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Research article

Existence of radial solutions for k-Hessian system

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Abstract: In this paper, we consider the existence of radial solutions to a k-Hessian system in a general form. The existence of radial solutions is obtained under the assumptions that the nonlinearities in the given system satisfy k-superlinear, k-sublinear or k-asymptotically linear at the origin and infinity, respectively. The results presented in this paper generalize some known results. Examples are given for the illustration of the main results.

Keywords: k-Hessian system; Guo-Krasnosel'skii fixed point theorem; existence; cone
Mathematics Subject Classification: 35A01, 35A09, 35A24, 35A35

1. Introduction

In this paper, we consider the existence of radial solutions for the following k-Hessian system

$$\begin{cases} S_k(\sigma(D^2 z_1)) = \lambda f(|x|, -z_1, -z_2), \text{ in } \Omega, \\ S_k(\sigma(D^2 z_2)) = \mu g(|x|, -z_1, -z_2), \text{ in } \Omega, \\ z_1 = z_2 = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ, μ are positive parameters, $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$, $k \le N < 2k$, the nonlinear terms f and g are nonnegative continuous functions, $S_k(\sigma(D^2 z))$ is the k-Hessian operator of z, $k = 1, 2, \cdots, N$. From a discrete perspective, the k-Hessian operator has the form

$$S_k(\sigma(D^2 z)) = \frac{1}{k} \sum_{i,j=1}^{N} (S_k^{ij} z_i)_j,$$

where $S_k^{ij} = \frac{\partial S_k(\sigma(D^2 z))}{\partial z_{ij}}$, see details in [1, 2].

In general, the k-Hessian operator is defined as

$$S_k(\sigma(D^2 z)) = P_k(\Lambda) = \sum_{1 \le j_1 < \cdots < j_k \le N} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k}, \quad k = 1, 2, \cdots, N,$$

where $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ are the eigenvalues of Hessian matrix D^2z. In particular, when $k = 1$, k-Hessian operator reduces to Laplace operator $S_1(\sigma(D^2z)) = \sum_{i=1}^{N} \lambda_i = \triangle z$; when $k = N$, k-Hessian operator is Monge-Ampère operator $S_N(\sigma(D^2z)) = \prod_{i=1}^{N} \lambda_i = \det(D^2z)$. About Laplace problem and Monge-Ampère problem, there are a lot of brilliant papers, see [3–7].

k-Hessian equations for $k \neq 1$ are fully nonlinear partial differential equations [8, 9], which have important applications in fluid mechanics, geometric analysis and other disciplines [10–12]. For single k-Hessian equation, in recent years, many interesting results have been obtained by different methods, such as the monotone iterative method [13–15], upper and lower solution method [16, 17] and different kinds of fixed point theorems [18–20]. However, there are few studies about k-Hessian system. In 2015, Zhang and Zhou [13] investigated the k-Hessian system

$$\begin{cases} S_k(\sigma(D^2z_1)) = p(|x|)f(z_2), \text{ in } \Omega, \\ S_k(\sigma(D^2z_2)) = q(|x|)g(z_1), \text{ in } \Omega, \end{cases}$$

where $p, q : [0, \infty) \to (0, \infty)$ are continuous, $f, g : [0, \infty) \to [0, \infty)$ are continuous and increasing. The existence of entire positive radial solutions was obtained by using monotone iterative method. Recently, Yang and Bai [21] considered a class of more general k-Hessian system with parameters

$$\begin{cases} S_k(\sigma(D^2z_1)) = \lambda p(|x|)f(-z_1, -z_2), \text{ in } \Omega, \\ S_k(\sigma(D^2z_2)) = \mu q(|x|)g(-z_1, -z_2), \text{ in } \Omega, \\ z_1 = z_2 = 0, \text{ on } \partial\Omega. \end{cases}$$

They were interested in the existence of at least one or two k-convex radial solutions by analysing the growth of f and g at the origin and infinity.

