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Research article

Existence result for the critical Klein-Gordon-Maxwell system involving steep potential well

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Abstract: The Klein-Gordon-Maxwell system has received great attention in the community of mathematical physics. Under a special superlinear condition on the nonlinear term, the existence of solution for the critical Klein-Gordon-Maxwell system with a steep potential well has been solved. In this paper, under two general superlinear conditions, we obtain the existence of ground state solution for the critical Klein-Gordon-Maxwell system with a steep potential well. The general superlinear conditions bring challenge in proving the boundedness of Cerami sequence, which is a key step in the proof of the existence. To solve this, we construct a Pohožaev identity and adopt some analytical techniques. Our results extend the previous results in the literature.

Keywords: Klein-Gordon-Maxwell system; critical growth; steep potential well; ground state solution; variational methods
Mathematics Subject Classification: 35J20, 35J62

1. Introduction

The Klein-Gordon-Maxwell (KGM) system [1, 2] describes the solitary waves for the nonlinear Klein-Gordon equation interacting with an electromagnetic field. It is widely employed in many mathematical physics contexts, such as quantum electrodynamics, semiconductor theory, nonlinear optics and plasma physics. In this paper, we will investigate the existence of solution for two cases of critical Klein-Gordon-Maxwell system with steep potential well.

To review the existing work, we start with the following KGM system:

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

where $m, \omega > 0$ are constants, standing for the particle's mass and the phase, respectively; $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ defines the electric potential and the nonlinear term of the particle's field u, respectively. The nonlinear term f describes the interaction between unknown particles or external nonlinear perturbations. If f does not explicitly depend on x, but only on u, we say f is autonomous.

When $f(x,u) = |u|^{q-2}u$, Benci and Fortunato [1] proved that (1.1) has infinitely many radially symmetric solutions if $4 < q < 6$ and $|m| > |\omega|$; D'Aprile and Mugnai [3] proved that (1.1) has no solution if $q \geq 6$ or $q \leq 2$; further, Azzollini and Pomponio [4] studied the existence of a ground state solution for (1.1) when one of the following conditions holds:
(i) $4 \leq q < 6$ and $m > \omega$;
(ii) $2 < q < 4$ and $m\sqrt{q-2} > \omega\sqrt{6-q}$.

Cassani [5] also considered (1.1), but with $f(x,u) = \mu|u|^{q-2}u + u^5$, where $\mu > 0$ is a constant, $4 \leq q < 6$. Cassani stated that a suffiently large μ plays an important role in ensuring the existence of solutions.

Some researchers studied the following critical KGM system with non-constant potentials. Carrião et al. [6] proved the existence of positive ground state solutions for the following critical KGM system:

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = \mu|u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\mu > 0$, $2 < q < 6$, $2^* = 6$, V is periodical in x and satisfies the following conditions:
(V1) *$V \in C(\mathbb{R}^3,\mathbb{R})$, $V(x) \geq V_0 > 0$, $x \in \mathbb{R}^3$, where $V_0 > \frac{2(4-q)}{q-2}\omega^2$ if $2 < q < 4$.*

Tang et al. [7] considered a similar system, but with a more general nonlinear term:

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = \mu f(u) + u^5, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

Suppose that V satisfies:
(V2) *$V \in C(\mathbb{R}^3,\mathbb{R})$, $V(x) \geq V_0 > 0$ and $V(x)$ is 1-periodic in x_1, x_2 and x_3;*
and f satisfies the following conditions:
(F1) *$f \in C(\mathbb{R},\mathbb{R})$, $f(t) = o(|t|)$ as $t \to 0$ and $f(t) = o(|t|^5)$ as $|t| \to \infty$;*
(F2) *there exists a constant $\theta \in (2,6)$ such that $f(t)t \geq \theta F(t)$ for $t \in \mathbb{R}$ and $F(t) \geq 0$ for $t \geq 0$, where $F(t) := \int_0^t f(s)ds$;*
(F3) *if $\theta \in (2,4]$ in (F2), then $F(t) \geq \alpha t^s$ for some $\alpha > 0$ and $s \in (2,4]$ and all $t \geq 1$.*

Then the system of (1.3) has a ground state solution provided one of the following conditions holds:
(i) $4 < \theta < 6$ *and* $\mu > 0$;
(ii) $\theta = 4$ *and* $\mu \geq \mu_0$;
(iii) $2 < \theta < 4, 0 < \omega < \frac{2\sqrt{2(\theta-2)V_0}}{4-\theta}$ *and $\mu \geq \mu_0$, where μ_0 is a positive constant determined by V, α and s.*

Liu et al. [8] considered the following KGM system:

$$\begin{cases} -\Delta u + (\lambda A(x)+1)u - (2\omega+\phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\lambda A(x) + 1$ is the steep potential well, and A satisfies the following conditions (originally introduced in [9]):
(A1) *$A \in C(\mathbb{R}^3,\mathbb{R})$, $A(x) \geq 0$ for all $x \in \mathbb{R}^3$ and $\Omega := A^{-1}(0)$ is nonempty;*
(A2) *There exists $M_0 > 0$ such that* $\text{meas}\left\{x \in \mathbb{R}^3 : A(x) \leq M_0\right\} < +\infty$.

If A satisfies (A1), (A2) and f satisfies
(f0) *there exists* $\theta \in (2, \infty)$ *such that* $0 < \theta F(x,t) \le f(x,t)t$, $\forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}$, $F(x,t) := \int_0^t f(s)ds$, Liu et al. [8] proved that (1.4) has a ground state solution if one of the following conditions holds:
(i) $\theta \in [4, \infty)$*; or*
(ii) $\theta \in (2,4)$ *and* $\omega \in (0, 2\sqrt{2(\theta-2)}/(4-\theta))$.

Zhang et al. [10] further studied (1.4) with an autonomous nonlinear term, e.g., $f(x,u) = f(u)$. By imposing an additional condition on A:
(A3) $\langle \nabla A(x), x\rangle \ge 0$ *for all* $x \in \mathbb{R}^3$ *and there exists* $\vartheta \in [0,1)$ *such that* $\langle \nabla A(x), x\rangle \le \frac{\vartheta}{2\lambda|x|^2}$ *for all* $x \in \mathbb{R}^3 \backslash \{0\}$.

Zhang et al. [10] extended the range of ω for which the ground state solution of (1.4) exists.

Zhang [11] also investigated a special case of (1.4), e.g., $f(x,u) = \mu f(u) + u^5$. The special system is

$$\begin{cases} -\Delta u + (\lambda A(x) + 1)u - (2\omega + \phi)\phi u = \mu f(u) + u^5, & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $\lambda, \mu > 0$ are positive parameters, $\omega > 0$ is a constant and $f : \mathbb{R} \to \mathbb{R}$ is a superlinear function, satisfying:
(f1) $f \in C(\mathbb{R}^+, \mathbb{R})$, $f(t) \ge 0$ *and* $\lim_{t\to 0^+} \frac{f(t)}{t} = \lim_{t\to+\infty} \frac{f(t)}{t^5} = 0$*;*
(f2) $f(t)t - 4F(t) \ge 0$*, where* $F(t) := \int_0^t f(s)ds$*. Moreover, there exist* $\theta \in (4,6)$*,* $D > 0$ *and* $\rho > 0$ *such that* $F(t) \ge \frac{D}{\rho^\theta} t^\theta$ *for* $t \ge \rho$*.*

If A satisfies (A1) and (A2), while f satisfies (f1) and (f2), Zhang [11] concluded that (1.5) has a ground state solution. For more results about KGM equctions, we refer to [12–15]. For more results about elliptic equations with critical growth or various potentials, we refer to [16–18].

