Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 26600–26613.
DOI: 10.3934/math.20231360
Received: 13 July 2023
Revised: 30 August 2023
Accepted: 05 September 2023
Published: 18 September 2023

Research article

Special non-lightlike ruled surfaces in Minkowski 3-space

Chang Sun, Kaixin Yao and Donghe Pei*

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

* **Correspondence:** Email: peidh340@nenu.edu.cn.

Abstract: In this paper, we give the existence and uniqueness theorems for non-lightlike framed surfaces and define a special non-lightlike ruled surface in Minkowski 3-space. It may have singularities. We give the conditions for identifying cross-caps and surfaces as developable and maximal. Besides, we demonstrate that if the spacelike ruled surface is developable, then the z-parameter curve is an asymptotic curve if and only if the ruled surface is maximal.

Keywords: non-lightlike framed surface; cross-cap; developable surface; maximal surface; asymptotic curve
Mathematics Subject Classification: 53A05, 57R45

1. Introduction

Ruled surfaces have a great effect on differential geometry and they are used in many fields, such as architecture, robotics, design and so on [1]. In real life, ruled surfaces can be seen everywhere, such as in most of the cooling tower structures of thermal power plants, the famous Mobius ring, and saddle-shaped potato chips. Ruled surfaces can be formed by a moving line in continuous motion. There are many kinds of special ruled surfaces. According to the different direction vectors, Kaya and Önder defined three kinds of ruled surfaces, i.e., the osculating-type ruled surface, the generalized normal ruled surface, and the generalized rectifying ruled surface whose base curves are regular curves in the Euclidean 3-space [2–4]. The direction vector of the osculating-type ruled surface is a combination of the tangent and the principal normal vectors. Changing the tangent vector to the binormal vector, we get the generalized normal ruled surface. Changing the principal normal vector to the binormal vector, we get the generalized rectifying ruled surface. Later, framed curves in Euclidean space were defined by Honda and Takahashi in order to research curves with singular points [5]. Many scholars applied this idea to study the ruled surface of a curve with singular points [6]. Most of the surfaces that are studied may have singular points. In order to address this matter, in 2019, Fukunaga and Takahashi gave the definition of framed surfaces in the Euclidean space [7]. A framed surface is a smooth surface with two unit orthogonal vector fields along it. The first vector is orthogonal to the tangent space of the surface anywhere. The theory of framed surfaces provides more possibilities for the study of singularities. Many researchers have used it. For details see [8–10]. The ruled surfaces have also been studied extensively in the Minkowski space. Kobayashi studied maximal surfaces and the Lorentz surfaces were differentiated by Kim and Yoon [11, 12]. Ruled surfaces have been studied extensively for a long time and are considered as regular points. See [13–17]. Using the theory of framed surfaces as a tool, we can extend the framed surface with singular points in Euclidean space to Minkowski space.

In the present paper, we give the existence and uniqueness theorems for non-lightlike framed surfaces and define special non-lightlike ruled surfaces with a spacelike regular base curve that has timelike principal normal vectors. The basic knowledge in Minkowski 3-space is reviewed in Section 2. In Section 3, a new special ruled surface is defined and the conditions for identifying cross-caps are given. Besides, the conditions for being cylindrical, stricture curves, as well as the connection between them are obtained. We give the fundamental theory for non-lightlike framed surfaces in Section 4. In Section 5, the ruled surfaces are considered to be non-lightlike framed surfaces. We obtain the fundamental invariants and curvatures. The conditions for being developable, and maximal, as well as the parameter curves on the surface are analyzed. And the relationship between them is found. Finally, we give an example of the special ruled surface with singular points in Section 6.

All maps and manifolds considered here are differentiable for class C^∞.

2. Preliminaries

Let $\mathbb{R}^3$ be the 3-dimensional real vector space. For any $\boldsymbol{c} = (c_1, c_2, c_3)$ and $\boldsymbol{d} = (d_1, d_2, d_3) \in \mathbb{R}^3$, the pseudo inner product is denoted by

$$\langle \boldsymbol{c}, \boldsymbol{d} \rangle = -c_1 d_1 + c_2 d_2 + c_3 d_3.$$

$(\mathbb{R}^3, \langle , \rangle)$ is called the Minkowski 3-space and is denoted by $\mathbb{R}^3_1$.

For any nonzero vector $\boldsymbol{c} \in \mathbb{R}^3_1 \backslash \{\boldsymbol{0}\}$, it is called spacelike, timelike or lightlike if $\langle \boldsymbol{c}, \boldsymbol{c} \rangle > 0$, $\langle \boldsymbol{c}, \boldsymbol{c} \rangle < 0$, or $\langle \boldsymbol{c}, \boldsymbol{c} \rangle = 0$, respectively. The norm of $\boldsymbol{c}$ is $\parallel \boldsymbol{c} \parallel = \sqrt{| \langle \boldsymbol{c}, \boldsymbol{c} \rangle |}$. The pseudo vector product of $\boldsymbol{c}$ and $\boldsymbol{d}$ is

$$\boldsymbol{c} \wedge \boldsymbol{d} = (-(c_2 d_3 - c_3 d_2), c_3 d_1 - c_1 d_3, c_1 d_2 - c_2 d_1).$$

There are three special subsets in $\mathbb{R}^3_1$:

$$S^2_1 = \{\boldsymbol{c} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{c}, \boldsymbol{c} \rangle = 1\},$$

$$H^2_+ = \{\boldsymbol{c} = (c_1, c_2, c_3) \in \mathbb{R}^3_1 \mid \langle \boldsymbol{c}, \boldsymbol{c} \rangle = -1, c_1 > 0\},$$

$$H^2_- = \{\boldsymbol{c} = (c_1, c_2, c_3) \in \mathbb{R}^3_1 \mid \langle \boldsymbol{c}, \boldsymbol{c} \rangle = -1, c_1 < 0\}.$$

We denote $H^2_0 = H^2_+ \cup H^2_-$.

Define a set $\triangle = \{(\nu_1, \nu_2) \in \mathbb{R}^3_1 \times \mathbb{R}^3_1 \mid \parallel \nu_1 \parallel = \parallel \nu_2 \parallel = 1, \langle \nu_1, \nu_2 \rangle = 0\}$.

