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Research article

On fixed proximal pairs of E_r-mappings

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Abstract: In this paper we introduce a Garcia-Falset-type of noncyclical mappings and study the convergence of the iterates generated by a Thakur-type iteration scheme to the fixed proximal pairs of the new class of mappings.

Keywords: proximal pairs; noncyclic mappings; Hadamard spaces
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1. Introduction

Fixed point theory revolves around the fixed point equation $x = Tx$, associated to some self-mapping $T: X \to X$ acting on a nonempty set X on a metric space. This research area is one of the major branches of mathematics, which has benefited from a dramatic development in the last century; for the pioneering source, see Caccioppoli [8]. One might say that there is actually an ongoing process at an increasing pace, which motivates efforts to investigate and generalize the results of Cacciooli in many other suitable frameworks [9–11, 18, 19, 28]. New and valuable results keep appearing in the literature dedicated to this field (for some recent ones, please see [2, 22, 25, 30]).

A particular extension, which is of interest for us in this paper, comes into discussion if instead of a self-mapping, i.e., a mapping whose domain and co-domain coincide, one considers a nonself mapping for which those are distinct (often disjoint). Thus, in this setting one no longer looks for solutions for the fixed point equation but for points which solve some minimization problem. More precisely, let (X, Y) be a pair of two nonempty subsets of a metric space (M, d). Given a noncyclic mapping, i.e., $T: X \cup Y \to X \cup Y$, such that $T(X) \subseteq X$ and $T(Y) \subseteq Y$, one can combine a fixed point problem with an optimization problem as follows: find $x \in X$ and $y \in Y$ such that

$$Tx = x, \quad Ty = y \quad \text{and} \quad d(x, y) = \text{dist}(X, Y), \tag{1.1}$$

where $\text{dist}(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}$. We shall call the solutions of (1.1) fixed proximal pairs.

This problem was first formulated by Eldred et al. in [4] for relatively nonexpansive mappings in strictly convex Banach spaces and Hilbert spaces. Also, it has been studied in a more recent paper [16] for relatively nonexpansive mappings as well but involving the Ishikawa iteration by Gabeleh et al.

Our goal in this paper is to extend the study of the problem (1.1) to the setting of CAT(0) spaces, involving a more general class of mappings, which we shall call noncyclic E_r mappings (shortly, E_r mappings), defined by a condition similar to condition (E) of Garcia-Falset et al. [17]. We are mainly concerned with the convergence of the iterative sequences towards solutions of the problem (1.1) for E_r mappings. The sequences of iterates are provided by an adapted version of the Thakur iterative scheme [29]. Besides convergence results, there are some auxiliary results which had to be adapted from the setting of Hilbert spaces to the setting of CAT(0) spaces and which we believe will be useful for other authors as well.

The study is partly motivated by the fact that CAT(0) spaces provide a suitable framework for obtaining many fixed point theoretic results and that many concepts valid in Banach spaces have exact counterparts in this setting as well; please, see Dhompongsa and Panyanak [13], Kirk and Panyanak [21] Abbas et al. [1], Bejenaru and Ciobănescu [5] and many others. For instance, the Opial's condition is readily satisfied in CAT(0) spaces. Moreover, as we shall see in the sequel, the proximal Opial's condition for proximinal pairs introduced in [16] is satisfied as well in CAT(0) spaces and is not required as a hypothesis for certain results. Another notable difference worth mentioning is that the approach from [16] for constructing sequences converging to fixed proximal pairs fails for E_r mappings. More precisely, it can be seen that in [16] the iterative process is initiated at two points which are each other's projection and by the relative nonexpansiveness condition their images and together with any convex combination will share the same property. In other words, the iterative sequences run in parallel on the subsets X and, respectively, Y. This is no longer the case if we consider E_r mappings as it can be seen in Example 4.2. Our proposed solution of circumventing this problem is to consider only the iterates which converge to the solution of the fixed point problem in only one of the two subsets, say $x \in X$, and then project that solution onto the second subset $P(x) \in Y$. Then we show that the projection is a fixed point for the mapping acting on the second subset, i.e., $(x, P(x))$ is a fixed proximal pair. It is worth mentioning that our approach reduces the number of iterations by half as compared to the approach from [16].

2. Preliminaries

Throughout this paper we shall use the following notations

$$\begin{aligned} X_0 &= \{x \in X : d(x, y) = \operatorname{dist}(X, Y) \text{ for some } y \in Y\}, \\ Y_0 &= \{y \in Y : d(x, y) = \operatorname{dist}(X, Y) \text{ for some } x \in X\}, \\ Prox_{X\times Y}(T) &= \{(x, y) \in X \times Y\ : x = Tx,\ y = Ty,\ d(x, y) = \operatorname{dist}(X, Y)\}. \end{aligned}$$

Definition 2.1 ([16]). *Let (X, Y) be a pair of nonempty subsets of a metric space (M, d) and let $T : X \cup Y \to X \cup Y$ be a noncyclic mapping. Then T is called noncyclic relatively nonexpansive if*

$$d(Tx, Ty) \leq d(x, y),$$

for all $(x, y) \in X \times Y$.

Definition 2.2 ([15]). *Let (M, d) be a metric space and let (X, Y) be a pair of nonempty subsets of M. A noncyclic mapping $T\colon X \cup Y \to X \cup Y$ is said to be quasi-noncyclic relatively nonexpansive if*

(1) $F(T) \cap X_0 \neq \emptyset$ and $F(T) \cap Y_0 \neq \emptyset$,
(2) For each $q \in F(T) \cap Y_0$ and $p \in F(T) \cap X_0$, we have for all $x \in X$ and $y \in Y$

$$d(Tx, q) \leq d(x, q) \ \text{ and } \ d(p, Ty) \leq d(p, y),$$

where $F(T)$ denotes the set of all fixed points of the mapping T.

The following definition introduces a class of mappings which is more general that the class of noncyclic relatively nonexpansive and, respectively, more restrictive than quasi-noncyclic relatively nonexpansive mappings.

