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*Research article*

# The generalized Kloosterman's sums and its fourth power mean

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**Abstract:** The main purpose of this article is to study the calculating problem of one kind fourth power mean of the generalized Kloosterman's sums and provide an accurate calculating formula for it utilizing analytical methods and character sums' properties. Simultaneously, the work also provides a fresh and valuable approach for researching the related power mean problem.

**Keywords:** generalized Kloosterman's sums; fourth power mean; analytic method; calculating formula
**Mathematics Subject Classification:** 11L03, 11L05

## 1. Introduction

Let $q$ be an integer with $q > 1$. The generalized Kloosterman's sums $S(m, n, \chi; q)$ are defined as follows for all integers $m$, $n$, and any Dirichlet character $\chi$ mod $q$:

$$S(m, n, \chi; q) = \sum_{a=1}^{q-1} \chi(a) e\left(\frac{ma + n\overline{a}}{q}\right),$$

where $\overline{a}$ denotes $a \cdot \overline{a} \equiv 1 \bmod q$, $e(y) = e^{2\pi i y}$, and $i^2 = -1$.

Classical Kloosterman's sums (see H. D. Kloosterman [8]) are $S(m, n, \chi_0; q) = S(m, n; q)$ if $\chi = \chi_0$ is the main character modulo $q$. It is common knowledge that for a prime $p$,

$$\sum_{a=1}^{p-1} e\left(\frac{a + n\overline{a}}{p}\right) = -2\sqrt{p}\cos(\theta(n)),$$

where the angles $\theta(n)$ is equi-distributed in $[0, \pi]$ with respected to the Sato-Tate measure $\frac{2}{\pi}\sin^2(\theta)d\theta$. Additional information can be found in the provided reference [7].

In the study of analytical number theory, Kloosterman's sums is crucial. As a result, some scholars have studied the various properties of $S(m,n;q)$, and obtained a series of significant results. For instance, S. Chowla [7] or T. Estermann [3] proved the upper bound estimate:

$$\sum_{a=1}^{q}{}' e\left(\frac{ma+n\overline{a}}{q}\right) \ll (m,n,q)^{\frac{1}{2}} \cdot d(q) \cdot q^{\frac{1}{2}},$$

where $\sum_{a=1}^{q}{}'$ denotes the summation over all $1 \le a \le q$ such that $(a,q)=1$, $d(n)$ denotes the Dirichlet divisor function, $(m,n,q)$ denotes the greatest common factor of $m$, $n$, and $q$.

H. Salié [11] demonstrated that for any odd prime $p$, one had the identity

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{a+m\overline{a}}{p}\right)\right|^4 = 2p^3-3p^2-3p.$$

The proofs of this result can also be found in H. Iwaniec [5].

W. P. Zhang [14] used the elementary methods to prove a generalized result. In other words, for any integer $n$ with $(n,q)=1$, one has the identity

$$\sum_{m=1}^{q}\left|\sum_{a=1}^{q}{}' e\left(\frac{ma+n\overline{a}}{q}\right)\right|^4 = 3^{\omega(q)}q^2\phi(q)\prod_{p\|q}\left(\frac{2}{3}-\frac{1}{3p}-\frac{4}{3p(p-1)}\right),$$

where $\phi(q)$ is Euler function, $\omega(q)$ denotes the number of all different prime divisors of $q$, $p\|q$ denotes the product over all prime divisors of $q$ with $p \mid q$ and $p^2 \nmid q$.

There are also some equivalent results for the generalized Kloosterman's sums. For instance, A. V. Malyshev [10] demonstrated that for any odd prime $p$, then

$$\sum_{a=1}^{p-1}\chi(a)\, e\left(\frac{ma+n\overline{a}}{p}\right) \ll (m,n,p)^{\frac{1}{2}} \cdot p^{\frac{1}{2}}.$$

W. P. Zhang [13] or J. H. Li and Y. N. Liu [9] used the different methods to prove the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)\, e\left(\frac{ma+n\overline{a}}{p}\right)\right|^4 = \begin{cases} 2p^3-3p^2-3p-1 & \text{if } \chi \text{ is the principal character modulo } p; \\ 3p^3-8p^2 & \text{if } \chi \text{ is the Legendre symbol modulo } p; \\ 2p^3-7p^2 & \text{if } \chi \text{ is a complex character modulo } p, \end{cases}$$

where $p$ is an odd prime.

Some of the other results related to Kloosterman's sums can also be found in references [2, 6, 12, 15, 16]. To save space, all the results are no longer listed here.

