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Research article

Distinguishing colorings of graphs and their subgraphs

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Abstract: In this paper, several distinguishing colorings of graphs are studied, such as vertex distinguishing proper edge coloring, adjacent vertex distinguishing proper edge coloring, vertex distinguishing proper total coloring, adjacent vertex distinguishing proper total coloring. Finally, some related chromatic numbers are determined, especially the comparison of the correlation chromatic numbers between the original graph and the subgraphs are obtained.

Keywords: vertex distinguishing edge coloring; adjacent vertex distinguishing edge coloring; vertex distinguishing total coloring; adjacent vertex distinguishing total coloring
Mathematics Subject Classification: 05C15

1. Introduction

Graph distinguishing coloring is a hot issue in the area of graph coloring theory. As an application, graph distinguishing colorings may be connected with the frequency assignment problem in wireless communication. The vertices (nodes) of graphs (networks) represent transmitters, the set $C[u,\pi]$ of colors assigned to a vertex u and the edges (links) incident to u under a total coloring π indicates the frequencies usable. For a pair of adjacent vertices u and v, the property $C[u,\pi] \neq C[v,\pi]$ ensures the corresponding two stations can operate on a wide range of frequencies without the danger of interfering with each other.

We use standard notation and terminology of graph theory [1]. Some additional notations will be used. The shorthand notation $[m,n]$ stands for a set $\{m, m+1, \dots, n\}$, where integers n and m hold $n > m \geq 0$. The set of vertices that are adjacent to a vertex u of G is denoted as $N(u)$, thus, the *degree* $d_G(u)$ of the vertex u is equal to the cardinality $|N(u)|$. $N(u)$ is also called the *neighborhood* of the vertex u. The *edge neighborhood* of the vertex u is defined as $N_e(u) = \{uv : v \in N(u)\}$. Let $\Delta(G)$ (or

Δ) and $\delta(G)$ denote the maximum degree and minimum degree of G, respectively. Let $n_d(G)$ be the number of vertices of degree d in G. In a graph, a *k-degree vertex* is a vertex of degree k, especially, a Δ-*vertex* is a vertex of maximum degree; a *leaf* is a vertex of degree one; a k-cycle is a cycle of k vertices.

Graphs mentioned here are undirected, connected and simple, and all graph colorings are proper. A *k-total coloring* π of a graph G from a nonempty subset $S \subset V(G) \cup E(G)$ to $[1,k]$ (also, $[1,k]$ is called a *color set*) satisfies that $\pi(x) \neq \pi(y)$ for any two adjacent/incident elements $x, y \in S$. The set $\{\pi(ux) : ux \in N_e(u)\}$ is called the *edge-color set* of the vertex u, and denoted by $C(u,\pi)$. Notice that an edge-color set $C(u,\pi)$ is never a multiset, that is, $|C(u,\pi)| = d_G(u)$. Let π be a total coloring of G, the set $C[u,\pi] = C(u,\pi) \cup \{\pi(u)\}$ is called a *closed color set* of vertex u. Clearly, $|C[u,\pi]| = d_G(u) + 1$. The following distinguishing constraints will be used in the following Definitions:

(A1) $C(u,\pi) \neq C(v,\pi)$ for distinct vertices $u, v \in V(G)$;
(A2) $C(u,\pi) \neq C(v,\pi)$ for each edge $uv \in E(G)$;
(A3) $C[u,\pi] \neq C[v,\pi]$ for distinct vertices $u, v \in V(G)$;
(A4) $C[u,\pi] \neq C[v,\pi]$ for each edge $uv \in E(G)$.

For the purpose of simplicity, two notations $\pi(V(G)) = \{\pi(x) : x \in V(G)\}$ and $\pi(E(G)) = \{\pi(xy) : xy \in E(G)\}$ will be used here. We reformulate the definitions of several known distinguishing colorings of graphs occurred in literature.

Definition 1.1. *[2, 3] Let $\pi : E(G) \to [1,k]$ be an edge coloring of G. If (A1) holds, then π is called a vertex distinguishing edge k-coloring (k-vdec) of G. The vdec chromatic number of G is the minimum number of k colors required for which G admits a k-vdec, and denoted by $\chi'_s(G)$.*

Definition 1.2. *[4] Let $\pi : E(G) \to [1,k]$ be an edge coloring of G. If (A2) holds, then π is called an adjacent vertex distinguishing edge k-coloring (k-avdec) of G. The avdec chromatic number of G is the minimum number of k colors required for which G admits a k-avdec, and denoted by $\chi'_{as}(G)$.*

Definition 1.3. *[5] Let $\pi : V(G) \cup E(G) \to [1,k]$ be a total coloring of G. If (A3) holds, then π is called a vertex distinguishing total k-coloring (k-vdtc) of G. The vdtc chromatic number of G is the minimum number of k colors required for which G admits a k-vdtc, and denoted by $\chi''_s(G)$.*

Definition 1.4. *[6] Let $\pi : V(G) \cup E(G) \to [1,k]$ be a total coloring of G. If (A4) holds, then π is called an adjacent vertex distinguishing total k-coloring (k-avdtc) of G. The avdtc chromatic number of G is the minimum number of k colors required for which G admits a k-avdtc, and denoted by $\chi''_{as}(G)$.*

A graph G is called a *vdec no-covered graph* (resp. a *vdtc no-covered graph*) if there exists a proper subgraph H of G such that $\chi'_s(H) > \chi'_s(G)$ (resp. $\chi''_s(H) > \chi''_s(G)$). Similarly, a graph G is called an *avdec no-covered graph* (resp. an *avdtc no-covered graph*) if G contains a proper subgraph H with $\chi'_{as}(H) > \chi'_{as}(G)$ (resp. $\chi''_{as}(H) > \chi''_{as}(G)$).

The following three lemmas will be used.

Lemma 1.1. *[7] For a graph G of order at least three, $\chi'_s(G) \leq |V(G)| + 1$.*

Lemma 1.2. *[8] For integers $n, m \geq 3$, $\chi'_s(C_m) = n$ if and only if either n is odd and $m \in \left[\frac{n^2-4n+5}{2}, \frac{n^2-n-6}{2}\right] \bigcup \left\{\frac{n^2-n}{2}\right\}$, or n is even and $m \in \left[\frac{n^2-3n-2}{2}, \frac{n^2-3n}{2}\right] \bigcup \left[\frac{n^2-3n+4}{2}, \frac{n^2-n}{2}\right]$.*

Lemma 1.3. *Let K_n, P_n and C_n be a complete graph, path and cycle on n vertices, respectively; and let $K_{m,n}$ be a complete bipartite graph on $m + n$ vertices. Then*

(1) *[4] $\chi'_{as}(K_{m,n}) = n$ for $n > m > 0$ and $\chi'_{as}(K_{n,n}) = n + 2$ for $n \geq 2$.*

(2) *[5] $\chi''_{s}(K_{m,n}) = n + 1$ for $n > m \geq 2$, and $\chi''_{s}(K_{n,n}) = n + 3$ for $n \geq 3$.*

(3) *[9] $\chi''_{as}(K_{2m+2}) = \chi''_{as}(K_{2m+1}) = 2m + 3$ for all integers $m \geq 1$.*

(4) *[6] $\chi''_{as}(P_n) = 4$ for $n \geq 4$; $\chi''_{as}(C_n) = 4$ for $n \geq 6$.*

In 1997, Burris and Schelp [2] studied the vertex distinguishing edge colorings (vdec) on graphs. Independently, as "observability" of a graph, the concept of the vdec was proposed by Černý, Horňák and Soták [3]. This graph coloring has absorbed researchers' intentions. Surprisingly, determining the vdec chromatic numbers of graphs seems to be quite difficult, even for some special graphs. In 2002, Balister, Bollobás and Schelp [10] proved that $k \leq \chi'_s(G) \leq k + 5$ for a graph G with $\Delta(G) = 2$, $n_1(G) \leq k$ and $n_2(G) \leq \binom{k}{2}$. Zhang, Liu and Wang [4] introduced the adjacent vertex distinguishing edge colorings (avdec) of graphs, a weaker version of a vdec. However, finding avdec chromatic number of graphs is not easy, even for special planar graphs [11]. Three conjectures on graph distinguishing colorings are listed as follows.