Definition 2.3. *Let (X, Y) be a pair of nonempty subsets of a metric space (M, d) and let $T\colon X \cup Y \to X \cup Y$ be a noncyclic mapping. We say that a mapping $T\colon X \cup Y \to X \cup Y$ satisfies the noncyclic relative condition (E) (shortly (E_r)-condition) if there exists $\mu \geq 1$ such that*

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y) \ \text{ and } \ d(y, Tx) \leq \mu d(y, Ty) + d(x, y),$$

for all $(x, y) \in X \times Y$.

This class is patterned after operators defined by the so-called condition (E), introduced in [17]. It is easy to see that any noncyclic relatively nonexpansive mapping satisfies the noncyclic (E_r)-condition for $\mu = 1$.

Proposition 2.1. *Every mapping satisfying (E_r)-condition which has a best proximity pair is a quasi-noncyclic relatively nonexpansive mapping.*

Proof. Suppose the mapping T satisfies the (E_r)-condition and let $(x_0, y_0) \in X \times Y$ be a best proximity pair. Since $x_0 = Tx_0$ and $y_0 = Ty_0$, from the (E_r)-condition, we get

$$d(x_0, Tx) \leq \mu d(x_0, Tx_0) + d(x_0, x) \leq d(x_0, x)$$

and

$$d(y_0, Ty) \leq \mu d(y_0, Ty_0) + d(y_0, y) \leq d(y_0, y),$$

for any $(x, y) \in X \times Y$, where $\mu \geq 1$. □

The metric structure alone is oftentimes insufficient for finding best proximal pairs for the problem (1.1). Besides Banach spaces with additional geometric properties adopted in the original paper by Eldred et al. [4], a suitable setting for studying the problem (1.1) is provided by CAT(0) spaces, as we shall see below. Let us first recall briefly the basic results regarding the CAT(0) spaces used in the sequel.

Let (M, d) be a metric space and let x, y be two distinct points in M. A continous mapping $c\colon [0, t] \to M$ having $c(0) = x$ and $c(t) = y$ is called a geodesic path which joins x and y if

$$d(c(\tau_1), c(\tau_2)) = |\tau_1 - \tau_2|,$$

for any $\tau_1, \tau_2 \in [0, t]$. Furthermore, $[x, y]$ represents its image and it is said to be a geodesic segment. If any pair of distinct points can be joined by a geodesic, we say that (M, d) is called geodesic space. Also, the space is said to be uniquely geodesic if joining an arbitrary pair of points the geodesic is unique.

A unique geodesic triangle in an uniquely geodesic metric space (M, d) is composed by three distinct points x, y, z of M and it is denoted by $\Delta(x, y, z)$. The geodesic segments $[x, y]$, $[y, z]$, $[z, x]$ represent the sides of $\Delta(x, y, z)$. On the other hand, a comparison triangle for $\Delta(x, y, z)$ is a triangle in the Euclidean plane $\overline{\Delta}(\overline{x}, \overline{y}, \overline{z})$ which fulfils such that

$$d(x, y) = d_E(\overline{x}, \overline{y}), \quad d(y, z) = d_E(\overline{y}, \overline{z}), \quad d(z, x) = d_E(\overline{z}, \overline{x}),$$

where d_E is the Euclidean metric.

Definition 2.4 ([7, 13]). *Let (M, d) be a geodesic space and let Δ be a geodesic triangle in M having $\overline{\Delta}$ its corresponding comparison triangle. Then the triangle Δ satisfies the* CAT(0) *inequality if*

$$d(x, y) \leq d_E(\overline{x}, \overline{y}),$$

for all $x, y \in \Delta$ and the corresponding $\overline{x}, \overline{y} \in \overline{\Delta}$. A geodesic space is said to be a CAT(0) *space if all its geodesic triangles satisfy the* CAT(0) *inequality.*

Below are some basic properties used in the sequel.

Lemma 2.1 ([13]). *Let (M, d) be a* CAT(0) *space. Then*

(i) *(M, d) is uniquely geodesic.*
(ii) *For a given pair of distinct points x, y in M and a some $t \in [0, 1]$, there exists a unique point $z \in [x, y]$, such that $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$. We denote this point by $z = tx \oplus (1 - t)y$.*
(iii) *$[x, y] = \{tx \oplus (1 - t)y\colon t \in [0, 1]\}$.*
(iv) *$d(x, z) + d(z, y) = d(x, y)$ if and only if $z \in [x, y]$.*
(v) *The mapping $f\colon [0, 1] \to [x, y]$, $f(t) = tx \oplus (1 - t)y$ is continuous and bijective.*

Lemma 2.2 ([13]). *Let (M, d) be a* CAT(0) *space. Then*

$$d(z, tx \oplus (1 - t)y) \leq td(z, x) + (1 - t)d(z, y),$$

$$d^2(z, tx \oplus (1 - t)y) \leq td^2(z, x) + (1 - t)d^2(z, y) - t(1 - t)d^2(x, y),$$

for all $x, y, z \in M$ and $t \in [0, 1]$.

Definition 2.5. *A bounded sequence $\{x_n\}$ in a* CAT(0) *space (M, d) defines a function*

$$r(\cdot, \{x_n\}) : M \to [0, \infty), \quad r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n),$$

which, in turn, defines the asymptotic radius

$$r(\{x_n\}) = \inf\{x \in M : r(x, \{x_n\}\}$$

and, respectively, the asymptotic center

$$A(\{x_n\}) = \{x \in M : r(x, \{x_n\}) = r(\{x_n\})\}$$

of the sequence $\{x_n\}$.

The asymptotic center plays a key role in the definition of a weaker notion of convergence on CAT(0) spaces, the so called Δ-convergence, which behaves in many aspects like weak convergence in Banach spaces.

