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Research article

Logarithmic Bergman-type space and a sum of product-type operators

Yan-fu Xue[1], **Zhi-jie jiang**[1,2,*], **Hui-ling Jin**[1] **and Xiao-feng Peng**[1]

[1] School of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, China

[2] South Sichuan Center for Applied Mathematics, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, China

* **Correspondence:** Email: matjzj@126.com.

Abstract: One of the aims of the present paper is to obtain some properties about logarithmic Bergman-type space on the unit ball. As some applications, the bounded and compact operators $\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{i=0}^m M_{u_i} C_\varphi \mathfrak{R}^i$ from logarithmic Bergman-type space to weighted-type space on the unit ball are completely characterized.

Keywords: product-type operator; boundedness; compactness; logarithmic Bergman-type space; weighted-type space
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1. Introduction

Let $\mathbb{C}$ denote the complex plane and $\mathbb{C}^n$ the n-dimensional complex Euclidean space with an inner product defined as $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$. Let $B(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}$ be the open ball of $\mathbb{C}^n$. In particular, the open unit ball is defined as $\mathbb{B} = B(0, 1)$.

Let $H(\mathbb{B})$ denote the set of all holomorphic functions on $\mathbb{B}$ and $S(\mathbb{B})$ the set of all holomorphic self-mappings of $\mathbb{B}$. For given $\varphi \in S(\mathbb{B})$ and $u \in H(\mathbb{B})$, the weighted composition operator on or between some subspaces of $H(\mathbb{B})$ is defined by

$$W_{u,\varphi} f(z) = u(z) f(\varphi(z)).$$

If $u \equiv 1$, then $W_{u,\varphi}$ is reduced to the composition operator usually denoted by C_φ. If $\varphi(z) = z$, then $W_{u,\varphi}$ is reduced to the multiplication operator usually denoted by M_u. Since $W_{u,\varphi} = M_u \cdot C_\varphi$, $W_{u,\varphi}$ can be regarded as the product of M_u and C_φ.

If $n = 1$, $\mathbb{B}$ becomes the open unit disk in $\mathbb{C}$ usually denoted by $\mathbb{D}$. Let D^m be the mth differentiation operator on $H(\mathbb{D})$, that is,

$$D^m f(z) = f^{(m)}(z),$$

where $f^{(0)} = f$. D^1 denotes the classical differentiation operator denoted by D. As expected, there has been some considerable interest in investigating products of differentiation and other related operators. For example, the most common products DC_φ and $C_\varphi D$ were extensively studied in [1, 10–13, 23, 25, 26], and the products

$$M_uC_\varphi D,\ C_\varphi M_u D,\ M_u DC_\varphi,\ C_\varphi DM_u,\ DM_uC_\varphi,\ DC_\varphi M_u \tag{1.1}$$

were also extensively studied in [14, 18, 22, 27]. Following the study of the operators in (1.1), people naturally extend to study the operators (see [5, 6, 30])

$$M_uC_\varphi D^m,\ C_\varphi M_u D^m,\ M_u D^m C_\varphi,\ C_\varphi D^m M_u,\ D^m M_u C_\varphi,\ D^m C_\varphi M_u.$$

Other examples of products involving differentiation operators can be found in [7, 8, 19, 32] and the related references.

As studying on the unit disk becomes more mature, people begin to become interested in exploring related properties on the unit ball. One method for extending the differentiation operator to $\mathbb{C}^n$ is the radial derivative operator

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

Naturally, replacing D by $\Re$ in (1.1), we obtain the following operators

$$M_uC_\varphi\Re,\ C_\varphi M_u\Re,\ M_u\Re C_\varphi,\ C_\varphi\Re M_u,\ \Re M_uC_\varphi,\ \Re C_\varphi M_u. \tag{1.2}$$

Recently, these operators have been studied in [31]. Other operators involving radial derivative operators have been studied in [21, 33, 34].

Interestingly, the radial derivative operator can be defined iteratively, namely, $\Re^m f$ can be defined as $\Re^m f = \Re(\Re^{m-1} f)$. Similarly, using the radial derivative operator can yield the related operators

$$M_uC_\varphi\Re^m,\ C_\varphi M_u\Re^m,\ M_u\Re^m C_\varphi,\ C_\varphi\Re^m M_u,\ \Re^m M_uC_\varphi,\ \Re^m C_\varphi M_u. \tag{1.3}$$

Clearly, the operators in (1.3) are more complex than those in (1.2). Since $C_\varphi M_u\Re^m = M_{u\circ\varphi}C_\varphi\Re^m$, the operator $M_uC_\varphi\Re^m$ can be regarded as the simplest one in (1.3) which was first studied and denoted as $\Re^m_{u,\varphi}$ in [24]. Recently, it has been studied again because people need to obtain more properties about spaces to characterize its properties (see [29]).

To reconsider the operator $C_\varphi\Re^m M_u$, people find the fact

$$C_\varphi\Re^m M_u = \sum_{i=0}^{m} C^i_m \Re^i_{(\Re^{m-i}u)\circ\varphi,\varphi}. \tag{1.4}$$

Motivated by (1.4), people directly studied the sum operator (see [2, 28])

$$\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{i=0}^{m} M_{u_i} C_\varphi \mathfrak{R}^i,$$

where $u_i \in H(\mathbb{B})$, $i = \overline{0,m}$, and $\varphi \in S(\mathbb{B})$. Particularly, if we set $u_0 \equiv \cdots \equiv u_{m-1} \equiv 0$ and $u_m = u$, then $\mathfrak{S}^m_{\vec{u},\varphi} = M_u C_\varphi \mathfrak{R}^m$; if we set $u_0 \equiv \cdots \equiv u_{m-1} \equiv 0$ and $u_m = u \circ \varphi$, then $\mathfrak{S}^m_{\vec{u},\varphi} = C_\varphi M_u \mathfrak{R}^m$. In [28], Stević et al. studied the operators $\mathfrak{S}^m_{\vec{u},\varphi}$ from Hardy spaces to weighted-type spaces on the unit ball and obtained the following results.

Theorem A. *Let $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, and μ a weight function on $\mathbb{B}$. Then, the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H^p \to H^\infty_\mu$ is bounded and*

$$\sup_{z\in\mathbb{B}} \mu(z)|u_j(\varphi(z))||\varphi(z)| < +\infty, \quad j = \overline{1,m}, \tag{1.5}$$

if and only if

$$I_0 = \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}}} < +\infty$$

and

$$I_j = \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}} < +\infty, \quad j = \overline{1,m}.$$

Theorem B. *Let $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, and μ a weight function on $\mathbb{B}$. Then, the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H^p \to H^\infty_\mu$ is compact if and only if it is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}}} = 0$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u_j(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}} = 0, \quad j = \overline{1,m}.$$

It must be mentioned that we find that the necessity of Theorem A requires (1.5) to hold. Inspired by [2, 28], here we use a new method and technique without (1.5) to study the sum operator $\mathfrak{S}^m_{\vec{u},\varphi}$ from logarithmic Bergman-type space to weighted-type space on the unit ball. To this end, we need to introduce the well-known Bell polynomial (see [3])

$$B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) = \sum \frac{m!}{\prod_{i=1}^{m-k-1} j_i!} \prod_{i=1}^{m-k-1} \left(\frac{x_i}{i!}\right)^{j_i},$$

where all non-negative integer sequences j_1, j_2, $\ldots$, j_{m-k+1} satisfy

$$\sum_{i=1}^{m-k+1} j_i = k \quad \text{and} \quad \sum_{i=1}^{m-k+1} i j_i = m.$$

In particular, when $k = 0$, one can get $B_{0,0} = 1$ and $B_{m,0} = 0$ for any $m \in \mathbb{N}$. When $k = 1$, one can get $B_{i,1} = x_i$. When $m = k = i$, $B_{i,i} = x_1^i$ holds.

