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Research article

An efficient linearly-implicit energy-preserving scheme with fast solver for the fractional nonlinear wave equation

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Abstract: The paper considers the Hamiltonian structure and develops efficient energy-preserving schemes for the nonlinear wave equation with a fractional Laplacian operator. To this end, we first derive the Hamiltonian form of the equation by using the fractional variational derivative and then applying the finite difference method to the original equation to obtain a semi-discrete Hamiltonian system. Furthermore, the scalar auxiliary variable method and extrapolation technique is used to approximate a semi-discrete system to construct an efficient linearly-implicit energy-preserving scheme. A fast solver for the proposed scheme is presented to reduce CPU consumption. Ample numerical results are given to finally confirm the efficiency and conservation of the developed scheme.

Keywords: fractional nonlinear wave equation; Hamiltonian system; scalar auxiliary variable; conservative scheme
Mathematics Subject Classification: 65M06, 65M70

1. Introduction

The wave equation is significant among the nonlinear evolution equations extensively utilized in nonlinear mathematical and physical sciences [13]. In recent years, the widespread application of fractional calculus theory in various physical fields has prompted scholars to introduce the fractional wave equation to simulate nonlocal physical phenomena [17]. This paper considers the following fractional nonlinear wave equation [18]

$$\frac{\partial^2 u(\boldsymbol{x},t)}{\partial t^2} + (-\Delta_x)^{\frac{\alpha}{2}} u(\boldsymbol{x},t) + (-\Delta_y)^{\frac{\alpha}{2}} u(\boldsymbol{x},t) + F'(u(\boldsymbol{x},t)) = 0, \quad \boldsymbol{x} = (x,y) \in \Omega \subset \mathbb{R}^2, \quad t \in (0,T], \tag{1.1}$$

$$u(\boldsymbol{x},0) = f(\boldsymbol{x}), \quad u_t(\boldsymbol{x},0) = g(\boldsymbol{x}), \quad \boldsymbol{x} = (x,y) \in \Omega \subset \mathbb{R}^2, \tag{1.2}$$

$$u(\boldsymbol{x},t)=0,\quad \boldsymbol{x}\in\mathbb{R}^2\setminus\Omega,\quad t\in(0,T], \tag{1.3}$$

here $1<\alpha\leq 2$, $(-\Delta_x)^{\frac{\alpha}{2}}$ represents the fractional Laplacian operator with α order for x [15, 16], and it is equivalent to the Riesz fractional differential operator [28], the nonlinear term $F^{'}(u(\boldsymbol{x},t))$ is the derivative of a smooth potential energy $F(u)$ with respect to u, $f(\boldsymbol{x})$, $g(\boldsymbol{x})$ are smooth functions. The equation is simplified to the classical wave equation when $\alpha=2$. We know that the classical wave equation can inherit the energy [2], and the fractional Laplacian operator has the same conjugation as the classical Laplacian operator. Therefore, we can derive the following energy conservation law for the fractional wave equation [18]

$$\frac{d}{dt}\int_{\Omega}\Big[u_t^2+\big((-\Delta_x)^{\frac{\alpha}{4}}u\big)^2+\big((-\Delta_y)^{\frac{\alpha}{4}}u\big)^2+2F(u)\Big]d\mathbf{x}=0. \tag{1.4}$$

Fractional calculus has gained significant attention recently due to its ability to describe non-local and memory-dependent phenomena. The concept of fractional order derivatives, extending traditional integer-order derivatives to non-integer orders, has applications in various fields such as physics, engineering, biology, and finance. While the theory of fractional calculus is well-established, the computation of fractional order derivatives poses challenges due to the non-local nature of these operators. Therefore, numerous numerical algorithms have been developed to approximate fractional order derivatives efficiently and accurately, such as Obtaining an analytic solution for the fractional wave equation is generally challenging due to the non-local nature of fractional differential operators. Consequently, numerous researchers have focused on developing algorithms to solve this equation in the past decade. For instance, Ran and Zhang [21] introduced a compact-difference scheme for the equation, which achieves fourth-order accuracy in spatial discretization. In reference [27], scholars proposed two finite difference schemes to solve the fourth-order strongly damped nonlinear wave equations. Liu et al. [14] employed spectral methods to tackle the time fractional wave equation. Unfortunately, these aforementioned numerical schemes do not preserve the energy of the equation during the solution process.

In recent years, researchers have extensively researched preserving energy for conservative equations, leading to significant research outcomes. These developed methods, known as structure-preserving algorithms [5, 10, 11], aim to retain inherent system properties during long-term calculations and demonstrate improved stability compared to traditional numerical methods [1, 4]. Many structure-preserving methods have also been devised for the fractional wave equation. For instance, Mac as-D az et al. [17] derived the energy conservation law for the wave equation and proposed relevant conservative schemes for solving equation (1.1) [19]. Hu constructed a fully-implicit dissipation-preserving scheme for the wave equation with integral fractional Laplacian [19]. Additional conservative schemes for the sine-Gordon equation can be found in references [13, 18]. However, most of these schemes require nonlinear iterative solutions, resulting in a significant computational burden.