Assume that U is an open domain and $\boldsymbol{\chi} : U \to \mathbb{R}^3_1$ is a surface. We say that $\boldsymbol{\chi}$ is spacelike, timelike or lightlike if the tangent plane is spacelike, timelike or lightlike at any point $(z, w) \in U$, respectively. We call both the spacelike surface and the timelike surface the non-lightlike surface. In this paper, we consider the non-lightlike surface which can have singularities, and the tangent plane is non-lightlike at its regular point.

We say that $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ is a regular spacelike curve with the arclength parameter z which has timelike principal normal vectors with the orthonormal frame $\{\boldsymbol{T}(z), \boldsymbol{N}(z), \boldsymbol{B}(z)\}$. The Frenet formulas are as follows

$$\begin{cases} \dot{\boldsymbol{T}}(z) = \kappa(z)\boldsymbol{N}(z), \\ \dot{\boldsymbol{N}}(z) = \kappa(z)\boldsymbol{T}(z) + \tau(z)\boldsymbol{B}(z), \\ \dot{\boldsymbol{B}}(z) = \tau(z)\boldsymbol{N}(z), \end{cases}$$

where $\kappa(z)$ and $\tau(z)$ are the curvature and torsion of $\boldsymbol{\gamma}$.

A ruled surface $\boldsymbol{\chi}(z, w)$ is given by

$$\boldsymbol{\chi}(z, w) : I \times \mathbb{R} \to \mathbb{R}^3_1,$$

$$\boldsymbol{\chi}(z, w) = \boldsymbol{\gamma}(z) + w\boldsymbol{q}(z),$$

where $\boldsymbol{\gamma} : I \subset \mathbb{R} \to \mathbb{R}^3_1$ is the base curve and $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^3_1$ is the direction vector. The ruled surface $\boldsymbol{\chi}(z, w)$ is cylindrical if only if $\dot{\boldsymbol{q}}(z) = \boldsymbol{0}$. We say that the curve $\boldsymbol{\beta} = \boldsymbol{\beta}(z)$ is the striction curve of $\boldsymbol{\chi}(z, w)$ if it satisfies the condition that $\langle \dot{\boldsymbol{\beta}}(z), \dot{\boldsymbol{q}}(z) \rangle = 0$.

An arbitrary spacelike surface is a developable surface if $K = 0$ and maximal if $H = 0$.

3. Ruled surface of regular spacelike curve with timelike principal normal vectors

Definition 3.1. Let $\boldsymbol{\gamma} : I \subset \mathbb{R} \to \mathbb{R}^3_1$ be a spacelike curve with timelike principal normal vectors. The ruled surface $\boldsymbol{\chi}(z, w) : I \times \mathbb{R} \to \mathbb{R}^3_1$ is defined by

$$\boldsymbol{\chi}(z, w) = \boldsymbol{\gamma}(z) + w\boldsymbol{q}(z),$$

where $\boldsymbol{q}(z) = \cos\theta(z)\boldsymbol{T}(z) + \sin\theta(z)\boldsymbol{B}(z)$ and $\theta(z)$ is a smooth function.

It is known that $\theta(z) = k\pi (k \in \mathbb{Z})$ for all $z \in I, \boldsymbol{q} = \pm\boldsymbol{T}$ and $\boldsymbol{\chi} = \boldsymbol{\gamma} \pm w\boldsymbol{T}$. When $\theta(z) = \frac{\pi}{2} + k\pi (k \in \mathbb{Z})$ for all $z \in I, \boldsymbol{q} = \pm\boldsymbol{B}$ and $\boldsymbol{\chi} = \boldsymbol{\gamma} \pm v\boldsymbol{B}$.

Let $P : I \subset \mathbb{R} \to \mathbb{R}$ and $Q : I \times \mathbb{R} \to \mathbb{R}$ be smooth functions that are respectively defined as

$$P = \sin\theta - w\dot{\theta},$$

$$Q = w(\kappa\cos\theta + \tau\sin\theta).$$

Proposition 3.2. The ruled surface $\boldsymbol{\chi}(z, w)$ has singular points if and only if $P = Q = 0$.

Proof. By taking the derivative directly, we have

$$\begin{aligned} \boldsymbol{\chi}_z &= (1 - w\dot{\theta}\sin\theta)\boldsymbol{T} + Q\boldsymbol{N} + w\dot{\theta}\cos\theta\boldsymbol{B}, \\ \boldsymbol{\chi}_w &= \cos\theta\boldsymbol{T} + \sin\theta\boldsymbol{B}. \end{aligned}$$

Then, from the cross product we obtain

$$\boldsymbol{\chi}_z \wedge \boldsymbol{\chi}_w = Q\sin\theta\boldsymbol{T} + P\boldsymbol{N} - Q\cos\theta\boldsymbol{B}$$

and we have that $P = Q = 0$ if and only if $\boldsymbol{\chi}_z \wedge \boldsymbol{\chi}_w = \boldsymbol{0}$. □

Corollary 3.3.
(1) If $\sin\theta = 0, \cos\theta = 1$ and $\boldsymbol{\chi}(z,w) = \boldsymbol{\gamma}(z) + w\boldsymbol{T}(z)$, the set of singular points is

$$U = \{(z,w) \in I \times \mathbb{R} \mid w\kappa(z) = 0\}.$$

(2) If $\sin\theta = 1, \cos\theta = 0$ and $\boldsymbol{\chi}(z,w) = \boldsymbol{\gamma}(z) + w\boldsymbol{B}(z)$, it has no singular points.