2. Preliminaries

In this section, we need to introduce logarithmic Bergman-type space and weighted-type space. Here, a bounded positive continuous function on $\mathbb{B}$ is called a weight. For a weight μ, the weighted-type space H_μ^∞ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < +\infty.$$

With the norm $\|\cdot\|_{H_\mu^\infty}$, H_μ^∞ becomes a Banach space. In particular, if $\mu(z) = (1-|z|^2)^\sigma$ $(\sigma > 0)$, the space H_μ^∞ is called classical weighted-type space usually denoted by H_σ^∞. If $\mu \equiv 1$, then space H_μ^∞ becomes the bounded holomorphic function space usually denoted by H^∞.

Next, we need to present the logarithmic Bergman-type space on $\mathbb{B}$ (see [4] for the unit disk case). Let dv be the standardized Lebesgue measure on $\mathbb{B}$. The logarithmic Bergman-type space $A^p_{w_{\gamma,\delta}}$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|^p_{A^p_{w_{\gamma,\delta}}} = \int_{\mathbb{B}} |f(z)|^p w_{\gamma,\delta}(z) dv(z) < +\infty,$$

where $-1 < \gamma < +\infty$, $\delta \le 0$, $0 < p < +\infty$ and $w_{\gamma,\delta}(z)$ is defined by

$$w_{\gamma,\delta}(z) = \Big(\log\frac{1}{|z|}\Big)^\gamma \Big[\log\Big(1 - \frac{1}{\log|z|}\Big)\Big]^\delta.$$

When $p \ge 1$, $A^p_{w_{\gamma,\delta}}$ is a Banach space. While $0 < p < 1$, it is a Fréchet space with the translation invariant metric $\rho(f,g) = \|f-g\|^p_{A^p_{\omega_{\gamma,\delta}}}$.

Let $\varphi \in S(\mathbb{B})$, $0 \le r < 1$, $0 \le \gamma < \infty$, $\delta \le 0$, and $a \in \mathbb{B}\backslash\{\varphi(0)\}$. The generalized counting functions are defined as

$$N_{\varphi,\gamma,\delta}(r,a) = \sum_{z_j(a)\in\varphi^{-1}(a)} w_{\gamma,\delta}\left(\frac{z_j(a)}{r}\right)$$

where $\left|z_j(a)\right| < r$, counting multiplicities, and

$$N_{\varphi,\gamma,\delta}(a) = N_{\varphi,\gamma,\delta}(1,a) = \sum_{z_j(a)\in\varphi^{-1}(a)} w_{\gamma,\delta}\left(z_j(a)\right).$$

If $\varphi \in S(\mathbb{D})$, then the function $N_{\varphi,\gamma,\delta}$ has the integral expression: For $1 \le \gamma < +\infty$ and $\delta \le 0$, there is a positive function $F(t)$ satisfying

$$N_{\varphi,\gamma,\delta}(r,u) = \int_0^r F(t) N_{\varphi,1}(t,u) \mathrm{d}t, \quad r \in (0,1), \quad u \ne \varphi(0).$$

When $\varphi \in S(\mathbb{D})$ and $\delta = 0$, the generalized counting functions become the common counting functions. Namely,

$$N_{\varphi,\gamma}(r,a) = \sum_{z\in\varphi^{-1}(a), |z|<r} \left(\log\frac{r}{|z|}\right)^\gamma,$$

and

$$N_{\varphi,\gamma}(a) = N_{\varphi,\gamma}(1,a) = \sum_{z\in\varphi^{-1}(a)} \left(\log\frac{1}{|z|}\right)^{\gamma}.$$

In [17], Shapiro used the function $N_{\varphi,\gamma}(1,a)$ to characterize the compact composition operators on the weighted Bergman space.

Let X and Y be two topological spaces induced by the translation invariant metrics d_X and d_Y, respectively. A linear operator $T : X \to Y$ is called bounded if there is a positive number K such that

$$d_Y(Tf, 0) \le K d_X(f, 0)$$

for all $f \in X$. The operator $T : X \to Y$ is called compact if it maps bounded sets into relatively compact sets.

In this paper, $j = \overline{k,l}$ is used to represent $j = k, ..., l$, where $k, l \in \mathbb{N}_0$ and $k \le l$. Positive numbers are denoted by C, and they may vary in different situations. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a positive number C such that $a \le Cb$ (resp. $a \ge Cb$). When $a \lesssim b$ and $b \gtrsim a$, we write $a \asymp b$.

3. Logarithmic Bergman-type space

In this section, we obtain some properties on the logarithmic Bergman-type space. First, we have the following point-evaluation estimate for the functions in the space.

Theorem 3.1. *Let $-1 < \gamma < +\infty$, $\delta \le 0$, $0 < p < +\infty$ and $0 < r < 1$. Then, there exists a positive number $C = C(\gamma, \delta, p, r)$ independent of $z \in K = \{z \in \mathbb{B} : |z| > r\}$ and $f \in A^p_{w_{\gamma,\delta}}$ such that*

$$|f(z)| \le \frac{C}{(1-|z|^2)^{\frac{\gamma+n+1}{p}}} \left[\log\left(1 - \frac{1}{\log|z|}\right)\right]^{-\frac{\delta}{p}} \|f\|_{A^p_{w_{\gamma,\delta}}}. \tag{3.1}$$

Proof. Let $z \in \mathbb{B}$. By applying the subharmonicity of the function $|f|^p$ to Euclidean ball $B(z,r)$ and using Lemma 1.23 in [35], we have

$$|f(z)|^p \le \frac{1}{v(B(z,r))} \int_{B(z,r)} |f(w)|^p dv(w) \le \frac{C_{1,r}}{(1-|z|^2)^{n+1}} \int_{B(z,r)} |f(w)|^p dv(w). \tag{3.2}$$

Since $r < |z| < 1$ and $1 - |w|^2 \asymp 1 - |z|^2$, we have

$$\log\frac{1}{|w|} \asymp 1 - |w| \asymp 1 - |z| \asymp \log\frac{1}{|z|} \tag{3.3}$$

and

$$\log\left(1 - \log\frac{1}{|w|}\right) \asymp \log\left(1 - \log\frac{1}{|z|}\right). \tag{3.4}$$