Comparing the conditions (f1)–(f2) (used in Zhang [11]) with (F1)–(F3) (used in Tang [7]), we find that, (f1) is essentially the same as (F1) for the positive ground state solutions; while the inequality $f(t)t - 4F(t) \ge 0$ in (f2) is a special case of the inequality $f(t)t \ge \theta F(t)$ with $\theta = 4$ in (F2), and the exponentially increasing property in (f2) is stronger than that in (F3). Therefore, we apply the conditions (F1)–(F3) rather than (f1)–(f2) to the nonlinear function f in (1.5) and its following extension:

$$\begin{cases} -\Delta u + (\lambda A(x) + 1)u - (2\omega + \phi)\phi u = K(x)f(u) + u^5, & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

and give results about the existence of ground state solution. Note that, in (1.6), we use a potential K, instead of the constant μ in (1.5). Assume $K : \mathbb{R}^3 \to \mathbb{R}$ satisfies the following assumptions:
(K1) $K \in C(\mathbb{R}^3, \mathbb{R})$, $0 < K_0 := \inf_{x\in\mathbb{R}^3} K(x) \le K(x) \le K_\infty := \sup_{x\in\mathbb{R}^3} K(x) < \infty$ *for all* $x \in \mathbb{R}^3$*;*
(K2) $\langle \nabla K(x), x\rangle \le 0$ *for all* $x \in \mathbb{R}^3$.
For A, instead of (A3), we apply the following weaker condition:
(A3') $\langle \nabla A(x), x\rangle \ge 0$ *for all* $x \in \mathbb{R}^3$.

Our first result is as follows.

Theorem 1.1. *Assume that A satisfies* (A1), (A2) *and* (A3'), f *satisfies* (F1)–(F3), *then problem (1.5) has a ground state solution when one of the following conditions holds:*
(i) $4 < \theta < 6$ *and* $\mu > 0$*;*
(ii) $3 \le \theta \le 4$ *and* $\mu \ge \mu_0$*;*
(iii) $2 < \theta < 3, 0 < \omega < \frac{\sqrt{(\theta-2)(4-\theta)}}{3-\theta}$ *and* $\mu \ge \mu_0$*,*

where μ_0 is a positive constant determined by A, α and s.

Theorem 1.2. *Assume that* A *satisfies* (A1), (A2) *and* (A3'), f *satisfies* (F1)–(F3) *and* K *satisfies* (K1)–(K2), *then problem (1.6) has a ground state solution when one of the following conditions holds:*
(i) $4 < \theta < 6$;
(ii) $3 \leq \theta \leq 4$;
(iii) $2 < \theta < 3$ *and* $0 < \omega < \frac{\sqrt{(\theta-2)(4-\theta)}}{3-\theta}$.

Remark 1.3. After we replaced (f2) with (F2) and (F3), the following function still satisfies (F2) and (F3) but not (f2):

$$f(u) = u^{2.5}.$$

Remark 1.4. Let $K(x) = \frac{a}{1+|x|^{\xi}} + 1$ with $a, \xi > 0$. It is easy to verify that K satisfies (K1) and (K2). There seems no results dealing with the nonlinearity which is combined with K and f in the existing results by using Pohožaev identity since it is difficult to prove the compactness of functional associated with problem (1.6).

The paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorem 1.1. At last, the proof of Theorem 1.2 is given in Section 4.

2. Preliminaries

In this section, we will introduce some notations and lemmas which will be used in the proof of our theorems.

C_i's denote positive constants used in different place; $B_R(x) = \{y \in \mathbb{R}^3 : |y - x| < R\}$ denotes a neighborhood (with radius R) of the point x; $H^1(\mathbb{R}^3)$ is the usual Sobolev space equipped with the inner product and norm $(u, v) = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + uv)dx$, $\|u\| = (u, u)^{1/2}$, $\forall u, v \in H^1(\mathbb{R}^3)$; $\|u\|_s = (\int_{\mathbb{R}^3} |u|^s dx)^{1/s}(1 \leq s < \infty)$ is the standard norm of the Lebesgue space $L^s(\mathbb{R}^3)$. Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^{\infty}(\mathbb{R}^3)$ endow with the norm $\|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 dx$. Define E as the variational space

$$E := \left\{u \in H^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} A(x)u^2 dx < +\infty\right\},$$

equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + (A(x) + 1)u^2\right] dx, \quad \forall\, u \in E, \tag{2.1}$$

where A satisfies (A1) and (A2). By (A1), (A2) and the Poincaré inequality, $E \hookrightarrow H^1(\mathbb{R}^3)$ is continuous for any $s \in [2, 6]$, and there exists $\gamma_s > 0$ such that

$$\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{1/s} \leq \gamma_s \|u\|, \quad \forall\, u \in E.$$

Lemma 2.1. ([3]) *For any* $u \in H^1\left(\mathbb{R}^3\right)$, *there exists a unique* $\phi = \phi_u \in D^{1,2}\left(\mathbb{R}^3\right)$ *which satisfies:*

$$-\Delta\phi + u^2\phi = -\omega u^2.$$

Moreover, the map $I : u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in D^{1,2}\left(\mathbb{R}^3\right)$ is continuously differentiable, and

(i) $-\omega \le \phi_u \le 0$ *on the set* $\left\{x \in \mathbb{R}^3 | u(x) \neq 0\right\}$;
(ii) $\|\phi_u\|_{D^{1,2}} \le C\|u\|_E^2$ *and* $\int_{\mathbb{R}^3} |\phi_u| u^2 dx \le C\|u\|_{12/5}^4 \le C\|u\|_E^4$.

Lemma 2.2. ([4]) *If* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$*, then passing to a subsequence,* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$*. As a consequence* $I'(u_n) \to I'(u)$ *in the sense of distributions.*

3. Proof of Theorem 1.1

Similar to the argument in [3], we define the functional $J_\lambda(u) : E \to \mathbb{R}$ associated with (1.5) by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + (\lambda A(x) + 1)u^2 - \omega\phi_u u^2\right] dx - \int_{\mathbb{R}^3} \mu F(u) dx - \frac{1}{6}\int_{\mathbb{R}^3} u^6 dx, \quad \forall u \in E. \tag{3.1}$$

By Lemmas 2.1 and 2.2, $J_\lambda \in C^1(E, \mathbb{R})$, we have

$$\langle J_\lambda'(u), v\rangle = \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + (\lambda A(x) + 1)uv] dx - \int_{\mathbb{R}^3} [(2\omega + \phi_u)\phi_u u + \mu f(u) + u^5] v dx, \ \forall\, u, v \in E. \tag{3.2}$$

Let $\mathcal{M} := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : J_\lambda'(u) = 0\}$ be the collection of the critical points of J_λ. Any critical point u of J_λ satisfies the following Pohožaev identity [3]:

$$\begin{aligned} P_\lambda(u) = &\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \left[3(\lambda A(x) + 1) + \lambda\langle\nabla A(x), x\rangle - 5\omega\phi_u - 2\phi_u^2\right] u^2 dx \\ &- \int_{\mathbb{R}^3} (6\mu F(u) + u^6) dx = 0. \end{aligned} \tag{3.3}$$

Let

$$\begin{aligned} I_\lambda(u) &= \langle J_\lambda'(u), u\rangle - \frac{1}{2}P_\lambda(u) \\ &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^3} [(\lambda A(x) + 1) + \lambda\langle\nabla A(x), x\rangle - \omega\phi_u] u^2 dx \\ &\quad + \mu\int_{\mathbb{R}^3} (3F(u) - f(u)u) dx - \frac{1}{2}\int_{\mathbb{R}^3} u^6 dx. \end{aligned} \tag{3.4}$$

Then, $I_\lambda(u) = 0$, $\forall\, u \in \mathcal{M}$.