The main work of the paper is to develop an efficient linearly-implicit energy-preserving scheme for the fractional wave equation. In our approach, we utilize the scalar auxiliary variable method [22–24] to address the fractional wave equation, establishing an equivalent system to facilitate our analysis. We then employ a combination of difference methods and the implicit midpoint technique to develop conservative schemes. To enhance computational efficiency, we introduced a fast solver based on the

properties of the difference matrices. The constructed scheme demonstrates second-order accuracy in the time and space direction, balancing accuracy and computational cost.

2. Hamiltonian formulation of the equation

In this section, we first reformulate the fractional nonlinear wave as a canonical Hamiltonian system based on the following lemmas.

Lemma 2.1. *[6] Let $\alpha > 0$, then for any real functions u, v with boundary conditions* (1.3)*, we have*

$$((-\Delta)^{\frac{\alpha}{2}}u, v) = ((-\Delta)^{\frac{\alpha}{4}}u, (-\Delta)^{\frac{\alpha}{4}}v) = (u, (-\Delta)^{\frac{\alpha}{2}}v). \tag{2.1}$$

Lemma 2.2. *[6] For a functional $\varphi[\rho]$ with the following form*

$$\varphi[\rho] = \int_{\Omega} f(\rho(\boldsymbol{\eta}), (-\Delta)^{\frac{\alpha}{4}}\rho(\boldsymbol{\eta}))d\boldsymbol{\eta}, \tag{2.2}$$

where ρ is a smooth function on the Ω, then the variational derivative of $F[\rho]$ is given as follows

$$\frac{\delta\varphi}{\delta\rho} = \frac{\partial f}{\partial\rho} + (-\Delta)^{\frac{\alpha}{4}}\frac{\partial f}{(-\Delta)^{\frac{\alpha}{4}}\rho}. \tag{2.3}$$

By setting $u_t = v$, and we can arrange system (1.1) as a first-order system

$$u_t = v, \tag{2.4}$$

$$v_t = -(-\Delta_x)^{\frac{\alpha}{2}}u - (-\Delta_y)^{\frac{\alpha}{2}}u - F'(u), \tag{2.5}$$

with the boundary conditions

$$u(\boldsymbol{x}, t) = 0, \quad v(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \Omega, \quad t \in [0, T]. \tag{2.6}$$

Taking the inner products of (2.4) and (2.5) with v_t, v, respectively, we derive the energy conservation law as follows

$$\frac{d}{dt}\mathcal{H} = 0, \tag{2.7}$$

where

$$\mathcal{H} = \frac{1}{2}\int_{\Omega}\Big[v^2 + ((-\Delta_x)^{\frac{\alpha}{4}}u)^2 + ((-\Delta_y)^{\frac{\alpha}{4}}u)^2 + 2F(u)\Big]d\boldsymbol{x}. \tag{2.8}$$

Based on the fractional variational derivative formula in Lemma 2.2, we obtain the following theorem.

Theorem 2.1. The systems (2.4) and (2.5) is an infinite-dimensional canonical Hamiltonian system

$$\begin{pmatrix} v_t \\ u_t \end{pmatrix} = J^{-1}\begin{pmatrix} \delta\mathcal{H}/\delta v \\ \delta\mathcal{H}/\delta u \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the Hamiltonian functional $\mathcal{H}$ is defined by

$$\mathcal{H} = \frac{1}{2}\int_{\Omega}\Big[v^2 + ((-\Delta)^{\frac{\alpha}{4}}u)^2 + 2F(u)\Big]d\boldsymbol{x}. \tag{2.9}$$

Proof. According to Lemma 2.2, we can deduce

$$\frac{\delta \mathcal{H}}{\delta v} = v, \tag{2.10}$$

$$\frac{\delta \mathcal{H}}{\delta u} = (-\Delta_x)^{\frac{\alpha}{2}} u + (-\Delta_y)^{\frac{\alpha}{2}} u + F^{'}(u). \tag{2.11}$$

Then, systems (2.4) and (2.5) can be rewritten as

$$\begin{pmatrix} v_t \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta \mathcal{H}/\delta v \\ \delta \mathcal{H}/\delta u \end{pmatrix}.$$

This completes the proof.

3. Semi-discrete Hamiltonian system

This section develops a structure-preserving scheme that can conserve discrete energy exactly to solve the fractional nonlinear wave equation based on the SAV approach.

Let us consider the system in $\Omega = (x_a, x_b) \times (y_a, y_b)$, and denote the mesh sizes $h_x = (x_b - x_a)/N_1$, $h_y = (y_b - y_a)/N_2$, N_1 and N_2 are positive integer. Then, a new vector space is given

$$P = (p_{1,1}, p_{2,1}, \cdots, p_{N_1-1,1}, \cdots, p_{N_1-1,N_2-1})^T. \tag{3.1}$$

Noting that the considered equation has the Dirichlet boundary conditions, therefore, here $p_{i,j} = p(x_i, y_j)$ with $x_i = x_a + ih_x, i = 1, \cdots, N_1 - 1$ and $y_j = y_a + jh_y, j = 1, \cdots, N_2 - 1$. Then, we define some notations

$$\|P\|_\infty = \max_{i,j} |p_{i,j}|, \qquad \Big(P, Q\Big) = h_x h_y \sum_{i=1}^{N_1-1} \sum_{j=1}^{N_2-1} p_{i,j} q_{i,j}, \qquad \|P\|^2 = \Big(P, P\Big).$$

and we introduce following operators

$$\delta_t U^n = \frac{U^{n+1} - U^n}{\tau}, \quad U^{n+\frac{1}{2}} = \frac{U^{n+1} + U^n}{2} \quad \widetilde{U}^{n+\frac{1}{2}} = \frac{3U^n - U^{n-1}}{2}.$$