Conjecture 1.1. *[2] For a graph G of order at least three, if k is the minimal integer such that $\binom{k}{d} \geq n_d(G)$ for all d with $\delta(G) \leq d \leq \Delta(G)$, then $\chi'_s(G) = k$ or $k + 1$.*

Conjecture 1.2. *[4] For a graph G of order at least three, if G is not a cycle on five vertices, then $\chi'_{as}(G) \leq \Delta(G) + 2$.*

Conjecture 1.3. *[6] For a graph G of order at least three, $\chi''_{as}(G) \leq \Delta(G) + 3$.*

In [10,12–15], some results on Conjecture 1.1 were obtained. Conjecture 1.2 was partially answered in [16] and [17]. A good result on Conjecture 1.2, due to Hatami [18], is that if G has no isolated edges and $\Delta(G) > 10^{20}$, then $\chi'_{as}(G) \leq \Delta(G) + 300$. Since $\Delta(G) \leq \chi'_{as}(G)$, the Hatami's result implies that $\lim\limits_{\Delta(G)\to\infty} \frac{\chi'_{as}(G)}{\Delta(G)} = 1$. Conjecture 1.3 has been verified for some special graphs in [6] and [19]. It is noticeable, Conjecture 1.2 implies that G is not an avdec no-covered graph if $\chi'_{as}(G) = \Delta(G) + 2$; and Conjecture 1.3 implies that G is not an avdtc no-covered graph if $\chi''_{as}(G) = \Delta(G) + 3$.

2. Vertex distinguishing edge and total colorings

Theorem 2.1. *Each connected graph is a proper subgraph of a vdec no-covered Hamilton graph.*

Proof. First of all, we prove the following claim.

Claim. *Let G be a Hamilton graph with $\delta(G) \geq 3$. Then G has a vdec no-covered Hamilton graph with $|E(G)| - 1$ edges.*

The proof of Claim. Suppose that a Hamilton graph G on n vertices admits a k-vdec π with $k = \chi'_s(G)$. Without loss of generality, $\pi(uv) = 1$ for a certain edge uv of a Hamilton cycle of G. By Lemma 1.1, two cases appear as follows:

For $k < n + 1$ we take a cycle C_m on $m = \binom{n}{2}$ vertices. Clearly, $\chi'_s(C_m) = n$ by Lemma 1.2. Let ϕ be a n-vdec of C_m, and $\phi(xy) = 1$ for a certain edge $xy \in E(C_m)$. We construct a new graph G^* by deleting two edges uv and xy from G and C_m respectively, and then join u with x, v with y together. Next, we define an edge coloring ψ of G^* as: $\psi(st) = \pi(st)$ if $st \in (E(G) \setminus \{uv\}) \subset E(G^*)$; $\psi(st) = \phi(st)$ if

$st \in (E(C_m) \setminus \{xy\}) \subset E(G^*)$, and $\psi(ux) = \psi(vy) = 1$. It is easy to see that $C(a,\psi) \neq C(b,\psi)$ for distinct vertices $a, b \in V(G^*)$, since degrees $d_{G^*}(x) \geq 3$ for $x \in V(G^*) \setminus V(C_m)$. Immediately, $\chi'_s(G^*) \leq n$, and furthermore $\chi'_s(G^*) = n$ since G^* has $\binom{n}{2}$ vertices of degree two. Notice that $C_{m+n} \subset G^*$, and $m + n = \binom{n}{2} + n = \binom{n+1}{2}$ as well as $\chi'_s(C_{m+n}) = n + 1$ according to Lemma 1.2. Therefore, G^* is a vdec no-covered graph.

If $k = n + 1$, we take a cycle C_m on $m = \binom{n+1}{2}$ vertices, and then we still have a graph H^* obtained from G and C_m by the same construction as the above one. Clearly, $\chi'_s(H^*) = n + 1$, and H^* is vdec no-covered, since H^* contains a Hamilton cycle C_{m+n} with $m + n = \binom{n+1}{2} + n > \binom{n+1}{2}$, which means $\chi'_s(C_{m+n}) > n + 1$. The proof of the claim is finished.

By adding vertices and edges to a given connected graph H, we can obtain a Hamilton graph G with $\delta(G) \geq 3$ and G has a Hamilton cycle containing an edge $uv \notin E(H)$ and $\pi(uv) = 1$ under a k-vdec π of G. The theorem follows from the above claim. □

The construction of the graph G^* in the proof of Theorem 2.1 enables us to obtain the following result:

Corollary 2.1. *For given positive integers M and N, there is a vdec no-covered Hamilton graph G having $\Delta(G) \geq M$ and $|V(G)| \geq N$.*

Theorem 2.2. *There exist Hamilton graphs $H_0, H_1, \ldots, H_m$ with $m \geq 1$ such that H_i is a proper subgraph of H_{i-1} and $\chi'_s(H_i) > \chi'_s(H_{i-1})$ for $i \in [1, m]$.*

Proof. For $m = 1$, the conclusion is deduced by Theorem 2.1. The case $m \geq 2$ is discussed as follows. Let N be a positive integer, and let $G_1, G_2, \ldots, G_m$ be m vertex-disjoint Hamilton graphs with $|G_i| = N + i - 1$, $2 \leq k_i = \chi'_s(G_i) \leq N$, $\Delta(G_{j-1}) < \delta(G_j)$ for $j \in [2, m]$, and $\delta(G_1) \geq 3$. Suppose that π_i is a k_i-vdec of G_i for $i \in [1, m]$. Since $k_i \geq 2$, we have two edges x_iy_i and u_iv_i of a Hamilton cycle T_i of G_i with $\pi_i(x_iy_i) = 1$ and $\pi_i(u_iv_i) = 2$, $i \in [1, m]$. We construct a graph G by means of $G_1, G_2, \ldots, G_m$ in the following two steps.

Step 1. Delete edges u_1v_1 of G_1, u_mv_m of G_m, and x_iy_i and u_iv_i of G_i for $i \in [2, m-1]$, respectively.

Step 2. Join x_i with y_{i-1}, y_i with x_{i-1}, u_i with v_{i-1} and v_i with u_{i-1}, respectively, $i \in [3, m-1]$; join u_2 with v_1, v_2 with u_1; and join u_m with v_{m-1}, v_m with u_{m-1}.