From (3.3) and (3.4), it follows that there is a positive constant $C_{2,r}$ such that $w_{\gamma,\delta}(z) \leq C_{2,r} w_{\gamma,\delta}(w)$ for all $w \in B(z, r)$. From this and (3.2), we have

$$\begin{aligned} |f(z)|^p &\leq \frac{C_{1,r}C_{2,r}}{(1-|z|^2)^{n+1} w_{\gamma,\delta}(z)} \int_{B(z,r)} |f(w)|^p w_{\gamma,\delta}(w) dv(w) \\ &\leq \frac{C_{1,r}C_{2,r}}{(1-|z|^2)^{n+1} w_{\gamma,\delta}(z)} \|f\|^p_{A^p_{w_{\gamma,\delta}}}. \end{aligned} \tag{3.5}$$

From (3.5) and the fact $\log \frac{1}{|z|} \asymp 1 - |z| \asymp 1 - |z|^2$, the following inequality is right with a fixed constant $C_{3,r}$

$$|f(z)|^p \leq \frac{C_{1,r}C_{2,r}C_{3,r}}{(1-|z|^2)^{n+1+\gamma}} \Big[\log\Big(1 - \frac{1}{\log|z|}\Big)\Big]^{-\delta} \|f\|^p_{A^p_{w_{\gamma,\delta}}}.$$

Let $C = \frac{C_{1,r}C_{2,r}C_{3,r}}{p}$. Then the proof is end. □

Theorem 3.2. *Let $m \in \mathbb{N}$, $-1 < \gamma < +\infty$, $\delta \leq 0$, $0 < p < +\infty$ and $0 < r < 1$. Then, there exists a positive constant $C_m = C(\gamma, \delta, p, r, m)$ independent of $z \in K$ and $f \in A^p_{w_{\gamma,\delta}}$ such that*

$$\Big| \frac{\partial^m f(z)}{\partial z_{i_1} \partial z_{i_2} \dots \partial z_{i_m}} \Big| \leq \frac{C_m}{(1-|z|^2)^{\frac{\gamma+n+1}{p}+m}} \Big[\log\Big(1 - \frac{1}{\log|z|}\Big)\Big]^{-\frac{\delta}{p}} \|f\|_{A^p_{w_{\gamma,\delta}}}. \tag{3.6}$$

Proof. First, we prove the case of $m = 1$. By the definition of the gradient and the Cauchy's inequality, we get

$$\Big| \frac{\partial f(z)}{\partial z_i} \Big| \leq |\nabla f(z)| \leq \tilde{C}_1 \frac{\sup_{w \in B(z, q(1-|z|))} |f(w)|}{1 - |z|}, \tag{3.7}$$

where $i = \overline{1, n}$. By using the relations

$$1 - |z| \leq 1 - |z|^2 \leq 2(1 - |z|),$$

$$(1-q)(1-|z|) \leq 1 - |w| \leq (q+1)(1-|z|),$$

and

$$\log\Big(1 - \frac{1}{\log|z|}\Big) \asymp \log\Big(1 - \frac{1}{\log|w|}\Big),$$

we obtain the following formula

$$|f(w)| \leq \frac{\breve{C}_1}{(1-|z|^2)^{\frac{\gamma+n+1}{p}}} \left[\log\left(1 - \frac{1}{\log|z|}\right)\right]^{-\frac{\delta}{p}} \|f\|_{A^p_{w_{\gamma,\delta}}}$$

for any $w \in B(z, q(1-|z|))$. Then,

$$\sup_{w \in B(z, q(1-|z|))} |f(w)| \leq \frac{\breve{C}_1}{(1-|z|^2)^{\frac{\gamma+n+1}{p}}} \left[\log\left(1 - \frac{1}{\log|z|}\right)\right]^{-\frac{\delta}{p}} \|f\|_{A^p_{w_{\gamma,\delta}}}.$$

From (3.1) and (3.2), it follows that

$$\left|\frac{\partial f(z)}{\partial z_i}\right| \le \frac{\hat{C}_1}{(1-|z|^2)^{\frac{\gamma+n+1}{p}+1}}\left[\log\left(1-\frac{1}{\log|z|}\right)\right]^{-\frac{\delta}{p}}\|f\|_{A^p_{w_{\gamma,\delta}}}. \tag{3.8}$$

Hence, the proof is completed for the case of $m = 1$.

We will use the mathematical induction to complete the proof. Assume that (3.6) holds for $m < a$. For convenience, let $g(z) = \frac{\partial^{a-1} f(z)}{\partial z_{i_1}\partial z_{i_2}\ldots\partial z_{i_{a-1}}}$. By applying (3.7) to the function g, we obtain

$$\left|\frac{\partial g(z)}{\partial z_i}\right| \le \tilde{C}_1 \frac{\sup_{w\in B(z,q(1-|z|))}|g(w)|}{1-|z|}. \tag{3.9}$$

According to the assumption, the function g satisfies

$$|g(z)| \le \frac{\hat{C}_{a-1}}{(1-|z|^2)^{\frac{\gamma+n+1}{p}+a-1}}\left[\log\left(1-\frac{1}{\log|z|}\right)\right]^{-\frac{\delta}{p}}\|f\|_{A^p_{w_{\gamma,\delta}}}.$$

By using (3.8), the following formula is also obtained

$$\left|\frac{\partial g(z)}{\partial z_i}\right| \le \frac{\hat{C}_{a}}{(1-|z|^2)^{\frac{\gamma+n+1}{p}+a}}\left[\log\left(1-\frac{1}{\log|z|}\right)\right]^{-\frac{\delta}{p}}\|f\|_{A^p_{w_{\gamma,\delta}}}.$$

This shows that (3.6) holds for $m = a$. The proof is end. □

As an application of Theorems 3.1 and 3.2, we give the estimate in $z = 0$ for the functions in $A^p_{\omega_{\gamma,\delta}}$.

Corollary 3.1. *Let* $-1 < \gamma < +\infty$*,* $\delta \le 0$*,* $0 < p < +\infty$*, and* $0 < r < 2/3$*. Then, for all* $f \in A^p_{w_{\gamma,\delta}}$*, it follows that*

$$|f(0)| \le \frac{C}{(1-r^2)^{\frac{\gamma+n+1}{p}}}\left[\log\left(1-\frac{1}{\log r}\right)\right]^{-\frac{\delta}{p}}\|f\|_{A^p_{w_{\gamma,\delta}}}, \tag{3.10}$$

and

$$\left|\frac{\partial^m f(0)}{\partial z_{l_1}\ldots\partial z_{l_m}}\right| \le C_m(1-r^2)^{\frac{\gamma+n+1}{p}+m}\left[\log\left(1-\frac{1}{\log r}\right)\right]^{-\frac{\delta}{p}}\|f\|_{A^p_{w_{\gamma,\delta}}}, \tag{3.11}$$

where constants C *and* C_m *are defined in Theorems 3.1 and 3.2, respectively.*

Proof. For $f \in A^p_{w_{\gamma,\delta}}$, from Theorem 3.1 and the maximum module theorem, we have

$$|f(0)| \le \max_{|z|=r}|f(z)| \le \frac{C}{(1-r^2)^{\frac{\gamma+n+1}{p}}}\left[\log\left(1-\frac{1}{\log r}\right)\right]^{-\frac{\delta}{p}}\|f\|_{A^p_{w_{\gamma,\delta}}},$$

which implies that (3.10) holds. By using the similar method, we also have that (3.11) holds. □

Next, we give an equivalent norm in $A^p_{w_{\gamma,\delta}}$, which extends Lemma 3.2 in [4] to $\mathbb{B}$.