3.1. Structure-preserving spatial discretization

In fact, the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}} u(x,t)$ is equivalent to the Riesz fractional derivative, and can be expressed in terms of the respective Riemann-Liouville fractional derivatives, namely

$$-(-\Delta)^{\frac{\alpha}{2}} u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = -\frac{1}{2\cos\frac{\alpha\pi}{2}} [{}_{-\infty}D_x^\alpha u(x,t) + {}_x D_{+\infty}^\alpha u(x,t)], \tag{3.2}$$

where ${}_{-\infty}D_x^\alpha u(x,t)$ and ${}_xD_{+\infty}^\alpha u(x,t)$ are the left and right side Riemann-Liouville fractional derivatives [13], respectively, which are defined as

$${}_{-\infty}D_x^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{x} (x-s)^{1-\alpha} u(s,t) ds, \tag{3.3}$$

$$ {}_xD^{\alpha}_{+\infty}u(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^{+\infty}(s-x)^{1-\alpha}u(s,t)ds, \tag{3.4} $$

where $\Gamma(z) = \int_0^{\infty} x^{z-1}\mathrm{e}^{-x}$. Next, we present the fractional centered difference method to approximate the Riesz fractional derivative.

Lemma 3.1. *[13] Let $u(x) \in C^5(\mathbb{R})$ and all derivatives up to order five belong to $L^1(\mathbb{R})$. Then, for $1 < \alpha \leq 2$, we have*

$$ \frac{\partial^{\alpha}u(x)}{\partial|x|^{\alpha}} = -\frac{1}{h^{\alpha}}\sum_{l=-\infty}^{+\infty} d_l^{(\alpha)}u(x-lh) + O(h^2), \tag{3.5} $$

where the coefficients $d_l^{(\alpha)} := \frac{(-1)^l\Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2}-l+1)\Gamma(\frac{\alpha}{2}+l+1)}$.

Since $u(x,t)$=0 for $x \in \mathbb{R} \setminus \Omega$, we can obtain

$$ \frac{\partial^{\alpha}u(x,t)}{\partial|x|^{\alpha}} = -\frac{1}{h^{\alpha}}\sum_{l=-(x_b-x)/h}^{-(x_a-x)/h} d_l^{(\alpha)}u(x-lh,t) + O(h^2), \tag{3.6} $$

and deduce

$$ (-\Delta)^{\frac{\alpha}{2}}u_j^n = \frac{1}{h^{\alpha}}\sum_{l=-M+j}^{j} d_l^{(\alpha)}u_{j-l}^n + O(h^2) = \frac{1}{h^{\alpha}}\sum_{l=1}^{M-1} d_{j-l}^{(\alpha)}u_l^n + O(h^2). \tag{3.7} $$

According to above discussions, we denote differential matrix D^{α} as

$$ D^{\alpha} = \begin{pmatrix} d_0^{(\alpha)} & d_{-1}^{(\alpha)} & \cdots & d_{-M+2}^{(\alpha)} \\ d_1^{(\alpha)} & d_0^{(\alpha)} & \cdots & d_{-M+3}^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{M-2}^{(\alpha)} & d_{M-3}^{(\alpha)} & \cdots & d_0^{(\alpha)} \end{pmatrix}, $$

where D^{α} is a real-value symmetric positive definite Toeplitz matrix, and we have

$$ (-\Delta)^{\frac{\alpha}{2}}u = D^{\alpha}U. $$

3.2. Semi-discrete conservative system

Applying the fractional centered difference method to approximate systems (2.4) and (2.5) in space, we can obtain a semi-discrete system

$$ U_t = V, \tag{3.8} $$

$$ V_t = -DU - F^{'}(U), \tag{3.9} $$

where $D = I_y \otimes D_x^{\alpha} + D_y^{\alpha} \otimes I_x$, and it is a symmetric positive definite matrix. Then, the semi-discrete system is reformulated as

$$\boldsymbol{Z}_t = S^{-1}\nabla H(Z), \tag{3.10}$$

where $\boldsymbol{Z} = (V, U)^T$, and

$$\nabla H = \begin{pmatrix} V \\ \mathbf{D}U + F'(U) \end{pmatrix}, \quad S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{3.11}$$

Theorem 3.1. The constructed scheme is a conservative system and can inherit the energy, namely

$$\frac{d}{dt}H(Z) = 0 \tag{3.12}$$

with

$$H(Z) = \Big(V, V\Big) + \Big(DU, U\Big) + 2\Big(F(U), \mathbf{1}\Big). \tag{3.13}$$

Proof. We can calculate it directly and obtain

$$\frac{d}{dt}\mathcal{H}(t) = \nabla\mathcal{H}z_t = \nabla\mathcal{H}S^{-1}\nabla\mathcal{H} = 0, \tag{3.14}$$

where antisymmetry of matrix S^{-1} is used.