Let C_n be a cycle on $n = \binom{N}{2}$ vertices. Suppose that an edge x_0y_0 of C_n is labelled as $\pi(x_0y_0) = 1$ under a N-vdec π of C_n from Lemma 1.2. Now, we have a graph H_0 obtained by joining $x_0 \in V(C_n)$ with $y_1 \in V(G)$, $y_0 \in V(C_n)$ with $x_1 \in V(G)$, and removing the edge x_0y_0. Clearly, H_0 is hamiltonian by the above construction. We define an edge coloring θ of H_0 as: $\theta(wz) = \pi_i(wz)$ when $wz \in E(G_i) \setminus \{x_iy_i, u_iv_i\}$ for $i \in [2, m-1]$; $\theta(x_iy_{i-1}) = \theta(y_ix_{i-1}) = 1$ for $i \in [3, m-1]$; $\theta(u_iv_{i-1}) = \theta(v_iu_{i-1}) = 2$ for $i \in [2, m]$; $\theta(x_1y_0) = \theta(y_1x_0) = 1$ and $\theta(wz) = \pi(wz)$ when $wz \in E(C_n) \setminus \{x_0y_0\}$; $\theta(u_mv_{m-1}) = \theta(v_mu_{m-1}) = 2$ and $\theta(wz) = \pi_m(wz)$ for $wz \in E(G_m) \setminus \{u_mv_m\}$. By the choices of C_n and $G_1, G_2, \ldots, G_m$, we have $C(w,\theta) \neq C(z,\theta)$ for distinct vertices $w, z \in V(H_0)$ and $\chi'_s(H_0) \leq N$.

Deleting every edge of $S_1 = E(H_0) \cap (E(G_1) \setminus E(T_1))$ from H_0 results in a graph H_1. Notice that H_1 is hamiltonian and has $|G_1| + n = N + \binom{N}{2} = \binom{N+1}{2}$ vertices of degree 2 that form a set $X_1^{(2)}$, which implies $\chi'_s(H_1) \geq N + 1 \geq 1 + \chi'_s(H_0)$. By Lemm 1.2 we have $\chi'_s(H_1) = N + 1$, since $d_{H_1}(z) \geq 3$ for $z \in V(H_1) \setminus X_1^{(2)}$. In general, we have Hamilton graphs $H_i = H_{i-1} - S_i$, where $S_i = E(H_{i-1}) \cap (E(G_i) \setminus E(T_i))$ with $\chi'_s(H_i) = N + i$, since H_i has $|G_i| + \binom{N+i-1}{2} = (N + i - 1) + \binom{N+i-1}{2} = \binom{N+i}{2}$ vertices of degree 2 that form a set $X_i^{(2)}$ such that $d_{H_i}(z) \geq 3$ for $z \in V(H_i) \setminus X_i^{(2)}$, $i \in [1, m]$. Continue with this method, finally, we obtain H_m that is a cycle of order $M = |G_m| + \binom{N+m-1}{2} = (N + m - 1) + \binom{N+m-1}{2} = \binom{N+m}{2}$ with $\chi''_s(H_m) = N + m$. The proof is completed. □

Theorem 2.3. *Each Hamilton graph is a proper subgraph of some vdtc no-covered Hamilton graph.*

Proof. By Lemma 1.3, $\chi''_s(K_{n,n-1}) = n + 1$ and $\chi''_s(K_{n-1,n-1}) = n + 2$ for $n \geq 4$. Let G be a graph having a Hamilton cycle $C_m = x_1x_2 \cdots x_mx_1$ and let n be so large such that $n - 1 \geq \max\{\Delta(G), 7\}$ and $n - 5 \geq k = \chi''_s(G)$. Since $K_{n-1,n-1} = K_{n,n-1} - \{u\}$ is hamiltonian, we have a Hamilton cycle $C_{2n-2} = u_1v_1u_2v_2 \cdots u_{n-1}v_{n-1}u_1$ of $K_{n-1,n-1}$, where u is adjacent to v_i for $i \in [1, n - 1]$ in $K_{n,n-1}$. Let $H = C_4 + \{w_2w_4\}$, where $C_4 = w_1w_2w_3w_4w_1$.

Let f be a $(n+1)$-vdtc of $K_{n,n-1}$, without loss of generality, $f(v_1u_2) = 1$, $f(u_1v_1) = 2$ and $f(u_1v_{n-1}) = 3$; and let g be a k-vdtc of G. Select an edge u_iv_i with $2 \leq i \leq n - 1$. We can modify the colors of elements of $V(G) \cup E(G)$ under the coloring g such that some edge x_jx_{j+1} of C_m holds $g(x_j) \neq f(v_i)$, $g(x_{j+1}) \neq f(u_i)$, $g(x_jx_{j+1}) = f(u_iv_i)$ (if $k < f(u_iv_i)$, we set $g(x_jx_{j+1}) = f(u_iv_i)$ directly). Now, we define a total coloring h of H as: $\{h(w_i) : i \in [1, 4]\} = \{n-2, n-1, n, n+1\}$ with $h(w_1) \neq f(v_1)$, $h(w_2) \neq f(u_2)$, $h(w_3) \neq f(v_{n-1})$ and $h(w_4) \neq f(u_1)$; $h(w_1w_2) = 1$, $h(w_2w_3) = 2$, $h(w_3w_4) = 3$, $h(w_1w_4) = 4$ and $h(w_2w_4) = 5$.

We do: (1) delete w_1w_2 of H and v_1u_2 of $K_{n,n-1}$, and join w_2 with u_2, join w_1 with v_1; (2) delete w_3w_4 of H and u_1v_{n-1} of $K_{n,n-1}$, and join w_3 with v_{n-1}, join w_4 with u_1; (3) delete the edge x_jx_{j+1} of G and the edge u_iv_i of $K_{n,n-1}$, and join x_j with u_i, join x_{j+1} with v_i. Through the above measures, we get a connected graph G^* and G^* has a total coloring θ as folows: $\theta(V(G^*)) = f(V(K_{n,n-1})) \cup g(V(G)) \cup h(V(H))$,

$$\begin{aligned}&\theta(E(G^*) \setminus \{w_2u_2, w_1v_1, w_3v_{n-1}, w_4u_1, x_ju_i, x_{j+1}v_i\})\\ &= f(E(K_{n,n-1}) \setminus \{v_1u_2, u_1v_{n-1}\}) \cup g(E(G) \setminus \{x_jx_{j+1}\}) \cup h(E(H) \setminus \{w_1w_2, w_3w_4\}),\end{aligned}$$

and $\theta(w_2u_2) = 1$, $\theta(w_1v_1) = 1$, $\theta(w_3v_{n-1}) = 3$, $\theta(w_4u_1) = 3$, $\theta(x_ju_i) = f(u_iv_i)$ and $\theta(x_{j+1}v_i) = f(u_iv_i)$. Notice that $C(a, \theta) \neq C(b, \theta)$ for distinct vertices $a, b \in V(G^*)$ according to the definitions of the total colorings f, g, h and θ. We conclude that θ is a $(n + 1)$-vdtc of G^*. Clearly, $\chi''_s(G^*) \leq n + 1$ and, G^* has a Hamilton cycle $uu_1v_1w_1w_4w_2w_3v_{n-1}u_{n-1} \cdots v_jx_{j+1}x_{j+2} \cdots x_mx_1 \cdots x_ju_iv_{i-1} \cdots v_2u_2u$.

Since G and $K_{n-1,n-1}$ both are the proper subgraphs of G^*, the theorem is covered. □

3. Adjacent vertex distinguishing edge and total colorings

Theorem 3.1. *For given integers $M \geq 3$ and $N \geq 5$, there is a connected graph G with $\Delta \geq M$ and $n \geq N$ vertices such that G contains a proper subgraph H with $\chi'_{as}(H) > \chi'_{as}(G)$.*

Proof. First of all, we define the *avd-linking operation* on two graphs. Suppose that each connected graph G_i has a k_i-avdec π_i with $k_i = \chi'_{as}(G_i)$ for $i = 1, 2$, and furthermore $\pi_1(u_1v_1) = \pi_2(u_2v_2)$ for $u_iv_i \in E(G_i)$, and $C(u_i, \pi_i) \neq C(v_{3-i}, \pi_{3-i})$, $i = 1, 2$. We obtain a new graph H^* by deleting edges u_iv_i from G_i, and join u_i to v_{3-i} for $i = 1, 2$. Clearly, H^* admits a k_0-avdec π, where $k_0 = \max\{k_1, k_2\}$, by defining an edge-coloring π as: $\pi(uv) = \pi_i(uv)$ for $uv \in E(G_i) \setminus \{u_iv_i\}$, and $\pi(u_iv_{3-i}) = \pi_i(u_iv_i)$ for $i = 1, 2$.