Definition 2.6 ([21]). *Let (M, d) be a* CAT(0) *space and let $\{x_n\}$ be a sequence in M. We say that $\{x_n\}$ Δ-converges to some point $x \in M$ if x is the unique asymptotic center for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.*

Lemma 2.3 ([13, 21]). *In a* CAT(0) *space (M, d) the following assertions are true.*

(1) Any bounded sequence in M has a Δ-convergent subsequence.
(2) If $\{x_n\}$ is a bounded sequence in a closed and convex subset $C \subseteq M$, then $A(\{x_n\}) \in C$.
(3) If $\{x_n\}$ is a bounded sequence in M with $A(\{x_n\}) = \{x\}$ and $\{p_n\}$ is a subsequence of $\{x_n\}$ with $A(\{p_n\}) = \{p\}$ and the sequence $\{d(x_n, p)\}$ converges, then $x = p$.

Lemma 2.4 ([23]). *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a* CAT(0) *space (M, d). Suppose there exists $\ell \geq 0$ and $p \in M$ such that*

$$\limsup_{n\to\infty} d(x_n, p) \leq \ell, \;\; \limsup_{n\to\infty} d(y_n, p) \leq \ell \;\; \text{and} \;\; \limsup_{n\to\infty} d(\alpha_n x_n \oplus (1-\alpha_n) y_n, p) = \ell,$$

where $\{\alpha_n\}$ is a sequence bounded away by 0 and 1, then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

The following geometric property plays an important role in obtaining best proximity point results and will be used in the sequel.

Definition 2.7 ([26, 27]). *Let X, Y be two nonempty convex subsets of a metric space (M, d). A pair (X, Y) of subsets in M is said to have the P-property if the following implication holds*

$$d(x_1, y_1) = \mathrm{dist}(X, Y) = d(x_2, y_2) \implies d(x_1, x_2) = d(y_1, y_2),$$

for $x_1, x_2 \in X_0$ and $y_1, y_2 \in Y_0$.

A remarkable example of pairs having the P-property is given below in Lemma 3.1, while for a numerical approach in this direction, we refer to [20].

In a complete CAT(0) space M, the metric projection onto closed convex subsets in uniquely defined (see for instance [7]). More precisely, given a closed and convex subset X in M, for any point $x \in M$, there is a unique point $p \in X$ such that

$$d(x, p) = \inf_{y\in X} d(x, y).$$

Some nice properties of the projection can be described using the quasilinearization map introduced on metric spaces by Berg and Nikolaev [6]. Le us briefly recall it. A pair $(a, b) \in M \times M$, where (M, d) is a metric space, is called a vector and is denoted by $\overrightarrow{ab}$. The map $\langle \cdot, \cdot \rangle : (M \times M) \times (M \times M) \to \mathbb{R}$,

$$\left\langle \overrightarrow{ab}, \overrightarrow{uv} \right\rangle = \frac{1}{2}\left(d^2(a, v) + d^2(b, u) - d^2(a, u) - d^2(b, v)\right), \;\; a, b, u, v \in M, \tag{2.1}$$

is called the quasilinearization map. Some basic properties of (2.1) are as follows

$$\begin{array}{ll} i) & \left\langle \overrightarrow{xy}, \overrightarrow{uv} \right\rangle = \left\langle \overrightarrow{uv}, \overrightarrow{xy} \right\rangle; \\ ii) & \left\langle \overrightarrow{xy}, \overrightarrow{uv} \right\rangle = -\left\langle \overrightarrow{yx}, \overrightarrow{xy} \right\rangle; \\ iii) & \left\langle \overrightarrow{xz}, \overrightarrow{uv} \right\rangle + \left\langle \overrightarrow{zy}, \overrightarrow{uv} \right\rangle = \left\langle \overrightarrow{xy}, \overrightarrow{uv} \right\rangle. \end{array} \tag{2.2}$$

As in the case of Hilbert spaces, it is possible to characterize the metric projection through the quasilinearization map as in the following lemma, which will be key in the sequel.

Lemma 2.5 ([12]). *Let M be a complete* CAT(0) *space and X be a nonempty closed and convex subset of M. Let $x \in M$ and $p \in X$. Then $p = P_X(x)$ if and only if $\langle \overrightarrow{xp}, \overrightarrow{py} \rangle \geq 0$, for all $y \in X$.*

Given the sets X and Y which are nonempty, closed and convex, there is a well defined mapping

$$P\colon X \cup Y \to X \cup Y, \qquad P(x) = \begin{cases} P_X(x), & x \in Y, \\ P_Y(x), & x \in X. \end{cases} \tag{2.3}$$

This mapping will play the role of the metric projection in the sequel and will serve as a base for certain definitions. The first being the adapted version of Opial's condition. Recall that a Banach space $(X, \|\cdot\|)$ is said to satisfy Opial's condition [24] if

$$\limsup_{n\to\infty} \|x_n - x\| \leq \limsup_{n\to\infty} \|x_n - y\|,$$

for any sequence $\{x_n\} \in X$ which converges weakly to some $x \in X$ and all $y \in X$ such that $y \neq x$.

The remarkable fact about this property is that it holds as well in the setting of CAT(0) spaces with respect to the Δ-convergence. Moreover, given a proximinal pair (X, Y), i.e., $X_0 = X$ and $Y_0 = Y$, following Gabeleh et al. [16], we define below the proximal Opial's condition, which, as we shall see in the sequel, is satisfied in any CAT(0) space.

Definition 2.8. *Let (M, d) be a* CAT(0) *space and (X, Y) be a nonempty, closed, convex and proximal pair of M. The proximinal pair (X, Y) is said to satisfy the proximal Opial's condition if for each sequence $\{x_n\} \subset X$ such that $x_n \xrightarrow{\Delta} q \in X$ (respectively $x_n \xrightarrow{\Delta} q \in Y$), we have*

$$\limsup_{n\to\infty} d(x_n, P(q)) < \limsup_{n\to\infty} d(x_n, y),$$

for all $y \neq P(q) \in Y$ (respectively for all $y \neq P(q) \in X$).

We end this section by recalling the iterative scheme proposed by Thakur et al. [29] to approximate fixed points of Suzuki's generalized nonexpansive mappings, which we adapt here for our purpose. Let $x_1 \in X_0$ (or Y_0) and take $\{a_n\}, \{b_n\}$ two sequences in $[a, b]$ such that $0 < a \leq b < 1$ and define the iterative step as

$$\begin{array}{lcl} z_n & = & (1-b_n)x_n \oplus b_n T x_n, \\ y_n & = & T((1-a_n)x_n \oplus a_n z_n), \\ x_{n+1} & = & T y_n, \end{array} \tag{2.4}$$

for all $n \geq 1$.