Theorem 3.3. *Let $r_0 \in [0,1)$. Then, for every $f \in A^p_{w_{\gamma,\delta}}$, it follows that*

$$\|f\|^p_{A^p_{w_{\gamma,\delta}}} \asymp \int_{\mathbb{B}\setminus r_0\mathbb{B}} |f(z)|^p w_{\gamma,\delta}(z)dv(z). \tag{3.12}$$

Proof. If $r_0 = 0$, then it is obvious. So, we assume that $r_0 \in (0,1)$. Integration in polar coordinates, we have

$$\|f\|^p_{A^p_{w_{\gamma,\delta}}} = 2n\int_0^1 w_{\gamma,\delta}(r)r^{2n-1}dr\int_{\mathbb{S}}|f(r\zeta)|^p d\sigma(\zeta).$$

Put

$$A(r) = w_{\gamma,\delta}(r)r^{2n-1} \quad \text{and} \quad M(r,f) = \int_{\mathbb{S}}|f(r\zeta)|^p d\sigma(\zeta).$$

Then it is represented that

$$\|f\|^p_{A^p_{w_{\gamma,\delta}}} \asymp \int_0^{r_0} + \int_{r_0}^1 M(r,f)A(r)dr. \tag{3.13}$$

Since $M(r,f)$ is increasing, $A(r)$ is positive and continuous in r on $(0,1)$ and

$$\lim_{r\to 0} A(r) = \lim_{x\to+\infty} x^\gamma\Big[\log(1+\frac{1}{x})\Big]^\delta e^{-(2n-1)x} = \lim_{x\to+\infty}\frac{x^{\gamma-\delta}}{e^{(2n-1)x}} = 0,$$

that is, there is a constant $\varepsilon > 0$ ($\varepsilon < r_0$) such that $A(r) < A(\varepsilon)$ for $r \in (0,\varepsilon)$. Then we have

$$\begin{aligned}\int_0^{r_0} M(r,f)A(r)dr &\le \frac{2r_0}{1-r_0}\max_{\varepsilon\le r\le r_0} A(r)\int_{r_0}^{\frac{1+r_0}{2}} M(r,f)dr \\ &\le \frac{2r_0}{1-r_0}\frac{\max_{\varepsilon\le r\le r_0}A(r)}{\min_{r_0\le r\le\frac{1+r_0}{2}}A(r)}\int_{r_0}^{\frac{1+r_0}{2}} M(r,f)A(r)dr \\ &\lesssim \int_{r_0}^1 M(r,f)A(r)dr. \end{aligned}\tag{3.14}$$

From (3.13) and (3.14), we obtain the inequality

$$\|f\|^p_{A^p_{w_{\gamma,\delta}}} \lesssim \int_{r_0}^1 M(r,f)A(r)dr.$$

The inequality reverse to this is obvious. The asymptotic relationship (3.12) follows, as desired. □

The following integral estimate is an extension of Lemma 3.4 in [4]. The proof is similar, but we still present it for completeness.

Lemma 3.1. *Let $-1 < \gamma < +\infty$, $\delta \le 0$, $\beta > \gamma - \delta$ and $0 < r < 1$. Then, for each fixed $w \in \mathbb{B}$ with $|w| > r$,*

$$\int_{\mathbb{B}}\frac{\omega_{\gamma,\delta}(z)}{|1-\langle z,w\rangle|^{n+\beta+1}}dv(z) \lesssim \frac{1}{(1-|w|)^{\beta-\gamma}}\left[\log\left(1-\frac{1}{\log|w|}\right)\right]^\delta.$$

Proof. Fix $|w|$ with $|w| > r_0$ $(0 < r_0 < 1)$. It is easy to see that

$$\log\frac{1}{r} \asymp 1 - r \quad \text{for } \; r_0 \le r < 1. \tag{3.15}$$

By applying Theorem 3.3 with

$$f_w(z) = \frac{1}{(1 - \langle z, w\rangle)^{n+\beta+1}}$$

and using (3.15), the formula of integration in polar coordinates gives

$$\begin{aligned}
&\int_{\mathbb{B}} \frac{1}{|1 - \langle z, w\rangle|^{n+\beta+1}} \omega_{\gamma,\delta}(z) dv(z) \\
\lesssim &\int_{r_0}^{1} M(r, f_w)(1 - r)^{\gamma} \left[\log\left(1 - \frac{1}{\log r}\right)\right]^{\delta} r^{2n-1} dr.
\end{aligned} \tag{3.16}$$

By Proposition 1.4.10 in [15], we have

$$M(r, f_w) \asymp \frac{1}{(1 - r^2|w|^2)^{\beta+1}}. \tag{3.17}$$

From (3.16) and (3.17), we have

$$\begin{aligned}
&\int_{\mathbb{B}} \frac{1}{|1 - \langle z, w\rangle|^{\beta+2n}} \omega_{\gamma,\delta}(z) dv(z) \\
\lesssim &\int_{r_0}^{1} \frac{1}{(1 - r^2|w|^2)^{\beta+1}} (1 - r)^{\gamma} \left[\log\left(1 - \frac{1}{\log r}\right)\right]^{\delta} r^{2n-1} dr \\
\lesssim &\int_{r_0}^{1} \frac{1}{(1 - r|w|)^{\beta+1}} (1 - r)^{\gamma} \left[\log\left(1 - \frac{1}{\log r}\right)\right]^{\delta} r^{2n-1} dr \\
\lesssim &\int_{r_0}^{|w|} \frac{1}{(1 - r|w|)^{\beta+1}} (1 - r)^{\gamma} \left[\log\left(1 - \frac{1}{\log r}\right)\right]^{\delta} r^{2n-1} dr \\
&+ \int_{|w|}^{1} \frac{1}{(1 - r|w|)^{\beta+1}} (1 - r)^{\gamma} \left[\log\left(1 - \frac{1}{\log r}\right)\right]^{\delta} r^{2n-1} dr \\
= &I_1 + I_2.
\end{aligned}$$

