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Research article

Value of first eigenvalue of some minimal hypersurfaces embedded in the unit sphere

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Abstract: We prove that the first nonzero eigenvalue of the Laplace-Beltrami operator of equator-like minimal submanifold embedded in the sphere S^{n+1} is equal to n. The proof uses the spectral properties of the heat kernel operator corresponding to the submanifold.

Keywords: minimal submanifolds; Laplace operator; eigenvalues; heat kernel
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1. Introduction

In the theory of minimal submanifolds in a sphere, an interesting question asks about the value of first nonzero eigenvalue of the Laplacian for a minimal hypersurface Σ embedded in $(n+2)$-unit sphere S^{n+1} in $\mathbb{R}^{n+2}$. In its list of famous problems, the following question has been raised by S. T. Yau (problem 100, [17]).

Conjecture: [17] Let Σ be a minimal hypersurface embedded in the $n+1$-unit sphere S^{n+1}. Then, $\lambda_1(\Sigma) = n$.

The upper bound $\lambda_1(\Sigma) \leq n$ is not obvious, and was obtained before the statement of the conjecture due to Takahashi [15]. Just after the conjecture was published, Choi and Wang proved that $\lambda_1(\Sigma) \geq n/2$. In fact, they proved a more general statement based on Reilly's formula, see [9]. Until this day, it was the best known lower bound in the general case. Many important steps towards this conjecture has been done by proving the conjecture for some minimal homogenous hypersurfaces due to Muto-Ohnita-Urakawa [12], Kotani [10] and Solomon [13,14]. Recently, Z. Tang and W. Yan proved that the conjecture is valid for closed minimal isoparametric hypersurfaces [16]. In a recent work, S Deshmukh has proved some results related to the conjecture in [6]. For the case when $\lambda_1(\Sigma) < n$, it is shown that one has the following alternative, either $\lambda_1(\Sigma) \leq (1+k_0)n$ or $\lambda_1(\Sigma) \geq n + (nk_0 - n + 1)\dfrac{n}{2}$. In the opposite case, when $\lambda_1(\Sigma) = n$, either Σ is isometric to the unit sphere S^n or otherwise $k_0 \leq n - 1/n$. A generalization of this work for pseudo-umbilical hypersurface in the unit sphere has been proved by M. A. Choudhary in [2].

The method we are going to use in the paper are very different to the previous works, which have studied in this topic. Indeed, we are going to focus on the spectral properties of the Laplacian of a special type of immersed minimal submanifolds in the unit sphere. One of the most important objects in spectral geometry is the heat kernel operator associated with a given Riemmanian manifold, which corresponds to the solution of the heat equation on the manifold. The first nonzero eigenvalue controls the rate of growth of the heat kernel when time tends to infinity.

1.1. The main result

Let Σ be the hypersurface given by the locus of vanishing of some smooth function ψ on the unit sphere S^{n+1} i.e.,

$$\Sigma = \left\{ x \in S^{n+1} \mid \psi(x) = 0 \right\}.$$

We assume that Σ is embedded in S^{n+1}, which amounts to say that the gradient of ψ never vanishes on Σ. Thus, Σ is Riemannian submanifold on S^{n+1}, in particular it has an induced metric which gives rise to the corresponding Laplace-Beltrami operator Δ_Σ which is also self-adjoint. The spectrum of Δ_Σ which is discrete, has a least nonzero eigenvalue, $\lambda_1(\Sigma)$. Let us consider the polar coordinates parametrization, (σ, θ) coming from the stereographic projection. Suppose ψ is chosen so that Σ is a minimal embedded hypersurface in the $n+1$-unit sphere S^{n+1}. Then, one has the following positive answer to the conjecture for minimal submanifolds satisfying some tranversality conditions.