Lemma 3.1. *Assume that* (F1)–(F2) *and* (A1)–(A2) *hold. Then there exist a sequence* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *satisfying:*

$$J_\lambda(u_n) \to c_\lambda > 0, \quad \|J_\lambda'(u_n)\|(1 + \|u_n\|) \to 0 \quad \text{and} \quad I_\lambda(u_n) \to 0, \tag{3.5}$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)), \quad \Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } J_\lambda(\gamma(1)) < 0\}. \tag{3.6}$$

Proof. The proof of Lemma 3.1 is similar to [19, Theorem 2.2], so we omit it here. □

In the following, we first estimate the upper bound of critical value c_λ and prove the mountain pass geometry of energy function J_λ by using Brézis-Nirenberg techique [19]. Then we give the proof of Theorem 1.1.

Lemma 3.2. *Assume that* (F1)–(F3) *and* (A1)–(A2) *hold. If one of the following conditions holds:*

(i) $4 < \theta < 6$ *and* $\mu > 0$*;*

(ii) $2 < \theta \leq 4$ *and* $\mu \geq \mu_0$,

Then, we have $c_\lambda < \frac{1}{3}S^{3/2}$, where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and μ_0 is a positive constant given in (3.28).

Proof. From (A1), there exists $e \in E\backslash\{0\}$ such that the support of e is in Ω. Hence, we get

$$c_\lambda \leq \max_{t\geq 0} J_\lambda(te) \leq \max_{t\geq 0}\left(\frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla e|^2 + e^2)dx - \frac{t^2}{2}\int_{\mathbb{R}^3}\omega\phi_{te}e^2dx - \mu\int_{\mathbb{R}^3}F(te)dx - \frac{t^6}{6}\int_{\mathbb{R}^3}e^6dx\right). \quad (3.7)$$

If $4 < \theta < 6$ in (F2), then (F1) and (F2) imply that, there exist constants $\beta_1, \beta_2 > 0$ such that

$$F(u) \geq \beta_1|u|^\theta - \beta_2u^2, \quad \forall u \in \mathbb{R}. \quad (3.8)$$

By (3.7), (3.8) and $B_R(0) \subset B_{2R}(0) \subset \Omega$, and the same deduction in [6, Lemma 3.5], we can easily prove the inequality $c_\lambda < \frac{1}{3}S^{3/2}$ if the condition (i) is true.

Next, we consider the condition (ii). For any $\epsilon > 0$, define the following extremal function

$$w_\epsilon(x) = \frac{(3\epsilon)^{1/4}}{(\epsilon + |x|^2)^{1/2}}, \quad x \in \mathbb{R}^3, \quad (3.9)$$

for the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

Let

$$S = \frac{\|\nabla w_\epsilon\|_2^2}{\|w_\epsilon\|_6^2} = \left(\frac{\sqrt{3}}{2}\right)^{2/3}\pi^2, \quad (3.10)$$

$$\phi(r) = \begin{cases} 1, & r \in [0,1), \\ 2 - r, & r \in [1,2), \quad \in C([0,\infty),\ [0,1]), \\ 0, & r \in [2,+\infty). \end{cases} \quad (3.11)$$

and $e_\epsilon(x) = \phi(|x|)w_\epsilon(x)$.

By simple computation, we have

$$\|\nabla e_\epsilon\|_2^2 \leq S^{3/2} + 4\pi(4\ln 2 - 1)\sqrt{3}\epsilon^{1/2} := S^{3/2} + C_1\epsilon^{1/2}, \quad (3.12)$$

$$\|e_\epsilon\|_2^2 \leq 4\pi(3\epsilon)^{1/2}\int_0^2 \frac{r^2}{\epsilon + r^2}dr \leq 8\sqrt{3}\pi\epsilon^{1/2}, \quad (3.13)$$

$$4 \cdot 3^{s/4}\pi\epsilon^{(6-s)/4}\int_0^1 \frac{r^2}{(1+r^2)^{s/2}}dr \geq (\frac{3}{4})^{(s-4)/4}\pi\epsilon^{(6-s)/4} := C_2\epsilon^{(6-s)/4} \quad (3.14)$$

and

$$\|e_\epsilon\|_6^6 \geq S^{3/2} - 9\sqrt{3}\pi\epsilon^{3/2}, \quad \forall 0 < \epsilon < 1. \quad (3.15)$$

By (F2), (3.12)–(3.15) and Lemma 2.1, we have

$$\begin{aligned}
J_\lambda(te_\epsilon) &\leq \frac{t^2}{2}\left[\|\nabla e_\epsilon\|_2^2 + \left(\lambda \sup_{x\in\mathbb{R}^3} A(x) + 1\right)\|e_\epsilon\|_2^2 + \omega^2\|e_\epsilon\|_2^2\right] - \mu\int_{\mathbb{R}^3} F(te_\epsilon)dx - \frac{t^6}{6}\|e_\epsilon\|_6^6 \\
&\leq \frac{t^2}{2}\left[S^{3/2} + \left(C_1 + 8\sqrt{3}\pi\left(\lambda\sup_{x\in\mathbb{R}^3} A(x) + 1 + \omega^2\right)\right)\epsilon^{1/2}\right] - \frac{t^6}{6}(S^{3/2} - 9\sqrt{3}\pi\epsilon^{3/2}) \\
&\quad - 4\pi\mu\epsilon^{3/2}\int_0^1 r^2 F\left(\frac{3^{1/4}t\epsilon^{-1/4}}{(1+r^2)^{1/2}}\right)dr \qquad (3.16)\\
&= \left(\frac{t^2}{2} - \frac{t^6}{6}\right)S^{3/2} + \frac{1}{2}\left(C_1 + 8\sqrt{3}\pi\left(\lambda\sup_{x\in\mathbb{R}^3} A(x) + 1 + \omega^2\right)\right)\epsilon^{1/2}t^2 + \frac{3\sqrt{3}\pi}{2}\epsilon^{3/2}t^6 \\
&\quad - 4\pi\mu\epsilon^{3/2}\int_0^1 r^2 F\left(\frac{3^{1/4}t\epsilon^{-1/4}}{(1+r^2)^{1/2}}\right)dr.
\end{aligned}$$

Set

$$\bar{A} := \lambda\sup_{x\in\mathbb{R}^3} A(x) + 1 + \omega^2, \quad \epsilon_0 := \frac{S^3}{(C_1 + 8\sqrt{3}\pi\bar{A} + 18\sqrt{3}\pi)^2}. \qquad (3.17)$$

Then (3.10) implies that $0 < \epsilon_0 < 1$.

Define the following function:

$$\varphi(t) := \frac{t^2}{2}\left[S^{3/2} + (C_1 + 8\sqrt{3}\pi\bar{A})\epsilon^{1/2}\right] - \frac{t^6}{6}\left(S^{3/2} - 9\sqrt{3}\pi\epsilon^{3/2}\right), \quad \forall t \geq 0. \qquad (3.18)$$

For any $0 < \epsilon < \epsilon_0$, we can easily prove that $\varphi(t)$ is increasing on $[0, 2^{-1/4}]$ and decreasing on $[2^{1/4}, \infty)$.