4. A linearly-implicit fully-discrete scheme

4.1. Equivalent system with the SAV approach

In the SAV approach, we introduce a scalar variable $W(t) = \sqrt{\mathcal{E}(t)}$, where

$$\mathcal{E} = (F(U) + C_0, 1), \tag{4.1}$$

where C_0 is a positive constant such that $F(U)+C_0 > 0$. Then, the energy function (3.13) is transformed into a modified formal

$$\mathcal{H} = (V, V) + (DU, U) + W^2. \tag{4.2}$$

Let $\mathcal{B}(u) = \frac{F'(U)}{\sqrt{\mathcal{E}(t)}}$, then, the original systems (2.4) and (2.5) can be written as

$$U_t = V, \tag{4.3}$$

$$V_t = -Du - \mathcal{B}(U)W, \tag{4.4}$$

$$W_t = \frac{1}{2}(\mathcal{B}(U), U_t), \tag{4.5}$$

where $(\cdot, \cdot)$ represents the discrete L^2-inner product.

Taking the inner products of (4.3) and (4.4) with V_t, V, respectively, and then multiplying (4.5) with $2W$, and summing them together, we can deduce a modified energy conservation law

$$\frac{d}{dt}\Big[(V,V)+(DU,U)+W^2\Big]=0. \tag{4.6}$$

We observe that equivalent systems (4.3) and (4.5) still preserves energy, but the energy is in terms of the new variables now.

Applying the Crank-Nicolson method for systems (4.3) and (4.5) in time, and utilizing the extrapolation technique on (4.4), which yield a second-order linearly implicit scheme for the fractional nonlinear wave equation, namely

$$\frac{U^{n+1}-U^n}{\tau}=V^{n+\frac{1}{2}}, \tag{4.7}$$

$$\frac{V^{n+1}-V^n}{\tau}=DU^{n+\frac{1}{2}}+\widetilde{\mathcal{B}}^nW^{n+\frac{1}{2}}, \tag{4.8}$$

$$W^{n+1}-W^n=\frac{1}{2}(\widetilde{\mathcal{B}}^n,U^{n+1}-U^n), \tag{4.9}$$

where $\widetilde{\mathcal{B}}^n=\mathcal{B}(\widetilde{U}^{n+\frac{1}{2}})$. The proposed schemes (4.7)–(4.9) is called SAV scheme.

Theorem 4.1. The SAV schemes (4.7)–(4.9) possesses the following discrete total energy conservation law

$$H^n=H^{n+1},\ \ 0\le n\le N-1,$$

where

$$H^n=\frac{1}{2}(\|V^n\|^2-(U^n)^TD^\alpha U^n)+\|W^n\|^2.$$

Proof. Taking the inner product of (4.7) and (4.8) with $\frac{V^{n+1}-V^n}{\tau}$, $\frac{U^{n+1}-U^n}{\tau}$, multiplying (4.9) with $\frac{W^{n+1}+W^n}{\tau}$, and adding them together, we deduce

$$\frac{1}{2}(\|V^{n+1}\|^2-(U^{n+1})^TD^\alpha U^{n+1})+\|W^{n+1}\|^2=\frac{1}{2}(\|V^n\|^2-(U^n)^TD^\alpha U^n)+\|W^n\|^2 \tag{4.10}$$

Thus, we have

$$H^n=H^{n+1},\ \ 0\le n\le N-1. \tag{4.11}$$

This ends the proof.

Remark 4.1. The proposed scheme only has second-order accuracy in time and space, which cannot meet the high-accuracy requirements of numerical simulation algorithms. In references [20, 25], scholars developed high-order conservative schemes for conservative equations by using the Runge-Kutta methods. The constructed schemes can conserve the quadratic invariant and arrive at high accuracy; these methods can also be applied to obtain high-accuracy structure-preserving schemes for the wave equation. In future research work, we will focus on constructing high-order accuracy structure-preserving numerical schemes for solving conservation-type differential equations.

5. A fast solver for the proposed scheme

The SAV scheme is linear implicit and highly efficient in that only decoupled equations with constant coefficients must be solved at each time step. To demonstrate the linear implicit advantage with a constant coefficient matrix of the SAV scheme, we first eliminate V^{n+1}, W^{n+1}, and can obtain

$$\frac{U^{n+1}-U^n}{\tau}=V^n+\frac{\tau}{2}\left(DU^{n+\frac{1}{2}}-\widetilde{\mathcal{B}}^nW^n-\frac{\widetilde{\mathcal{B}}^n}{4}\left(\widetilde{\mathcal{B}}^n,U^{n+1}-U^n\right)\right), \tag{5.1}$$

which can be further rewritten as

$$\begin{aligned}\left(I-\frac{\tau^2}{4}D^\alpha\right)U^{n+1}+\frac{\tau^2}{8}\widetilde{\mathcal{B}}^n\left(\widetilde{\mathcal{B}}^n,U^{n+1}\right)=&\left(I+\frac{\tau^2}{4}D\right)U^n+\frac{\tau^2}{8}\widetilde{\mathcal{B}}^n\left(\overline{B}^n,U^n\right)\\&-\frac{\tau^2}{2}\widetilde{\mathcal{B}}^nW^n+\tau V^n.\end{aligned} \tag{5.2}$$