Theorem 1.1. *Let Σ a minimal embedded hypersurface in the $n+1$-unit sphere S^{n+1} and assume that the normal bundle of Σ is a one-dimensional subspace of $T(S^{n+1})$ generated by the vector field ∂_θ coming from the stereographic projection. Then,*

$$\lambda_1(\Sigma) = n.$$

2. Useful facts

2.1. Spectral expansion of the Laplacian on Riemannian manifolds

We begin by some classical results that we are going to need. These can be found in many places (see e.g., [1, 8]). Given an n-dimensional Riemannian manifold (M, g), one can define the Laplace-Beltrami operator which acts on smooth functions over M. In the local coordinate around the point $x = (x_1, \ldots, x_n)$ with associated frame $(\partial_1, \ldots, \partial_n)$ which forms a basis of the tangent space $T_x(M)$, the Laplace-Beltrami operator takes the following form

$$\Delta_{M.g} = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \partial_i (\sqrt{g} \sum_{j=1}^{n} g^{ij} \partial_j), \tag{2.1}$$

where $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. Assuming that M is a compact makes the operator $-\Delta_M$ being a self-adjoint operators in $L^2(M)$. In particular, it has a discrete spectrum given by

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \ldots$$

The spectral decompositon of the Hilbert space $L^2(M)$ with respect to Δ_M allows to write any function $f \in L^2(M)$ as

$$f = \sum_{k\geq 0} \langle f, \Phi_k \rangle_{L^2(M)} \Phi_k,$$

where $(\Phi_k)_{k\geq 0}$ is a basis of eigenfunctions of $L^2(M)$. Associated to this, one can strongly continous operator in $L^2(M)$, $P_t = e^{-t\Delta_M}$ satisfying the property $P_{s+t} = P_s \circ P_t$ such that $\|P_t\| \leq 1$. It can be proved see e.g., [3] that the operator P_t has a kernel $K_t : M \times M \to \mathbb{R}$ for all $t \geq 0$. This means that, for any function $L^2(M)$

$$P_t f(x) = \int_M K_t(x, y) f(y) \mathrm{vol}_g(dy).$$

The heat kernel characterizes the heat operator, and it can be obtained to perform the following evaluation

$$P_t(\delta_y)(x) = K_t(x, y).$$

The latter evaluation is allowed since one can identify the regular distribution with the function whenever it is continous, which is the case for the kernel operator. In can be shown that the function $u(t, x) := P_t f(x)$ satisfies the following equation with initial Dirichlet boundary condition

$$\begin{cases} \Delta_M u(t, x) + \dfrac{\partial}{\partial t} u(t, x) = 0, \\ \\ u(0, x) = f(x) \ \text{ on } \ \partial M. \end{cases} \tag{2.2}$$

The spectral decomposition of the heat kernel is given by

$$K_t(x, y) = \sum_{k\geq 0} e^{\lambda_k(M)t} \Phi_k(x) \Phi_k(y), \tag{2.3}$$

for every $x, y \in M$. The exponential growth of $K_t(x, y)$ is controlled by the first eigenvalue $\lambda_1(M)$, which is nonzero for compact Riemannian manifolds with Dirichlet initial value condition. A fundamental result for long time behavior of the heat kernel it that for every $x, y \in M$ one has (see e.g., [11])

$$\lim_{t\to\infty} \frac{\log K_t(x, y)}{t} = \lambda_1(M). \tag{2.4}$$

The non-nullity of the first eigenvalue is granted by the fact that we consider the heat equation on a bounded domain of the sphere with Dirichlet boundary conditions. In fact, we can say much more about it since we are going to work with embedded closed minimal hypersurfaces in the unit sphere S^{n+1}. For this class of domains, the first eigenvalue is quite large in a certain sense, since it was proved by Choi and Wang that $\lambda_1(M) \geq \frac{n}{2}$ [9]. In particular, the first eigenvalue is not zero.

Yau's conjecture predicts that this value is maximal, in that $\lambda_1(M) = n$ for such hypersurfaces. Thus, the minimality condition for an hypersurface on the unit sphere implies maximality of the first eigenvalue.

For the sphere $M = S^{n+1}$, the eigenvalues of the operator $(-\Delta_{S^{n+1}})$ acting on $L^2(S^{n+1})$ are given by

$$\lambda_l(S^{n+1}) = l(n + l), \tag{2.5}$$

in particular the first eigenvalue is given by

$$\lambda_1(S^{n+1}) = n + 1. \tag{2.6}$$

In particular, using 2.4 and 2.6 for the unit sphere one has asymptotic estimate as t tends to infinity

$$K_t^{S^{n+1}}(x, x) \sim e^{-(n+1)t}\phi_1(x)^2. \tag{2.7}$$

2.2. Polar coordinates of the unit sphere

Let us consider the stereographic projection π of the sphere S^{n+1} on $\mathbb{R}^{n+1}$ relatively to the north pole $N = e_{n+2} = (0, \ldots, 0, 1) \in S^{n+1}$. It is given by the rule

