0^t e^{\alpha s}|X_s|^2 ds + \epsilon(1 + K(\gamma))\int_0^t \alpha e^{\alpha s}.
\end{aligned}
$$

Taking the expectation on the both sides of the above inequality, according to Gronwall inequality by Theorem 2.4 in [25], then, we obtain

$$
\begin{aligned}
e^{\alpha t}E|X_t|^2 &\le |X_0|^2 + (\alpha - \eta(1 + K(\gamma)))\int_0^t e^{\alpha s}E|X_s|^2 ds + \epsilon(1 + K(\gamma))\int_0^t \alpha e^{\alpha s}ds \\
&\le |X_0|^2 exp(\int_0^t (\alpha - \eta(1 + K(\gamma))))ds + \epsilon(1 + K(\gamma))\int_0^t \alpha e^{\alpha s}ds \qquad (4.5)\\
&\le |X_0|^2 exp(\alpha - \eta(1 + K(\gamma)))t + \epsilon(1 + K(\gamma))e^{\alpha t}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
E|X_t|^2 &\le |X_0|^2 exp(\alpha - \eta(1 + K(\gamma)))t + \epsilon(1 + K(\gamma)) \\
&\le |X_0|^2 exp(-\eta(1 + K(\gamma))t) + \epsilon(1 + K(\gamma)) \qquad (4.6)\\
&\le |X_0|^2 exp(-\eta t) + \epsilon(1 + K(\gamma)).
\end{aligned}
$$

Hence, we deduce that S is a global attracting set of the nonlinear uncertain differential equations. The proof is complete. □

5. Some examples

Example 5.1. *To illustrate further the effectiveness of the obtained result, we consider nonlinear uncertain differential equation*

$$dX_t = -\alpha e^t X_t dt + (\int_{-1}^0 e^{-2s}X_{t+s}ds)dt + (\int_{-1}^0 e^{-s}X_{t+s}ds)dC_t, \qquad t \ge 0, \qquad (5.1)$$

where $\alpha > 0$ stands for a parameter and C_t is the one-dimensional canonical process.

Chearly, in nonlinear uncertain differential Eq (5.1),

$$f_0(t, X_t) = -\alpha e^t X_t, f_1(t, X_t) = \int_{-1}^0 e^{-2s}X_{t+s}ds, g(t, X_t) = \int_{-1}^0 e^{-s}X_{t+s}ds.$$

Then,

$$< X_t - Y_t, f_0(t, X_t) - f_0(t, Y_t)) >\le -\alpha|X_t - Y_t|^2, \qquad t \ge 0, \ x, y \in \mathbb{R},$$

$$|f_1(t,X_t)-f_1(t,Y_t)| \leq \int_{-1}^{0} e^{-2s}|X_{t+s}-Y_{t+s}|ds, \qquad t\geq 0,\ x,y\in\mathbb{R},$$

and

$$|g(t,X_t)-g(t,Y_t| \leq \int_{-1}^{0} e^{-s}|X_{t+s}-Y_{t+s}|ds, \qquad t\geq 0,\ x,y\in\mathbb{R},$$

$$< X_t-Y_t, f_1(t,X_t)-f_1(t,Y_t)) >\leq (-\alpha+\frac{1}{2}\int_{-1}^{0} e^{-2s}ds)|X_t-Y_t|^2+\frac{1}{2}\int_{-1}^{0} e^{-2s}|X_{t+s}-Y_{t+s}|^2ds, \tag{5.2}$$

and

$$< X_t-Y_t, g(t,X_t)-g(t,Y_t)) >\leq (\frac{1}{2}\int_{-1}^{0} e^{-s}ds)|X_t-Y_t|^2+\frac{1}{2}\int_{-1}^{0} e^{-s}|X_{t+s}-Y_{t+s}|^2ds, \tag{5.3}$$

for all $t\in\mathbb{R}_+,\ x,y\in\mathbb{R}$. *By view of (5.2) and (5.3), we have*