In the sequel by specifying a property related to a pair we shall mean that it applies to both elements of the pair simultaneously.

3. Main results

First, we shall establish the P-property for any pair of nonempty, closed and convex subsets in a CAT(0) space. We first extend Lemma 4.1 and, respectively, Lemma 4.2 of Eldred and Raj [3] from the setting of Hilbert spaces to CAT(0) spaces.

Lemma 3.1. *Let (X, Y) be a pair of closed, convex sets in a complete CAT(0) space (M, d). If $x \in X_0$ and $y, z \in Y_0$ are such that $d(x, y) \le d(x, z)$, then $d(x, P_X(y)) \le d(x, P_X(z))$.*

Proof. Applying Lemma 2.1 [12], yields

$$\langle \overrightarrow{P_X(y)x},\ \overrightarrow{yP_X(y)} \rangle \ge 0, \tag{3.1}$$

$$\langle \overrightarrow{P_X(y)x},\ \overrightarrow{xP_Y(x)} \rangle \ge 0, \tag{3.2}$$

$$\langle \overrightarrow{P_Y(x)y},\ \overrightarrow{xP_Y(x)} \rangle \ge 0, \tag{3.3}$$

$$\langle \overrightarrow{P_Y(x)y},\ \overrightarrow{yP_X(y)} \rangle \ge 0. \tag{3.4}$$

Rewriting (3.1) and (3.2) as

$$\langle \overrightarrow{P_X(y)x},\ \overrightarrow{yP_Y(x)} + \overrightarrow{P_Y(x)P_X(y)} \rangle \ge 0 \text{ and } \langle \overrightarrow{P_X(y)x},\ \overrightarrow{xP_X(y)} + \overrightarrow{P_X(y)P_Y(x)} \rangle \ge 0$$

and using iii) of (2.2), it follows that

$$\langle \overrightarrow{P_X(y)x},\ \overrightarrow{xP_X(y)} + \overrightarrow{yP_Y(x)} \rangle \ge 0. \tag{3.5}$$

Similarly, from (3.3) and (3.4), one obtains

$$\langle \overrightarrow{P_Y(x)y},\ \overrightarrow{xP_X(y)} + \overrightarrow{P_X(y)P_Y(x)} \rangle \ge 0 \ \text{ and } \langle \overrightarrow{P_Y(x)y},\ \overrightarrow{yP_Y(x)} + \overrightarrow{P_Y(x)P_X(y)} \rangle \ge 0,$$

implying

$$\langle \overrightarrow{P_Y(x)y},\ \overrightarrow{xP_X(y)} + \overrightarrow{yP_Y(x)} \rangle \ge 0. \tag{3.6}$$

Taking into account iii) of (2.2) while combining (3.5) and (3.6), yields

$$\langle \overrightarrow{P_X(y)x} + \overrightarrow{P_Y(x)y},\ \overrightarrow{xP_X(y)} + \overrightarrow{yP_Y(x)} \rangle \ge 0,$$

which, due to i) and ii), is actually an equality

$$\langle \overrightarrow{P_X(y)x} + \overrightarrow{P_Y(x)y},\ \overrightarrow{xP_X(y)} + \overrightarrow{yP_Y(x)} \rangle = 0,$$

implying that the right hand terms of (3.1)–(3.4) are actually equal to 0. Using (2.1) in (3.1), we get

$$d^2(x, y) = d^2(P_X(y), y) + d^2(x, P_X(y)). \tag{3.7}$$

In a similar manner, we obtain

$$d^2(x, z) = d^2(P_X(z), z) + d^2(x, P_X(z)) \tag{3.8}$$

and the conclusion follows by noticing that $d(P_X(y), y) = \mathrm{dist}(X, Y) = d(P_X(z), z)$. □

Lemma 3.2. *Let (X, Y) be a pair of closed, convex sets in a complete* CAT(0) *space (M, d) and let $x, y \in X_0$. Then $d(x, P_Y(y)) = d(y, P_Y(x))$.*

Proof. Replacing y by $P_Y(y)$ in (3.7), gives

$$d^2(x, P_Y(y)) = d^2(P_X(P_Y(y)), P_Y(y)) + d^2(x, P_X(P_Y(y))) = d^2(x, y) + d^2(y, P_Y(y)).$$

Similarly, it follows that

$$d^2(y, P_Y(x)) = d^2(P_X(P_Y(x)), P_Y(x)) + d^2(y, P_X(P_Y(x))) = d^2(y, x) + d^2(x, P_Y(x)).$$

The conclusion follows again by noticing that $d(y, P_Y(y)) = \mathrm{dist}(X, Y) = d(x, P_Y(x))$. □

Theorem 3.1. *Any pair (X, Y) of nonempty, closed and convex subsets in a* CAT(0) *space (M, d) has the P-property.*

Proof. Let $x_1, x_2 \in X_0$ and $y_1, y_2 \in Y_0$ such that $d(x_1, y_1) = \mathrm{dist}(X, Y) = d(x_2, y_2)$. From the uniqueness of the metric projection, it follows that $y_1 = P_Y(x_1)$, $x_1 = P_X(y_1)$, $y_2 = P_Y(x_2)$ and $x_2 = P_X(y_2)$. Rewriting (3.7) for x_1 and y_2, and, respectively, for y_1 and x_2 gives

$$d^2(x_1, y_2) = d^2(P_X(y_2), y_2) + d^2(x_1, P_X(y_2))$$

and

$$d^2(y_1, x_2) = d^2(P_Y(x_2), x_2) + d^2(y_1, P_Y(x_2))$$

which, in the view of Lemma 3.2 and the fact that $d(P_Y(x_2), x_2) = d(P_X(y_2), y_2) = \mathrm{dist}(X, Y)$, implies that $d(x_1, x_2) = d(y_1, y_2)$, completing the proof. □

Certain results in the sequel will use projections on the subsets X_0 and Y_0. Therefore, in order for the projections to be well defined, it is required that the sets X_0 and Y_0 be nonempty, closed and convex. In this respect, as a corollary to Proposition 3.1 of [14], in the view of the fact that CAT(0) spaces are Busemann convex and reflexive (for more details, please see [14]), we have the following.