Since $[\log(1 - \frac{1}{\log r})]^{\delta}$ is decreasing in r on $[|w|, 1]$, we have

$$\begin{aligned}
I_2 = &\int_{|w|}^{1} \frac{1}{(1 - r|w|)^{\beta+1}} (1 - r)^{\gamma} \left[\log\left(1 - \frac{1}{\log r}\right)\right]^{\delta} r^{2n-1} dr \\
\lesssim &\frac{1}{(1 - |w|)^{\beta+1}} \left[\log\left(1 - \frac{1}{\log |w|}\right)\right]^{\delta} \int_{|w|}^{1} (1 - r)^{\gamma} dr \\
\asymp &\frac{1}{(1 - |w|)^{\beta-\gamma}} \left[\log\left(1 - \frac{1}{\log |w|}\right)\right]^{\delta}.
\end{aligned} \tag{3.18}$$

On the other hand, we obtain

$$\begin{aligned} I_1 &= \int_{r_0}^{|w|} \frac{1}{(1-r|w|)^{\beta+1}}(1-r)^\gamma \left[\log\left(1-\frac{1}{\log r}\right)\right]^\delta r^{2n-1}dr \\ &\lesssim \int_{r_0}^{|w|} (1-r)^{\gamma-\beta-1}\left(\log\frac{2}{1-r}\right)^\delta dr. \end{aligned}$$

If $\delta = 0$ and $\beta > \gamma$, then we have

$$I_1(0) \lesssim (1-|w|)^{\gamma-\beta}.$$

If $\delta \neq 0$, then integration by parts gives

$$\begin{aligned} I_1(\delta) =& -\frac{1}{\gamma-\beta}(1-|w|)^{\gamma-\beta}\left(\log\frac{2}{1-|w|}\right)^\delta \\ &+ \frac{1}{\gamma-\beta}(1-r_0)^{\gamma-\beta}\left(\log\frac{2}{1-r_0}\right)^\delta + \frac{\delta}{\gamma-\beta}I_1(\delta-1). \end{aligned}$$

Since $\delta < 0$, $\gamma-\beta < 0$ and

$$\left(\log\frac{2}{1-r}\right)^{\delta-1} \leq \left(\log\frac{2}{1-r}\right)^\delta \quad \text{for } r_0 < r < |w| < 1,$$

we have

$$I_1(\delta) \leq -\frac{1}{\gamma-\beta}(1-|w|)^{\gamma-\beta}\left(\log\frac{2}{1-|w|}\right)^\delta + \frac{\delta}{\gamma-\beta}I_1(\delta)$$

and from this follows

$$I_1(\delta) \lesssim (1-|w|)^{\gamma-\beta}\left(\log\frac{2}{1-|w|}\right)^\delta \asymp (1-|w|)^{\gamma-\beta}\left[\log\left(1-\frac{1}{\log|w|}\right)\right]^\delta$$

provided $\gamma-\beta-\delta < 0$. The proof is finished. □

The following gives an important test function in $A^p_{w_{\gamma,\delta}}$.

Theorem 3.4. *Let $-1 < \gamma < +\infty$, $\delta \leq 0$, $0 < p < +\infty$ and $0 < r < 1$. Then, for each $t \geq 0$ and $w \in \mathbb{B}$ with $|w| > r$, the following function is in $A^p_{w_{\gamma,\delta}}$*

$$f_{w,t}(z) = \left[\log\left(1-\frac{1}{\log|w|}\right)\right]^{-\frac{\delta}{p}} \frac{(1-|w|^2)^{-\frac{\delta}{p}+t+1}}{(1-\langle z,w\rangle)^{\frac{\gamma-\delta+n+1}{p}+t+1}}.$$

Moreover,

$$\sup_{\{w\in\mathbb{B}:|w|>r\}} \|f_{w,t}\|_{A^p_{w_{\gamma,\delta}}} \lesssim 1.$$

Proof. By Lemma 3.1 and a direct calculation, we have

$$\begin{aligned}\|f_{w,t}\|^p_{A^p_{w_{\gamma,\delta}}} &= \int_{\mathbb{B}} \left| \left[\log\left(1-\frac{1}{\log|w|}\right)\right]^{-\frac{\delta}{p}} \frac{(1-|w|^2)^{-\frac{\delta}{p}+t+1}}{(1-\langle z,w\rangle)^{\frac{\gamma-\delta+n+1}{p}+t+1}}\right|^p w_{\gamma,\delta}(z)dA(z)\\ &= (1-|w|^2)^{p(t+1)-\delta}\left[\log\left(1-\frac{1}{\log|w|}\right)\right]^{-\delta}\\ &\quad\times\int_{\mathbb{B}}\frac{1}{|1-\langle z,w\rangle|^{\gamma-\delta+p(t+1)+n+1}}w_{\gamma,\delta}(z)dA(z)\\ &\lesssim 1.\end{aligned}$$

The proof is finished. □

4. Boundedness and compactness of the operator $\mathfrak{S}^m_{\vec{u},\varphi}: A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$

In this section, for simplicity, we define

$$B_{i,j}(\varphi(z)) = B_{i,j}(\varphi(z), \varphi(z), \dots, \varphi(z)).$$

In order to characterize the compactness of the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$, we need the following lemma. It can be proved similar to that in [16], so we omit here.

Lemma 4.1. *Let $-1 < \gamma < +\infty$, $\delta \le 0$, $0 < p < +\infty$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, and $\varphi \in S(\mathbb{B})$. Then, the bounded operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is compact if and only if for every bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $A^p_{w_{\gamma,\delta}}$ such that $f_k \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $k \to \infty$, it follows that*

$$\lim_{k\to\infty} \|\mathfrak{S}^m_{\vec{u},\varphi} f_k\|_{H^\infty_\mu} = 0.$$

The following result was obtained in [24].

Lemma 4.2. *Let $s \ge 0$, $w \in \mathbb{B}$ and*

$$g_{w,s}(z) = \frac{1}{(1-\langle z,w\rangle)^s}, \quad z \in \mathbb{B}.$$

Then,

$$\mathfrak{R}^k g_{w,s}(z) = s\frac{P_k(\langle z,w\rangle)}{(1-\langle z,w\rangle)^{s+k}},$$

where $P_k(w) = s^{k-1}w^k + p^{(k)}_{k-1}(s)w^{k-1} + ... + p^{(k)}_2(s)w^2 + w$, and $p^{(k)}_j(s)$, $j = \overline{2,k-1}$, are nonnegative polynomials for s.

We also need the following result obtained in [20].

Lemma 4.3. *Let $s > 0$, $w \in \mathbb{B}$ and*

$$g_{w,s}(z) = \frac{1}{(1 - \langle z, w\rangle)^s}, \quad z \in \mathbb{B}.$$

Then,

$$\mathfrak{R}^k g_{w,s}(z) = \sum_{t=1}^{k} a_t^{(k)} \Big(\prod_{j=0}^{t-1} (s + j) \Big) \frac{\langle z, w\rangle^t}{(1 - \langle z, w\rangle)^{s+t}},$$

where the sequences $(a_t^{(k)})_{t \in \overline{1,k}}$, $k \in \mathbb{N}$, are defined by the relations

$$a_k^{(k)} = a_1^{(k)} = 1$$

for $k \in \mathbb{N}$ and

$$a_t^{(k)} = t a_t^{(k-1)} + a_{t-1}^{(k-1)}$$

for $2 \le t \le k - 1, k \ge 3$.