To obtain the desired conclusion, we consider the following three cases:
1) $0 \leq t \leq 2^{-1/4}$; (2) $t \geq 2^{1/4}$; (3) $2^{-1/4} < t < 2^{1/4}$.

Case 1: $0 \leq t \leq 2^{-1/4}$. By (3.16) and (3.18), we have

$$\begin{aligned}
J_\lambda(te_\epsilon) &< \varphi(2^{-1/4}) \\
&= \frac{1}{3}S^{3/2} - \left(\frac{1}{3} - \frac{5}{12\sqrt{2}}\right)S^{3/2} + \frac{C_1 + 8\sqrt{3}\pi\bar{A}}{2\sqrt{2}}\epsilon^{1/2} + \frac{3\sqrt{3}\pi}{4\sqrt{2}}\epsilon^{3/2} \qquad (3.19)\\
&:= \frac{1}{3}S^{3/2} - \left(\frac{1}{6} - \frac{5}{24\sqrt{2}}\right)S^{3/2} + h_1(\epsilon), \quad \forall\, 0 < \epsilon \leq \epsilon_0.
\end{aligned}$$

Set

$$\epsilon_1 := \frac{2\left(\frac{1}{3} - \frac{5}{12\sqrt{2}}\right)^2 S^3}{\left(C_1 + 8\sqrt{3}\pi\bar{A} + \frac{3\sqrt{3}\pi}{2}\right)^2}. \qquad (3.20)$$

We can easily prove that $h_1(\epsilon) \leq 0$ for all $0 < \epsilon \leq \min\{\epsilon_0, \epsilon_1\}$. This result, together with (3.19), imply that

$$\sup_{t\in[0,2^{-1/4}]} J_\lambda(te_\epsilon) < \frac{1}{3}S^{3/2}, \quad \forall\, 0 < \epsilon \leq \min\{\epsilon_0, \epsilon_1\}. \qquad (3.21)$$

Case 2: $t \geq 2^{1/4}$. By (3.16) and (3.18), we have

$$\begin{aligned} J_\lambda(te_\epsilon) &< \varphi(2^{1/4}) \\ &= \frac{1}{3}S^{3/2} - \left(\frac{1}{3} - \frac{\sqrt{2}}{6}\right)S^{3/2} + \frac{\sqrt{2}(C_1 + 8\sqrt{3}\pi\bar{A})}{2}\epsilon^{1/2} + 3\sqrt{6}\pi\epsilon^{3/2} \\ &:= \frac{1}{3}S^{3/2} - \left(\frac{1}{6} - \frac{\sqrt{2}}{12}\right)S^{3/2} + h_2(\epsilon), \quad \forall\, 0 < \epsilon \leq \epsilon_0. \end{aligned} \tag{3.22}$$

Set

$$\epsilon_2 := \frac{2\left(\frac{1}{6} - \frac{\sqrt{2}}{12}\right)^2 S^3}{\left(C_1 + 8\sqrt{3}\pi\bar{A} + 6\sqrt{3}\pi\right)^2}. \tag{3.23}$$

We can easily prove that $h_2(\epsilon) \leq 0$ for all $0 < \epsilon \leq \min\{\epsilon_0, \epsilon_2\}$. This result, together with (3.22), implies

$$\sup_{t\in[2^{1/4},\infty)} J_\lambda(te_\epsilon) < \frac{1}{3}S^{3/2}, \quad \forall\, 0 < \epsilon \leq \min\{\epsilon_0, \epsilon_2\}. \tag{3.24}$$

Case 3: $2^{-1/4} < t < 2^{1/4}$. From (F3), (3.14) and (3.16), we obtain

$$\begin{aligned} J_\lambda(te_\epsilon) &< \frac{1}{3}S^{3/2} + \frac{\sqrt{2}(C_1 + 8\sqrt{3}\pi\bar{A})}{2}\epsilon^{1/2} + 3\sqrt{6}\pi\epsilon^{3/2} \\ &\quad - 4\cdot\left(\frac{3}{2}\right)^{s/4}\pi\alpha\mu\epsilon^{(6-s)/4}\int_0^1 \frac{r^2}{(1+r^2)^{s/2}}dr \\ &< \frac{1}{3}S^{3/2} + \frac{\sqrt{2}(C_1 + 8\sqrt{3}\pi\bar{A}) + 6\sqrt{6}\pi}{2}\epsilon^{1/2} - \frac{C_2\mu\alpha}{2^{s/4}}\epsilon^{(6-s)/4}, \\ &\quad \forall\, 2^{-1/4} < t < 2^{1/4},\ 0 < \epsilon \leq \frac{3}{8}. \end{aligned} \tag{3.25}$$

Set

$$\epsilon_3 := \frac{2\left(\frac{1}{3} - \frac{5}{12\sqrt{2}}\right)^2 S^3}{\left(C_1 + 8\sqrt{3}\pi\bar{A} + 18\sqrt{3}\pi\right)^2}. \tag{3.26}$$

Hence, we have $0 < \epsilon_3 < \min\{3/8, \epsilon_0, \epsilon_1, \epsilon_2\} < 1$. By (3.25), we have

$$\sup_{t\in(2^{-1/4},2^{1/4})} J_\lambda(te_{\epsilon_3}) < \frac{1}{3}S^{3/2} \tag{3.27}$$

provided

$$\mu \geq \mu_0 := \frac{\sqrt{2}(C_1 + 8\sqrt{3}\pi\bar{A}) + 6\sqrt{6}\pi}{C_2\alpha}(2\epsilon_3)^{(s-4)/4}, \tag{3.28}$$

where positive constants C_1, C_2 and ϵ_3 are given by (3.12), (3.14) and (3.26). From (3.21), (3.24), (3.27) and the definition of c_λ, we have

$$c_\lambda \leq \sup_{t\geq 0} J_\lambda(te_{\epsilon_3}) < \frac{1}{3}S^{3/2}, \quad \forall\, \mu \geq \mu_0. \tag{3.29}$$

□

Now we prove that the Cerami sequence obtained in Lemma 3.1 is bounded.

Lemma 3.3. *Suppose that* (F2), (A1) *and* (A3') *hold. Then any Cerami sequence* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *given in (3.5) is bounded.*

Proof. If $\theta \in [4,6)$, by (3.1), (3.2), (3.5) and Lemma 2.1, we have

$$\begin{aligned} c_\lambda + o(1) &= J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3}\left[|\nabla u_n|^2 + (\lambda A(x)+1)u_n^2\right]dx \\ &\quad + \int_{\mathbb{R}^3}\left[\left(\frac{2}{\theta}-\frac{1}{2}\right)\omega\phi_{u_n} + \frac{1}{\theta}\phi_{u_n}^2\right]u_n^2dx + \mu\int_{\mathbb{R}^3}\left[\frac{1}{\theta}f(u_n)u_n - F(u_n)\right]dx \\ &\quad + \left(\frac{1}{\theta}-\frac{1}{6}\right)\int_{\mathbb{R}^3}u_n^6dx \\ &\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3}\left[|\nabla u_n|^2 + (\lambda A(x)+1)u_n^2\right]dx. \end{aligned} \tag{3.30}$$

From (2.1), (3.30) and $\lambda \geq 1$, we conclude that $\{u_n\}$ is bounded in E when $\theta \in [4,6)$ in (F2).