There are many ways to construct the desired graph G mentioned in this theorem. Here, we take a complete graph $K_{\Delta+1,\Delta}$ with $\Delta \geq M \geq 3$. Clearly, a proper subgraph $K_{\Delta,\Delta} \subset K_{\Delta+1,\Delta}$ holds $\chi'_{as}(K_{\Delta,\Delta}) = \Delta + 2 > \Delta + 1 = \chi'_{as}(K_{\Delta+1,\Delta})$.

Let $K'_{\Delta+1,\Delta}$ be a copy of $K_{\Delta+1,\Delta}$, and let the edge $u'v' \in E(K'_{\Delta+1,\Delta})$ be isomorphic to an edge $uv \in E(K_{\Delta+1,\Delta})$. By the avd-linking operation on two graphs $K_{\Delta+1,\Delta}$ and $K'_{\Delta+1,\Delta}$, we obtain a connected graph $H_1(2(2\Delta+1))$ that contains two copies of $K_{\Delta,\Delta}$ and has $2(2\Delta+1)$ vertices. Since $\chi'_{as}(H_1(2(2\Delta+1))) = \chi'_{as}(K_{\Delta+1,\Delta}) = \Delta+1$, so $\chi'_{as}(K_{\Delta,\Delta}) > \chi'_{as}(H_1(2(2\Delta+1)))$. Consequently, by implementing the avd-linking operation, we take a copy $H'_1(2(2\Delta+1))$ of $H_1(2(2\Delta+1))$, and then construct a connected graph $H_2(2^2(2\Delta+1))$ such that $\chi'_{as}(K_{\Delta,\Delta}) = \Delta+2 > \Delta+1 = \chi'_{as}(H_2(2^2(2\Delta+1)))$ for a proper subgraph $K_{\Delta,\Delta}$ of $H_2(2^2(2\Delta+1))$. Go on in this way, we can construct a connected graph $H_k(2^k(2\Delta+1))$ having a proper subgraph $K_{\Delta,\Delta}$, and furthermore, $\chi'_{as}(H_k(2^k(2\Delta+1))) = \Delta+1$ and $2^k(2\Delta+1) \geq N \geq 5$. □

An example for illustrating the avd-linking operation used in the proof of Theorem 3.1 is given in Figures 1 and 2.

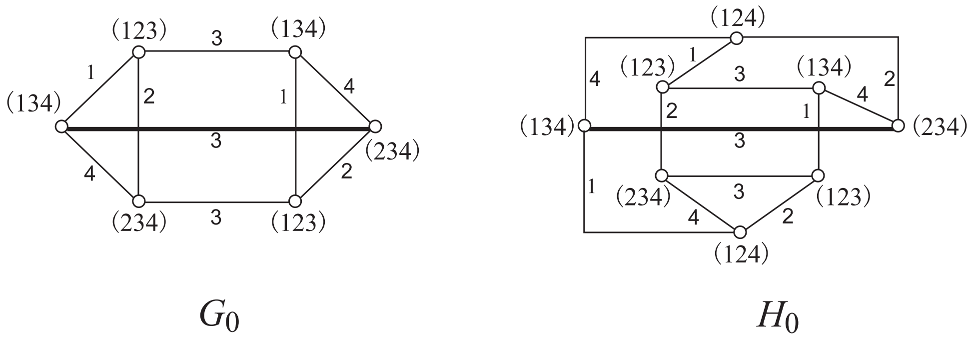

Figure 1. Two avdec no-covered, 3-regular and hamiltonian graphs G_0 and H_0.

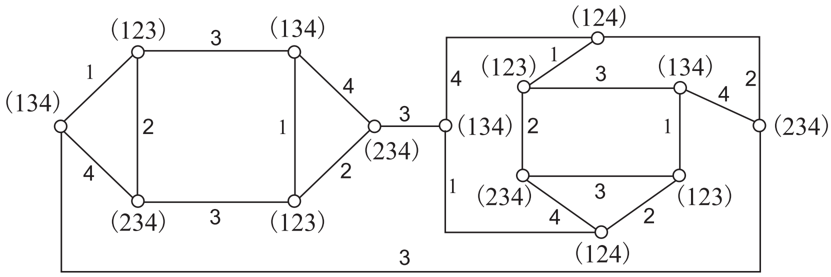

Figure 2. An avdec no-covered, 3-regular and hamiltonian graph obtained by using the avd-linking operation on two graphs G_0 and H_0 shown in Figure 1.

Theorem 3.2. *Each connected graph is a proper subgraph of an avdec no-covered graph.*

Proof. Let the bipartition (X, Y) of vertex set of $K_{n+1,n}$ for $n \geq 3$ be defined as $X = \{u_i : i \in [1, n+1]\}$ and $Y = \{v_i : i \in [1, n]\}$, and let f be a $(n+1)$-avdec of $K_{n+1,n}$ by Lemma 1.3. Suppose that each given connected graph H_j has a k_j-avdec g_j with $k_j = \chi'_{as}(H_j) \leq n-2$ and $\Delta(H_j) \leq n-2$ for $j = 1, 2$.

We delete the edge u_1v_1 from $K_{n+1,n}$, without loss of generality, set $f(u_1v_1) = n+1$. Next we join u_1 with a vertex w_1 of H_1, join v_1 with a vertex w_2 of H_2, the resultant graph is denoted by G. Notice that $\chi'_{as}(G) \geq n+1$ since $d_G(v_i) = n+1$ for $i \in [2, n]$. We define an edge coloring h of G as: $h(xy) = f(xy)$ for $xy \in (E(K_{n+1,n}) \setminus \{u_1v_1\}) \subset E(G)$; $h(u_1w_1) = f(u_1v_1)$ and $h(v_1w_2) = f(u_1v_1)$; $h(xy) = g_j(xy)$ for $xy \in E(H_j)$, $j = 1, 2$. Furthermore, $\Delta(G) = n+1 \leq \chi'_{as}(G) \leq n+1 = \max\{\chi'_{as}(K_{n+1,n}), k_1, k_2\}$. However,

$\chi'_{as}(K_{n,n}) = n+2 > n+1 = \chi'_{as}(G)$ since $K_{n,n} \subset G$. The theorem is covered since each given connected graph H_j is a proper subgraph of G for $j = 1, 2$. □

Theorem 3.3. *There are infinitely many connected 3-regular graphs G on $n(\geq 6)$ vertices that are avdec no-covered, planar and hamiltonian, and furthermore $\chi'_{as}(G) = \chi''(G) = 4$.*

Proof. For the sake of simplicity, we define Property (I) as follows: A graph G has Property (I) if (i) G is connected, 3-regular, planar and hamiltonian; (ii) G is embedded well in the plane; (iii) G has a particular edge uv locating in the bound of the outer-face and a Hamilton cycle of G, $G - \{uv\}$ contains at least a 5-cycle; and (iv) $\chi'_{as}(G) = 4$.