The final lemma of this section was obtained in [24].

Lemma 4.4. If $a > 0$, then

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n-1 \\ a(a+1) & (a+1)(a+2) & \cdots & (a+n-1)(a+n) \\ \vdots & \vdots & \cdots & \vdots \\ \prod_{k=0}^{n-2}(a+k) & \prod_{k=0}^{n-2}(a+k+1) & \cdots & \prod_{k=0}^{n-2}(a+k+n-1) \end{vmatrix} = \prod_{k=1}^{n-1} k!.$$

Theorem 4.1. *Let $-1 < \gamma < +\infty$, $\delta \le 0$, $0 < p < +\infty$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, and $\varphi \in S(\mathbb{B})$. Then, the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is bounded if and only if*

$$M_0 := \sup_{z \in \mathbb{B}} \frac{\mu(z)|u_0(z)|}{(1 - |\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}} \Big[\log\Big(1 - \frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} < +\infty \tag{4.1}$$

and

$$M_j := \sup_{z \in \mathbb{B}} \frac{\mu(z)| \sum_{i=j}^{m} u_i(z) B_{i,j}(\varphi(z))|}{(1 - |\varphi(z)|^2)^{\frac{\gamma+n+1}{p}+j}} \Big[\log\Big(1 - \frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} < +\infty \tag{4.2}$$

for $j = \overline{1,m}$.

Moreover, if the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is bounded, then

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_{w_{\gamma,\delta}} \to H^\infty_\mu} \asymp \sum_{j=0}^{m} M_j. \tag{4.3}$$

Proof. Suppose that (4.1) and (4.2) hold. From Theorem 3.1, Theorem 3.2, and some easy calculations, it follows that

$$
\begin{aligned}
&\mu(z)\Big|\sum_{i=0}^{m} u_i(z)\mathfrak{R}^i f(\varphi(z))\Big| \le \mu(z)\sum_{i=0}^{m}\big|u_i(z)\big|\big|\mathfrak{R}^i f(\varphi(z))\big| \\
&= \mu(z)|u_0(z)||f(\varphi(z))| \\
&\quad + \mu(z)\Big|\sum_{i=1}^{m}\sum_{j=1}^{i}\Big(u_i(z)\sum_{l_1=1}^{n}\cdots\sum_{l_j=1}^{n}\Big(\frac{\partial^j f}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_j}}(\varphi(z))\sum_{k_1,\dots,k_j} C^{(i)}_{k_1,\dots,k_j}\prod_{t=1}^{j}\varphi_{l_t}(z)\Big)\Big)\Big| \\
&= \mu(z)|u_0(z)f(\varphi(z))| \\
&\quad + \mu(z)\Big|\sum_{j=1}^{m}\sum_{i=j}^{m}\Big(u_i(z)\sum_{l_1=1}^{n}\cdots\sum_{l_j=1}^{n}\Big(\frac{\partial^j f}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_j}}(\varphi(z))\sum_{k_1,\dots,k_j} C^{(i)}_{k_1,\dots,k_j}\prod_{t=1}^{j}\varphi_{l_t}(z)\Big)\Big)\Big| \\
&\lesssim \frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}}\|f\|_{A^p_{w_{\gamma,\delta}}} \\
&\quad + \sum_{j=1}^{m}\frac{\mu(z)|\sum_{i=j}^{m} u_i(z)B_{i,j}(\varphi(z))|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}+j}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}}\|f\|_{A^p_{w_{\gamma,\delta}}} \\
&= M_0\|f\|_{A^p_{w_{\gamma,\delta}}} + \sum_{j=1}^{m} M_j\|f\|_{A^p_{w_{\gamma,\delta}}}.
\end{aligned} \tag{4.4}
$$

By taking the supremum in inequality (4.4) over the unit ball in the space $A^p_{w_{\gamma,\delta}}$, and using (4.1) and (4.2), we obtain that the operator $\mathfrak{S}^m_{\vec{u},\varphi}: A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is bounded. Moreover, we have

$$
\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_{w_{\gamma,\delta}}\to H^\infty_\mu} \le C\sum_{j=0}^{m} M_j, \tag{4.5}
$$

where C is a positive constant.

Assume that the operator $\mathfrak{S}^m_{\vec{u},\varphi}: A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is bounded. Then there exists a positive constant C such that

$$
\|\mathfrak{S}^m_{\vec{u},\varphi} f\|_{H^\infty_\mu} \le C\|f\|_{A^p_{w_{\gamma,\delta}}} \tag{4.6}
$$

for any $f \in A^p_{w_{\gamma,\delta}}$. First, we can take $f(z) = 1 \in A^p_{w_{\gamma,\delta}}$, then one has that

$$
\sup_{z\in\mathbb{B}} \mu(z)|u_0(z)| < +\infty. \tag{4.7}
$$

Similarly, take $f_k(z) = z_k^j \in A^p_{w_{\gamma,\delta}}$, $k = \overline{1,n}$ and $j = \overline{1,m}$, by (4.7), then

$$
\mu(z)\Big|u_0(z)\varphi_k(z)^j + \sum_{i=j}^{m}\Big(u_i(z)B_{i,j}(\varphi_k(z))\Big)\Big| < +\infty \tag{4.8}
$$

for any $j \in \{1, 2, \ldots, m\}$. Since $\varphi(z) \in \mathbb{B}$, we have $|\varphi(z)| \le 1$. So, one can use the triangle inequality (4.7) and (4.8), the following inequality is true

$$\sup_{z\in\mathbb{B}} \mu(z) \Big| \sum_{i=j}^{m} u_i(z) B_{i,j}(\varphi(z)) \Big| < +\infty. \tag{4.9}$$

Let $w \in \mathbb{B}$ and $d_k = \frac{\gamma+n+1}{p} + k$. For any $j \in \{1, 2, \ldots, m\}$ and constants $c_k = c_k^{(j)}$, $k = \overline{0, m}$, let

$$h_w^{(j)}(z) = \sum_{k=0}^{m} c_k^{(j)} f_{w,k}(z), \tag{4.10}$$

where $f_{w,k}$ is defined in Theorem 3.4. Then, by Theorem 3.4, we have

$$L_j = \sup_{w\in\mathbb{B}} \|h_w^{(j)}\|_{A^p_{w_{\gamma,\delta}}} < +\infty. \tag{4.11}$$