To identify the condition of being cross caps, singular points are divided into two classes U_1 and U_2. From Proposition 3.2 singular points satisfy the following:

$$\sin\theta - w\dot{\theta} = 0,$$

$$w(\kappa\cos\theta + \tau\sin\theta) = 0.$$

Let us assume that $w = 0$. Then, the class U_1 can be given by

$$U_1 = \{(z,0) \in I \times \mathbb{R} \mid \sin\theta = 0\}.$$

If $w \neq 0$, the class U_2 can be given by

$$U_2 = \{(z,w) \in I \times \mathbb{R} \mid \kappa\cos\theta + \tau\sin\theta = 0, \sin\theta - w\dot{\theta} = 0, w \neq 0\}.$$

Theorem 3.4. Let $\boldsymbol{\chi}(z,w)$ be a ruled surface of the regular spacelike base curve $\boldsymbol{\gamma}(z)$ with timelike principal normal vectors. Then apply the following:
(1) If $(z_1, 0) \in U_1$, then $\boldsymbol{\chi}(z,w)$ has no cross-cap.
(2) If $(z_2, w_2) \in U_2$ and $\dot{\theta}(z_2)\dot{f}(z_2) \neq 0$, where $f(z) = \kappa(z)\cos\theta(z) + \tau(z)\sin\theta(z)$, then $\boldsymbol{\chi}(z,w)$ has a cross-cap at the point (z_2, w_2).
Proof. If $(z_1, 0) \in U_1$, we can get

$$\begin{aligned}
\boldsymbol{\chi}_w(z_1,0) &= \cos\theta(z_1)\boldsymbol{T}(z_1),\\
\boldsymbol{\chi}_{wz}(z_1,0) &= \kappa(z_1)\cos\theta(z_1)\boldsymbol{N}(z_1) + \dot{\theta}(z_1)\cos\theta(z_1)\boldsymbol{B}(z_1),\\
\boldsymbol{\chi}_{zz}(z_1,0) &= \kappa(z_1)\boldsymbol{N}(z_1).
\end{aligned}$$

And

$$\det(\boldsymbol{\chi}_w, \boldsymbol{\chi}_{wz}, \boldsymbol{\chi}_{zz}) = -\dot{\theta}(z_1)\kappa(z_1).$$

But, from cross-cap judging theorem in [18], we know that $\boldsymbol{\chi}_z(z_1,0) = \boldsymbol{T}(z_1) \neq \boldsymbol{0}$, so if $(z_1,0) \in U_1$, then $\boldsymbol{\chi}(z,w)$ has no cross-cap.

If $(z_2, w_2) \in U_2$, we can get

$$\begin{aligned}
\boldsymbol{\chi}_w(z_2,w_2) &= \cos\theta(z_2)\boldsymbol{T}(z_2) + \sin\theta(z_2)\boldsymbol{B}(z_2),\\
\boldsymbol{\chi}_{wz}(z_2,w_2) &= -\dot{\theta}(z_2)\sin\theta(z_2)\boldsymbol{T}(z_2) + \dot{\theta}(z_2)\cos\theta(z_2)\boldsymbol{B}(z_2),\\
\boldsymbol{\chi}_{zz}(z_2,w_2) &= (-w_2\dot{\theta}^2(z_2)\cos\theta(z_2) - w_2\ddot{\theta}(z_2)\sin\theta(z_2))\boldsymbol{T}(z_2) + w_2\dot{f}(z_2)\boldsymbol{N}(z_2)\\
&\quad + (w_2\ddot{\theta}(z_2)\cos\theta(z_2) - w_2\dot{\theta}^2(z_2)\sin\theta(z_2))\boldsymbol{B}(z_2),
\end{aligned}$$

where $f(z) = \kappa(z)\cos\theta(z) + \tau(z)\sin\theta(z)$.

And

$$\det(\boldsymbol{\chi}_w, \boldsymbol{\chi}_{wz}, \boldsymbol{\chi}_{zz}) = -w_2\dot{\theta}(z_2)\dot{f}(z_2).$$

Then, from cross-cap judging theorem in [18], we know that, if $(z_2, w_2) \in U_2$ and $\dot{\theta}(z_2)\dot{f}(z_2) \neq 0$, $\boldsymbol{\chi}(z,w)$ is a cross-cap at the point (z_2, w_2). □

Proposition 3.5. Let $\boldsymbol{\chi}(z,w)$ be a ruled surface of a regular spacelike base curve $\gamma(z)$ with timelike principal normal vectors and $\sin\theta(z)\cos\theta(z) \neq 0$, for any $z \in I$. The following conclusions are obtained:

(1) The ruled surface $\boldsymbol{\chi}(z,w)$ is cylindrical if and only if $\gamma(z)$ is a helix.
(2) The base curve $\gamma(z)$ of $\boldsymbol{\chi}(z,w)$ is its striction curve if and only if $\theta(z)$ is constant.
Proof. (1) By differentiating $\boldsymbol{q} = \cos\theta\boldsymbol{T} + \sin\theta\boldsymbol{B}$, we can get

$$\dot{\boldsymbol{q}} = -\dot{\theta}\sin\theta\boldsymbol{T} + (\kappa\cos\theta + \tau\sin\theta)\boldsymbol{N} + \dot{\theta}\cos\theta\boldsymbol{B}.$$

If the ruled surface $\boldsymbol{\chi}(z,w)$ is cylindrical, $\dot{\boldsymbol{q}} = \boldsymbol{0}$. Then

$$\begin{aligned} \dot{\theta}\sin\theta &= 0, \\ \kappa\cos\theta + \tau\sin\theta &= 0, \\ \dot{\theta}\cos\theta &= 0. \end{aligned}$$

We know that θ is constant and $\frac{\kappa}{\tau} = -\tan\theta$, that is γ is a helix.
(2) The expression of the striction curve of $\boldsymbol{\chi}(z,w)$ is obtained as follows:

$$\begin{aligned} \boldsymbol{\beta}(z) &= \gamma - \frac{\langle\dot{\gamma},\dot{\boldsymbol{q}}\rangle}{\langle\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}\rangle}\boldsymbol{q} \\ &= \gamma + \frac{\dot{\theta}\sin\theta}{\dot{\theta}^2 - (\kappa\cos\theta + \tau\sin\theta)^2}\boldsymbol{q}. \end{aligned}$$

It is clear that $\gamma(z)$ of $\boldsymbol{\chi}(z,w)$ is its striction curve if and only if $\theta(z)$ is constant. □

Proposition 3.6. Let the base curve $\gamma(z)$ be a striction curve of $\boldsymbol{\chi}(z,w)$ and $\sin\theta(z)\cos\theta(z) \neq 0$, for any $z \in I$. $\boldsymbol{\chi}(z,w)$ is a developable surface if and only if $\gamma(z)$ is a helix.
Proof. We get

$$\det(\dot{\gamma},\boldsymbol{q},\dot{\boldsymbol{q}}) = -\sin\theta(\kappa\cos\theta + \tau\sin\theta).$$

$\boldsymbol{\chi}(z,w)$ is developable if and only if $\det(\dot{\gamma},\boldsymbol{q},\dot{\boldsymbol{q}}) = 0$. Then

$$\kappa(z)\cos\theta(z) + \tau(z)\sin\theta(z) = 0.$$

If the base curve $\gamma(z)$ is a striction curve, $\theta(z)$ is constant. Finally, we can get the result. □

Corollary 3.7. Let the base curve $\gamma(z)$ be a striction curve of $\boldsymbol{\chi}(u,w)$ and $\sin\theta(z)\cos\theta(z) \neq 0$, for any $z \in I$. $\boldsymbol{\chi}(z,w)$ is a developable surface if and only if $\boldsymbol{\chi}(z,w)$ is cylindrical.