Obviously, a graph having Property (I) is avdec no-covered. Let G_0 and H_0 be two vertex-disjoint graphs shown in Figure 1. Clearly, G_0 and H_0 both possess Property (I). Let $u_0v_0 \in E(G_0)$ and $x_0y_0 \in E(H_0)$ be the particular edges described in Property (I). And let C_0 be a Hamilton cycle of G_0 such that $u_0v_0 \in E(C_0)$ and C'_0 be a Hamilton cycle of H_0 such that $x_0y_0 \in E(C'_0)$. Since $\chi'_{as}(G_0) = 4 = \chi'_{as}(H_0)$, we have a 4-avdec π_0 of G_0 and a 4-avdec π'_0 of H_0 such that $\pi_0(u_0v_0) = \pi'_0(x_0y_0)$ (we can modify the colors used in π'_0 to achieve the desired requirement, since G_0 and H_0 are vertex-disjoint). We construct a graph G_1 by deleting the edges u_0v_0 and x_0y_0 from G_0 and H_0, respectively, and then join u_0 with y_0, join v_0 with x_0. Clearly, G_1 has a Hamilton cycle $(C_0 - u_0v_0) + u_0y_0 + (C'_0 - x_0y_0) + x_0v_0$, and has Property (I). We select another graph H_1 having Property (I), and then construct a graph G_2 from G_1 and H_1 by the above way such that G_2 has Property (I). Go on in this way, we have graphs G_m having Property (I) for all integers $m \geq 1$. Notice that $C(s, f_i) \neq C(t, f_i)$ for every edge $st \in E(G_i)$ under a 4-avdec f_i of G_i, so $[1, 4] \setminus C(s, f_i) \neq [1, 4] \setminus C(t, f_i)$, $i \geq 1$. We can use one color in $[1, 4] \setminus C(s, f_i)$ to color the vertex s, and use one color in $[1, 4] \setminus C(t, f_i)$ to color the vertex t for every edge st of G_i, thus, $\chi''(G_i) = 4$ for $i \geq 1$, as desired. □

For a graph G with $\Delta(G) = 3$, let $P = ux_1x_2 \cdots x_mv$ be a (u, v)-path of G if $d_G(v) = 3$ and $d_G(x_i) = 2$ for $i \in [1, m]$. We call P a $(k, 3; m)$*-path* if $d_G(u) = k$ for $k = 1, 3$.

Lemma 3.1. *If $\chi''_{as}(G) = 3$, then G contains no proper subgraph H such that $\chi''_{as}(H) \geq 4$.*

Proof. Since $\chi''_{as}(G) = 3$, we have $\Delta(G) \leq 2$. By Lemma 1.3, G is a path of length at most two, which implies that G has no proper subgraph H with $\chi''_{as}(H) \geq 4$. □

Lemma 3.2. *Suppose that a graph H has maximum degree $\Delta = 3$ and no adjacent Δ-vertices. If each 2-degree vertex is adjacent to two 3-degree vertices in H, then $\chi''_{as}(H) = 4$.*

Proof. It is trivial for $|V(H)| = 4$. For $|V(H)| = 5$, $H = K_{2,3}$ and $\chi''_{as}(K_{2,3}) = 4$. The case $|V(H)| \geq 6$ is considered as follows. Notice that $d_H(x) \neq d_H(y)$ for every edge xy of H, which means that each proper total coloring of H is also an avdtc, and vice versa. Thereby, we can use the induction on orders of graphs.

Case 1. H has leaves.

Let $N(w) = \{w_1, w_2, w_3\}$ be the neighborhood of a 3-degree vertex w of H. If w_1, w_2 both are leaves, and w_3 is adjacent to two 3-degree vertices w and v. We know that $\chi''_{as}(H - \{w_1, w_2, w\}) = 4$ by induction hypothesis. Suppose that $H - \{w_1, w_2, w\}$ has a 4-avdtc f such that $f(v) = 1$, $f(w_3v) = 2$ and $f(w_3) = 3$. We can extend f to a 4-avdtc of H described in Figure 3(a).

If w_1, w_2 both are 2-degree vertices, where w_1 is adjacent to a 3-degree vertex x, w_2 is adjacent to a 3-degree vertex y, and w_3 is a leaf. We have a graph G_1 obtained by deleting w and all vertices in $N(w)$, and adding a new vertex u to join x and y respectively. Clearly, G_1 is a graph holding the hypothesis of the lemma. Let f_1 be a 4-avdtc of G_1 by induction hypothesis. Without loss of generality, $f_1(xu) = 1$, $f_1(u) = 2$ and $f_1(uy) = 3$. Thereby, we can extend f_1 to a 4-avdtc of H, see Figure 3(b).

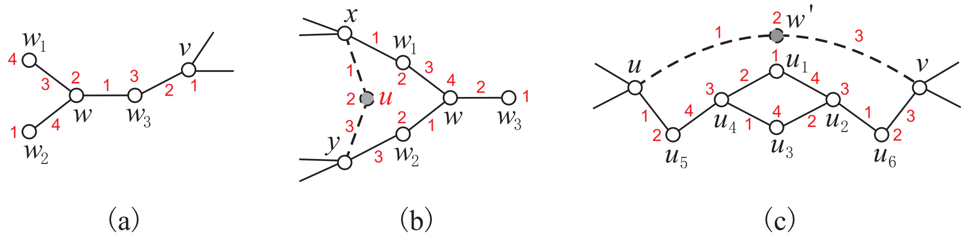

Figure 3. The first diagram for the proof of Lemma 3.2.

Case 2. H has no leaves.

Subcase 2.1. H has 4-cycles. $H \neq K_{2,3}$ since $|V(H)| \geq 6$. Let $u_1u_2u_3u_4$ be a 4-cycle of H. Notice that $\delta(H) = 2$ and each 2-degree vertex of H is adjacent to two 3-degree vertices. Hence, a local part of H that contains the 4-cycle $u_1u_2u_3u_4$ is shown in Figure 3(c). Let $X = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Thereby, a graph G_2 can be obtained by deleting every vertex of X and add a new vertex w' to join u and v, respectively. Clearly, G_2 is a graph holding the hypothesis of the lemma. By induction hypothesis, G_2 has a 4-avdtc f_2 such that $f_2(uw') = 1$, $f_2(w') = 2$ and $f_2(w'v) = 3$. It is not hard to extend f_2 to a 4-avdtc of H (see Figure 3(c)).

If u is adjacent to u_6, refereing Figure 3(c), we have a graph $G'_2 = H - \{u_1, u_2, u_3, u_4\}$ having leaves. So $\chi''_{as}(G'_2) = 4$ by the proof in Case 1. Then we can extend a 4-avdtc of G'_2 to a 4-avdtc of G.

Subcase 2.2. H has no 4-cycles. If H has the exact four 3-degree vertices, then it is an edge-subdivision of K_4, so we are done, see Figure 4(a). We, now, consider that H has at least six 3-degree vertices in the following. H has a subgraph H^* with vertex set $V(H^*) = \{u, v, w, u_1, u_2, v_1, v_2\}$ by referring to Figure 4(b). In H, two vertices u, v both have 3-degree, each vertex of $V(H^*) \setminus \{u, v\}$ has 2-degree, where $N(u_1) = \{u, x\}$, $N(u_2) = \{u, y\}$, $N(v_1) = \{v, s\}$ and $N(v_2) = \{v, t\}$ (see Figure 4(b)). Now, we have a graph G_3 obtained by deleting $V(H^*)$ from H and adding a new vertex x_0 to join x and y, and adding another new vertex s_0 to join s and t, respectively. So G_3 has a 4-avdtc f_3 by induction hypothesis, since G_3 satisfies the lemma's hypothesis. Without loss of generality, $f_3(xx_0) = 1$, $f_3(x_0) = 2$, $f_3(x_0y) = 3$. We define a proper total coloring h_3 of H in the following two steps.