Let B^n denote the the righthand side of (5.2). Multiplying (5.2) with $(I-\frac{\tau^2}{4}D)^{-1}$, then taking the discrete inner product with $\widetilde{\mathcal{B}}^n$, we can deduce

$$\left(\widetilde{\mathcal{B}}^n,U^{n+1}\right)+\frac{\tau^2}{8}\chi^n\left(\widetilde{\mathcal{B}}^n,U^{n+1}\right)=\left(\widetilde{\mathcal{B}}^n,\left(I-\frac{\tau^2}{4}D\right)^{-1}B^n\right), \tag{5.3}$$

where $\chi^n=\left(\widetilde{\mathcal{B}}^n,\left(I-\frac{\tau^2}{4}D\right)^{-1}\widetilde{\mathcal{B}}^n\right)$. Hence, we can obtain

$$\left(\widetilde{\mathcal{B}}^n,U^{n+1}\right)=\frac{\left(\widetilde{\mathcal{B}}^n,\left(I-\frac{\tau^2}{4}D\right)^{-1}B^n\right)}{1+\frac{\tau^2}{8}\chi^n}. \tag{5.4}$$

Finally, substituting the result into (5.2) will give the solution of U^{n+1}.

To summarize, the manners of the SAV scheme computing the U^{n+1} are given as

Manners: Compute the U^{n+1}

(1) Compute $\chi^n=(\widetilde{\mathcal{B}}^n,(I-\frac{\tau^2}{4}D)^{-1}\widetilde{\mathcal{B}}^n)$. This can be completed by solving a algebraic system, which has constant coefficients.

(2) Compute $(\widetilde{\mathcal{B}}^n,U^{n+1})$ using (5.4). This requires solving another system $(I-\frac{\tau^2}{4}D)^{-1}B^n$ with constant coefficients.

(3) With $(\widetilde{\mathcal{B}}^n,(I-\frac{\tau^2}{4}D)^{-1}\widetilde{\mathcal{B}}^n)$, $(\widetilde{\mathcal{B}}^n,U^{n+1})$ and $(I-\frac{\tau^2}{4}D)^{-1}B^n$ known, we can obtain U^{n+1} from (5.2).

In summary, we only need to solve two linear systems of (5.3) and (5.4) successively with constant matrix $\left(I-\frac{\tau^2}{4}D\right)$.

Noting that D is a Toeplitz matrix. Using the special properties of the matrix, we can design a fast algorithms for the proposed scheme. We consider the following Toeplitz matrix C_N

$$C_N=\begin{pmatrix}c_0&c_1&c_2&\dots&c_{N-1}\\c_{-1}&c_0&c_1&\ddots&c_{N-2}\\\vdots&c_{-1}&c_0&\ddots&\vdots\\c_{2-N}&\ddots&\ddots&\ddots&c_1\\c_{1-N}&c_{2-N}&\dots&\dots&c_0\end{pmatrix}.$$

Further, we construct a new $2N \times 2N$ circulant matrix C_{2N} with the form

$$C_{2N} = \begin{pmatrix} C_N & \mathcal{A}_N \\ \mathcal{A}_N & C_N \end{pmatrix}, \quad \mathcal{A}_N = \begin{pmatrix} 0 & c_{1-N} & c_{2-N} & \dots & c_{-1} \\ c_{N-1} & 0 & c_{1-N} & \ddots & c_{-2} \\ \vdots & c_{N-1} & 0 & \ddots & \vdots \\ c_2 & \ddots & \ddots & \ddots & c_{1-N} \\ c_1 & c_2 & \dots & \dots & 0 \end{pmatrix}.$$

Based on this, we can rewrite $C_N\mathbf{b}$ as

$$C_{2N}\begin{pmatrix} \mathbf{b} \\ \mathbf{0}_{N\times 1} \end{pmatrix} = \begin{pmatrix} C_N\mathbf{b} \\ \mathcal{A}_N\mathbf{b} \end{pmatrix}.$$

Then, the FFT technique can be applied to diagonaliz C_{2N}, namely

$$C_{2n} = \mathcal{F}^{-1}\,\mathrm{diag}(\mathcal{F}\mathbf{b})\mathcal{F},$$

with

$$\mathcal{F}_{i,j} = \left[\frac{1}{\sqrt{2N}} e^{\frac{-2\pi\mathrm{i}(i-1)(j-1)}{2N}}\right]_{i,j=1,2,\dots,2N}.$$

From discussions, we can use the FFT technique to compute the $\mathcal{F}$ and implement the proposed linear scheme efficiently.

6. Numerical examples

In this section, some numerical examples are presented to demonstrate the effectiveness of the proposed scheme in the conservation of discrete energy for the two-dimensional nonlinear fractional wave equation. The relative errors of energy are defined as

$$RH^n = |(H^n - H^0)/H^0|,$$

where H^n denotes the energy at $t = n\tau$.