Corollary 3.8. If the ruled surface $\boldsymbol{\chi}(z,w)$ is cylindrical, its parameter expression is

$$\boldsymbol{\chi}(z,w) = \gamma(z) + w(c\boldsymbol{T}(z) + \sqrt{1-c^2}\boldsymbol{B}(z)),$$

where $c \in (-1,1)$ is constant.
Proof. Let the ruled surface $\boldsymbol{\chi}(z,w)$ of the base curve $\gamma(z)$ be cylindrical. According to Proposition 3.5, $\theta(z)$ is constant. We can write $\sin\theta(z) = c$ and $\cos\theta(z) = \sqrt{1-c^2}$. So $\boldsymbol{\chi}(z,w) = \gamma(z) + w(c\boldsymbol{T}(z) + \sqrt{1-c^2}\boldsymbol{B}(z))$. □

Proposition 3.9. The trajectory of the singular points of $\chi(z,w)$ can be expressed as follows:
(1) If the singular points belongs to U_1, the trajectory of the singular points of $\chi(z,w)$ is $\gamma(z)$.
(2) If the singular points belongs to U_2, the trajectory of the singular points of $\chi(z,w)$ can be expressed as $\boldsymbol{\rho}(z) = \gamma(z) + \frac{\sin\theta(z)}{\dot{\theta}(z)}(\cos\theta(z)\boldsymbol{T}(z) + \sin\theta(z)\boldsymbol{B}(z))$.

4. Non-lightlike framed surface

Definition 4.1. Let U be an open domain of $\mathbb{R}^2$. We call $(\chi, \psi_1, \psi_2) : U \to \mathbb{R}^3_1 \times \triangle$ a non-lightlike framed surface if $\langle \chi_z(z,w), \psi_1(z,w)\rangle = \langle \chi_w(z,w), \psi_1(z,w)\rangle = 0$ for all $(z,w) \in U$, where $\chi_z = \frac{\partial \chi}{\partial z}$ and $\chi_w = \frac{\partial \chi}{\partial w}$. A surface $\chi : U \to \mathbb{R}^3_1$ is a non-lightlike framed base surface if there exists $(\psi_1, \psi_2) : U \to \triangle$ such that (χ, ψ_1, ψ_2) is a non-lightlike framed surface.

Define $\psi_3 = \psi_1 \wedge \psi_2$. $\{\psi_1, \psi_2, \psi_3\}$ is a moving frame on the surface χ and the Frenet type formulas are

$$\begin{pmatrix} \chi_z \\ \chi_w \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix}, \tag{4.1}$$

$$\begin{pmatrix} \psi_{1z} \\ \psi_{2z} \\ \psi_{3z} \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -\sigma\delta e_1 & 0 & g_1 \\ \delta f_1 & \sigma g_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{4.2}$$

$$\begin{pmatrix} \psi_{1w} \\ \psi_{2w} \\ \psi_{3w} \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -\sigma\delta e_2 & 0 & g_2 \\ \delta f_2 & \sigma g_2 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{4.3}$$

where $\sigma = \mathrm{sign}(\psi_1(z,w))$, $\delta = \mathrm{sign}(\psi_2(z,w))$.

The functions $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}$ $(i = 1,2)$ are called the basic invariants of (χ, ψ_1, ψ_2).

Remark 4.2. χ is singular at $(z_0, w_0) \in U$ if and only if the determinant $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$.

Because (χ, ψ_1, ψ_2) is smooth, $\chi_{zw} = \chi_{wz}, \psi_{1zw} = \psi_{1wz}$ and $\psi_{2zw} = \psi_{2wz}$. So we can get the following integrability conditions

$$\begin{cases} -\sigma a_1 e_2 + b_1 f_2 = -\sigma a_2 e_1 + b_2 f_1, \\ a_{1w} + \sigma b_1 g_2 = a_{2z} + \sigma b_2 g_1, \\ b_{1w} + a_1 g_2 = b_{2z} + a_2 g_1, \end{cases} \tag{4.4}$$

$$\begin{cases} e_{1w} - e_{2z} = \sigma(f_2 g_1 - f_1 g_2), \\ f_{1w} - f_{2z} = (e_2 g_1 - e_1 g_2), \\ g_{1w} - g_{2z} = \sigma\delta(e_1 f_2 - e_2 f_1). \end{cases} \tag{4.5}$$

Theorem 4.3 (Existence)**.** *Give two groups of smooth functions $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}$ $(i = 1,2)$, which satisfy the integrability conditions (4.4) and (4.5). Then, there exists a non-lightlike framed surface $(\chi, \psi_1, \psi_2) : U \to \mathbb{R}^3_1 \times \Delta$, whose invariants are $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}$, $i = 1,2$.*

Proof. Since a_i, b_i, e_i, f_i, g_i, $i = 1,2$ are smooth and (4.5) holds, there exists a pseudo-orthogonal moving frame $\{\psi_1, \psi_2, \psi_3\}$ that satisfies (4.2) and (4.3). And (4.4) holds; then, there exists a surface $\chi : U \to \mathbb{R}^3_1$ that satisfies (4.1).