$$\begin{aligned}
&< X_t-Y_t, f_1(t,X_t)-f_1(t,Y_t)) > + < X_t-Y_t, g(t,X_t)-g(t,Y_t)) >\\
&\leq (-\alpha+\frac{1}{2}\int_{-1}^{0} e^{-2s}ds)|X_t-Y_t|^2+\frac{1}{2}\int_{-1}^{0} e^{-2s}|X_{t+s}-Y_{t+s}|^2ds\\
&+(\frac{1}{2}\int_{-1}^{0} e^{-s}ds)|X_t-Y_t|^2+\frac{1}{2}\int_{-1}^{0} e^{-s}|X_{t+s}-Y_{t+s}|^2ds\\
&\leq (-\alpha+\frac{1}{2}\int_{-1}^{0} e^{-2s}ds+\frac{1}{2}\int_{-1}^{0} e^{-s}ds)|X_t-Y_t|^2+\frac{1}{2}\int_{-1}^{0} e^{-2s}|X_{t+s}-Y_{t+s}|^2ds\\
&+\frac{1}{2}\int_{-1}^{0} e^{-s}|X_{t+s}-Y_{t+s}|^2ds.
\end{aligned}$$

So,we deduce that the zero solution of (5.1) is exponentially stable in mean square if

$$\alpha > \frac{e^2}{2}+e-\frac{3}{2}.$$

Example 5.2. *Consider the following nonlinear uncertain differential equation*

$$dX_t = (-2e^{-t}X_t - X_t + \int_{-1}^{0} e^{-t+s}X_{t+s}ds)dt + (a\int_{-1}^{0} e^{-t+s}X_{t+s}ds)dC_t \tag{5.4}$$

for $t\geq t_0\geq 0$, *where* $a\in\mathbb{R}$ *is a parameter and* C_t *a one-dimensional Liu process.*

Obviously, (5.6) is from (1.1) where

$$f(t,X_t) = -2e^{-t}X_t - X_t + \int_{-1}^{0} e^{-t+s}X_{t+s}ds,$$

$$g(t,X_t) = a\int_{-1}^{0} e^{-t+s}X_{t+s}ds,$$

for $t\geq 0$. *Clearly, we have for* $t\geq 0$ *that*

$$\begin{aligned}
< X_t, f(t,X_t) > &= \left\langle X_t, (-2e^{-t}-1)X_t + \int_{-1}^{0} e^{-t+s}X_{t+s}ds\right\rangle\\
&\leq (-2e^{-t}-1)|X_t|^2 + \frac{1}{2}\int_{-1}^{0} e^{-t+s}X_{t+s}ds|X_t|^2 + \frac{1}{2}\int_{-1}^{0} e^{-t+s}X_{t+s}ds|X_{t+s}|^2\\
&= (-2e^{-t}-1+\frac{1}{2}e^{-t}-\frac{1}{2}e^{-t-1})|X_t|^2 + \frac{1}{2}\int_{-1}^{0} e^{-t+s}X_{t+s}|X_{t+s}|^2ds
\end{aligned}$$

and

$$< X_t, g(t, X_t) > \leq (\frac{a}{2}e^{-t} - \frac{a}{2}e^{-t-1})|X_t|^2 + \frac{a}{2}\int_{-1}^{0} e^{-t+s}X_{t+s}|X_{t+s}|^2 ds.$$

Then, for any $\beta \geq 0$ *we have*

$$\begin{aligned} 2 < X_t, f(t, X_t) > + 2 < X_t, g(t, X_t) > \\ &\leq [-2 + (3 + \frac{a+1}{e})e^{-t}]|X_t|^2 + (a+1)\int_{-1}^{0} e^{-t+s}X_{t+s}|X_{t+s}|^2 ds + \beta \\ &\leq (1 + \frac{a+1}{e})|X_t|^2 + (a+1)\int_{-1}^{0} e^{s}X_{t+s}|X_{t+s}|^2 ds + \beta. \end{aligned}$$

Thus, there exist a global attracting set of (5.4) if $a < \frac{-3e-1}{1+e}$.

6. Conclusions

Uncertain differential equation is a type of differential equation driven by canonical Liu process. In this paper, we proposed a sufficient condition for the nonlinear uncertain differential equation being exponentially stable in mean square. Besides, we introduced the nonlinear uncertain differential equation have a global attracting set if it satisfies a sufficient condition. Furthermore, some examples are given to illustrate the stability and global attracting sets.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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