If $\theta \in (2,4)$, by (3.1), (3.2), (3.4) and (3.5), we have

$$\begin{aligned} c_\lambda + o(1) &= J_\lambda(u_n) + \frac{\theta-4}{6-\theta}\langle J'_\lambda(u_n), u_n\rangle + \frac{2-\theta}{6-\theta}I_\lambda(u_n) \\ &= \frac{\theta-2}{6-\theta}\int_{\mathbb{R}^3}(\lambda A(x)+1)u_n^2dx + \frac{2\mu}{6-\theta}\int_{\mathbb{R}^3}\left[f(u_n)u_n - \theta F(u_n)\right]dx \\ &\quad + \frac{1}{6-\theta}\int_{\mathbb{R}^3}\left[2(3-\theta)\omega\phi_{u_n} + (4-\theta)\phi_{u_n}^2\right]u_n^2dx + \frac{1}{3}\int_{\mathbb{R}^3}u_n^6dx \\ &\quad + \frac{\theta-2}{2(6-\theta)}\int_{\mathbb{R}^3}\lambda\langle\nabla A(x), x\rangle u_n^2dx. \end{aligned} \tag{3.31}$$

To prove the boundedness of $\int_{\mathbb{R}^3}(\lambda A(x)+1)u_n^2dx$, we consider the following two cases:
Case 1: $3 \leq \theta < 4$. From Lemma 2.1, we have

$$2(3-\theta)\omega s + (4-\theta)s^2 \geq 0, \quad \forall -\omega \leq s \leq 0. \tag{3.32}$$

Then by (3.31), (3.32), (F2) and (A3'), we conclude that $\int_{\mathbb{R}^3}(\lambda A(x)+1)u_n^2dx$ is bounded.
Case 2: $\theta \in (2,3)$ and $\omega \in (0, \sqrt{(\theta-2)(4-\theta)}/(3-\theta))$. A direct computation leads to the following inequalities:

$$\begin{aligned} 2(3-\theta)\omega s + (4-\theta)s^2 + \theta - 2 &\geq -\frac{(3-\theta)^2}{4-\theta}\omega^2 + \theta - 2 \\ &= \frac{(\theta-2)(4-\theta)-(3-\theta)^2\omega^2}{4-\theta} > 0 \quad \text{for } s \in [-\omega, 0]. \end{aligned} \tag{3.33}$$

Then from (3.31), (3.33), (F2) and (A3'), we have

$$c_\lambda + o(1) \geq \min\left\{\frac{\theta-2}{6-\theta}, \frac{(\theta-2)(4-\theta)-(3-\theta)^2\omega^2}{(4-\theta)(6-\theta)}\right\}\int_{\mathbb{R}^3}(\lambda A(x)+1)u_n^2dx. \tag{3.34}$$

Hence, $\int_{\mathbb{R}^3}(\lambda A(x)+1)u_n^2dx$ is bounded when $\theta \in (2,4)$.

From the derivation above, we can also conclude that $\int_{\mathbb{R}^3} u_n^2dx$ is bounded. Therefore, from Lemma 2.1, there exists a constant $C_3 > 0$ such that

$$0 \le -\int_{\mathbb{R}^3} \omega\phi_{u_n} u_n^2 dx \le C_3. \tag{3.35}$$

From (3.1), (3.2), (3.5) and (3.35), we get

$$\begin{aligned} c_\lambda + o(1) &= J_\lambda(u_n) - \frac{1}{\theta}\langle J'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3}\left[|\nabla u_n|^2 + (\lambda A(x)+1)u_n^2\right]dx \\ &\quad + \int_{\mathbb{R}^3}\left[\left(\frac{2}{\theta}-\frac{1}{2}\right)\omega\phi_{u_n} + \frac{1}{\theta}\phi_{u_n}^2\right]u_n^2dx + \mu\int_{\mathbb{R}^3}\left[\frac{1}{\theta}f(u_n)u_n - F(u_n)\right]dx \\ &\quad + \left(\frac{1}{\theta}-\frac{1}{6}\right)\int_{\mathbb{R}^3} u_n^6 dx \\ &\ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3}\left[|\nabla u_n|^2 + (\lambda A(x)+1)u_n^2\right]dx - \left(\frac{2}{\theta}-\frac{1}{2}\right)C_3. \end{aligned} \tag{3.36}$$

It follows from (3.36) that $\int_{\mathbb{R}^3}|\nabla u_n|^2dx$ is bounded. Thus, $\{u_n\}$ is bounded in E when $\theta \in (2,4)$. □

Remark 3.4. *In the case of $\theta \in (2,4)$, it is difficult to prove the boundedness of Cerami sequence by using the Pohožaev identity due to the presence of the non-constant potential A in (1.5). Thus, we solve this difficulty by applying the condition* (A3') *and some analytical skills.*

Using Lemmas 3.1–3.3, we can prove Theorem 1.1.

Proof of Theorem 1.1. First, we prove that $\mathcal{M} \neq \emptyset$. From Lemma 3.1, there exist a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ satisfying (3.5). By Lemma 3.3, we have $\{u_n\}$ is bounded in E. Assume that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|u_n|^2dx = 0 \tag{3.37}$$

hold. By Lions' concentration compactness principle [20, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for all $s \in (2,6)$. From (F1), (3.2) and Lemma 2.1, one has

$$\begin{aligned} o(1) = \langle J'_\lambda(u_n), u_n\rangle &\ge \int_{\mathbb{R}^3}\left[|\nabla u_n|^2 + (\lambda A(x)+1)u_n^2\right]dx - \mu\int_{\mathbb{R}^3} f(u_n)u_n dx - \int_{\mathbb{R}^3} u_n^6dx + o(1) \\ &\ge \int_{\mathbb{R}^3}\left[|\nabla u_n|^2 + (\lambda A(x)+1)u_n^2\right]dx - \int_{\mathbb{R}^3} u_n^6 dx + o(1). \end{aligned} \tag{3.38}$$

Then, from Lemma 3.3 and $c_\lambda > 0$, we assume that there exists a constant $l > 0$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\left[|\nabla u_n|^2 + (\lambda A(x)+1)u_n^2\right]dx = l, \tag{3.39}$$

together with (3.38), we have $\lim_{n\to\infty}\int_{\mathbb{R}^3} u_n^6dx \ge l$. By (3.38), (3.39) and Sobolev inequality, we get that

$$l \ge S^{3/2}. \tag{3.40}$$

Then, by (3.1), (3.2), (3.5), (3.40), Lemmas 2.1 and 3.2, we have

$$\frac{1}{3}S^{3/2} > c_\lambda = \lim_{n\to\infty}\left[J_\lambda(u_n) - \frac{1}{6}\langle J'_\lambda(u_n), u_n\rangle\right] \geq \frac{1}{3}l \geq \frac{1}{3}S^{3/2}, \tag{3.41}$$

This is a contradiction. Then there exists $\kappa > 0$ such that $\lim_{n\to\infty}\sup_{y_n\in\mathbb{R}^3}\int_{B_1(y_n)}|u_n|^2dx = \kappa > 0$. For $R > 0$, set

$$A_R := \{x \in \mathbb{R}^3 : |x| \geq R, A(x) \geq M_0\}, \quad D_R := \{x \in \mathbb{R}^3 : |x| \geq R, A(x) < M_0\}.$$