$$\pi(x) = \frac{1}{1 - x_{n+2}}(x_1, \ldots, x_{n+1}),$$

provided $x = (x_1, \ldots, x_{n+2})$ is not N. Let us set $\sigma(x) = \dfrac{\pi(x)}{\|\pi(x)\|}$, this defines an element of S^n. Also one defines a map $\theta : S^{n+1} \to [0, \pi]$ by assigning to each $x \in S^{n+1}$, the angle $\theta(x) = 2(\widehat{\overrightarrow{NO}, \overrightarrow{Nx}})$. One can explicit a formula for θ, indeed one has

$$\|\overrightarrow{NO}\|\|\overrightarrow{Nx}\| \cos(\frac{\theta}{2}) = \langle \overrightarrow{N0}, \overrightarrow{Nx} \rangle.$$

In terms of the coordinates $x = (x_1, \ldots, x_{n+2})$ and $\overrightarrow{NO} = -e_{n+2} = (0, \ldots, -1)$, the previous equality gives

$$\sqrt{x_1^2 + \ldots + x_{n+1}^2 + (x_{n+2} - 1)^2} \cos(\frac{\theta(x)}{2}) = 1 - x_{n+2}.$$

Thus,

$$\theta(x) = 2\cos^{-1}\left(\frac{1 - x_{n+2}}{\sqrt{x_1^2 + \ldots + x_{n+1}^2 + (1 - x_{n+2})^2}}\right) \in [0, \pi).$$

We are able to product a diffeomorphism $\psi : S^{n+1} - \{N\} \to S^n \times [0, \pi]$ by setting

$$\psi(x) = (\frac{\pi(x)}{\|\pi(x)\|}, \theta(x)).$$

This gives the well-known realization of the unit sphere S^{n+1} minus N as the product $S^n \times [0, \pi]$ with the polar coordinates $x = (\sigma, \theta)$.

By now on, we use the parametrization of the unit sphere minus the north pole using the change of coordinates $(x_1, \ldots, x_{n+1}) \leftrightarrow (\sigma_1, \ldots, \sigma_{n+1})$ where $\sigma_{n+1} = \theta$. The length element is given by

$$dx^2_{|S^{n+1}} = \sin^2\theta d\sigma^2_{|S^n} + d\theta^2.$$

The metric $g_{|S^{n+1}}$ in the local coordinates $(\sigma_1, \ldots, \sigma_{n+1}, \theta)$ is given by the diagonal matrix

$$(g_{|S^{n+1}})_{ij} = \begin{bmatrix} \sin^2\theta & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \sin^2\theta & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix}.$$

The metric $g_{|S^{n+1}}$ gives rise to the Christoffel symbols given by

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{1 \leqslant l \leqslant n+2} g^{kl} \{\partial_j(g_{il}) + \partial_i(g_{jl}) - \partial_l(g_{ij})\} \quad (1 \leqslant i, j, k \leqslant n + 1).$$

We can use this coefficients to define a connection on the tangent bundle of S^{n+1}. Let $x \in S^{n+1} - N$ with local coordinates $x = (\sigma_1, \ldots, \sigma_n, \theta)$ and corresponding orthonormal frame $\{\partial_1, \ldots, \partial_n, \partial_\theta\}$ with respect to the metric $g_{|S^{n+1}}$ that is, for every $i, j = 1, \ldots, n + 1$

$$g_{|S^{n+1}}(\partial_i, \partial_j) = \delta_i^j.$$

Using the basis $\{\partial_1, \ldots, \partial_n, \partial_\theta\}$ of $T_x(S^{n+1})$ we are able to define a bilinear map

$$\nabla = \nabla^{S^{n+1}} \;:\; T(S^{n+1}) \times T(S^{n+1}) \to T(S^{n+1}),$$

by assigning the values taken by this form at the elements of the basis of $T(S^{n+1})$ by introducing the coefficients,

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n+1} \Gamma_{ij}^{k} \partial_k,$$

where we have denoted $\partial_{n+1} = \partial_\theta$. The operator ∇ therefore defines a connection on the tangent bundle $T(S^{n-1})$. Basic computations show that the Christoffel symbols relative to the metric g are symmetric in the sense $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all $1 \leqslant i, j, k \leqslant n + 1$. This implies that ∇ is torsion-free i.e., $\nabla_X Y - \nabla_Y X = [X, Y]$ and by uniqueness, ∇ is the Levi-Civita connection on $T(S^{n+1})$.