In this paper, we consider the calculating problem of the fourth power mean

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right|^2 \cdot \left|\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right|^2, \tag{1.1}$$

where $p$ is an odd prime, $\chi_1$ and $\chi_2$ are two Dirichlet characters modulo $p$.

This issue is undoubtedly a generalization of [13]. In reality, if $\chi_1 = \chi_2$ be the principal characters modulo $p$, then (1.1) becomes [11]. When $\chi_1 = \chi_2$ or $\overline{\chi}_2$, then (1.1) corresponds to [13]. As a result, the mean value (1.1), which extends the fourth power means of the generalized Kloosterman's sums, is significant.

The primary objective of this research is to investigate the calculating issues in (1.1) and provide a precise formula utilizing the analytic methods and the properties of the classical Gauss sums. The specific content of the argument is as follows.

**Theorem 1.** *Let $p > 3$ be an odd prime, $\chi_1$ and $\chi_2$ are characters modulo $p$ (not all principal character modulo $p$). Then we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right|^2$$
$$= \begin{cases} p^2(3p-8) & \text{if } \chi_1=\chi_2 \text{ is the Legendre symbol modulo } p;\\ p^2(2p-7) & \text{if } \chi_1=\chi_2 \text{ or } \overline{\chi}_2 \text{ is not the Legendre symbol modulo } p;\\ p^2(p-6) & \text{if all } \chi_1\neq\chi_2 \text{ and } \chi_1\chi_2 \text{ not } \chi_0 \text{ and } \chi_1(-1)=\chi_2(-1);\\ p^2(p-4) & \text{if all } \chi_1\neq\chi_2 \text{ and } \chi_1\chi_2 \text{ not } \chi_0 \text{ and } \chi_1(-1)\neq\chi_2(-1);\\ p(p^2-4p-2) & \text{if one of } \chi_1 \text{ and } \chi_2 \text{ is } \chi_0 \text{ and } \chi_1(-1)=\chi_2(-1);\\ p(p^2-2p-2) & \text{if one of } \chi_1 \text{ and } \chi_2 \text{ is } \chi_0 \text{ and } \chi_1(-1)\neq\chi_2(-1). \end{cases}$$

**Some notes:** It is evident that our result is a broadening and extension of the findings from [13]. This theorem merely took into account the fact that $p$ is an odd prime. The question of whether there is a calculation formula that corresponds to the universal composite number $q$ is still unresolved.

## 2. Several Lemmas

In this section, in order to make the structure of the theorem proof clear and the content complete, we first introduce three simple lemmas. Hereinafter, the properties of the classical Gauss sums and character sums, as well as some understanding of analytic number theory, elementary number theory, and those topics in general may all be found in numerous number theory textbooks, including [1] and [4]. Therefore, we have omitted more detailed content. The first is as follows.

**Lemma 1.** *Let $p$ be a prime, $\chi_1$ and $\chi_2$ be two fixed Dirichlet characters modulo $p$ (not all principal characters). Then we have the identity*

$$\left|\sum_{m=1}^{p-1}\chi_0(m)\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2$$
$$= \begin{cases} p^2(p-2)^2 & \text{if } \chi_1\chi_2=\chi_0;\\ p^2 & \text{if all } \chi_1,\chi_2 \text{ and } \chi_1\chi_2 \text{ not the principal characters modulo } p;\\ p & \text{if one of } \chi_1 \text{ and } \chi_2 \text{ is the principal character modulo } p, \end{cases}$$

*where $\chi_0$ denotes the principal character modulo $p$.*

*Proof.* For any integer $n$, note that the trigonometric identity

$$\sum_{m=0}^{p-1} e\left(\frac{mn}{p}\right) = \begin{cases} p & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n. \end{cases}$$

From the definition of the classical Gauss sums and note that the identity

$$\tau\left(\overline{\chi}_1\right)\tau\left(\overline{\chi}_2\right) = \chi_1(-1)p,$$

if $\chi_1\chi_2 = \chi_0$, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\\
=\ & \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)\chi_2(b)\sum_{m=1}^{p-1}e\left(\frac{m(a+b)+\overline{a}+\overline{b}}{p}\right)\\
=\ & \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)\chi_2(b)\sum_{m=1}^{p}e\left(\frac{m(a+b)+\overline{a}+\overline{b}}{p}\right) - \tau\left(\overline{\chi}_1\right)\tau\left(\overline{\chi}_2\right)\\
=\ & p\sum_{a=1}^{p-1}\sum_{\substack{b=1\\ p|(a+b)}}^{p-1}\chi_1(a)\chi_2(b) - \tau\left(\overline{\chi}_1\right)\tau\left(\overline{\chi}_2\right)\\
=\ & p\sum_{a=1}^{p-1}\chi_1(a)\chi_2(p-a) - \tau\left(\overline{\chi}_1\right)\tau\left(\overline{\chi}_2\right)\\
=\ & \begin{cases} \chi_2(-1)p(p-1) - \chi_1(-1)p & \text{if } \chi_1\chi_2 = \chi_0, \\ -\tau\left(\overline{\chi}_1\right)\tau\left(\overline{\chi}_2\right) & \text{if } \chi_1\chi_2 \neq \chi_0, \end{cases}
\end{aligned} \tag{2.1}$$