6.1. Accuracy tests

In this experiment, we consider the one-dimensional nonlinear fractional wave equation with $F(u) = u - u^3$, and initial conditions

$$u(x, 0) = \sqrt{2}\,\mathrm{sech}(\lambda x),$$

$$u_t(x, 0) = \sqrt{2}c\lambda\,\mathrm{sech}(\lambda x)\tanh(\lambda x),$$

and the boundary condition

$$u(x, t) = 0, \quad x \in \mathbb{R} \setminus \Omega, \quad t \in (0, T].$$

When $\alpha = 2$, the system reduces to classical nonlinear wave equation, and the analytical solution is given by

$$u(x,t) = \sqrt{2}\,\mathrm{sech}(\lambda(x-ct)),$$

where $\lambda = \sqrt{\frac{1}{1-c^2}}$ and $1-c^2 > 0$. The error in the spatial direction with sufficiently small τ

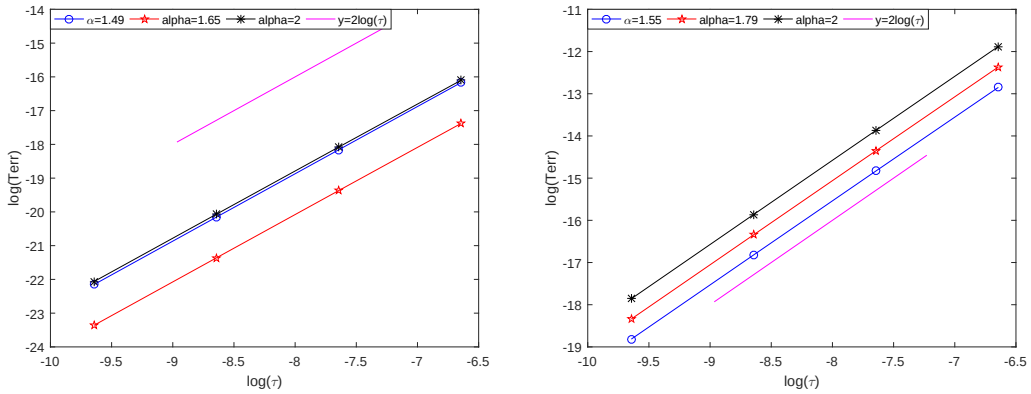
$$\mathrm{Serr} = \begin{cases} \max|u(\tau,h) - U(\tau,h/2)|, & \alpha = 2, \\ \max\left|U_j^N(\tau,h) - U_{2j}^N(\tau,h/2)\right|, & 1 < \alpha < 2, \end{cases}$$

and the error in the temporal direction with sufficiently small h

$$\mathrm{Terr} = \begin{cases} \max|u(\tau,h) - U(\tau/2,h)|, & \alpha = 2, \\ \max\left|U_j^N(\tau,h) - U_j^{2N}(\tau/2,h)\right|, & 1 < \alpha < 2. \end{cases}$$

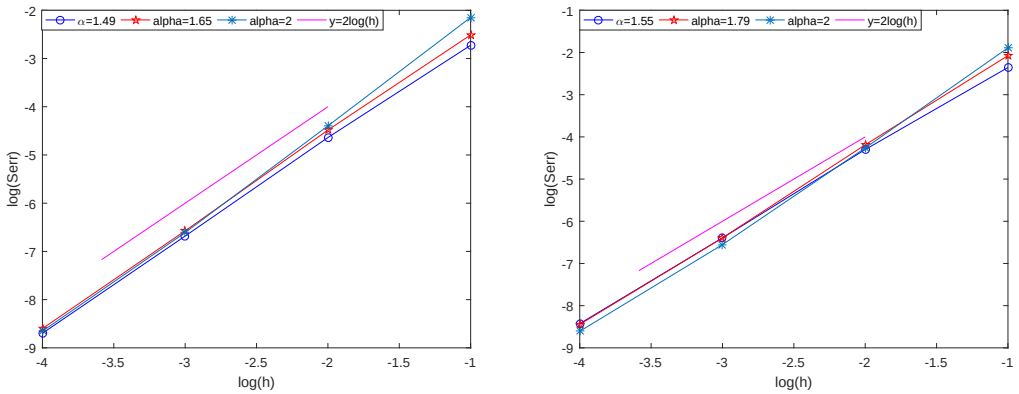
Without loss of generality, Here we take c=0.1 with $\alpha = 1.49, 1.65, 2$, and c=0.9 with $\alpha = 1.55, 1.79, 2$, and set the space interval $\Omega = [-20, 20]$.

Firstly, we choose $T = 1$ and verify the convergence order in the time and space. Figures 1 and 2 show maximum norm convergence order in space and time respectively with different fractional order α and initial value. We can find that SAV scheme show about second-order experimental accuracy in time and space.