2.3. The hypersurface Σ in S^{n+1} in the polar coordinate system

We consider the unit sphere equipped with the polar coordinates system (σ, θ) introduced in the previous paragraph. Thus, the hypersurface Σ in the coordinate system (σ, θ) is defined as follows

$$\Sigma = \{(\sigma_1, \ldots, \sigma_n, \theta) \in S^n \times [0, \pi] \;\mid\; \psi(\sigma_1, \ldots, \sigma_n, \theta) = 0\}.$$

Since Σ is embedded in S^{n+1}, the chain rule implies that gradient of ψ satisfies $\nabla\psi(x) \neq 0$ for any $x \in \Sigma$ in the polar coordinates. The hypersurface Σ inherits a structure of Riemannian manifold given by a metric g_Σ which the one induced by $g_{|S^{n+1}}$ with associated volume Riemmanian form $d\mathrm{vol}_\Sigma = \sqrt{g_\Sigma}(\sigma, \theta)\, d\sigma \wedge d\theta$ which we do not need to explicit. In the local coordinates, we can assume that $\{\frac{\partial}{\partial\sigma_1}, \ldots, \frac{\partial}{\partial\sigma_n}\}$ is an orthonormal frame of $T(\Sigma)$ whereas $\xi = \frac{\partial}{\partial\theta}$ generates the normal bundle $N(\Sigma)$. The vector fields $(\frac{\partial}{\partial\sigma_i})$ are simply denoted ∂_i for $1 \leqslant i \leqslant n$ and sometimes we will denote either ∂_{n+1} or ∂_θ the vector field $\frac{\partial}{\partial\theta}$. With these notations, we have the orthonormal frame for $T(S^{n+1}) = \{\partial_1, \ldots, \partial_n, \partial_\theta\}$ which extends the tangent bundle $T(\Sigma) = \{\partial_1, \ldots, \partial_n\}$. The tangent space of $T(S^{n+1})$ can be splited as folllows:

$$T(S^{n+1}) = T(\Sigma) \oplus N(\Sigma).$$

Since Σ is a smooth hypersurface, namely of codimension one, $N(\Sigma)$ is a line bundle over Σ which is generated by the normal vector field $\xi = \partial_\theta$. One can give an explicit expression for ξ, the metric g_Σ

and the mean curvature of Σ in function of the derivatives of u with respect to the frame $\{\partial_1, \ldots, \partial_n\}$, but we will not need it. Instead, we will use the general expression of the mean curvature in terms of the connections on Σ and S^{n+1}. Let us denote $\nabla^{S^{n+1}}$ (resp. ∇^{Σ}) the Levi-Civita connection of S^{n+1} (resp. Σ) relative to the metric g and the induced metric $g_{|\Sigma}$ given in polar coordinates. The second fundamental form II_Σ of Σ in S^{n+1} is defined by

$$\nabla^{S^{n+1}}_X Y = \nabla^{\Sigma}_X Y + \mathrm{II}_\Sigma(X, Y),$$

for any two vectors fields in $X, Y \in T(S^{n+1})$. In particular taking $\{\partial_1, \ldots, \partial_n\}$ as an orthonormal basis for $T(\Sigma)$, the previous relation applied to $X = Y = \partial_i$ gives us

$$\nabla^{S^{n+1}}_{\partial_i} \partial_i = \nabla^{\Sigma}_{\partial_i} \partial_i + \mathrm{II}_\Sigma(\partial_i, \partial_i).$$

Taking the sum we obtain the fundamental relation

$$\sum_{i=1}^{n} \nabla^{S^{n+1}}_{\partial_i} \partial_i = \sum_{i=1}^{n} \nabla^{\Sigma}_{\partial_i} \partial_i + H_\Sigma, \tag{2.8}$$

where $H_\Sigma = \sum_{i=1}^{n} \mathrm{II}_\Sigma(\partial_i, \partial_i)$ is the mean curvature vector of Σ.