Lemma 3.3. *Let (M, d) be a* CAT(0) *space and let X, Y be two nonempty, closed, convex subsets of M. Suppose additionally that Y is bounded. Then the subsets X_0 and Y_0 are nonempty, closed, convex and bounded.*

It is worth emphasizing that, in the above results, only one of the sets X or Y has to be bounded.

Now, given that the sets X_0 and Y_0 are nonempty, closed and convex, there is a well defined mapping $P : X_0 \cup Y_0 \to X_0 \cup Y_0$, as in (2.3).

Lemma 3.4. *Let (X, Y) be a pair of nonempty, closed and convex subsets of a* CAT(0) *space (M, d) such that at least one of the sets X_0 or Y_0 is bounded. Then*

(1) $d(x, P(x)) = \mathrm{dist}(X, Y)$, for any $x \in X_0 \cup Y_0$;
(2) $P(X_0) \subseteq Y_0$ and $P(Y_0) \subseteq X_0$;
(3) P is isometry, i. e., $d(P(x), P(\overline{x})) = d(x, \overline{x})$, for all $x, \overline{x} \in X_0$ and $d(P(y), P(\overline{y})) = d(y, \overline{y})$ for all $y, \overline{y} \in Y_0$.

Proof. According to Lemma 3.3, the sets X_0 and Y_0 are nonempty, closed and convex. Thus, i) and ii) follow directly from the definition of the sets X_0 and Y_0 and the definition of the mapping P. Taking into account Theorem 3.1, the pair (X_0, Y_0) has the P-property which implies both assertions at iii). □

In the following results, we shall denote the set of fixed points of a given mapping T by $F(T)$.

Lemma 3.5. *Let X, Y be two nonempty subsets of a* CAT(0) *space* (M, d) *and let $T: X \cup Y \to X \cup Y$ be a mapping which satisfies the noncyclic (E_r)-condition. For arbitrary chosen $x_1 \in X$ consider the sequence $\{x_n\}$, generated by Algorithm* (2.4)*. Then, for all $p \in F(T) \cap Y_0$ the limit $\lim_{n\to\infty} d(x_n, p)$ exists. Furthermore, the sequence $\{x_n\}$ is bounded.*

Proof. Since (X, Y) is a pair of nonempty, closed and convex subsets of M, using the iteraive process (2.4) we have for any $p \in F(T) \cap Y_0$, according to the (E_r)-condition,

$$\begin{aligned}
d(x_{n+1}, p) &= d(Ty_n, p) \le d(y_n, p) \\
&= d(T((1-a_n)x_n \oplus a_n z_n), p) \\
&\le d((1-a_n)x_n \oplus a_n z_n, p) \\
&\le (1-a_n)d(x_n, p) + a_n d(z_n, p) \\
&\le (1-a_n)d(x_n, p) + a_n d((1-b_n)x_n \oplus b_n T x_n, p) \\
&\le (1-a_n)d(x_n, p) + a_n(1-b_n)d(x_n, p) + a_n b_n d(T x_n, p) \\
&\le (1-a_n)d(x_n, p) + a_n(1-b_n)d(x_n, p) + a_n b_n d(x_n, p) \\
&\le d(x_n, p).
\end{aligned}$$

This implies that the sequence $\{d(x_n, p)\}$ is non-increasing and bounded, so $\lim_{n\to\infty} d(x_n, p)$ exists. Moreover, the sequence $\{x_n\}$ is bounded. □

A similar result is obtained if we interchange the roles of the sets X and Y.

Lemma 3.6. *Let (M, d) be a* CAT(0) *space and X, Y two nonempty subsets of M and let $T: X \cup Y \to X \cup Y$ be a mapping which satisfies the noncyclic (E_r)-condition. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by Algorithm* (2.4)*, then $\lim_{n\to\infty} d(x_n, z_n) = 0$ and $\lim_{n\to\infty} d(x_n, T x_n) = 0$.*

Proof. Let $p \in F(T) \cap Y_0$. According to Lemma 3.5, the limit $\ell := \lim_{n\to\infty} d(x_n, p)$ exists. The (E_r)-condition implies that $\lim_{n\to\infty} d(T x_n, p) \le \lim_{n\to\infty} d(x_n, p) = \ell$.

As

$$\begin{aligned}
d(z_n, p) &= d((1-b_n)x_n \oplus b_n T x_n, p) \\
&\le (1-b_n)d(x_n, p) + b_n d(T x_n, p) \\
&\le (1-b_n)d(x_n, p) + b_n d(T x_n, p) \\
&\le d(x_n, p),
\end{aligned}$$

it follows that

$$\limsup_{n\to\infty} d(T z_n, p) \le \limsup_{n\to\infty} d(z_n, p) \le \limsup_{n\to\infty} d(x_n, p) = \ell. \tag{3.9}$$

Letting $n \to \infty$ in

$$d(x_{n+1}, p) \leq d(Ty_n, p) \leq d((1 - a_n)x_n \oplus a_n z_n, p) \leq d(x_n, p),$$

we obtain

$$\limsup_{n\to\infty} d((1 - a_n)x_n \oplus a_n z_n, p) = \ell$$

and according to Lemma 2.4, we get

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.10}$$

From the triangle inequality, we have

$$d(x_n, p) \leq d(x_n, z_n) + d(z_n, p),$$

in which taking $n \to \infty$, yields

$$\ell \leq \limsup_{n\to\infty} d(z_n, p). \tag{3.11}$$

Then, by (3.9) and (3.11), we get $\limsup_{n\to\infty} d(z_n, p) = \ell$. Thus

$$\limsup_{n\to\infty} d(z_n, p) = \limsup_{n\to\infty} d((1 - b_n)x_n \oplus b_n T x_n, p) = \ell.$$

Again, by Lemma 2.4, one obtains

$$\lim_{n\to\infty} d(x_n, T x_n) = 0,$$

and this completes the proof. □

In the following, we prove Δ and strong convergence results for which proximal Opial's condition plays an important role. So, let us first establish that it holds in a CAT(0) space.