(a) $c = 0.1$ (a) $c = 0.9$

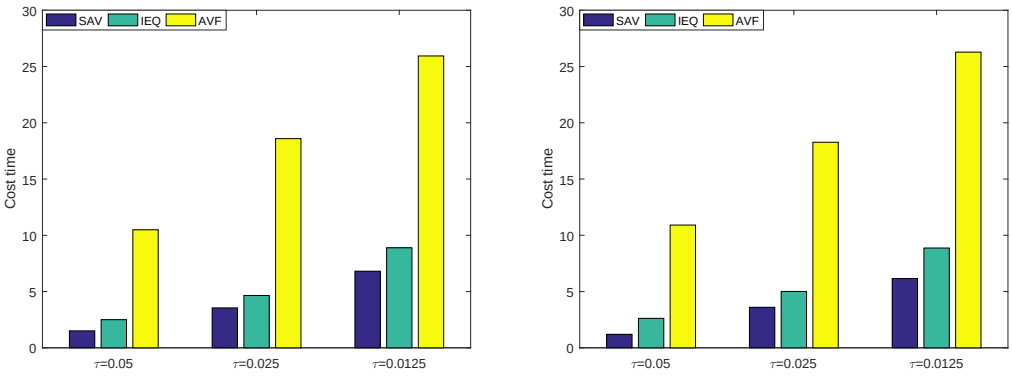
Figure 1. Maximal norm convergence order in time for different α.



(a) $c = 0.1$ (a) $c = 0.9$

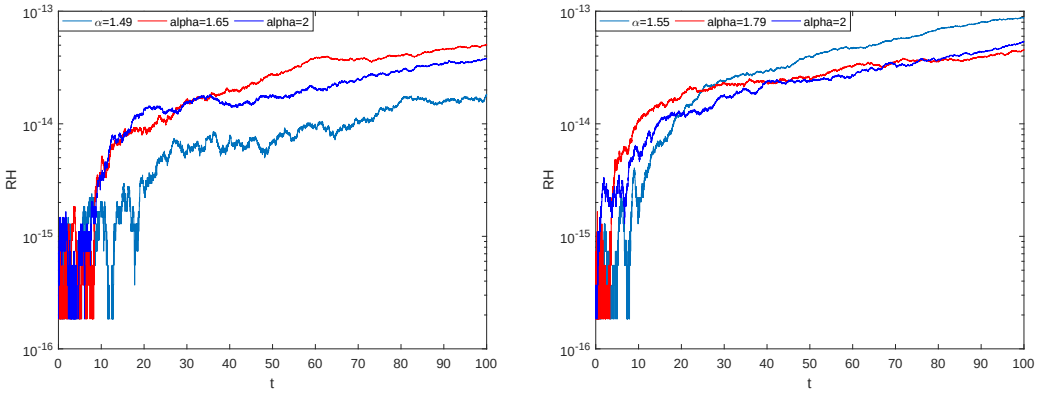
Figure 2. Maximal norm convergence order in space for different α.

In section 3, we demonstrate the SAV scheme only needs to solve decoupled equations with constant coefficients at each time step, which leads that the scheme is more efficient than the linear implicit energy-preserving scheme through the invariant energy quadratization (IEQ) method [6]. While the energy-preserving scheme based on the averaged vector field (AVF) method is fully-implicit and needs nonlinear iterations to obtain the numerical solution, which can approximately represent the general implicit method regarding the computational cost. In Figure 3, we choose $T = 100$ and present comparisons of the computational costs of the three schemes with different time steps. We can conclude that the SAV scheme significantly reduces the computational cost. We study the conservative property of the proposed scheme. Here, we run a long-time simulation till $T = 100$ and plot the energy deviation in Figure 4, corresponding to different values of fractional order α. Numerical results show that the SAV scheme can preserve the discrete energy very well. Therefore, it is preferable to construct linearly implicit schemes through the SAV approach for large-scale simulations, keeping the system energy preserved exactly. The evolution of solitons with different α is shown in Figures 5 and 6, and the results show that α has a significant impact on the waveform of particle evolution.



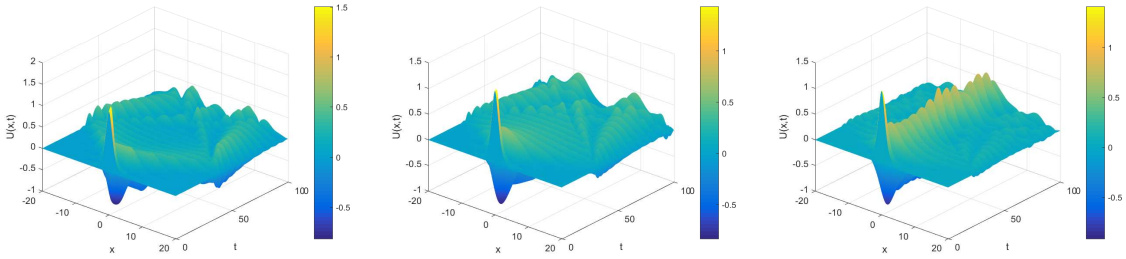
(a) $\alpha = 1.55$ (b) $\alpha = 2$

Figure 3. Computational cost of SAV, IEQ and AVF scheme with different time step when $c = 0.1$.



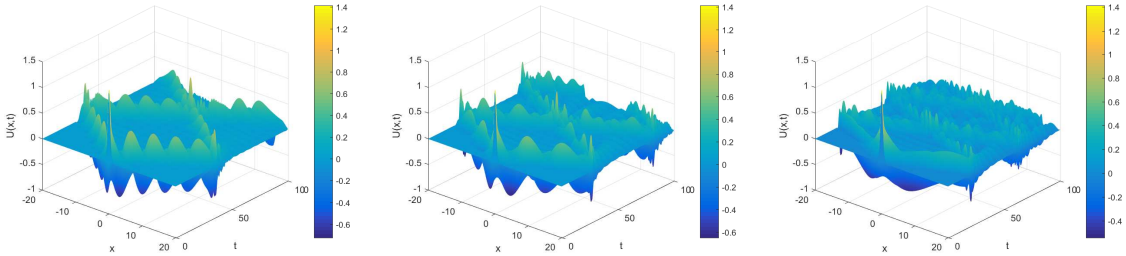
(a) $\lambda = 0.1$ (b) $\lambda = 0.9$

Figure 4. The relative energy error with $h = 0.1$, $\tau = 0.01$ for different order α.