Some researchers focused on the blow-up radial solution of the k-Hessian equations, such as, Covei [14] established the necessary and sufficient conditions for the coupled k-Hessian system

$$\begin{cases} S_k^{\frac{1}{k}}(\sigma(D^2z_1)) = p(|x|)f(z_1, z_2), \text{ in } \Omega, \\ S_k^{\frac{1}{k}}(\sigma(D^2z_2)) = q(|x|)g(z_1, z_2), \text{ in } \Omega. \end{cases}$$

For more results about blow-up solutions, we refer the reader to [22–25].

Based on the above discussions, in this paper, we consider the existence of the radial solutions of the k-Hessian system (1.1). Three cases are considered for the growth of the nonlinear terms f and g, that is, k-superlinear, k-sublinear and k-asymptotically linear at the origin and infinity, respectively.

The results obtained in this paper generalize and develop some known results from two aspects. The one is that the nonlinearities f and g have such more general form as $f(|x|, -z_1, -z_2)$ and $g(|x|, -z_1, -z_2)$ than the nonlinearity in [21], in which the nonlinear term is separable with the form $H(|x|)f(-z_1, -z_2)$. The other one is, to the best of our knowledge, there are few results about the existence of the entire radial solutions for k-Hessian equations, especially for k-Hessian system. In [26], the authors only discussed the existence of solutions for the single k-Hessian equation. From this perspective, parts of our main results generalize the results in [26] .

The rest of the paper is organized as follows. In Section 2, we provide some preliminary results, which are useful in the following proof. In Section 3, under the conditions of k-superlinear or k-sublinear for nonlinear terms, the existence of solutions of (1.1) is obtained for any positive parameters

λ and μ. For the case of k-asymptotically linear growth, the range of parameters is determined to ensure the existence of radial solutions of system (1.1). Four examples are given to verify some of our results in Section 4.

2. Preliminaries

In this section we will provide basic lemmas which are necessary for the understanding of subsequent results.

Lemma 2.1. *[27] Assume $y(r) \in C^2[0,1)$, with $y'(0) = 0$. Then for $z(|x|) = y(r)$, $r = |x| < 1$, we have that $z(|x|) \in C^2(\Omega)$, and*

$$\sigma(D^2 z) = \begin{cases} \left(y''(r), \dfrac{y'(r)}{r}, \cdots, \dfrac{y'(r)}{r}\right), & r \in (0,1), \\ (y''(0), y''(0), \cdots, y''(0)), & r = 0, \end{cases}$$

and then

$$S_k(\sigma(D^2 z)) = \begin{cases} C_{N-1}^{k-1} y''(r) \left(\dfrac{y'(r)}{r}\right)^{k-1} + C_{N-1}^{k} \left(\dfrac{y'(r)}{r}\right)^{k}, & r \in (0,1), \\ C_N^k (y''(0))^k, & r = 0, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$ *and* $C_N^k = \frac{N!}{k!(N-k)!}$.

Set space $X = C[0,1] \times C[0,1]$ with the norm $\|(y_1, y_2)\| = \|y_1\|_\infty + \|y_2\|_\infty$, where $\|\cdot\|_\infty$ denotes the max norm in $C[0,1]$. It is well known that the space $(X, \|\cdot\|)$ is a Banach space. For any $\theta \in (0, \frac{1}{2})$, the cone K is defined by

$$K = \left\{(y_1, y_2) \in X \mid \ y_1, y_2 \geq 0, \ \min_{r \in [\theta, 1-\theta]} (y_1(r) + y_2(r)) \geq \theta(\|y_1\|_\infty + \|y_2\|_\infty)\right\}.$$

Throughout the paper, we assume that
(H) $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous.