So $(\chi, \psi_1, \psi_2) : U \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed surface and its invariants are $a_i, b_i, e_i, f_i, g_i, i = 1, 2$. $\square$

Definition 4.4. *Let* $(\chi, \psi_1, \psi_2) : U \to \mathbb{R}^3_1 \times \Delta$ *and* $(\tilde{\chi}, \tilde{\psi}_1, \tilde{\psi}_2) : U \to \mathbb{R}^3_1 \times \Delta$ *be two non-lightlike framed surfaces.* (χ, ψ_1, ψ_2) *and* $(\tilde{\chi}, \tilde{\psi}_1, \tilde{\psi}_2)$ *are congruent through a Lorentz motion if there exists a matrix A and a constant vector* $\boldsymbol{c} \in \mathbb{R}^3_1$

$$\tilde{\chi}(z, w) = A(\chi(z, w)) + \boldsymbol{c}, \tilde{\psi}_1(z, w) = A(\psi_1(z, w)), \tilde{\psi}_2(z, w) = A(\psi_2(z, w)),$$

for any $(z, w) \in U$, *where A satisfies the following:*

$$A^T G A = G, \det(A) = 1, G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 4.5. If two non-lightlike framed surfaces (χ, ψ_1, ψ_2) and $(\tilde{\chi}, \tilde{\psi}_1, \tilde{\psi}_2)$ are congruent through a Lorentz motion, then their basic invariants are equal, that is $(a_i, b_i, e_i, f_i, g_i) = (\tilde{a}_i, \tilde{b}_i, \tilde{e}_i, \tilde{f}_i, \tilde{g}_i), i = 1, 2$.

Theorem 4.6 (Uniqueness)**.** *Let* (χ, ψ_1, ψ_2) *and* $(\tilde{\chi}, \tilde{\psi}_1, \tilde{\psi}_2)$ *be two non-lightlike framed surfaces which have the same basic invariants* $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}$ $(i = 1, 2)$. *Assume that two surfaces have the same time orientation. Then* (χ, ψ_1, ψ_2) *and* $(\tilde{\chi}, \tilde{\psi}_1, \tilde{\psi}_2)$ *are congruent through a Lorentz motion.*

Proof. Fix a point $(z_0, w_0) \in U$; then,there exists a Lorentz motion $A \in \mathbb{R}^{3\times3}$ such that $\tilde{\psi}_1(z_0, w_0) = A\psi_1(z_0, w_0)$ and $\tilde{\psi}_2(z_0, w_0) = A\psi_2(z_0, w_0)$. Then we have that $\tilde{\psi}_3(z_0, w_0) = A\psi_3(z_0, w_0)$.

Note that $\{\psi_1, \psi_2, \psi_3\}$ and $\{\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3\}$ are both solutions of (4.2) and (4.3), so $\{\psi_1, \psi_2, \psi_3\} = \{\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3\}$.

Take $\boldsymbol{c} = \tilde{\chi}(z_0, w_0) - A\chi(z_0, w_0)$. Since $\tilde{\chi}(z, w)$ and $A\chi(z, w) + \boldsymbol{c}$ both satisfy the Eq (4.1), then $\tilde{\chi}(z, w) = A\chi(z, w) + \boldsymbol{c}$. Thus (χ, ψ_1, ψ_2) and $(\tilde{\chi}, \tilde{\psi}_1, \tilde{\psi}_2)$ are congruent through a Lorentz motion. $\square$

Definition 4.7. Let $(\chi, \psi_1, \psi_2) : U \to \mathbb{R}^3_1 \times \triangle$ be a non-lightlike framed surface. Define the following functions

$$\begin{aligned} E &= \langle \chi_z, \chi_z \rangle = \delta a_1^2 - \sigma\delta b_1^2, \\ F &= \langle \chi_z, \chi_w \rangle = \delta a_1 a_2 - \sigma\delta b_1 b_2, \\ G &= \langle \chi_w, \chi_w \rangle = \delta a_2^2 - \sigma\delta b_2^2, \\ L &= -\langle \chi_z, \psi_{1z} \rangle = -\delta a_1 e_1 + \sigma\delta b_1 f_1, \\ M &= -\langle \chi_z, \psi_{1w} \rangle = -\delta a_1 e_2 + \sigma\delta b_1 f_2, \\ N &= -\langle \chi_w, \psi_{1w} \rangle = -\delta a_2 e_2 + \sigma\delta b_2 f_2. \end{aligned}$$

We call E, F, G the first fundamental invariants of (χ, ψ_1, ψ_2) and L, M, N the second fundamental invariants of (χ, ψ_1, ψ_2).

Definition 4.8. Let $(\chi, \psi_1, \psi_2) : U \to \mathbb{R}^3_1 \times \triangle$ be a non-lightlike framed surface. Define

$$J_\chi = \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, K_\chi = \det\begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix},$$

$$H_\chi = \frac{1}{2}\left(-\det\begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} + \det\begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}\right).$$

We call (J_χ, K_χ, H_χ) the curvature of the non-lightlike framed surface (χ, ψ_1, ψ_2).

Proposition 4.9. Let $\chi : U \to \mathbb{R}^3_1$ be a non-lightlike regular surface. The Gauss curvature and the mean curvature of (χ, ψ_1, ψ_2) are respectively denoted by

$$K = \frac{K_\chi}{J_\chi}, H = \frac{H_\chi}{J_\chi}.$$

5. Special spacelike framed ruled surface

Definition 5.1. Assume that $(\chi, \psi_1, \psi_2) : U \to \mathbb{R}^3_1 \times \triangle$ is a non-lightlike framed surface. $\gamma(s) = \chi(z(s), w(s))$ is a curve on χ.
(1) If the principal normal vector of γ is parallel to ψ_1, γ is called the geodesic curve.
(2) If the principal normal vector of γ is perpendicular to ψ_1, γ is called the asymptotic curve.
(3) If the tangent vector of the curve γ is parallel to $\frac{d}{ds}\psi_1$, γ is called the line of curvature.

Proposition 5.2. Let $\gamma(z)$ be a spacelike base curve with timelike principal normal vectors. $\chi(z, w)$ is the ruled surface of $\gamma(z)$. If there exist two smooth functions $\varphi, \phi : I \times \mathbb{R} \to \mathbb{R}$ satisfying $\langle \chi_z, \psi_1 \rangle = 0$, $\langle \chi_w, \psi_1 \rangle = 0$ and

$$w \cosh\varphi(\kappa\cos\theta + \tau\sin\theta) = -\sinh\varphi(\sin\theta - w\dot{\theta}),$$

then $\chi : I \times \mathbb{R} \to \mathbb{R}^3_1$ is a spacelike framed base surface, where

$$\begin{aligned}
\psi_1 &= -\sinh\varphi\sin\theta\boldsymbol{T} + \cosh\varphi\boldsymbol{N} + \sinh\varphi\cos\theta\boldsymbol{B}, \\
\psi_2 &= (\sin\phi\cos\theta - \cos\phi\sin\theta\cosh\varphi)\boldsymbol{T} + \cos\phi\sinh\varphi\boldsymbol{N} + (\sin\phi\sin\theta + \cos\phi\cos\theta\cosh\varphi)\boldsymbol{B}, \\
\psi_3 &= (\cos\phi\cos\theta + \sin\phi\sin\theta\cosh\varphi)\boldsymbol{T} - \sin\phi\sinh\varphi\boldsymbol{N} + (\cos\phi\sin\theta - \sin\phi\cos\theta\cosh\varphi)\boldsymbol{B}.
\end{aligned}$$