Proposition 3.1. *Let (M, d) be a* CAT(0) *space and (X, Y) be a nonempty, closed, convex and proximal pair of M. Then any proximinal pair (X, Y) satisfies proximal Opial's condition.*

Proof. Take a sequence $\{x_n\} \subset X = X_0$ such that $x_n \xrightarrow{\Delta} q \in X$ and let $y \in Y = Y_0$. From the Opial's condition for M we have

$$\limsup_{n\to\infty} d(x_n, q) < \limsup_{n\to\infty} d(x_n, P_X(y)),$$

implying

$$\limsup_{n\to\infty} d^2(x_n, q) + d^2(q, P_Y(q)) < \limsup_{n\to\infty} d^2(x_n, P_X(y)) + d^2(y, P_X(y)),$$

which, according to (3.7), rewrites as

$$\limsup_{n\to\infty} d^2(x_n, Pq) < \limsup_{n\to\infty} d^2(x_n, y),$$

proving the Opial's property for the first case. The second case follows similarly. □

Following is a demiclosedness-type result for E_r-mappings in the noncyclic setting.

Theorem 3.2. *Let X, Y be two nonempty, bounded, closed and convex subsets of* CAT(0) *space* (M, d)*. Assume that $T: X \cup Y \to X \cup Y$ is a (E_r)-mapping and that the sequence $\{x_n\}$ Δ-converges to $x \in X$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $(x, P(x)) \in Prox_{X\times Y}(T)$.*

Proof. According to our assumption, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Writing the noncyclic (E_r)-condition for T for an arbitrary element of the sequence and $P(x)$, yields

$$d(x_n, TP(x)) \leq \mu d(x_n, Tx_n) + d(x_n, P(x))$$

and letting $n \to \infty$, it follows

$$\limsup_{n\to\infty} d(x_n, TP(x)) \leq \limsup_{n\to\infty} d(x_n, P(x)),$$

and from proximal Opial's condition, it follows that

$$TP(x) = P(x).$$

Moreover, from

$$d(P(x), Tx) \leq \mu d(P(x), TP(x)) + d(P(x), x).$$

we get that $x = Tx$. Thus, we have the fixed proximal pair $(x, P(x)) \in Prox_{X\times Y}(T)$. □

Theorem 3.3. *Let (M, d) be a* CAT(0) *space and X, Y be two nonempty, bounded, closed and convex subsets of M. Suppose $T: X\cup Y \to X\cup Y$ is a mapping which satisfies the noncyclic (E_r)-condition and $\{x_n\}$ is a sequences generated by the iterative scheme* (2.4)*. Then the sequence $\{(x_n, P(x_n))\}$ Δ-converges to a fixed proximal pair of T.*

Proof. According to Lemma 3.5, the sequence $\{x_n\}$ is bounded and, thus, it contains a Δ-convergent subsequence. Assume that $\{x_{n_k}\}$ and $\{x_{n_m}\}$ are two sequences which Δ-converge to p and q, respectively, such that $p \neq q$. Lemma 3.6 implies that $\lim_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$ and $\lim_{n\to\infty} d(x_{n_m}, Tx_{n_m}) = 0$ and according to Theorem 3.2, $Tp = p$, $Tq = q$, $TP(p) = P(p)$ and $TP(q) = P(q)$.

Moreover, by Lemma 3.5, the limits $\lim_{n\to\infty} d(x_n, P(p))$ and $\lim_{n\to\infty} d(x_n, P(q))$ exist and, due to the fact that (X_0, Y_0) satisfies the proximal Opial's condition, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, P(p)) &= \limsup_{k\to\infty} d(x_{n_k}, P(p)) \\
&< \limsup_{k\to\infty} d(x_{n_k}, P(q)) \\
&= \limsup_{n\to\infty} d(x_{n_m}, P(q)) \\
&< \limsup_{j\to\infty} d(x_{n_m}, P(p)) \\
&= \limsup_{n\to\infty} d(x_n, P(p)),
\end{aligned}$$

which is a contradiction. Hence, $p = q$ and the sequence $\{x_n\}$ is Δ-convergent to a point $p \in X_0$. Moreover, according to Theorem 3.2 it follows that $(p, P(p)) \in Prox_{X\times Y}(T)$. It remains to show that $P(x_n)$ Δ-converges to $P(p)$. Indeed, this follows from the fact that $d(x_n, p) = d(P(x_n), P(p))$ for any $n \geq 1$, which is implied by (3.7), (3.8) and Lemma 3.2. □

Let us notice that a similar result follows if we interchange the roles of the sets X and Y. Moreover, in our approach, besides iterating only over one set, we don't need to bother about the initial projection as compared to the approach from [16].

Theorem 3.4. *Let (X, Y) be a nonempty, closed and convex pair of subsets in a* CAT(0) *space (M, d) such that at least one of the subsets is compact and let $T : X \cup Y \to X \cup Y$ be a (E_r)-mapping. Let $\{x_n\}$ be a sequence generated by Algorithm* (2.4)*. Then the sequence $\{(x_n, P(x_n))\}$ converges strongly to a fixed proximal pair of T.*

Proof. Suppose that X is a compact (and hence X_0 as well). Then the sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ converging strongly to some element p, i.e., $\lim_{n_k \to \infty} d(x_{n_k}, p) = 0$. Applying Lemma 3.6 and taking into account Opial's condition (clearly x_{n_k} Δ-converges to p) in

$$d(x_{n_k}, TP(p)) \leq \mu d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, P(p))$$

yields $TP(p) = P(p)$. On the other hand, according to Lemma 3.5, the limit $\lim_{n\to\infty} d(x_n, P(p))$ exists and equals to $\lim_{n_k\to\infty} d(x_{n_k}, P(p)) = d(p, P(p)) = \mathrm{dist}(X, Y)$. According to the proof of Lemma 3.2, we have

$$d^2(x_n, P(p)) = d^2(x_n, p) + d^2(p, P(p)), \quad \text{for all } \ n \geq 1,$$

in which taking $n \to \infty$, yields

$$\lim_{n\to\infty} d(x_n, p) = 0.$$

As above, $d(x_n, p) = d(P(x_n), P(p))$, for all $n \geq 1$, implying that

$$\lim_{n\to\infty} d(P(x_n), P(p)) = 0,$$

which completes the proof. □

Remark that the last three results have natural corollaries for the more particular class of relatively nonexpansive mappings. Indeed, in this case we obtain two sequences running in parallel in the sense described above. The same pair of sequences will be obtained by using either of the two approaches described above (projecting or running two parallel iterations).