From (A2) and Lemma 3.3, there exists a constant $C_4 > 0$ such that

$$\int_{A_R} u_n^2 dx \leq \frac{1}{1+\lambda M_0}\int_{\mathbb{R}^3}(1+\lambda A(x))u_n^2 dx \leq \frac{C_4}{1+\lambda M_0},$$

when $n \to \infty$. Taking $\lambda \geq \frac{4C_4}{\kappa M_0}$, we have

$$\int_{A_R} u_n^2 dx \leq \frac{C_4}{1+\lambda M_0} \leq \frac{C_4}{\lambda M_0} \leq \frac{\kappa}{4}, \tag{3.42}$$

uniformly in n. Combining Hölder and Sobolev inequality, we obtain

$$\int_{D_R} u_n^2 dx = \left(\int_{D_R}|u_n|^s dx\right)^{2/s}\left(\int_{D_R} 1dx\right)^{(s-2)/s} \leq C_4(\text{meas}(D_R))^{(s-2)/s},$$

where $s \in (2, 6]$. Since $\text{meas}(D_R) \to 0$ as $R \to 0$ by (A2), for any $\delta > 0$ and taking $\delta < \frac{\kappa}{4}$, there exists R^* such that $R > R^*$, and we obtain

$$\int_{D_R} u_n^2 dx \leq \frac{\kappa}{4}, \tag{3.43}$$

uniformly in n. From $u_n \to 0$ in L^s_{loc} with $s \in (2, 6)$, (3.42) and (3.43), we have

$$\begin{aligned}\kappa = \lim_{n\to\infty}\sup_{y_n\in\mathbb{R}^3}\int_{B_1(y_n)}|u_n|^2dx &\leq \lim_{n\to\infty}\int_{\mathbb{R}_3}|u_n|^2dx \\ &= \lim_{n\to\infty}\left(\int_{B_R}|u_n|^2dx + \int_{B_R^c}|u_n|^2dx\right) \\ &= \lim_{n\to\infty}\left(\int_{A_R}u_n^2dx + \int_{D_R}u_n^2dx\right) \leq \frac{\kappa}{2}.\end{aligned}$$

This is a contradiction. Hence, there exists $u \in E\backslash\{0\}$ such that $J'_\lambda(u) = 0$. Then $u \in \mathcal{M}$.

Set $m := \inf_{u\in\mathcal{M}} J_\lambda(u)$. Let $\{\breve{u}_n\} \subset \mathcal{M}$ be such that $J_\lambda(\breve{u}_n) \to m$ and $J'_\lambda(\breve{u}_n) = 0$ as $n \to \infty$. Passing to a subsequence, we have $\breve{u}_n \rightharpoonup \breve{u}$ in E, $\breve{u}_n \to \breve{u}$ in $L^s_{\text{loc}}(\mathbb{R}^3)$, $s \in [1, 6)$ and $\breve{u}_n \to \breve{u}$ a.e. in $\mathbb{R}^3$. Moreover, by Lemma 2.2 and a standard argument, we have $J'_\lambda(\breve{u}) = I_\lambda(\breve{u}) = 0$ and $J_\lambda(\breve{u}) \geq m$.

Now, we should prove $J_\lambda(\breve{u}) = m$. If $\theta \in [4,6)$, from (3.1), (3.2), Lemma 2.1 and Fatou's Lemma, we have

$$\begin{aligned} m &= \lim_{n\to\infty}\left[J_\lambda(\breve{u}_n) - \frac{1}{\theta}\langle J'_\lambda(\breve{u}_n), \breve{u}_n\rangle\right] \\ &= \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3}\left[|\nabla \breve{u}_n|^2 dx + (\lambda A(x)+1)\breve{u}_n^2\right]dx + \left(\frac{1}{\theta}-\frac{1}{6}\right)\int_{\mathbb{R}^3}\breve{u}_n^6 dx \right.\\ &\quad \left. + \int_{\mathbb{R}^3}\left[\frac{1}{\theta}\phi_{\breve{u}_n}^2 - \left(\frac{1}{2}-\frac{2}{\theta}\right)\omega\phi_{\breve{u}_n}\right]\breve{u}_n^2 dx + \mu\int_{\mathbb{R}^3}\left[\frac{1}{\theta}f(\breve{u}_n)\breve{u}_n - F(\breve{u}_n)\right]dx\right\} \\ &\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3}\left[|\nabla \breve{u}|^2 dx + (\lambda A(x)+1)\breve{u}^2\right]dx + \left(\frac{1}{\theta}-\frac{1}{6}\right)\int_{\mathbb{R}^3}\breve{u}^6 dx \\ &\quad + \int_{\mathbb{R}^3}\left[\frac{1}{\theta}\phi_{\breve{u}}^2 - \left(\frac{1}{2}-\frac{2}{\theta}\right)\omega\phi_{\breve{u}}\right]\breve{u}^2 dx + \mu\int_{\mathbb{R}^3}\left[\frac{1}{\theta}f(\breve{u})\breve{u} - F(\breve{u})\right]dx \\ &= J_\lambda(\breve{u}) - \langle J'_\lambda(\breve{u}), \breve{u}\rangle \geq m. \end{aligned} \tag{3.44}$$

If $\theta \in (2,4)$, thanks to (3.1), (3.2), (3.4), (3.32), (3.33), Lemma 2.1 and Fatou's Lemma, we obtain

$$\begin{aligned} m &= \lim_{n\to\infty}\left[J_\lambda(\breve{u}_n) + \frac{\theta-4}{6-\theta}\langle J'_\lambda(\breve{u}_n), \breve{u}_n\rangle + \frac{2-\theta}{6-\theta}I_\lambda(\breve{u}_n)\right] \\ &= \lim_{n\to\infty}\left\{\frac{\theta-2}{6-\theta}\int_{\mathbb{R}^3}(\lambda A(x)+1)\breve{u}_n^2 dx + \frac{\theta-2}{2(6-\theta)}\int_{\mathbb{R}^3}\lambda\langle\nabla A(x), x\rangle\breve{u}_n^2 dx + \frac{1}{3}\int_{\mathbb{R}^3}\breve{u}_n^6 dx \right.\\ &\quad \left. + \frac{1}{6-\theta}\int_{\mathbb{R}^3}\left[2(3-\theta)\omega\phi_{\breve{u}_n} + (4-\theta)\phi_{\breve{u}_n}^2\right]\breve{u}_n^2 dx + \frac{2\mu}{6-\theta}\int_{\mathbb{R}^3}\left[f(\breve{u}_n)\breve{u}_n - \theta F(\breve{u}_n)\right]dx\right\} \\ &\geq \frac{\theta-2}{6-\theta}\int_{\mathbb{R}^3}(\lambda A(x)+1)\breve{u}^2 dx + \frac{\theta-2}{2(6-\theta)}\int_{\mathbb{R}^3}\lambda\langle\nabla A(x), x\rangle\breve{u}^2 dx + \frac{1}{3}\int_{\mathbb{R}^3}\breve{u}^6 dx \\ &\quad + \frac{1}{6-\theta}\int_{\mathbb{R}^3}\left[2(3-\theta)\omega\phi_{\breve{u}} + (4-\theta)\phi_{\breve{u}}^2\right]\breve{u}^2 dx + \frac{2\mu}{6-\theta}\int_{\mathbb{R}^3}\left[f(\breve{u})\breve{u} - \theta F(\breve{u})\right]dx \\ &= J_\lambda(\breve{u}) + \frac{\theta-4}{6-\theta}\langle J'_\lambda(\breve{u}), \breve{u}\rangle + \frac{2-\theta}{6-\theta}I_\lambda(\breve{u}) \geq m. \end{aligned} \tag{3.45}$$

By (3.44) and (3.45), we have $J_\lambda(\breve{u}) = m = \inf_{\mathcal{M}} J_\lambda(u)$. That is to say $\breve{u}$ is a ground state solution for (1.5). □

4. Proof of Theorem 1.2

The proof is easy for case (i). For the proof of cases (ii) and (iii), due to the non-constant potential K in the nonlinearity of (1.6), it is challenging in obtaining the boundness of the Cerami sequence through the Pohožaev identity. We tackle this by applying the condition (K2) and some analytical techniques.