Step 1. (1) Set $h_3(z) = f_3(z)$ for $z \in (V(G_3) \cup E(G_3)) \setminus \{xx_0, x_0, x_0y, ss_0, s_0, s_0t\}$; (2) $h_3(xu_1) = 1$, $h_3(u_1) = 2$, $h_3(yu_2) = 3$, $h_3(u_2) = 2$; (3) $h_3(u_1u) = 3$, $h_3(u) = 4$, $h_3(u_2u) = 1$ and $h_3(uw) = 2$ (see Figure 4(c)).

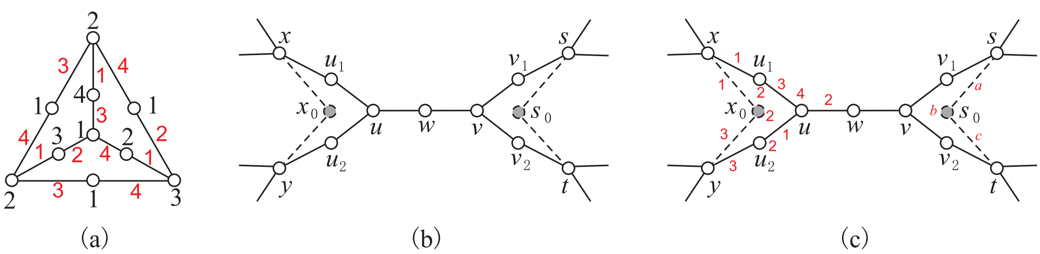

Figure 4. The second diagram for the proof of Lemma 3.2.

Step 2. To color every element of $S = \{sv_1, v_1, v_1v, tv_2, v_2, v_2v, v, vw, w\}$, we consider the colors on ss_0, s_0, s_0t under f_3. Let $f_3(ss_0) = a$, $f_3(s_0) = b$, $f_3(s_0t) = c$ (see Figure 4(c)).

Step 2.1. $(a, b, c) = (1, 2, 3)$. Thus, we need to consider two cases $f_3(t) = 4$ and $f_3(t) = 1$, respectively.

(i) When $f_3(t) = 4$, we set $h_3(sv_1) = 1$, $h_3(v_1) = 2$, $h_3(v_1v) = 3$, $h_3(tv_2) = 3$, $h_3(v_2) = 1$, $h_3(v_2v) = 2$, $h_3(v) = 4$ and $h_3(vw) = 1$ and $h_3(w) = 3$. Thereby, h_3 is a desired 4-avdtc of H.

(ii) If $f_3(t) = 1$, we set $h_3(sv_1) = 1$, $h_3(v_1) = 2$, $h_3(v_1v) = 3$, $h_3(tv_2) = 3$, $h_3(v_2) = 4$, $h_3(v_2v) = 2$, $h_3(v) = 1$ and $h_3(vw) = 4$ and $h_3(w) = 3$. In this situation, h_3 is a desired 4-avdtc of H.

Step 2.2. $(a, b, c) = (a, 2, c)$ and $(a, b, c) \neq (1, 2, 3)$, where distinct $a, c \in [1, 4]\backslash\{2\}$. The procedure of coloring properly every element of S is very similar to that in Step 2.1. We still obtain a desired 4-avdtc of H.

Step 2.3. $b \neq 2$. We set $h_3(sv_1) = a$, $h_3(v_1) = b$, $h_3(tv_2) = c$, $h_3(v_2) = b$, $h_3(vw) = b$, $h_3(v) = 2$ and $h_3(w) \in [1, 4] \setminus \{2, b, 4\}$ (see Figure 3(c)). For the last two edges v_1v and v_2v, we set $h_3(v_1v) = c$ and $h_3(v_2v) = a$ if $a, c \in [1, 4] \setminus \{2, b\}$; and if $a = 2$, without loss of generality, we set $h_3(v_1v) = c$ and $h_3(v_2v) \in [1, 4] \setminus \{2, b, c\}$. Eventually, we can get a desired 4-avdtc of H.

The proof is completed. □

Since $\chi''_{as}(P_n) = 4$ for $n \geq 4$, it is not hard to get the following result.

Lemma 3.3. *Let H be a graph with $\Delta(H) = 3$, no adjacent Δ-vertices and $\chi''_{as}(H) = 4$. If H has an edge uv with $d_H(u) = 1$ and $d_H(v) = 3$, replace the edge uv with a path $P = ux_1x_2 \cdots x_mv$ ($m \geq 1$), where each x_i is out of H for $i \in [1, m]$, the resultant graph is denoted by H_2, then $\chi''_{as}(H_2) = 4$.*

Theorem 3.4. *For a graph G with $\Delta = 3$ and no adjacent Δ-vertices, $\chi''_{as}(G) = 4$.*

Proof. Obviously, the conclusion holds when $|E(G)| = 3$. Next we consider the case that $|E(G)| \geq 4$. If G contains some $(3, 3; m)$-(u, v)-paths with $m \geq 2$. For every $(3, 3; m)$-(u, v)-path $P = ux_1x_2 \cdots x_mv$ of G, we delete x_i for $i \in [1, m]$, and add a new vertex w out of G by joining w with u, v respectively. The resulting graph is denoted as G'. If G' contains some $(1, 3; m)$-paths $Q = uy_1y_2 \cdots y_mv$ with $m \geq 1$, $d_G(u) = 1$ and $d_G(v) = 3$. To each $(1, 3; m)$-path Q of G', we delete y_j for $j \in [1, m]$ and then join u with v. Eventually, we obtain a graph G'' that contains no adjacent Δ-vertices, and each 2-degree vertex is adjacent to two 3-degree vertices.

According to Lemma 3.2, $\chi''_{as}(G'') = 4$, and then $\chi''_{as}(G') = 4$ by Lemma 3.3. Thereby, we can use the induction on orders of G' in the following argument.

Let G' have a 2-degree vertex w that is adjacent to two 3-degree vertices u, v, after replacing some $(3, 3; m)$-(u, v)-path $P = ux_1x_2 \cdots x_mv$ of G with $m \geq 2$ by a path uwv.

Case 1. Replace the path uwv by a path ux_1x_2v.

If one of two vertices u, v is in a 4-cycle of G', without loss of generality, G' has a 4-cycle $vu_1u_2u_3v$ shown in Figure 5(a). We delete the set $\{w, v, u_1, u_2, u_3, u_4\}$ from G', and then add a new vertex s out of G' by joining s with u, u_5, respectively. The resulting graph is written as H_1. Clearly, H_1 admits 4-avdtcs by induction hypothesis, since H_1 has $\Delta(H_1) = 3$ and no adjacent Δ-vertices. Next, we substitute by a path ux_1x_2v the path uwv of G' to obtain another graph H_2. By means of a 4-avdtc of H_1, we can get a desired 4-avdtc of H_2 shown in Figure 5(a).

Suppose that two vertices u, v are not in any 4-cycle of G'. If G' is the graph shown in Figure 4(a), we are done. Otherwise, G' has a part shown in Figure 5(b) (no dashing lines and black vertices).

We have a graph H_3 obtained by deleting every vertex of $\{u_1, u_2, u, w, v, v_1, v_2\}$ from G' and adding a new vertex x_0 out of G' to join x and y, and adding another new vertex s_0 out of G' to join s and t, respectively. Since H_3 has $\Delta(H_3) = 3$ and no adjacent Δ-vertices that is a 4-avdtc f_3 of H_3 by induction hypothesis. Thus, we construct a graph H_4 in the way of replacing by a path ux_1x_2v the path uwv of G', and extend f_3 to a proper total coloring h_4 of H_4. Clearly, H_4 has maximum degree 3 and no adjacent Δ-vertices.