3. The proof of Theorem 1.1

By translating Σ using a rotation $k \in \mathrm{SO}(n+1)$, one can sufficiently rotate the hypersurface Σ so that $N \notin \Sigma$, that is, $\Sigma \subset S^{n+1} - \{N\}$. The hypothesis of Theorem 1.1 tells us that $T(\Sigma) \perp \partial_\theta$ and Σ is the graph of a smooth real valued function $u : S^n \to [0, \pi]$,

$$\Sigma = \{(\sigma, \theta) \in S^n \times [0, \pi] : \ \theta = u(\sigma)\}. \tag{3.1}$$

Since Σ is embedded in S^{n+1}, the gradient of ψ does not vanish, the implicit function theorem shows that

$$(\sigma, \theta) \in \Sigma - C \text{ if and only if } \theta = u(\sigma), \tag{3.2}$$

in the coordinates (σ, θ). In particular, for any $x = (\sigma, \theta) \in V \cap \Sigma - C$, one has $\psi(\sigma, u(\sigma)) = 0$ and at such point x, the hypersurface Σ only depends on the coordinates $\sigma_1, \ldots, \sigma_n$. Thus, $T_x(\Sigma)$ is a hyperplane in $T_x(S^{n+1})$ with orthonormal basis $\{\partial_1, \ldots, \partial_n\}$. This basis extends to a local frame $\{\partial_1, \ldots, \partial_n, \partial_\theta\}$ of $T_x(S^{n+1})$ where $\partial_n = \partial_\theta$. Thus, the normal direction is given by the line $N_x(\Sigma)$ generated by ∂_θ.

We give an explicit expression for the Laplacian of Σ in our setting which can also be found in [4,5].

Lemma 3.1. *For every $(\sigma, \theta) \mapsto f(\sigma, \theta)$ smooth function on S^{n+1}, one has*

$$\Delta_\Sigma f_{|\Sigma} = (\Delta_{S^{n+1}} f)_{|\Sigma} - \partial^2_\theta f.$$

Proof. The local orthonormal frame $\{\partial_1, \ldots, \partial_n\}$ of Σ gives the following expression for the Laplacian of Σ

$$\Delta_\Sigma = \sum_{i=1}^{n} (\partial_i^2 - \nabla^{\Sigma}_{\partial_i} \partial_i).$$

By (2.8), one has

$$\sum_{i=1}^{n} \nabla^{\Sigma}_{\partial_i}\partial_i = \sum_{i=1}^{n} \nabla^{S^{n+1}}_{\partial_i}\partial_i - H_\Sigma.$$

Therefore,

$$\Delta_\Sigma = \sum_{i=1}^{n}(\partial_i^2 - \nabla^{S^{n+1}}_{\partial_i}\partial_i) + H_\Sigma.$$

Let us write,

$$\Delta_\Sigma = \sum_{i=1}^{n+1}(\partial_i^2 - \nabla^{S^{n+1}}_{\partial_i}\partial_i) + H_\Sigma - (\partial_\partial^2 - \nabla^{S^{n+1}}_{\partial_\theta}\partial_\theta).$$

The choice of the system of coordinates tells us that $\partial_{n+1} = \partial_\theta$. We claim that $\nabla^{S^{n+1}}_{\partial_\theta}\partial_\theta = 0$. Indeed one has,

$$\nabla^{S^{n+1}}_{\partial_\theta}\partial_\theta = \sum_{j=1}^{n+1}\Gamma^j_{\theta\theta}\partial_j,$$

where

$$\Gamma^j_{\theta\theta} = \frac{1}{2}\sum_{k=1}^{n} g^{kj}\{\partial_\theta(g_{\theta k}) + \partial_\theta(g_{\theta k}) - \partial_k(g_{\theta\theta})\}.$$

Since the inverse metric tensor $(g_{S^{n+1}})^{kj}$ of S^{n+1} is diagonal with $g^{\theta\theta} = 1$ and $g^{kk} = 1/\sin^2\theta$ for $k = 1, \ldots, n$. Thus one has, for all $1 \leqslant j \leqslant n+1$

$$\Gamma^j_{\theta\theta} = \frac{1}{2}g^{jj}\{\partial_\theta(g_{\theta j}) + \partial_\theta(g_{\theta j}) - \partial_j(g_{\theta\theta})\} = 0.$$