Proof. According to Proposition 3.2,

$$\chi_z \wedge \chi_w = w\sin\theta(\kappa\cos\theta + \tau\sin\theta)\boldsymbol{T} + (\sin\theta - w\dot{\theta})\boldsymbol{N} - w\cos\theta(\kappa\cos\theta + \tau\sin\theta)\boldsymbol{B}$$

Since ψ_1 is parallel to $\chi_z \wedge \chi_w$ at the regular points, we obtain

$$w\cosh\varphi(\kappa\cos\theta + \tau\sin\theta) = -\sinh\varphi(\sin\theta - w\dot{\theta}).$$

By calculation,

$$\begin{aligned}
\langle \chi_z, \psi_1 \rangle &= -\sinh\varphi\sin\theta(1 - w\dot{\theta}\sin\theta) - w\cosh\varphi(k\cos\theta + \tau\sin\theta) + w\dot{\theta}\cos^2\theta \\
&= w\dot{\theta}\sinh\varphi - \sin\theta\sinh\varphi - w\cosh\varphi(k\cos\theta + \tau\sin\theta) \\
&= 0, \\
\langle \chi_w, \psi_1 \rangle &= -\sinh\varphi\sin\theta\cos\theta + \sinh\varphi\sin\theta\cos\theta \\
&= 0.
\end{aligned}$$

Finally, we get the proposition. □

Remark 5.3. If there exist two smooth functions $\varphi, \phi : I\times\mathbb{R} \to \mathbb{R}$ satisfying $\langle \chi_z, \psi_1\rangle = 0, \langle \chi_w, \psi_1\rangle = 0$ and

$$w\sinh\varphi(\kappa\cos\theta + \tau\sin\theta) = -\cosh\varphi(\sin\theta - w\dot{\theta}),$$

$\chi : I\times\mathbb{R} \to \mathbb{R}^3_1$ is a timelike framed base surface. It is similar to the spacelike framed base surface in the context of the differential geometry property. So we take the spacelike framed surface as an example.

After this part, for convenience, let

$$y = -w\dot{\theta}\cosh\varphi + w\kappa\sinh\varphi\cos\theta + \cosh\varphi\sin\theta + w\tau\sinh\varphi\sin\theta.$$

Proposition 5.4. The basic invariants of (χ, ψ_1, ψ_2) can be given by

$$\begin{aligned}
a_1 &= \sin\phi\cos\theta - \cos\phi y,\\
b_1 &= \cos\phi\cos\theta + \sin\phi y,\\
a_2 &= \sin\phi,\\
b_2 &= \cos\phi,\\
e_1 &= \sin\phi\sin\theta\cosh\varphi\tau - \dot{\theta}\sinh\varphi\sin\phi + \cos\phi\cos\theta\tau\\
&\quad + \kappa\cosh\varphi\sin\phi\cos\theta + \varphi_z\cos\phi - \kappa\sin\theta\cos\phi,\\
f_1 &= \cos\phi\sin\theta\cosh\varphi\tau - \dot{\theta}\sinh\varphi\cos\phi - \sin\phi\cos\theta\tau\\
&\quad + \kappa\cosh\varphi\cos\phi\cos\theta - \varphi_z\sin\phi + \kappa\sin\theta\sin\phi,\\
g_1 &= \kappa\sinh\varphi\cos\theta - \dot{\theta}\cosh\varphi + \phi_z + \tau\sinh\varphi\sin\theta,\\
e_2 &= \varphi_w\cos\phi,\\
f_2 &= -\varphi_w\sin\phi,\\
g_2 &= \phi_w.
\end{aligned}$$

Proposition 5.5. The curvature (J_χ, K_χ, H_χ) of (χ, ψ_1, ψ_2) is obtained

$$\begin{aligned}
J_\chi &= -y,\\
K_\chi &= \dot{\theta}\varphi_w\sinh\varphi - \kappa\varphi_w\sinh\varphi\cos\theta - \tau\varphi_w\cosh\varphi\sin\theta,\\
H_\chi &= \frac{1}{2}(\varphi_w\cos\theta - \tau\cos\theta + \kappa\sin\theta - \varphi_z).
\end{aligned}$$

Proposition 5.6. Let $\chi : I\times\mathbb{R} \to \mathbb{R}^3_1$ be a regular surface. The coefficients of the fundamental forms are calculated as follows:

$$\begin{aligned}
E &= \cos^2\theta + y^2,\\
F &= \cos\theta,\\
G &= 1,\\
L &= \dot{\theta}\sinh\varphi\cos\theta - \tau\sin\theta\cos\theta\cosh\varphi - \kappa\sin\theta y + \varphi_z y - \kappa\cosh\varphi\cos^2\theta + \tau\cos\theta y,\\
M &= \varphi_w y,\\
N &= 0.
\end{aligned}$$

Proposition 5.7. The Gauss curvature and the mean curvature of the regular surface $\chi(z,w)$ can be respectively expressed as

$$K = -\frac{\dot{\theta}\varphi_w \sinh\varphi - \kappa\varphi_w \sinh\varphi\cos\theta - \tau\varphi_w\cosh\varphi\sin\theta}{y},$$
$$H = -\frac{\varphi_w\cos\theta - \tau\cos\theta + \kappa\sin\theta - \varphi_z}{2y}.$$

Corollary 5.8. The regular surface $\chi(z,w) : I\times\mathbb{R}\to\mathbb{R}^3_1$ is developable if and only if $\varphi_w = 0$.
Proof. Because $\chi(z,w)$ is smooth, we obtain

$$\varphi_w y = \dot{\theta}\sinh\varphi - \kappa\sinh\varphi\cos\theta - \tau\cosh\varphi\sin\theta.$$

So $K = -\varphi_w^2$. Therefore, $\chi(z,w)$ is developable if and only if $\varphi_w = 0$. □

Corollary 5.9. The regular surface $\chi(z,w) : I\times\mathbb{R}\to\mathbb{R}^3_1$ is a maximal surface if and only if $\varphi_w\cos\theta - \tau\cos\theta + \kappa\sin\theta - \varphi_z = 0$.