4. Examples

We start this section by presenting some examples in order to emphasize the practical value of the formal analysis revealed above. In the first example we have a mapping which does not satisfy the (E)-condition but satisfies the (E_r)-condition. Moreover, this example illustrates the idea that in some cases we can enlarge a given class of mappings by restricting the set on which a certain condition is imposed.

Example 4.1. *Consider the subsets in the Euclidean plane $X = \{a = (0, 1), b = (2, 1), c = (4, 1)\}$, $Y = \{a' = (1, 0), b' = (3, 0), c' = (5, 1)\}$ and the noncyclic mapping $T\colon X \cup Y \to X \cup Y$,*

$$T(a) = b,\ T(b) = a,\ T(c) = c,\ T(a') = a',\ T(b') = c',\ T(c') = b'.$$

Clearly, as $d(c,b) = 2$ and $d(c,Tb) = d(c,a) = 4$, the mapping is not quasi-nonexpansive and hence does not satisfy the condition (E). On the other hand, it satisfies the (E_r)-condition. Indeed, we have $d(c,Ta') = d(c,a'), d(c,Tb') = d(c,c') = d(c,b') = d(c,Tc'), d(b,Tb) = d(a,Ta) = 2$ which verifies the first part of the condition E_r. The second part follows in a similar manner.

Example 4.2. *Let $X_0 = [0,1]\times\{1\}$ and $Y_0 = [0,1]\times\{0\}$ be two subsets of $\mathbb{R}^2$ endowed with the usual Euclidean metric and let $T\colon X_0\cup Y_0 \to X_0\cup Y_0$ be a mapping given by*

$$T(x,1) = \left(\frac{x+1}{2}, 1\right) \text{ and } T(y,0) = \left(\frac{y+2}{3}, 0\right).$$

Then T satisfies the (E_r)-condition, but is not noncyclical relatively nonexpansive.

Let $(x,y) \in X_0 \times Y_0$ and $\mu \geq 1$. For $(x,1) \in X_0$ and $(y,0) \in Y_0$, one obtains

$$d((x,1),T(y,0)) = d\left((x,1),\left(\frac{y+2}{3},0\right)\right) = \sqrt{\left(x-\frac{y+2}{3}\right)^2+1},$$

$$d((x,1),T(x,1)) = d\left((x,1),\left(\frac{x+1}{2},1\right)\right) = \frac{1-x}{2},$$

$$d((x,1),(y,0)) = \sqrt{(x-y)^2+1}.$$

It is sufficient to check whether the equation

$$d^2((x,1),T(y,0)) \leq \mu^2 d^2((x,1),T(x,1)) + d^2((x,1),(y,2))$$

holds for all $(x,1) \in X_0$ and $(y,0) \in Y_0$, which becomes

$$\left(x-\frac{y+2}{3}\right)^2 + 1 \leq \mu^2\left(\frac{1-x}{2}\right)^2 + (x-y)^2 + 1,$$

implying

$$\frac{1}{2} \leq \frac{\mu^2}{4}(1-x)^2 + \frac{8}{9}\left(y-\frac{1}{4}\right)^2 + \frac{4x}{3}(1-y).$$

We will consider the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \frac{\mu^2}{4}(1-x)^2 + \frac{8}{9}\left(y-\frac{1}{4}\right)^2 + \frac{4x}{3}(1-y)$$

and look for its local minima. The system defining the stationary points is

$$\begin{array}{rcl} -\dfrac{\mu^2}{2}(1-x) + \dfrac{4}{3}(1-y) & = & 0, \\ \dfrac{16}{9}\left(y-\dfrac{1}{4}\right) - \dfrac{4x}{3} & = & 0. \end{array} \tag{4.1}$$

From the second equation we have

$$x = \frac{4}{3}y - \frac{1}{3},$$

which substituted in the first equation of (4.1) yields

$$\frac{4}{3}(1-y)\left(1-\frac{\mu^2}{2}\right)=0.$$

For $\mu \geq 1$ and $\mu \neq \sqrt{2}$, we get $y = 1$ and $x = 1$. According to the second derivative test, in order for the pair (1,1) to be a local minima, it is enough to take $\mu > \sqrt{2}$. The minimal value, in this case is $f(1,1) = \dfrac{1}{2}$, completing the first case.

For the second case, take $(x, 0) \in Y_0$ and $(y, 1) \in X_0$ and we have

$$d((x,0),T(y,1)) = d\left((x,0),\left(\frac{y+1}{2}+1\right)\right) = \sqrt{\left(x-\frac{y+1}{2}\right)^2+1},$$

$$d(x,0),T(x,0)) = d(\left((x,0),\left(\frac{x+2}{3},0\right)\right) = \frac{2}{3}(1-x),$$

$$d((x,0),(y,1)) = \sqrt{(x-y)^2+1}.$$

Similarly, it is sufficient to check if the inequality

$$d^2((x,0),T(y,1)) \leq \mu^2 d^2((x,0),T(x,0)) + d^2((x,0),(y,1)),$$

holds for all $(x, 0) \in Y_0$ and $(y, 1) \in X_0$ and which is equivalent to

$$\left(x-\frac{y+1}{2}\right)^2+1 \leq \left(\frac{2\mu}{3}\right)^2(1-x)^2+(x-y)^2+1,$$

which rewrites as

$$\frac{1}{3} \leq \left(\frac{2\mu}{3}\right)^2(1-x)^2+\frac{3}{4}\left(y-\frac{1}{3}\right)^2+x(1-y).$$