Hence, as expected one has

$$\nabla^{S^{n-1}}_{\partial_\theta}\partial_\theta = 0.$$

Finally, one obtains the Laplace operator on the hypersurface Σ expressed in the local coordinates $(\sigma_1, \ldots, \sigma_n, \theta)$ in S^{n+1}

$$\Delta_\Sigma = \Delta_{S^{n+1}} + H_\Sigma - \partial_\theta^2. \tag{3.3}$$

Since Σ is minimal (i.e., $H_\Sigma = 0$), the previous equality gives us the expected decompostion of the Laplacian

$$\Delta_\Sigma = \Delta_{S^{n+1}} - \partial_\theta^2.$$

Now, let us restrict our attention to the spectrum of Δ_Σ. The main task is to find an explicit form of the heat kernel $K_t^\Sigma, (t > 0)$ of Σ. The previous lemma gives rise to the following relation between the heat operators of Σ, S^{n+1} and $[0, \pi]$

$$P_t^\Sigma = e^{-t\Delta_\Sigma} = e^{-t\Delta_{S^{n-1}} + t\partial_\theta^2}. \tag{3.4}$$

The fact that the two operators $\Delta_{S^{n+1}}$ and ∂_θ^2 commutes, i.e., $[\Delta_{S^{n+1}}, \partial_\theta^2] = 0$ implies the following:

$$P_t^\Sigma = e^{-t\Delta_{S^{n+1}} + t\partial_\theta^2} = e^{-t\Delta_{S^{n+1}}}e^{t\partial_\theta^2} = P_t^{S^{n+1}}P_t^{S^1}. \tag{3.5}$$

Note that we have denoted $P_t^{S^1} = e^{t\partial_\theta^2}$ the heat operator acting on $L^2(0,\pi)$ in order to emphasis with the fact that the operator ∂_θ^2 acts isopectrally either on $L^2(S^1)$ and $L^2(0,\pi)$ meaning that their eigenvalues are the same, $\lambda_l = -l^2$ for $l = 0, 1, 2, \ldots$ We arrive to the key lemma which gives a formula of the heat kernel of the hypersurface Σ in the (σ,θ)-coordinates of the unit sphere S^{n+1}.

Lemma 3.2. *For any $x = (\tau,\alpha)$ and (σ,θ) in Σ, one has*

$$K_t^{\Sigma}(x;(\sigma,\theta)) = K_t^{S^{n+1}}(x;(\sigma,\theta))\left(\int_{\beta=0}^{\pi} K_t^{S^1}(\theta,\beta)d\beta\right).$$

Proof. Let us denote by $(K_t^{S^{n+1}})_{t>0}$ (resp. $(K_t^{S^{n+1}})_{t>0}$) the heat kernel of $P_t^{S^{n+1}}$ (resp. $P_t^{S^1}$).

Let us consider $f \in L^2(S^{n+1})$ with support in Σ, then the factorization (3.5) yields

$$P_t^{\Sigma} f(x) = \int_{(\sigma,\theta)\in\Sigma} K_t^{S^{n+1}}(x;(\sigma,\theta))\,(P_t^{S^1} f)(\sigma,\theta)\ d\mathrm{vol}_{\Sigma}(\sigma,\theta).$$

Therefore,

$$P_t^{\Sigma} f(x) = \int_{(\sigma,\theta)\in\Sigma} K_t^{S^{n+1}}(x;(\sigma,\theta))\left(\int_{\beta=0}^{\pi} K_t^{S^1}(\theta,\beta) f(\sigma,\beta)d\beta\right) d\mathrm{vol}_{\Sigma}(\sigma,\theta).$$

Sard's Theorem [7] tells us that $\mathrm{vol}_{\Sigma}(C) = 0$, in other words, one can restrict the integral to $\Sigma - C$

$$P_t^{\Sigma} f(x) = \int_{(\sigma,\theta)\in\Sigma-C} K_t^{S^{n+1}}(x;(\sigma,\theta))\left(\int_{\beta=0}^{\pi} K_t^{S^1}(\theta,\beta) f(\sigma,\beta)d\beta\right) d\mathrm{vol}_{\Sigma}(\sigma,\theta).$$

Using the implicit function u as in (3.2), locally, $(\sigma,\theta) \in \Sigma - C$ means that $u(\sigma) = \theta$ or equivalently $\sigma \in u^{-1}(\theta)$. In other words, $(\sigma,\theta) \in \Sigma$ implies that for every $\beta \in [0,\pi]$

$$f(\sigma,\beta) = f(\sigma,\theta).$$

The latter fact comes from the fact u is a function, in that, it takes an unique value at each element of S^n.