Define the operator $T : X \to X$ as follows

$$T(y_1, y_2)(r) = (T_1(y_1(r), y_2(r)), T_2(y_1(r), y_2(r))),$$

for any $(y_1, y_2) \in X$, and $T_i : X \to C[0,1]$, $(i = 1, 2)$ be the operators

$$T_1(y_1(r), y_2(r)) = \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds\right)^{\frac{1}{k}} dt,$$

$$T_2(y_1(r), y_2(r)) = \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \mu g(s, y_1(s), y_2(s)) ds\right)^{\frac{1}{k}} dt.$$

Now we can claim that the radial solutions of system (1.1) is equivalent to the fixed point of the operator T in Banach space X. In fact, set $(z_1(|x|), z_2(|x|)) = (-y_1(r), -y_2(r))$ with $y_1'(0) = y_2'(0) = 0$. According to Lemma 2.1, system (1.1) can be transformed into the ordinary differential boundary value problems

$$
\begin{cases}
-C_{N-1}^{k-1}y_1''(r)\left(\dfrac{-y_1'(r)}{r}\right)^{k-1}+C_{N-1}^{k}\left(\dfrac{-y_1'(r)}{r}\right)^{k}=\lambda f(r,y_1(r),y_2(r)),\ 0<r<1,\\
-C_{N-1}^{k-1}y_2''(r)\left(\dfrac{-y_2'(r)}{r}\right)^{k-1}+C_{N-1}^{k}\left(\dfrac{-y_2'(r)}{r}\right)^{k}=\mu g(r,y_1(r),y_2(r)),\ 0<r<1,\\
y_1'(0)=0,\ y_2'(0)=0,\ y_1(1)=0,\ y_2(1)=0.
\end{cases}
\tag{2.1}
$$

Noticed that $C_{N-1}^{k}=\frac{N-k}{k}C_{N-1}^{k-1}$, the left side of the first equation of (2.1) can be reduced to

$$
-C_{N-1}^{k-1}y_1''(r)\left(\frac{-y_1'(r)}{r}\right)^{k-1}+C_{N-1}^{k}\left(\frac{-y_1'(r)}{r}\right)^{k}=\frac{C_{N-1}^{k-1}}{r^{N-1}}\left\{\frac{r^{N-k}}{k}(-y_1'(r))^{k}\right\}',
$$

then the first equation of (2.1) can be rewritten as

$$
\left\{\frac{r^{N-k}}{k}(-y_1'(r))^{k}\right\}'=\frac{r^{N-1}}{C_{N-1}^{k-1}}\lambda f(r,y_1(r),y_2(r)).
$$

Furthermore, integrating the above equation twice, we have

$$
y_1(r)=\int_r^1\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\lambda f(s,y_1(s),y_2(s))ds\right)^{\frac{1}{k}}dt,\quad 0<r<1.
\tag{2.2}
$$

Similarly, from the second equation of (2.1) and boundary value conditions, we can get

$$
y_2(r)=\int_r^1\left(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\mu g(s,y_1(s),y_2(s))ds\right)^{\frac{1}{k}}dt,\quad 0<r<1.
\tag{2.3}
$$

Combining (2.2), (2.3) and the definition of operators T_1, T_2 and T, we can conclude that the radial solutions of system (1.1) is equivalent to the fixed point of the operator T in Banach space X. In addition, standard arguments show that the operator $T:K\to K$ is completely continuous.

Lemma 2.2. *[28] Let X be a Banach space and $K\subset X$ be a cone in X. Assume Ω_1,Ω_2 are open subsets of X with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$ and*

$$
T:K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K
$$

is a completely continuous operator such that either

(1) $\|Ty\|\le\|y\|$, $y\in K\cap\partial\Omega_1$ and $\|Ty\|\ge\|y\|$, $y\in K\cap\partial\Omega_2$

or

(2) $\|Ty\|\ge\|y\|$, $y\in K\cap\partial\Omega_1$ and $\|Ty\|\le\|y\|$, $y\in K\cap\partial\Omega_2$.