Proposition 5.10. Assume that Π is a spacelike framed base surface. The base curve $\gamma(z)$ of Π is a geodesic curve if and only if $\sinh\varphi = 0$ or $\kappa = 0$.
Proof. According to the definition of a geodesic curve in Definition 5.1, $\ddot{\gamma}\wedge\psi_1 = 0$ is a sufficient and necessary condition of a geodesic curve:

$$\ddot{\gamma}\wedge\psi_1 = \kappa\sinh\varphi\cos\theta\boldsymbol{T} + \kappa\sinh\varphi\sin\theta\boldsymbol{B}.$$

Then, we get the result. □

Proposition 5.11. Assume that Π is a spacelike framed base surface. The base curve $\gamma(z)$ of Π is an asymptotic curve if and only if $\kappa = 0$.
Proof. According to the definition of an asymptotic curve in Definition 5.1, $\langle\ddot{\gamma},\psi_1\rangle = 0$ is a sufficient and necessary condition of the asymptotic curve:

$$\langle\ddot{\gamma},\psi_1\rangle = -\kappa\cosh\varphi.$$

So $\gamma(z)$ is an asymptotic curve if and only if $\kappa = 0$. □

Proposition 5.12. Let Π be a spacelike framed base surface. The results are as follows.
(1) The z-parameter curve of Π is an asymptotic curve if and only if

$$w\ddot{\theta}\sinh\varphi - w\kappa f\sinh\varphi\sin\theta + w\dot{\theta}\kappa\sin\theta\cosh\varphi - \kappa\cosh\varphi$$
$$- w\dot{f}\cosh\varphi - w\tau\dot{\theta}\cos\theta\cosh\varphi + w\tau f\sinh\varphi\cos\theta = 0.$$

(2) The w-parameter curve of Π is always an asymptotic curve.
Proof. (1) According to the definition of an asymptotic curve in Definition 5.1, $\langle\chi_{zz},\psi_1\rangle = 0$ is a sufficient and necessary condition for the z-parameter curve to be an asymptotic curve. We can get

$$\langle\chi_{zz},\psi_1\rangle = w\ddot{\theta}\sinh\varphi - w\kappa f\sinh\varphi\sin\theta + w\dot{\theta}\kappa\sin\theta\cosh\varphi - \kappa\cosh\varphi$$

$$- w\dot{f}\cosh\varphi - w\tau\dot{\theta}\cos\theta\cosh\varphi + w\tau f\sinh\varphi\cos\theta.$$

Then, we get the result.
(2) $\langle \chi_{ww}, \psi_1 \rangle = 0$ is a sufficient and necessary condition for the w-parameter curve to be an asymptotic curve. We can get

$$\langle \chi_{ww}, \psi_1 \rangle = 0.$$

Then we get the result. □

Proposition 5.13. Let Π be a spacelike framed base surface; the z-parameter curve and the w-parameter curve of Π are lines of curvature if and only if
(i) $\cos\theta = 0, \varphi_w = 0$,
or
(ii) $\cos\theta = 0, \cosh\varphi\sin\theta + w\tau\sinh\varphi\sin\theta = 0$.
Proof. According to the definition of a line of curvature in Definition 5.1, $\langle \chi_z, \chi_w \rangle = \langle \chi_z, \psi_{1w} \rangle = 0$ is a sufficient and necessary condition for the parameter curves to be lines of curvature. That means that $F = M = 0$. We know, from in Proposition 5.6, that

$$F = \cos\theta, M = \varphi_v y.$$

Then, we get the result. □

Theorem 5.14. Let Π be a developable spacelike framed base surface given by (χ, ψ_1, ψ_2). Π is a maximal surface if and only if the z-parameter curve of Π is an asymptotic curve.
Proof. From Proposition 5.6, we know that $N = 0$ and $G = 1$. So according to the expression of a developable surface, if Π is developable, then $M = 0$. The z-parameter curve of Π is an asymptotic curve; then, $L = 0$. Thus, $H = 0$. Finally, we can get the result. □

Proposition 5.15. Let the z-parameter curve and the w-parameter curves of Π be lines of curvature of the spacelike framed base surface given by (χ, ψ_1, ψ_2). The z-parameter curve of Π is an asymptotic curve if and only if Π is a maximal surface.
Proof. According to the definition of a line of curvature, $F = M = 0$ is a sufficient and necessary condition for parameter curves to be lines of curvature. And we know that $N = 0$ and $G = 1$. Assume that the z-parameter curve of Π is an asymptotic curve, which means that $L = 0$. Thus, $H = 0$. Finally, we can get the result. □

6. Example

This special ruled surface which we consider in Section 5 may have singular points; we will give an example. See Figure 1.

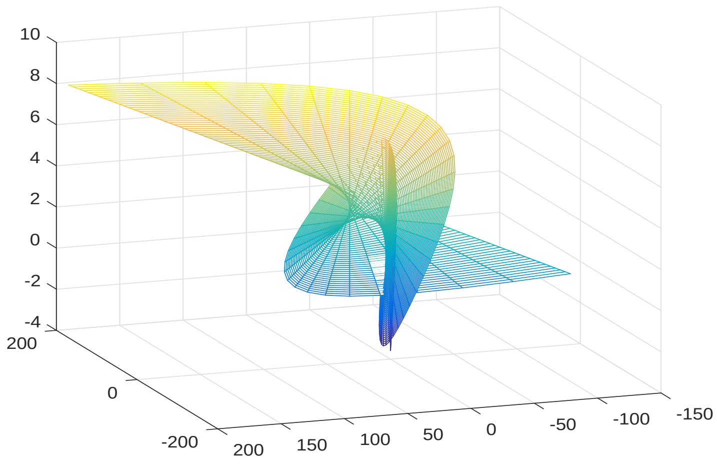

Figure 1. $\chi(z, w)$ with $z \in [0, 2\pi)$ and $w \in (-5, 5)$.