We look as above for the minima of the function

$$g\colon \mathbb{R}^2 \to \mathbb{R}, \quad g(x,y) = \left(\frac{2\mu}{3}\right)^2(1-x)^2+\frac{3}{4}\left(y-\frac{1}{3}\right)^2+x(1-y)$$

by solving the system

$$\begin{array}{rcl} -\dfrac{8}{9}\mu^2(1-x)+(1-y) & = & 0, \\ \dfrac{3}{2}\left(y-\dfrac{1}{3}\right)-x & = & 0. \end{array} \tag{4.2}$$

From the second equation we get

$$x = \frac{3}{2}\left(y-\frac{1}{3}\right)$$

and substitute in the first equation of (4.2), yielding

$$(1-y)\left(\frac{4}{3}\mu^2-1\right)=0,$$

which implies that $y = 1$ due to the fact that $\mu \geq 1$ and the pair $(1, 1)$ is the critical point of (4.2). The second partial derivative test shows that $(1, 1)$ is a local minimum and $g(1, 1) = \frac{1}{3}$, completing the second case.

Example 4.3. *Consider* $X_0 = [-2,1]\times\{-1\}$ *and* $Y_0 = [-2,1]\times\{1\}$ *with the usual Euclidean metric on* $\mathbb{R}^2$ *and let the mapping*

$$T\colon X_0\cup Y_0 \to X_0\cup Y_0, \qquad T(x,y) = \begin{cases} \left(\frac{|x|}{2}, y\right), & x\in[-2,1)\,, \\ \left(-\frac{1}{2}, y\right), & x=1. \end{cases}$$

Then T *satisfies the* (E_r)*-condition on* $X\cup Y$ *for* $\mu\geq 3$.

Indeed, we have the following cases:

Case I. Let $x\in[-2,0]$, $y=-1$ and $z\in[-2,1]$. Then

$$d((x,-1),T(x,-1)) = d\left((x,-1),\left(\frac{|x|}{2},-1\right)\right) = \frac{3}{2}|x|$$

so, we have

$$d((x,-1),T(z,-1)) \leq |x| + \frac{1}{2}|z| \leq \frac{3}{2}|x| + \frac{1}{2}|x-z| \leq d((x,-1),T(x,-1)) + d((x,-1),(z,-1)).$$

Case II. Let $x\in[0,1)$, $y=-1$ and $z\in[-2,1]$. In this case,

$$d((x,-1),T(x,-1)) = d\left((x,-1),\left(\frac{|x|}{2},-1\right)\right) = \frac{|x|}{2}$$

and one obtains

$$d((x,-1),T(z,-1)) \leq |x| + \frac{1}{2}|z| \leq \frac{3}{2}|x| + \frac{1}{2}|x-z| \leq 3d((x,-1),T(x,-1)) + d((x,-1),(z,-1)).$$

Case III. Let $x=1$, $y=-1$ and $z\in[-2,1)$. Thus,

$$d((1,-1),T(1,-1)) = d\left((1,-1),\left(-\frac{1}{2},-1\right)\right) = \frac{3}{2}.$$

Further

$$d((1,-1),T(z,-1)) \leq 1 + \frac{1}{2}|z| \leq \frac{1}{2} + \frac{1}{2}(1-|z|) \leq d((1,-1),T(1,-1)) + d((1,-1),(z,-1)).$$

Similarly, we take the previous cases for $x,z\in[-2,1]$ and $y=1$. Hence, for all $x,z\in[-2,1]$ and $y\in\{-1,1\}$, we get

$$d((x,y),T(z,y)) \leq \mu d((x,y),T(x,y)) + d((x,y),(z,y)),$$

for $\mu\geq 3$.

Example 4.4. *Let*

$$X_0 = \{(0,0,0,-1),(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\},$$

$$Y_0 = \{(0,0,0,1),(1,0,0,1),(0,1,0,1),(0,0,1,1)\}$$

be two subsets of $\mathbb{R}^4$ *endowed with the usual Euclidean metric and let* $T\colon X_0\cup Y_0\to X_0\cup Y_0$ *be a mapping given by*

$$\begin{pmatrix} (0,0,0,-1) & (1,0,0,-1) & (0,1,0,-1) & (0,0,1,-1) & (0,0,0,1) & (1,0,0,1) & (0,1,0,1) & (0,0,1,1) \\ (1,0,0,-1) & (0,0,0,-1) & (0,0,1,-1) & (0,1,0,-1) & (1,0,0,1) & (0,0,0,1) & (0,0,1,1) & (0,1,0,1) \end{pmatrix}.$$

Then T *satisfies the* (E_r)*-condition.*

In order to see that T satisfies the (E_r)-condition on $X \cup Y$, for all $x \in X, y \in Y$ we have

$$d(x, Tx) \geq 1, d(y, Ty) \geq 1 \;\; and \;\; d(x, y) = d(y, x) \geq 2.$$

Therefore

$$d(x, Ty) \leq \sqrt{6} \;\; and \;\; d(y, Tx) \leq \sqrt{6},$$

then

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y) \;\; and\,(y, Tx) \leq \mu d(y, Ty) + d(y, x),$$

for $\mu \geq 1$ and the proof is complete.

5. Conclusions

In this paper we have studied the problem of fixed proximal pairs for noncyclic mappings which satisfy a Garcia-Falset - type of generalized nonexpansiveness condition in the setting of CAT(0) spaces. We propose a new approach of studying this problem as the techniques used so far fail for this class of mappings. Main results are related to the Δ and strong convergence of the iterates of a Thakur-type scheme to fixed proximal pairs. Also, we present some auxiliary results which extend certain properties from Hilbert spaces to the setting of CAT(0) spaces. We expect that the approach adopted in this paper can be extended to uniformly convex Banach spaces.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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