Then T has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. Existence results

For the convenience, we firstly introduce some notation

$$
\begin{aligned}
f_0 &= \liminf_{y_1+y_2\to 0} \min_{r\in[\theta,1-\theta]} \frac{f(r,y_1,y_2)}{(y_1+y_2)^k}, \quad & f_\infty &= \liminf_{y_1+y_2\to \infty} \min_{r\in[\theta,1-\theta]} \frac{f(r,y_1,y_2)}{(y_1+y_2)^k},\\
g_0 &= \liminf_{y_1+y_2\to 0} \min_{r\in[\theta,1-\theta]} \frac{g(r,y_1,y_2)}{(y_1+y_2)^k}, \quad & g_\infty &= \liminf_{y_1+y_2\to \infty} \min_{r\in[\theta,1-\theta]} \frac{g(r,y_1,y_2)}{(y_1+y_2)^k},\\
f^0 &= \limsup_{y_1+y_2\to 0} \max_{r\in[0,1]} \frac{f(r,y_1,y_2)}{(y_1+y_2)^k}, \quad & f^\infty &= \limsup_{y_1+y_2\to \infty} \max_{r\in[0,1]} \frac{f(r,y_1,y_2)}{(y_1+y_2)^k},\\
g^0 &= \limsup_{y_1+y_2\to 0} \max_{r\in[0,1]} \frac{g(r,y_1,y_2)}{(y_1+y_2)^k}, \quad & g^\infty &= \limsup_{y_1+y_2\to \infty} \max_{r\in[0,1]} \frac{g(r,y_1,y_2)}{(y_1+y_2)^k}.
\end{aligned}
$$

Here we are in the position of our main results.

Theorem 3.1. *Assume (H) holds, $f^0 = g^0 = 0$, $f_\infty = \infty$ or $g_\infty = \infty$. Then, for all $\lambda, \mu \in (0,\infty)$, the k-Hessian system (1.1) has at least one radial solution.*

Proof. Let $L_1 = \frac{k}{2k-N}\left(\frac{N}{C_N^k}\right)^{\frac{1}{k}}$. Since $f^0 = g^0 = 0$, we can choose $J_1 > 0$ such that

$$
f(r,y_1,y_2) \le \varepsilon(y_1+y_2)^k, \quad g(r,y_1,y_2) \le \varepsilon(y_1+y_2)^k,
$$

for $r \in [0,1]$, $0 < y_1 + y_2 \le J_1$, where $\varepsilon > 0$ satisfies

$$
L_1\left(\frac{\lambda\varepsilon}{N}\right)^{\frac{1}{k}} \le \frac{1}{2}, \quad L_1\left(\frac{\mu\varepsilon}{N}\right)^{\frac{1}{k}} \le \frac{1}{2}.
$$

Set

$$
\Omega_1 = \{\, (y_1,y_2) \in X \mid \|(y_1,y_2)\| < J_1 \,\}.
$$

Then, for any $(y_1,y_2) \in K \cap \partial\Omega_1$, we have

$$
\begin{aligned}
T_1(y_1,y_2)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s,y_1(s),y_2(s))ds\right)^{\frac{1}{k}} dt\\
&\le \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^1 \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s,y_1(s),y_2(s))ds\right)^{\frac{1}{k}} dt\\
&= \frac{k}{2k-N}\left(\frac{N}{C_N^k}\right)^{\frac{1}{k}} \left(\int_0^1 s^{N-1}\lambda f(s,y_1(s),y_2(s))ds\right)^{\frac{1}{k}}\\
&\le L_1\left(\lambda\varepsilon\int_0^1 s^{N-1}ds\right)^{\frac{1}{k}} \cdot (\|y_1\|_\infty + \|y_2\|_\infty)\\
&\le \frac{\|(y_1,y_2)\|}{2}, \; r \in [0,1].
\end{aligned}
$$