where $\tau(\chi) = \sum_{a=1}^{p-1}\chi(a)e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums.

Note that for any non-principal character $\chi$ mod $p$, we have $|\tau(\chi)| = \sqrt{p}$. From (2.1) we may immediately deduce Lemma 1. □

**Lemma 2.** *Let $p$ be a prime, $\chi_1$ and $\chi_2$ be two fixed Dirichlet characters modulo $p$ (not all principal character modulo $p$). Then we have the identity*

$$\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}\left((a+1)(\overline{a}+1)\right)\right|^2\\
=\ & \begin{cases} (p-1)(2p-5) & \text{if } \chi_1 = \chi_2; \\ (p-1)(p-4) & \text{if } \chi_1 \neq \chi_2 \text{ and } \chi_1(-1) = \chi_2(-1); \\ (p-1)(p-2) & \text{if } \chi_1 \neq \chi_2 \text{ and } \chi_1(-1) \neq \chi_2(-1). \end{cases}
\end{aligned}$$

*Proof.* From the orthogonality of the characters modulo $p$ we have

$$
\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}((a+1)(\overline{a}+1))\right|^2\\
=\ &(p-1)\sum_{\substack{a=1\\(a+1)(\overline{a}+1)\equiv(b+1)(\overline{b}+1) \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}_1(b+1)\overline{\chi}_2\left(\overline{b}+1\right)\\
=\ &(p-1)\sum_{\substack{a=1\\(a-b)(\overline{a}-b)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}_1(b+1)\overline{\chi}_2\left(\overline{b}+1\right)\\
=\ &(p-1)(p-2)+(p-1)\sum_{a=1}^{p-2}\chi_1(a)\overline{\chi}_2(a)-(p-1). \qquad (2.2)
\end{aligned}
$$

If $\chi_1=\chi_2$, then $\chi_1\overline{\chi}_2=\chi_0$, in this time, we have

$$
\sum_{a=1}^{p-2}\chi_1(a)\overline{\chi}_2(a)=p-2. \qquad (2.3)
$$

If $\chi_1\neq\chi_2$, then $\chi_1\overline{\chi}_2\neq\chi_0$, in this time, we have

$$
\begin{aligned}
\sum_{a=1}^{p-2}\chi_1(a)\overline{\chi}_2(a) &= \sum_{a=1}^{p-1}\chi_1(a)\overline{\chi}_2(a)-\chi_1(-1)\chi_2(-1)\\
&= \begin{cases} -1 & \text{if } \chi_1(-1)=\chi_2(-1),\\ 1 & \text{if } \chi_1(-1)\neq\chi_2(-1). \end{cases} \qquad (2.4)
\end{aligned}
$$

Combining formula (2.2)–(2.4) we have the identity

$$
\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}((a+1)(\overline{a}+1))\right|^2\\
=\ &\begin{cases} (p-1)(2p-5) & \text{if } \chi_1=\chi_2;\\ (p-1)(p-4) & \text{if } \chi_1\neq\chi_2 \text{ and } \chi_1(-1)=\chi_2(-1);\\ (p-1)(p-2) & \text{if } \chi_1\neq\chi_2 \text{ and } \chi_1(-1)\neq\chi_2(-1). \end{cases}
\end{aligned}
$$

This proves Lemma 2. □

**Lemma 3.** *Let $p$ be a prime, $\chi_1$ and $\chi_2$ be two fixed Dirichlet characters modulo $p$. Then for any non-principal Dirichlet character $\chi$ mod $p$, we have the identity*

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\chi(m)\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\\
=\ &\tau(\chi)\tau\left(\overline{\chi_1\chi_2}\chi\right)\sum_{a=1}^{p-1}\chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}((a+1)(\overline{a}+1)).
\end{aligned}
$$