From (4.6), (4.11), and some easy calculations, it follows that

$$\begin{aligned}
& L_j \|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_{w_{\gamma,\delta}} \to H^\infty_\mu} \ge \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(j)}_{\varphi(w)}\|_{H^\infty_\mu} \\
& = \sup_{z\in\mathbb{B}} \mu(z) \Big| \sum_{i=0}^{m} u_0(z) h^{(j)}_{\varphi(w)}(\varphi(z)) \Big| \\
& \ge \mu(w) \Big| u_0(w) h^{(j)}_{\varphi(w)}(\varphi(w)) + \sum_{i=1}^{m} \Big(u_i(w) \mathfrak{R}^i h^{(j)}_{\varphi(w)}(\varphi(w)) \Big) \Big| \\
& = \mu(w) \Big| u_0(w) h^{(j)}_{\varphi(w)}(\varphi(w)) + \sum_{i=1}^{m} u_i(w) \sum_{k=0}^{m} c_k^{(j)} f_{\varphi(w),k}(\varphi(w)) \Big| \\
& = \mu(w) \Big| u_0(w) \frac{c_0 + c_1 + \cdots + c_m}{(1 - |\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}} + \Big\langle \sum_{i=1}^{m} u_i(w) B_{i,1}(\varphi(w)), \varphi(w) \Big\rangle \frac{(d_0 c_0 + \cdots + d_m c_m)}{(1 - |\varphi(w)|^2)^{\frac{\gamma+n+1}{p}+1}} + \cdots \\
& \quad + \Big\langle \sum_{i=j}^{m} u_i(w) B_{i,j}(\varphi(w)), \varphi(w)^j \Big\rangle \frac{(d_0 \cdots d_{j-1} c_0 + \cdots + d_m \cdots d_{m+j-1} c_m)}{(1 - |\varphi(w)|^2)^{\frac{\gamma+n+1}{p}+j}} + \cdots \\
& \quad + \langle u_m(w) B_{m,m}(\varphi(w)), \varphi(w)^m \rangle \frac{(d_0 \cdots d_{m-1} c_0 + \cdots + d_m \cdots d_{2m-1} c_m)}{(1 - |\varphi(w)|^2)^{\frac{\gamma+n+1}{p}+m}} \Big| \Big[\log\Big(1 - \frac{1}{\log|\varphi(w)|}\Big) \Big]^{-\frac{\delta}{p}}.
\end{aligned} \tag{4.12}$$

Since $d_k > 0$, $k = \overline{0, m}$, by Lemma 4.4, we have the following linear equations

$$\begin{pmatrix}
1 & 1 & \cdots & 1 \\
d_0 & d_1 & \cdots & d_m \\
\vdots & \vdots & \ddots & \vdots \\
\prod_{k=0}^{j-1} d_k & \prod_{k=0}^{j-1} d_{k+m} & \cdots & \prod_{k=0}^{j-1} d_{k+m} \\
\vdots & \vdots & \ddots & \vdots \\
\prod_{k=0}^{m-1} d_k & \prod_{k=0}^{m-1} d_{k+m} & \cdots & \prod_{k=0}^{m-1} d_{k+m}
\end{pmatrix}
\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_j \\ \vdots \\ c_m \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.13}$$

From (4.12) and (4.13), we have

$$L_j\|\mathfrak{S}^l_{\vec{u},\varphi}\|_{A^p_{w_{\gamma,\delta}}\to H^\infty_\mu} \geq \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|\sum_{i=j}^m u_i(z)B_{i,j}(\varphi(z))||\varphi(z)|^j}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}+j}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}}$$
$$\gtrsim \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|\sum_{i=j}^m u_i(z)B_{i,j}(\varphi(z))|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}+j}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}}. \tag{4.14}$$

On the other hand, from (4.9), we have

$$\sup_{|\varphi(z)|\leq 1/2} \frac{\mu(z)|\sum_{i=j}^m u_i(z)B_{i,j}(\varphi(z))|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}+j}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}}$$
$$\leq \sup_{z\in\mathbb{B}}\Big(\frac{4}{3}\Big)^{\frac{\gamma+n+1}{p}+j}\Big[\log\Big(1-\frac{1}{\log\frac{1}{2}}\Big)\Big]^{-\frac{\delta}{p}}\mu(z)\Big|\sum_{i=j}^m u_i(z)B_{i,j}(\varphi(z))\Big| < +\infty. \tag{4.15}$$

From (4.14) and (4.15), we get that (4.2) holds for $j=\overline{1,m}$.

For constants $c_k = c_k^{(0)}$, $k=\overline{0,m}$, let

$$h_w^{(0)}(z) = \sum_{k=0}^m c_k^{(0)} f_{w,k}(z). \tag{4.16}$$

By Theorem 3.4, we know that $L_0 = \sup_{w\in\mathbb{B}}\|h_w^{(0)}\|_{A^p_{w_{\gamma,\delta}}} < +\infty$. From this, (4.12), (4.13) and Lemma 4.4, we get

$$L_0\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_{w_{\gamma,\delta}}\to H^\infty_\mu} \geq \frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}}.$$

So, we have $M_0 < +\infty$. Moreover, we have

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{A^p_{w_{\gamma,\delta}}\to H^\infty_\mu} \geq \sum_{j=0}^m M_j. \tag{4.17}$$

From (4.5) and (4.17), we obtain (4.3). The proof is completed. □

From Theorem 4.1 and (1.4), we obtain the following result.

Corollary 4.1. *Let $m\in\mathbb{N}$, $u\in H(\mathbb{B})$, $\varphi\in S(\mathbb{B})$ and μ is a weight function on $\mathbb{B}$. Then, the operator $C_\varphi\mathfrak{R}^m M_u : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is bounded if and only if*

$$I_0 := \sup_{z\in\mathbb{B}} \frac{\mu(z)|\mathfrak{R}^m u\circ\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} < +\infty$$

and

$$I_j := \sup_{z\in\mathbb{B}} \frac{\mu(z)|\sum_{i=j}^m \mathfrak{R}^{m-i}u\circ\varphi(z)B_{i,j}(\varphi(z))|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}+j}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} < +\infty$$

for $j = \overline{1, m}$.

Moreover, if the operator $C_\varphi \mathfrak{R}^m M_u : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ *is bounded, then*

$$\|C_\varphi \mathfrak{R}^m M_u\|_{A^p_{w_{\gamma,\delta}} \to H^\infty_\mu} \asymp \sum_{j=0}^{m} I_j.$$

Theorem 4.2. *Let* $-1 < \gamma < +\infty$, $\delta \le 0$, $0 < p < +\infty$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0, m}$, *and* $\varphi \in S(\mathbb{B})$. *Then, the operator* $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ *is compact if and only if the operator* $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ *is bounded,*

$$\lim_{|\varphi(z)| \to 1} \frac{\mu(z)|\sum_{i=j}^m (u_i(z) B_{i,j}(\varphi(z))|}{(1 - |\varphi(z)|^2)^{\frac{\gamma+n+1}{p} + j}} \Big[\log\Big(1 - \frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} = 0 \tag{4.18}$$

for $j = \overline{1, m}$, *and*

$$\lim_{|\varphi(z)| \to 1} \frac{\mu(z)|u_0(z)|}{(1 - |\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}} \Big[\log\Big(1 - \frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} = 0. \tag{4.19}$$

Proof. Assume that the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is compact. It is obvious that the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is bounded.