Similar to the proof of Theorem 1.1, we define the function Φ_λ corresponding to (1.6) as

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla u|^2 + (\lambda A(x)+1)u^2 - \omega\phi_u u^2\right]dx - \int_{\mathbb{R}^3}K(x)F(u)dx - \frac{1}{6}\int_{\mathbb{R}^3}u^6 dx, \quad \forall u \in E. \tag{4.1}$$

Then,

$$\langle \Phi'_\lambda(u), v\rangle = \int_{\mathbb{R}^3}[\nabla u\cdot\nabla v + (\lambda A(x)+1)uv]dx - \int_{\mathbb{R}^3}[(2\omega+\phi_u)\phi_u u + K(x)f(u) + u^5]v dx, \ \forall\, u, v \in E. \tag{4.2}$$

Define the set of the critical points as $\breve{\mathcal{M}} := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \Phi'_\lambda(u) = 0\}$. We construct the following Pohožaev equality:

$$\begin{aligned} Q_\lambda(u) = &\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \left[3(\lambda A(x) + 1) + \lambda\langle\nabla A(x), x\rangle - 5\omega\phi_u - 2\phi_u^2\right] u^2 dx \\ &- \int_{\mathbb{R}^3} \left[(6K(x) + 2\langle\nabla K(x), x\rangle)F(u) + u^6\right] dx = 0. \end{aligned} \tag{4.3}$$

Let

$$\begin{aligned} \Psi_\lambda(u) &= \langle\Phi'_\lambda(u), u\rangle - \frac{1}{2}Q_\lambda(u) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} \left[(\lambda A(x) + 1) - \omega\phi_u\right] u^2 dx - \int_{\mathbb{R}^3} \lambda\langle\nabla A(x), x\rangle u^2 dx \\ &\quad + \int_{\mathbb{R}^3} \left[K(x)(3F(u) - f(u)u)\right] dx + \int_{\mathbb{R}^3} \langle\nabla K(x), x\rangle F(u) dx - \frac{1}{2}\int_{\mathbb{R}^3} u^6 dx. \end{aligned} \tag{4.4}$$

Then, $\Psi_\lambda(u) = 0$ for all $u \in \breve{\mathcal{M}}$.

Lemma 4.1. *Assume that* (F1)–(F2), (A1)–(A2) *and* (K1) *hold. Then there exist a sequence* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *satisfying:*

$$\Phi_\lambda(u_n) \to c_\lambda > 0, \quad \|\Phi'_\lambda(u_n)\|(1 + \|u_n\|) \to 0 \quad \text{and} \quad \Psi_\lambda(u_n) \to 0, \tag{4.5}$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi_\lambda(\gamma(t)), \quad \Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } \Phi_\lambda(\gamma(1)) < 0\}. \tag{4.6}$$

Proof. The proof is similar to that of Lemma 3.1, so we omit it here. □

Similar to the proof of Theorem 1.1, we need estimate the upper bound of critical value c_λ.

Lemma 4.2. *Assume that* (F1)–(F3), (A1)–(A2) *and* (K1) *hold. Then* $c_\lambda < \frac{1}{3}S^{3/2}$.

Proof. Using the fact that $K(x) > K_0$, then by choosing $\mu_0 = K_0$ in (3.28), and similar derivation in Lemma 3.2, we can easily obtain the upper bound of c_λ. □

Lemma 4.3. *Suppose that* (F2), (A1), (A3') *and* (K1)–(K2) *hold. Then the Cerami sequence obtained in Lemma 4.1 is bounded.*

Proof. If $\theta \in [4, 6)$, by (4.1), (4.2), (4.5) and Lemma 2.1, we have

$$\begin{aligned} c_\lambda + o(1) &= \Phi_\lambda(u_n) - \frac{1}{\theta}\langle\Phi'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3} \left[|\nabla u_n|^2 + (\lambda A(x) + 1)u_n^2\right] dx \\ &\quad + \int_{\mathbb{R}^3} \left[\left(\frac{2}{\theta} - \frac{1}{2}\right)\omega\phi_{u_n} + \frac{1}{\theta}\phi_{u_n}^2\right] u_n^2 dx + K(x)\int_{\mathbb{R}^3} \left[\frac{1}{\theta}f(u_n)u_n - F(u_n)\right] dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{6}\right)\int_{\mathbb{R}^3} u_n^6 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3} \left[|\nabla u_n|^2 + (\lambda A(x) + 1)u_n^2\right] dx. \end{aligned} \tag{4.7}$$

From (2.1), (4.7) and $\lambda \geq 1$, we obtain that $\{u_n\}$ is bounded in E when $\theta \in [4, 6)$ in (F2).

If $\theta \in (2, 4)$, by (4.1), (4.2), (4.4) and (4.5), we have

$$\begin{aligned} c_\lambda + o(1) &= \Phi_\lambda(u_n) + \frac{\theta-4}{6-\theta}\langle \Phi'_\lambda(u_n), u_n\rangle + \frac{2-\theta}{6-\theta}\Psi_\lambda(u_n) \\ &= \frac{\theta-2}{6-\theta}\int_{\mathbb{R}^3}(\lambda A(x)+1)u_n^2dx + \frac{2}{6-\theta}\int_{\mathbb{R}^3}K(x)\left[f(u_n)u_n - \theta F(u_n)\right]dx \\ &\quad + \frac{1}{6-\theta}\int_{\mathbb{R}^3}\left[2(3-\theta)\omega\phi_{u_n} + (4-\theta)\phi_{u_n}^2\right]u_n^2dx + \frac{1}{3}\int_{\mathbb{R}^3}u_n^6dx \\ &\quad + \frac{\theta-2}{2(6-\theta)}\int_{\mathbb{R}^3}\lambda\langle\nabla A(x), x\rangle u_n^2dx + \frac{2-\theta}{6-\theta}\int_{\mathbb{R}^3}\langle\nabla K(x), x\rangle F(u_n)dx \\ &\geq \frac{\theta-2}{6-\theta}\int_{\mathbb{R}^3}(\lambda A(x)+1)u_n^2dx + \frac{1}{6-\theta}\int_{\mathbb{R}^3}\left[2(3-\theta)\omega\phi_{u_n} + (4-\theta)\phi_{u_n}^2\right]u_n^2dx. \end{aligned} \tag{4.8}$$

The proof of rest part is similar to that in Lemma 3.3. So, $\{u_n\}$ is bounded in E when $\theta \in (2, 4)$. □

The following lemma is used to prove that the sequence obtained in Lemma 4.2 is non-vanishing.

Lemma 4.4. *Under assumptions of Theorem 1.2, problem (1.6) admits a nontrivial solution.*

Proof. Since $K(x) > 0$ and it is bounded, by using the deduction in the proof of Theorem 1.1, we can easily prove this lemma. So we omit here. □

Now, Theorem 1.2 can be proved by using Lemmas 4.1–4.4.