*Proof.* From the properties of the classical Gauss sums we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\chi(m)\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\\
=\ &\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)\chi_2(b)\sum_{m=1}^{p-1}\chi(m)e\left(\frac{m(a+b)+\overline{a}+\overline{b}}{p}\right)\\
=\ &\tau(\chi)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)\chi_2(b)\overline{\chi}(a+b)e\left(\frac{\overline{a}+\overline{b}}{p}\right)\\
=\ &\tau(\chi)\sum_{a=1}^{p-1}\chi_1(a)\overline{\chi}(a+1)\sum_{b=1}^{p-1}\chi_1(b)\chi_2(b)\overline{\chi}(b)e\left(\frac{\overline{b}\,(\overline{a}+1)}{p}\right)\\
=\ &\tau(\chi)\tau\left(\overline{\chi_1\chi_2\chi}\right)\sum_{a=1}^{p-1}\chi_1(a)\overline{\chi}(a+1)\chi_1\left(\overline{a}+1\right)\chi_2\left(\overline{a}+1\right)\overline{\chi}\left(\overline{a}+1\right)\\
=\ &\tau(\chi)\tau\left(\overline{\chi_1\chi_2\chi}\right)\sum_{a=1}^{p-1}\chi_1\left(a+1\right)\chi_2\left(\overline{a}+1\right)\overline{\chi}\left(\left(a+1\right)\left(\overline{a}+1\right)\right).
\end{aligned}$$

This proves Lemma 3. □

## 3. Proof of the theorem

This section is the most important part of the article, which is the proof of the main theorem. First from the orthogonality of the characters modulo $p$ we have

$$\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2\\
=\ &(p-1)\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right|^2. \qquad (3.1)
\end{aligned}$$

On the other hand, note that if $\chi=\chi_0$, then $|\tau(\chi)|=1$. If $\chi\neq\chi_0$, then $|\tau(\chi)|=\sqrt{p}$. So if $\chi_1\chi_2=\chi_0$ and $\chi_1=\chi_2$, then $\chi_1=\chi_2=\left(\frac{*}{p}\right)$ is the Legendre symbol modulo $p$. From Lemmas 1–3 we have

$$\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2\\
=\ &\left|\sum_{m=1}^{p-1}\chi_0(m)\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2\\
&+\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\left|\sum_{m=1}^{p-1}\chi(m)\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2
\end{aligned}$$

$$
\begin{aligned}
&= p^2(p-2)^2 + \sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} p^2 \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}\left((a+1)(\overline{a}+1)\right) \right|^2 \\
&= p^2(p-2)^2 + p^2 \cdot \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}\left((a+1)(\overline{a}+1)\right) \right|^2 \\
&\quad - p^2 \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_2(\overline{a}+1) \right|^2 \\
&= p^2(p-2)^2 + p^2 \cdot (p-1)(2p-5) - p^2 = p^2(p-1)(3p-8). \qquad (3.2)
\end{aligned}
$$

If $\chi_1\chi_2 = \chi_0$ and $\chi_1$ is not the Legendre symbol modulo $p$, then we have

$$
\begin{aligned}
&\sum_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \chi(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left( \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{mb+\overline{b}}{p}\right)\right) \right|^2 \\
&= p^2(p-2)^2 + p^2 \cdot (p-1)(p-4) - p^2 = p^2(p-1)(2p-7). \qquad (3.3)
\end{aligned}
$$

If $\chi_1 = \chi_2$ is a complex character modulo $p$, then we have

$$
\begin{aligned}
&\sum_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \chi(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left( \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{mb+\overline{b}}{p}\right)\right) \right|^2 \\
&= \left| \sum_{m=1}^{p-1} \chi_0(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left( \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{mb+\overline{b}}{p}\right)\right) \right|^2 \\
&\quad + \sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0,\, \chi_1^2}} \left| \sum_{m=1}^{p-1} \chi(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left( \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{mb+\overline{b}}{p}\right)\right) \right|^2 \\
&\quad + p \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_1(\overline{a}+1)\overline{\chi}_1^2\left((a+1)(\overline{a}+1)\right) \right|^2 \\
&= p^2 + p^2 \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_1(\overline{a}+1)\overline{\chi}\left((a+1)(\overline{a}+1)\right) \right|^2 \\
&\quad - p^2 \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_1(\overline{a}+1)\chi_0\left((a+1)(\overline{a}+1)\right) \right|^2 \\
&\quad - p^2 \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_1(\overline{a}+1)\overline{\chi}_1^2\left((a+1)(\overline{a}+1)\right) \right|^2 \\
&\quad + p \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_1(\overline{a}+1)\overline{\chi}_1^2\left((a+1)(\overline{a}+1)\right) \right|^2
\end{aligned}
$$