Except $h_4(x_1) = 1$, $h_4(x_1x_2) = 2$ and $h_4(x_2) = 3$, each $h_4(z)$ for $z \in (V(H_4) \cup E(H_4)) \setminus \{x_1, x_1x_2, x_2\}$ are determined well based on f_3 (see Figure 5(b)) such that $h_4(u) \neq h_4(v)$ since $h_4(u_1u) = c$ or d, and $h_4(u) = d$ or c, where $d \in [1, 4] \setminus \{a, b, c\}$. Notice that $d' \in [1, 4] \setminus \{a', b', c'\}$. By exhaustion we can select appropriately colors $1, 2, 3$ such that h_4 is a 4-avdtc of H_4.

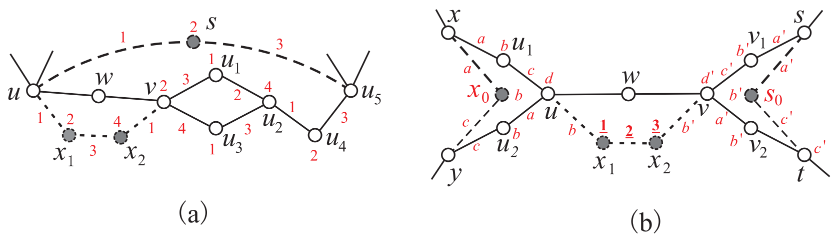

Figure 5. A diagram for the proof of Theorem 3.4.

Case 2. Let H' be a graph having maximum degree 3, a $(3, 3; m)$-path $P = ux_1x_2 \cdots x_mv$ with $m \geq 2$ and no adjacent Δ-vertices as well as $\chi''_{as}(H') = 4$. Our goal is to construct a new graph G^* having maximum degree 3 and no adjacent Δ-vertices by replacing an edge x_ix_{i+1} of the path P for some $i \in [1, m-1]$ by a path x_iyx_{i+1}, where y is a new vertex out of H', and then show $\chi''_{as}(G^*) = 4$. Note that G^* contains a path $(3, 3; m)$-path $Q = ux_1x_2 \cdots x_iyx_{i+1} \cdots x_mv$ with $m \geq 2$ and $|G^*| > |G|$.

Subcase 2.1. For $m = 2$, H' contains a $(3, 3; 2)$-path ux_1x_2v.

Let η_{21} be a 4-avdtc of H'. Without loss of generality, $\eta_{21}(ux_1) = 2$, $\eta_{21}(x_1) = 1$, $\eta_{21}(x_1x_2) = 3$. Hence, $\eta_{21}(x_2) \in \{2, 4\}$ and $\eta_{21}(x_2v) \in \{1, 2, 4\}$, since $C(x, \eta_{21}) \neq C(y, \eta_{21})$ for each edge xy of H'. Now, we delete the edge x_1x_2 from H' and add a new vertex z out of H' and two edges zx_1, zx_2. The resulting graph is denoted as H_{21}. We can define a 4-avdtc λ_{21} of H_{21} by means of η_{21}. Let $Z = \{x_1z, zx_2, x_2v\} \cup \{z, x_2\}$. We define $\lambda_{21}(s) = \eta_{21}(s)$ for $s \in (V(H_{21}) \cup E(H_{21})) \setminus Z$; and $\lambda_{21}(x_2) = \eta_{21}(x_2)$, $\lambda_{21}(x_2v) = \eta_{21}(x_2v)$.

If $\eta_{21}(x_2) = 2$ and $\eta_{21}(x_2v) = 4$, set $\lambda_{21}(x_1z) = 4$, $\lambda_{21}(z) = 3$ and $\lambda_{21}(zx_2) = 1$.

If $\eta_{21}(x_2) = 4$ and $\eta_{21}(x_2v) = 2$, define $\lambda_{21}(x_1z) = 4$, $\lambda_{21}(z) = 1$ and $\lambda_{21}(zx_2) = 3$.

If $\eta_{21}(x_2) = 4$ and $\eta_{21}(x_2v) = 1$, then let $\lambda_{21}(x_1z) = 4$, $\lambda_{21}(z) = 3$ and $\lambda_{21}(zx_2) = 2$.

If $\eta_{21}(x_2) = 2$ and $\eta_{21}(x_2v) = 1$, so $\eta_{21}(v) \in \{3, 4\}$. If $\eta_{21}(v) = 3$, we reset $\lambda_{21}(x_2) = 4$, and let $\lambda_{21}(x_2z) = 2$, $\lambda_{21}(z) = 3$ and $\lambda_{21}(zx_1) = 4$; if $\eta_{21}(v) = 4$, we reset $\lambda_{21}(x_2) = 3$, and let $\lambda_{21}(x_2z) = 2$, $\lambda_{21}(z) = 4$ and $\lambda_{21}(zx_1) = 3$. Thereby, λ_{21} is really a 4-avdtc of H_{21}.

Subcase 2.2. For $m = 3$, let $N(u) = \{u', u'', x_1\}$ and $N(v) = \{x_3, v', v''\}$. Let η_{22} be a 4-avdtc of H'. Without loss of generality, assume that $\eta_{22}(u'u) = 2$, $\eta_{22}(u''u) = 3$, $\eta_{22}(u) = 4$, $\eta_{22}(ux_1) = 1$, $\eta_{22}(x_1) = 2$, $\eta_{22}(x_1x_2) = 3$. So, $\eta_{22}(x_2) \in \{1, 4\}$ and $\eta_{22}(x_2x_3) \in \{1, 2, 4\}$. We replace the edge ux_1 of H' by a path uzx_1, where z is a new vertex out of H', and the resulting graph is denoted as H_{22}. We will extend η_{22} to a 4-avdtc λ_{22} of H_{22}.

First, we define $\lambda_{22}(s) = \eta_{22}(s)$ for $s \in (V(H_{22}) \cup E(H_{22})) \setminus Z'$, where $Z' = \{z, zx_1, x_1, x_1x_2, x_2,$

$x_2x_3\}$. If $\eta_{22}(x_2) = 4$ and $\eta_{22}(x_2x_3) = 2$, we define $\lambda_{22}(z) = 2$, $\lambda_{22}(zx_1) = 4$, $\lambda_{22}(x_1) = 1$; $\lambda_{22}(y) = \eta_{22}(y)$ for $y \in \{x_1x_2, x_2, x_2x_3\}$.

We have two subcases for colors $\eta_{22}(x_2)$ and $\eta_{22}(x_2x_3)$ in the following. (i) If $\eta_{22}(x_3) = 2$ and $\eta_{22}(x_3v) = 4$, then define $\lambda_{22}(z) = 3$, $\lambda_{22}(zx_1) = 4$, $\lambda_{22}(x_1) = 2$, $\lambda_{22}(x_1x_2) = 1$, $\lambda_{22}(x_2) = 4$ and $\lambda_{22}(x_2x_3) = 3$. If $\eta_{22}(x_3) = 3$ and $\eta_{22}(x_3v) = 2$ (resp. $\eta_{22}(x_3) = 2$ and $\eta_{22}(x_3v) = 3$), thus, we set $\lambda_{22}(z) = 2$, $\lambda_{22}(zx_1) = 4$, $\lambda_{22}(x_1) = 3$, $\lambda_{22}(x_1x_2) = 2$, $\lambda_{22}(x_2) = 4$ and $\lambda_{22}(x_2x_3) = 1$. (ii) If $\eta_{22}(x_2) = 1$ and $\eta_{22}(x_3) = 2$ or 4, we can extend η_{22} to a 4-avdtc λ_{22} of H_{22} by the way similarly to the above one.