In particular, provided $(\sigma,\theta) \in \Sigma - C$ we infer that

$$\int_{\beta=0}^{\pi} K_t^{S^1}(\theta,\beta) f(\sigma,\beta)d\beta = \left(\int_{\beta=0}^{\pi} K_t^{S^1}(\theta,\beta)d\beta\right) f(\sigma,\theta).$$

Therefore, we obtain the following form of the heat operator on Σ

$$P_t^{\Sigma} f(x) = \int_{(\sigma,\theta)\in\Sigma-C} K_t^{S^{n+1}}(x;(\sigma,\theta))\left(\int_{\beta=0}^{\pi} K_t^{S^1}(\theta,\beta)d\beta\right) f(\sigma,\theta) d\mathrm{vol}_{\Sigma}(\sigma,\theta).$$

Hence, using duality, we obtain the heat kernel for the hypersurface Σ

$$K_t^{\Sigma}(x;(\sigma,\theta)) = K_t^{S^{n+1}}(x;(\sigma,\theta))\left(\int_{\beta=0}^{\pi} K_t^{S^1}(\theta,\beta)d\beta\right). \tag{3.6}$$

The lemma is proved.

We are ready to prove the theorem, using Lemma 3.2 we get

$$\lim_{t\to 0^+}\frac{\log K_t^{\Sigma}(x;(\sigma,\theta))}{t}=\lim_{t\to 0^+}\frac{\log K_t^{S^{n+1}}(x;(\sigma,\theta))}{t}+\lim_{t\to 0^+}\frac{1}{t}\log\left(\int_{\beta=0}^{\pi}K_t^{S^1}(\theta,\beta)d\beta\right),$$

for every $x=(\tau,\alpha)$ and (σ,θ) in Σ. In view of (2.4), we obtain the equality

$$\lambda_1(\Sigma)=\lambda_1(S^{n+1})+\lim_{t\to 0^+}\frac{1}{t}\log\left(\int_{\beta=0}^{\pi}K_t^{S^1}(\theta,\beta)d\beta\right). \tag{3.7}$$

The spectral expansion of ∂_θ^2 with respect to the orthonormal eigenfunctions in $L^2(0,\pi)$ given by $(\Phi_k(x))_{n\geq 1}=(\frac{1}{\sqrt{2\pi}}\sin(kx))_{k\geq 1}$ is given by the uniformly convergent series

$$K_t^{S^1}(\theta,\beta)=\frac{2}{\pi}\sum_{k\geq 1}e^{-k^2t}\sin(k\theta)\sin(k\beta).$$

Therefore, using Fubini-Tonelli we can write

$$\int_{\beta=0}^{\pi}K_t^{S^1}(\theta,\beta)\,d\beta=\frac{2}{\pi}\sum_{k\geq 1}e^{-k^2t}\sin(k\theta)\int_{\beta=0}^{\pi}\sin(k\beta)d\beta.$$

Since $\int_{\beta=0}^{\pi}\sin(k\beta)d\beta=\frac{1}{k}(1-(-1)^k)$, we get

$$\int_{\beta=0}^{\pi}K_t^{S^1}(\theta,\beta)\,d\beta=\frac{2}{\pi}\sum_{m\geq 0}e^{-(2m+1)^2t}\frac{\sin((2m+1)\theta)}{2m+1}.$$

The long time asymptotic behaviour of the previous series is controlled by its first term, namely $m=0$. Thus, we obtain the estimate as $t\to\infty$,

$$\int_{\beta=0}^{\pi}K_t^{S^1}(\theta,\beta)\,d\beta\sim\frac{2}{\pi}e^{-t}\sin\theta.$$

Note that since $\theta\in(0,\pi)$, $\sin(\theta)>0$ we are allowed to take the logarithm in order to get

$$\lim_{t\to 0^+}\frac{1}{t}\log\left(\int_{\beta=0}^{\pi}K_t^{S^1}(\theta,\beta)d\beta\right)=-1.$$

Thus, (3.7) gives the equality

$$\lambda_1(\Sigma)=\lambda_1(S^{n+1})-1. \tag{3.8}$$

The fact that $\lambda_1(S^{n+1})=n+1$ gives the required and finishes the proof of Theorem 1.1.

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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