Proof of Theorem 1.2. From the proof above, we derive that the critical point set $\breve{\mathcal{M}}$ is nonempty. Set $\widetilde{m} := \inf_{\breve{\mathcal{M}}}\Phi_\lambda(u)$. Taking $\{\widetilde{u}_n\} \subset \breve{\mathcal{M}}$ such that $\Phi_\lambda(\widetilde{u}_n) \to \widetilde{m}$, $\Phi'_\lambda(\widetilde{u}_n) = 0$ and $\Psi(\widetilde{u}_n) = 0$ as $n \to \infty$. Passing to a subsequence, we have $\widetilde{u}_n \rightharpoonup \widetilde{u}_0 \neq 0$ in H, $\widetilde{u}_n \to \widetilde{u}_0$ in $L^q_{loc}(\mathbb{R}^3)$, $q \in [2, 6)$ and $\widetilde{u}_n \to \widetilde{u}_0$ a.e. in $\mathbb{R}^3$. Moreover, from Lemma 2.2, we obtain $\phi_{\widetilde{u}_n} \rightharpoonup \phi_{\widetilde{u}_0}$ in $D^{1,2}(\mathbb{R}^3)$, $\phi_{\widetilde{u}_n} \to \phi_{\widetilde{u}_0}$ in $L^q_{loc}(\mathbb{R}^3)$ for $q \in [2, 6)$ and $\phi_{\widetilde{u}_n} \to \phi_{\widetilde{u}_0}$ a.e. in $\mathbb{R}^3$. Hence, we have

$$\Phi'_\lambda(\widetilde{u}_0) = \Psi(\widetilde{u}_0) = 0 \quad \text{and} \quad \Phi_\lambda(\widetilde{u}_0) \geq \widetilde{m}. \tag{4.9}$$

Now, we need to prove $\Phi_\lambda(\widetilde{u}_0) = \widetilde{m}$. If $\theta \in [4, \infty)$, from (4.1), (4.2), (4.9), Lemma 2.1 and Fatou's Lemma, we have

$$\begin{aligned} \widetilde{m} &= \lim_{n\to\infty}[\Phi_\lambda(\widetilde{u}_n) - \frac{1}{\theta}\langle\Phi'_\lambda(\widetilde{u}_n), \widetilde{u}_n\rangle] \\ &= \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3}[|\nabla\widetilde{u}_n|^2dx + (\lambda A(x)+1)\widetilde{u}_n^2]dx + \left(\frac{1}{\theta}-\frac{1}{6}\right)\int_{\mathbb{R}^3}\widetilde{u}_n^6dx \right.\\ &\quad \left. + \int_{\mathbb{R}^3}\left[\frac{1}{\theta}\phi_{\widetilde{u}_n}^2 - \left(\frac{1}{2}-\frac{2}{\theta}\right)\omega\phi_{\widetilde{u}_n}\right]\widetilde{u}_n^2dx + \int_{\mathbb{R}^3}K(x)\left[\frac{1}{\theta}f(\widetilde{u}_n)\widetilde{u}_n - F(\widetilde{u}_n)\right]dx\right\} \\ &\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3}[|\nabla\widetilde{u}_0|^2dx + (\lambda A(x)+1)\widetilde{u}_0^2]dx + \left(\frac{1}{\theta}-\frac{1}{6}\right)\int_{\mathbb{R}^3}\widetilde{u}_0^6dx \\ &\quad + \int_{\mathbb{R}^3}\left[\frac{1}{\theta}\phi_{\widetilde{u}_0}^2 - \left(\frac{1}{2}-\frac{2}{\theta}\right)\omega\phi_{\widetilde{u}_0}\right]\widetilde{u}_0^2dx + \int_{\mathbb{R}^3}K(x)\left[\frac{1}{\theta}f(\widetilde{u}_0)\widetilde{u}_0 - F(\widetilde{u}_0)\right]dx \\ &= \Phi_\lambda(\widetilde{u}_0) - \frac{1}{\theta}\langle\Phi'_\lambda(\widetilde{u}_0), \widetilde{u}_0\rangle \geq \widetilde{m}. \end{aligned} \tag{4.10}$$

If $\theta \in (2,4)$, from Lemma 2.1, Fatou's Lemma, (3.32), (3.33), (4.1), (4.2), (4.4) and (4.9), we have

$$
\begin{aligned}
\widetilde{m} = & \lim_{n\to\infty}\left[\Phi_\lambda(\widetilde{u}_n) + \frac{\theta-4}{6-\theta}\langle\Phi'_\lambda(\widetilde{u}_n),\widetilde{u}_n\rangle + \frac{2-\theta}{6-\theta}\Psi_\lambda(\widetilde{u}_n)\right] \\
= & \lim_{n\to\infty}\left\{\frac{\theta-2}{6-\theta}\int_{\mathbb{R}^3}(\lambda A(x)+1)\widetilde{u}_n^2dx + \frac{\theta-2}{2(6-\theta)}\int_{\mathbb{R}^3}\lambda\langle\nabla A(x),x\rangle\widetilde{u}_n^2dx\right. \\
& + \frac{1}{6-\theta}\int_{\mathbb{R}^3}\left[2(3-\theta)\omega\phi_{\widetilde{u}_n} + (4-\theta)\phi_{\widetilde{u}_n}^2\right]\widetilde{u}_n^2dx + \frac{2}{6-\theta}\int_{\mathbb{R}^3}K(x)\left[f(\widetilde{u}_n)\widetilde{u}_n - \theta F(\widetilde{u}_n)\right]dx \\
& \left. + \frac{2-\theta}{6-\theta}\int_{\mathbb{R}^3}\langle\nabla K(x),x\rangle F(\widetilde{u}_n)dx + \frac{1}{3}\int_{\mathbb{R}^3}\widetilde{u}_n^6dx\right\} \\
\geq & \frac{\theta-2}{6-\theta}\int_{\mathbb{R}^3}(\lambda A(x)+1)\widetilde{u}_0^2dx + \frac{\theta-2}{2(6-\theta)}\int_{\mathbb{R}^3}\lambda\langle\nabla A(x),x\rangle\widetilde{u}_0^2dx + \frac{1}{3}\int_{\mathbb{R}^3}\widetilde{u}_0^6dx \\
& + \frac{1}{6-\theta}\int_{\mathbb{R}^3}\left[2(3-\theta)\omega\phi_{\widetilde{u}_0} + (4-\theta)\phi_{\widetilde{u}_0}^2\right]\widetilde{u}_0^2dx + \frac{2}{6-\theta}\int_{\mathbb{R}^3}K(x)\left[f(\widetilde{u}_0)\widetilde{u}_0 - \theta F(\widetilde{u}_0)\right]dx \\
& + \frac{2-\theta}{6-\theta}\int_{\mathbb{R}^3}\langle\nabla K(x),x\rangle F(\widetilde{u}_0)dx + \frac{1}{3}\int_{\mathbb{R}^3}\widetilde{u}_0^6dx \\
= & \Phi_\lambda(\widetilde{u}_0) + \frac{\theta-4}{6-\theta}\langle\Phi'_\lambda(\widetilde{u}_0),\widetilde{u}_0\rangle + \frac{2-\theta}{6-\theta}\Psi_\lambda(\widetilde{u}_0) \geq \widetilde{m}.
\end{aligned} \tag{4.11}
$$

Equations (4.10) and (4.11) imply that $\Phi_\lambda(\widetilde{u}_0) = \widetilde{m} = \inf_{\widetilde{\mathcal{M}}}\Phi_\lambda(u)$ with $\widetilde{u}_0$ being a ground state solution. The proof of Theorem 1.2 is completed. □

5. Conclusions

In this paper, we investigate a ground state solution for the critical KGM system with a steep potential well and its extension using general conditions and Pohožaev identity. Obviously, the techniques we use have been successfully applied to find the solution of the critical KGM system, and hope that these results can be widely used in other systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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