Subcase 2.3. For $m \geq 4$, we let $N(u) = \{u', u'', x_1\}$ and let η_{23} be a 4-avdtc of H'. Without loss of generality, we assume that $\eta_{23}(u'u) = 2$, $\eta_{23}(u''u) = 3$, $\eta_{23}(u) = 4$, $\eta_{23}(ux_1) = 1$, $\eta_{23}(x_1) = 2$, $\eta_{23}(x_1x_2) = 3$. So, $\eta_{23}(x_2) \in \{1, 4\}$ and $\eta_{23}(x_2x_3) \in \{1, 2, 4\}$, since $C(x, \eta_{23}) \neq C(y, \eta_{23})$ for each edge $xy \in E(H')$.

In order to obtain a new graph H_{23} we replace the edge ux_1 of H' by a path uzx_1, where z is a new vertex out of H'. Next, we extend η_{23} to a 4-avdtc λ_{23} of H_{23}. We set $\lambda_{23}(s) = \eta_{23}(s)$ for $s \in (V(H_{23}) \cup E(H_{23})) \setminus Z''$, where $Z'' = \{z, zx_1, x_1, x_1x_2, x_2\}$. Suppose that $\eta_{23}(x_2) = 4$, $\eta_{23}(x_2x_3) = 1$ (resp. $\eta_{23}(x_2x_3) = 2$). If $\eta_{23}(x_3) = 2$ and $\eta_{23}(x_3x_4) = 4$, define $\lambda_{23}(z) = 2$, $\lambda_{23}(zx_1) = 3$, $\lambda_{23}(x_1) = 4$, $\lambda_{23}(x_1x_2) = 2$ and $\lambda_{23}(x_2) = 3$. So $C(s, \lambda_{23}) \neq C(t, \lambda_{23})$ for every edge $st \in E(H_{23})$. If $\eta_{23}(x_3) = 2$ and $\eta_{23}(x_3x_4) = 3$ (resp. $\eta_{23}(x_3) = 3$ and $\eta_{23}(x_3x_4) = 4$), which imply $\eta_{23}(x_2x_3) = 1$ or 4, so we define $\lambda_{23}(z) = 2$, $\lambda_{23}(zx_1) = 4$, $\lambda_{23}(x_1) = 3$, $\lambda_{23}(x_1x_2) = 2$, and $\lambda_{23}(x_2) = 4$ when $\eta_{23}(x_2x_3) = 1$, or $\lambda_{23}(x_2) = 1$ if $\eta_{23}(x_2x_3) = 4$.

For the other choices of colors $\eta_{23}(u'u)$, $\eta_{23}(u''u)$, $\eta_{23}(u)$, $\eta_{23}(ux_1)$, $\eta_{23}(x_1)$ and $\eta_{23}(x_1x_2)$ under the distinguishing constraint $C(s, \eta_{23}) \neq C(t, \eta_{23})$ for each edge $st \in E(H')$, by the above ways we can obtain a 4-avdtc of H_{23}.

Based on the proofs in Case 1 and Case 2 above, we can reconstruct G from G' step by step such that $\chi''_{as}(G) = 4$. The proof is covered. □

Theorem 3.4 implies that a connected graph G with $\Delta = 3$ and no adjacent Δ-vertices holds $\chi''_{as}(G) = \chi''(G) = 4$ and $\chi'_{as}(G) = \chi'(G) = 3$, and furthermore $\chi(G) = 3$ if G contains odd-cycles.

Corollary 3.1. *There are infinitely many avdtc no-covered graphs.*

Proof. The result follows from Lemmas 3.1, 3.2, 3.3 and Theorem 3.4. □

4. Further works

Motivated from the avd-linking operation introduced in the proof of Theorem 3.1, we define the *avd-equivalent operation* on a graph as below. Let π be a k-avdec of graph G. If there are two edges u_1v_1, u_2v_2 of G such that $\pi(u_1v_1) = \pi(u_2v_2)$, $C(u_i, \pi) \neq C(v_{3-i}, \pi)$ and $u_iv_{3-i} \notin E(G)$ for $i = 1, 2$, then we have an *avd-equivalent graph* G' obtained by deleting u_1v_1, u_2v_2 from G and join u_i to v_{3-i} for $i = 1, 2$. Clearly, the avd-equivalent graph G' has a k-avdec generated from π. Notice that $V(G) = V(G')$, $|E(G)| = |E(G')|$, and $\chi'_{as}(G') \leq \chi'_{as}(G)$. The *avd-equivalent class* $\mathcal{G}_{as}(G)$ is the set of avd-equivalent graphs generated from G. In other words, each $H \in \mathcal{G}_{as}(G)$ is an avd-equivalent graph of a certain graph of $\mathcal{G}_{as}(G)$, and $\chi'_{as}(H) \leq \chi'_{as}(G)$. It is noticeable, $|\mathcal{G}_{as}(G_0)| = 1$, where G_0 is the graph shown in Figure 1. We want to figure out $\mathcal{G}_{as}(G)$ for a simple and connected graph G.

Observe that some connected graph G having a proper subgraph H with $\chi'_s(H) > \chi'_s(G)$ may hold $|\Delta(G) - \Delta(H)| \geq M$ for given integers $M \geq 1$. However, for cases $\chi'_{as}(H) > \chi'_{as}(G)$ and $\chi''_{as}(H) >$

$\chi''_{as}(G)$, can we say $|\Delta(G) - \Delta(H)| \leq 1$?

In [9], the author point out that no simple graph G having maximum degree three and $\chi''_{as}(G) = 6$ has been discovered, although all graphs H of maximum degree three obey $\chi''_{as}(H) \leq 6$. For a graph G with $\Delta(G) \geq 4$, we do not find a proper subgraph H of G such that $\chi''_{as}(G) < \chi''_{as}(H)$. Therefore, we propose:

Conjecture 4.1. (1) *Let H be a proper subgraph of G. Then $\chi'_{as}(H) \leq \chi'_{as}(G) + 1$ and $\chi''_{as}(H) \leq \chi''_{as}(G) + 1$.*

(2) *Let H be a proper subgraph of G. Then there is no proper subgraph H^* of H such that $\chi'_{as}(H^*) > \chi'_{as}(H) > \chi'_{as}(G)$, or $\chi''_{as}(H^*) > \chi''_{as}(H) > \chi''_{as}(G)$.*

(3) *Let H be a common proper subgraph of two graphs G_1 and G_2. If $\chi'_{as}(H) > \chi'_{as}(G_i)$ for $i = 1, 2$, then $\chi'_{as}(G_1) = \chi'_{as}(G_2)$.*

(4) *Let G be a graph having adjacent Δ-vertices and $\Delta = 3$. Then G is not avdtc no-covered.*

As known, $\chi'(G) \leq \chi''(G)$ is true in the traditional colorings of graph theory. But we can see $\chi''_{as}(C_5) = 4 < 5 = \chi'_{as}(C_5)$ and $\chi''_{s}(C_{13}) = 6 < 7 = \chi'_{s}(C_{13})$. We call G an *avdtc no-covering avdec* (resp. *vdtc no-covering vdec*) if $\chi''_{as}(G) < \chi'_{as}(G)$ (resp. $\chi''_{s}(G) < \chi'_{s}(G)$). It may be interesting to characterize avdtc no-covering avdec graphs or vdtc no-covering vdec graphs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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