If $\|\varphi\|_\infty < 1$, then it is clear that (4.18) and (4.19) are true. So, we suppose that $\|\varphi\|_\infty = 1$. Let $\{z_k\}$ be a sequence in $\mathbb{B}$ such that

$$\lim_{k \to 1} |\mu(z_k)| \to 1 \quad \text{and} \quad h^{(j)}_k = h^{(j)}_{\varphi(z_k)},$$

where $h^{(j)}_w$ are defined in (4.10) for a fixed $j \in \{1, 2, \dots, l\}$. Then, it follows that $h^{(j)}_k \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $k \to \infty$. Hence, by Lemma 4.1, we have

$$\lim_{k \to \infty} \|\mathfrak{S}^m_{\vec{u},\varphi} h_k\|_{H^\infty_\mu} = 0.$$

Then, we can find sufficiently large k such that

$$\frac{\mu(z_k)|\sum_{i=j}^m (u_i(z_k) B_{i,j}(\varphi(z_k))|}{(1 - |\varphi(z_k)|^2)^{\frac{\gamma+n+1}{p} + j}} \Big[\log\Big(1 - \frac{1}{\log|\varphi(z_k)|}\Big)\Big]^{-\frac{\delta}{p}} \le L_k \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(j)}_k\|_{H^\infty_\mu}. \tag{4.20}$$

If $k \to \infty$, then (4.20) is true.

Now, we discuss the case of $j = 0$. Let $h^{(0)}_k = h^{(0)}_{\varphi(z_k)}$, where $h^{(0)}_w$ is defined in (4.16). Then, we also have that $\|h^{(0)}_k\|_{A^p_{w_{\gamma,\delta}}} < +\infty$ and $h^{(0)}_k \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $k \to \infty$. Hence, by Lemma 4.1, one has that

$$\lim_{k \to \infty} \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(0)}_k\|_{H^\infty_\mu(\mathbb{B})} = 0. \tag{4.21}$$

Then, by (4.21), we know that (4.18) is true.

Now, assume that $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is bounded, (4.18) and (4.19) are true. One has that

$$\mu(z)|u_0(z)| \le C < +\infty \tag{4.22}$$

and

$$\mu(z)\Big|\sum_{i=j}^{m}(u_i(z)B_{i,j}(\varphi(z)))\Big| \le C < +\infty \tag{4.23}$$

for any $z \in \mathbb{B}$. By (4.18) and (4.19), for arbitrary $\varepsilon > 0$, there is a $r \in (0,1)$, for any $z \in K$ such that

$$\frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} < \varepsilon. \tag{4.24}$$

and

$$\frac{\mu(z)\Big|\sum_{i=j}^{m}(u_i(z)B_{i,j}(\varphi(z)))\Big|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}+j}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} < \varepsilon. \tag{4.25}$$

Assume that $\{f_s\}$ is a sequence such that $\sup_{s\in\mathbb{N}}\|f_s\|_{A^p_{w_{\gamma,\delta}}} \le M < +\infty$ and $f_s \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $s \to \infty$. Then by Theorem 3.1, Theorem 3.2 and (4.22)–(4.25), one has that

$$\begin{aligned}
\|\mathfrak{S}^m_{\vec{u},\varphi} f_s\|_{H^\infty_\mu(\mathbb{B})} &= \sup_{z\in\mathbb{B}}\mu(z)\Big|u_0(z)f(\varphi(z)) + \sum_{i=1}^{m}u_i(z)\mathfrak{R}^i f(\varphi(z))\Big| \\
&= \sup_{z\in K}\mu(z)\Big|u_0(z)f(\varphi(z)) + \sum_{i=1}^{m}u_i(z)\mathfrak{R}^i f(\varphi(z))\Big| \\
&\quad + \sup_{z\in\mathbb{B}\setminus K}\mu(z)\Big|u_0(z)f(\varphi(z)) + \sum_{i=1}^{m}u_i(z)\mathfrak{R}^i f(\varphi(z))\Big| \\
&\lesssim \sup_{z\in K}\frac{\mu(z)|u_0(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}}\|f_s\|_{A^p_{w_{\gamma,\delta}}} \\
&\quad + \sup_{z\in K}\frac{\mu(z)\Big|\sum_{i=j}^{m}(u_i(z)B_{i,j}(\varphi(z)))\Big|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}+j}}\Big[\log\Big(1-\frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}}\|f_s\|_{A^p_{w_{\gamma,\delta}}} \\
&\quad + \sup_{z\in\mathbb{B}\setminus K}\mu(z)|u_0(z)||f_s(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{B}\setminus K}\sum_{j=1}^{m}\mu(z)\Big|\sum_{i=j}^{m}(u_i(z)B_{i,j}(\varphi(z)))\Big|\max_{\{l_1,l_2,\dots,l_j\}}\Big|\frac{\partial^j f_s}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_j}}(\varphi(z))\Big| \\
&\le M\varepsilon + C\sup_{|w|\le\delta}\sum_{j=0}^{m}\max_{\{l_1,l_2,\dots,l_j\}}\Big|\frac{\partial^j f_s}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_j}}(w)\Big|.
\end{aligned}\tag{4.26}$$

Since $f_s \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $s \to \infty$. By Cauchy's estimates, we also have that $\frac{\partial^j f_s}{\partial z_{l_1}\partial z_{l_2}\cdots\partial z_{l_j}} \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $s \to \infty$. From this and using the fact that $\{w \in \mathbb{B} : |w| \le \delta\}$ is a compact subset of $\mathbb{B}$, by letting $s \to \infty$ in inequality (4.26), one get that

$$\limsup_{s\to\infty}\|\mathfrak{S}^m_{\vec{u},\varphi} f_s\|_{H^\infty_\mu} \lesssim \varepsilon.$$

Since ε is an arbitrary positive number, it follows that

$$\lim_{s\to\infty} \|\mathfrak{S}^m_{\vec{u},\varphi} f_s\|_{H^\infty_\mu} = 0.$$

By Lemma 4.1, the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is compact. □

As before, we also have the following result.

Corollary 4.2. *Let $m \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and μ is a weight function on $\mathbb{B}$. Then, the operators $C_\varphi \mathfrak{R}^m M_u : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is compact if and only if the operator $C_\varphi \mathfrak{R}^m M_u : A^p_{w_{\gamma,\delta}} \to H^\infty_\mu$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\mathfrak{R}^m u \circ \varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{\gamma+n+1}{p}}} \Big[\log\Big(1 - \frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} = 0$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum_{i=j}^m (\mathfrak{R}^{m-i} u \circ \varphi(z) B_{i,j}(\varphi(z))\right|}{\Big(1-|\varphi(z)|^2\Big)^{\frac{\gamma+n+1}{p}+j}} \Big[\log(1 - \frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta}{p}} = 0$$

for $j = \overline{1, m}$.

5. Conclusions

In this paper, we study and obtain some properties about the logarithmic Bergman-type space on the unit ball. As some applications, we completely characterized the boundedness and compactness of the operator

$$\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{i=0}^m M_{u_i} C_\varphi \mathfrak{R}^i$$

from the logarithmic Bergman-type space to the weighted-type space on the unit ball. Here, one thing should be pointed out is that we use a new method and technique to characterize the boundedness of such operators without the condition (1.5), which perhaps is the special flavour in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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