Example 6.1. Let $\gamma : [0, 2\pi) \to \mathbb{R}^3_1$ be a regular spacelike curve defined by

$$\gamma(z) = \left(\cosh \frac{\sqrt{2}}{2} z, \sinh \frac{\sqrt{2}}{2} z, \frac{\sqrt{2}}{2} z\right).$$

Through calculation, we obtain

$$\boldsymbol{T}(z) = \left(\frac{\sqrt{2}}{2} \sinh \frac{\sqrt{2}}{2} z, \frac{\sqrt{2}}{2} \cosh \frac{\sqrt{2}}{2} z, \frac{\sqrt{2}}{2}\right),$$

$$\boldsymbol{N}(z) = \left(\cosh \frac{\sqrt{2}}{2} z, \sinh \frac{\sqrt{2}}{2} z, 0\right)$$

and

$$\boldsymbol{B}(z) = \left(\frac{\sqrt{2}}{2} \sinh \frac{\sqrt{2}}{2} z, \frac{\sqrt{2}}{2} \cosh \frac{\sqrt{2}}{2} z, -\frac{\sqrt{2}}{2}\right),$$

respectively. And $\kappa = \tau = \frac{\sqrt{2}}{2}$. Let us assume that $\cos\theta(z) = \cos z$ and $\sin\theta(z) = \sin z$. So $\chi(z, w) = \gamma(z) + w\boldsymbol{q}(z)$, where

$$\boldsymbol{q}(z) = \left(\frac{\sqrt{2}}{2} \sinh \frac{\sqrt{2}}{2} z(\cos z + \sin z), \frac{\sqrt{2}}{2} \cosh \frac{\sqrt{2}}{2} z(\cos z + \sin z), \frac{\sqrt{2}}{2}(\cos z - \sin z)\right).$$

(1) If $w = 0$ and $\sin z_1 = 0$, the surface is singular at $(0, 0)$ and $(\pi, 0)$.

(2) $\kappa = \tau = \frac{\sqrt{2}}{2}$ is constant and $\sin\theta - w\dot{\theta} \neq 0$. So for any $(z_2, w_2) \in U_2$ is not a singular point, where $U_2 = \{(z_2, w_2) \in I \times \mathbb{R} \mid \kappa\cos\theta + \tau\sin\theta = 0, \sin\theta - w\dot{\theta} = 0, w \neq 0\}$.

7. Conclusions

In this paper, we have investigated the singular properties of a special ruled surface which is generated by spacelike straight lines and given the basic theory of non-lightlike framed surfaces. Regarding the ruled surface as a non-lightlike framed base surface, we have discussed its differential geometric properties.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 11671070).

Conflict of interest

The authors declare that there are no conflicts of interest that may influence the publication of this work.

References

1. M. Emmer, *Imagine math*, Milano: Springer, 2012. https://doi.org/10.1007/978-88-470-2427-4
2. O. Kaya, M. Önder, Generalized normal ruled surface of a curve in the Euclidean 3-space, *Acta U. Sapientiae-Ma.*, **13** (2021), 217–238. https://doi.org/10.2478/ausm-2021-0013
3. O. Kaya, T. Kahraman, M. Önder, Osculating-type ruled surfaces in the Euclidean 3-space, *Facta Univ-Ser. Math.*, **36** (2021), 939–959.
4. M. Önder, T. Kahraman, On rectifying ruled surfaces, *Kuwait J. Sci.*, **47** (2020), 1–11.
5. S. Honda, M. Takahashi, Framed curves in the Euclidean space, *Adv. Geom.*, **16** (2016), 265–276. https://doi.org/10.1515/advgeom-2015-0035
6. Ö. G. Yıldız, M. Akyiğit, M. Tosun, On the trajectory ruled surfaces of framed base curves in the Euclidean space, *Math. Meth. Appl. Sci.*, **44** (2021), 7463–7470. https://doi.org/10.1002/mma.6267
7. T. Fukunaga, M. Takahashi, Framed surfaces in the Euclidean space, *Bull. Braz. Math. Soc. New Series*, **50** (2019), 37–65. https://doi.org/10.1007/s00574-018-0090-z
8. J. Huang, D. Pei, Singularities of non-developable surfaces in three-dimensional Euclidean space, *Mathematics*, **7** (2019), 1106. https://doi.org/10.3390/math7111106
9. B. D. Yazıcı, Z. Işbilir, M. Tosun, Generalized osculating-type ruled surfaces of singular curves, *Math. Meth. Appl. Sci.*, **46** (2023), 85320–8545. https://doi.org/10.1002/mma.8997
10. K. Eren, Ö. G. Yıldız, M. Akyiğit, Tubular surfaces associated with framed base curves in Euclidean 3-space, *Math. Meth. Appl. Sci.*, **45** (2022), 12110–12118. https://doi.org/10.1002/mma.7590
11. O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space L^3, *Tokyo J. Math.*, **6** (1983), 297–309. https://doi.org/10.3836/tjm/1270213872
12. Y. H. Kim, D. W. Yoon, Classification of ruled surfaces in Minkowski 3-spaces, *J. Geom. Phys.*, **49** (2004), 89–100. https://doi.org/10.1016/S0393-0440(03)00084-6

13. A. T. Vanli, H. H. Hacisalihoglu, On the distribution parameter of timelike ruled surfaces in the Minkowski 3-space, *Far East J. Math. Sci.*, **5** (1997), 321–328.

14. N. Yüksel, The ruled surfaces according to Bishop frame in Minkowski 3-space, *Abstr. Appl. Anal.*, **2013** (2013), 810640. https://doi.org/10.1155/2013/810640

15. A. Turgut, H. H. Hacısalihoğlu, Spacelike ruled surfaces in the Minkowski 3-Space, *Commun. Fac. Sci. Univ. Ank. Series A1*, **46** (1997), 83–91.

16. Y. Yayli, S. Saracoglu, On developable ruled surfaces in Minkowski space, *Adv. Appl. Clifford Algebras*, **22** (2012), 499–510. https://doi.org/10.1007/s00006-011-0305-5

17. H. Liu, Ruled surfaces with lightlike ruling in 3-Minkowski space, *J. Geom. Phys.*, **59** (2009), 74–78. https://doi.org/10.1016/j.geomphys.2008.10.003

18. S. Izumiya, N. Takeuchi, Geometry of ruled surfaces, In: *Applicable mathematics in the golden age*, New Delhi: Narosa Publishing House, 2003.

AIMS Press