$$= \quad p^2(p-1)(2p-5) + 2p^2 - 2p^3 = p^2(p-1)(2p-7). \tag{3.4}$$

If all $\chi_1$, $\chi_2$, and $\chi_1\chi_2$ are not principal character modulo $p$, $\chi_1 \neq \chi_2$ and $\chi_1(-1) = \chi_2(-1)$, then we have

$$\begin{aligned}
&\sum_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \chi(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2 \\
= &\left| \sum_{m=1}^{p-1} \chi_0(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2 \\
&+ \sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0,\, \chi_1\chi_2}} \left| \sum_{m=1}^{p-1} \chi(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2 \\
&+p \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi_1\chi_2}((a+1)(\overline{a}+1)) \right|^2 \\
= &\ p^2 + p^2 \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi}((a+1)(\overline{a}+1)) \right|^2 \\
&-p^2 \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_2(\overline{a}+1)\chi_0((a+1)(\overline{a}+1)) \right|^2 \\
&-p^2 \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi_1\chi_2}((a+1)(\overline{a}+1)) \right|^2 \\
&+p \cdot \left| \sum_{a=1}^{p-1} \chi_1(a+1)\chi_2(\overline{a}+1)\overline{\chi_1\chi_2}((a+1)(\overline{a}+1)) \right|^2 \\
= &\ p^2 + p^2(p-1)(p-4) + p^2 - 2p^3 = p^2(p-1)(p-6).
\end{aligned} \tag{3.5}$$

If all $\chi_1$, $\chi_2$, and $\chi_1\chi_2$ are not principal character modulo $p$, $\chi_1 \neq \chi_2$ and $\chi_1(-1) \neq \chi_2(-1)$, then we have

$$\begin{aligned}
&\sum_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \chi(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2 \\
= &\ p^2 + p^2(p-1)(p-2) + p^2 - 2p^3 = p^2(p-1)(p-4).
\end{aligned} \tag{3.6}$$

Similarly, if one of $\chi_1$ and $\chi_2$ is principal character modulo $p$ and $\chi_1(-1)\chi_2(-1) = 1$, then we also have

$$\begin{aligned}
&\sum_{\chi \bmod p} \left| \sum_{m=1}^{p-1} \chi(m) \left( \sum_{a=1}^{p-1} \chi_1(a) e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2 \\
= &\ p + p^2 \cdot \left(p^2 - 5p + 4\right) + p - p^2 - p^2 = p(p-1)\left(p^2 - 4p - 2\right).
\end{aligned} \tag{3.7}$$

If one of $\chi_1$ and $\chi_2$ is principal character modulo $p$ and $\chi_1(-1)\chi_2(-1) = -1$, then we also have

$$\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\chi(m)\left(\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right)\left(\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right)\right|^2 \\
= \;& p + p^2\cdot(p-1)(p-2) + p - p^2 - p^2 = p(p-1)\left(p^2-2p-2\right). \qquad (3.8)
\end{aligned}$$

Now combining (3.1)–(3.8) we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right|^2$$

$$= \begin{cases} p^2(3p-8) & \text{if } \chi_1 = \chi_2 \text{ is the Legendre symbol modulo } p; \\ p^2(2p-7) & \text{if } \chi_1 = \chi_2 \text{ or } \overline{\chi}_2 \text{ is not the Legendre symbol modulo } p; \\ p^2(p-6) & \text{if all } \chi_1 \neq \chi_2 \text{ and } \chi_1\chi_2 \text{ not } \chi_0 \text{ and } \chi_1(-1) = \chi_2(-1); \\ p^2(p-4) & \text{if all } \chi_1 \neq \chi_2 \text{ and } \chi_1\chi_2 \text{ not } \chi_0 \text{ and } \chi_1(-1) \neq \chi_2(-1); \\ p(p^2-4p-2) & \text{if one of } \chi_1 \text{ and } \chi_2 \text{ is } \chi_0 \text{ and } \chi_1(-1) = \chi_2(-1); \\ p(p^2-2p-2) & \text{if one of } \chi_1 \text{ and } \chi_2 \text{ is } \chi_0 \text{ and } \chi_1(-1) \neq \chi_2(-1). \end{cases}$$

This completes the proof of our theorem.

## 4. Conclusions

The primary result of this research is to provide an exact calculating formula for the fourth power mean of one kind generalized Kloosterman's sums

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{ma+\overline{a}}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{mb+\overline{b}}{p}\right)\right|^2,$$

where $p$ be an odd prime, $\chi_1$ and $\chi_2$ is any fixed Dirichlet characters modulo $p$.

This proof strategies of the paper are not only novel, but they also serve as a good point of reference for problems that may arise in subsequent study.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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