(a) $\alpha = 1.49$ (b) $\alpha = 1.65$ (b) $\alpha = 2$

Figure 5. Evolution of the solitons with $c = 0.1$ for different α ($h = 0.1, \tau = 0.01$).



(a) $\alpha = 1.55$ (b) $\alpha = 1.79$ (b) $\alpha = 2$

Figure 6. Evolution of the solitons with $c = 0.9$ for different α ($h = 0.1, \tau = 0.01$).

Example 5.2. We consider the equation with the initial conditions

$$f(x, y) = 4 \tan^{-1}\left[\exp\left(\frac{4 - \sqrt{(x+3)^2 + (y+7)^2}}{0.436}\right)\right], \tag{6.1}$$

$$g(x, y) = 4.13 \operatorname{sech}\left(\frac{4 - \sqrt{(x+3)^2 + (y+7)^2}}{0.436}\right), -30 \le x \le 10, -21 \le y \le 7. \tag{6.2}$$

In this example, we take $\alpha = 1.4, 1.7, 1.9, 2$ to test our theoretical analysis. The results of Figure 7 show the SAV scheme can conserve the energy very well for different α with $h = 0.25$, $\tau = 0.01$. The surface (left) and contours (right) plots of numerical solutions for different α at time $T = 5$. As α approaches 2, the shape of the soliton is also similar to that of the classical wave equation.

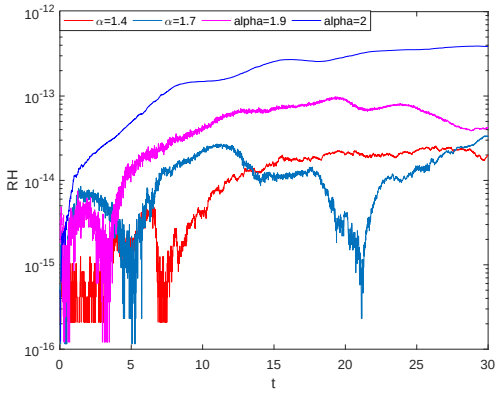


Figure 7. The relative energy error with $h = 0.25$, $\tau = 0.01$ for different order α.

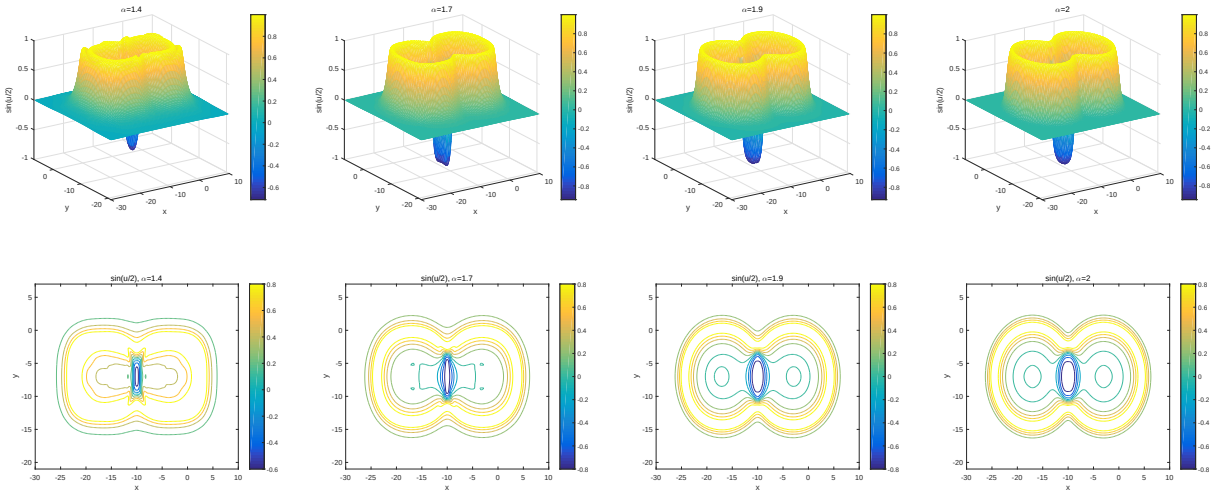


Figure 8. Surface (left) and contours (right) plots of numerical solutions for different α at time $T = 5$, in terms of $\sin(u/2)$. Spatial and temporal step sizes are taken as $h = 0.25$, $\tau = 0.01$.

7. Conclusions

In this paper, we construct a new difference scheme for solving the fractional wave equation based on the scalar auxiliary variable approach. The scheme is linearly implicit, can be solved efficiently, and can conserve the modified energy of the equation. In addition, this method also can be extended to fractional conservative system equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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