Therefore, we have

$$
\|T_1(y_1,y_2)\|_\infty \le \frac{\|(y_1,y_2)\|}{2}. \tag{3.1}
$$

Similarly, we can get

$$\|T_2(y_1, y_2)\|_\infty \leq \frac{\|(y_1, y_2)\|}{2}. \tag{3.2}$$

Combining inequalities (3.1) and (3.2), for $(y_1, y_2) \in K \cap \partial\Omega_1$, we have

$$\|T(y_1, y_2)\| = \|T_1(y_1, y_2)\|_\infty + \|T_2(y_1, y_2)\|_\infty \leq \|(y_1, y_2)\|.$$

For any $\theta \in (0, \frac{1}{2})$, denote

$$L_2 = \frac{k}{2k - N}\left(\frac{N}{C_N^k}\right)^{\frac{1}{k}}\left(1 - (1 - \theta)^{\frac{2k-N}{k}}\right).$$

Next, considering $f_\infty = \infty$, an analogous estimate holds if $g_\infty = \infty$. There is $\hat{J} > 0$ such that

$$f(r, y_1, y_2) \geq \delta(y_1 + y_2)^k, \tag{3.3}$$

for any $r \in [\theta, 1 - \theta],\ y_1 + y_2 \geq \hat{J}$, where $\delta > 0$ satisfies

$$\theta L_2\left(\lambda\delta \int_\theta^{1-\theta} s^{N-1} ds\right)^{\frac{1}{k}} > 1.$$

Let $J_2 = \max\{2J_1, \frac{1}{\theta}\hat{J}\}$ and

$$\Omega_2 = \{(y_1, y_2) \in X \mid \|(y_1, y_2)\| < J_2\}.$$

If $(y_1, y_2) \in K \cap \partial\Omega_2$, then

$$\min_{r\in[\theta,1-\theta]}(y_1(r) + y_2(r)) \geq \theta(\|y_1\|_\infty + \|y_2\|_\infty).$$

Furthermore, from the above inequality and (3.3), we have

$$\begin{aligned}
T_1(y_1, y_2)(1 - \theta) &= \int_{1-\theta}^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds\right)^{\frac{1}{k}} dt \\
&\geq \int_{1-\theta}^1 \left(\frac{k}{t^{N-k}} \int_\theta^{1-\theta} \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds\right)^{\frac{1}{k}} dt \\
&= \frac{k}{2k - N}\left(\frac{N}{C_N^k}\right)^{\frac{1}{k}}\left(1 - (1 - \theta)^{\frac{2k-N}{k}}\right)\left(\int_\theta^{1-\theta} s^{N-1}\lambda f(s, y_1(s), y_2(s)) ds\right)^{\frac{1}{k}} \\
&\geq \theta L_2\left(\lambda\delta \int_\theta^{1-\theta} s^{N-1} ds\right)^{\frac{1}{k}} \cdot (\|y_1\|_\infty + \|y_2\|_\infty) \\
&\geq \|(y_1, y_2)\|.
\end{aligned}$$

Therefore, if $(y_1, y_2) \in K \cap \partial\Omega_2$, then

$$\|T(y_1, y_2)\| = \|T_1(y_1, y_2)\|_\infty + \|T_2(y_1, y_2)\|_\infty \geq \|(y_1, y_2)\|.$$

By part (1) of Lemma 2.2, system (1.1) has at least one radial solution. □

Theorem 3.2. *Assume (H) holds, $f^\infty = g^\infty = 0$, $f_0 = \infty$ or $g_0 = \infty$. Then, for all $\lambda, \mu \in (0, \infty)$, the k-Hessian system (1.1) has at least one radial solution.*

Proof. Considering $f_0 = \infty$, an analogous estimate holds if $g_0 = \infty$. We can choose $J_3 > 0$ such that

$$f(r, y_1, y_2) \geq \eta(y_1 + y_2)^k, \quad r \in [\theta, 1-\theta],\ 0 < y_1 + y_2 \leq J_3, \tag{3.4}$$

where $\eta > 0$ satisfies

$$\theta L_2 \left(\lambda\eta \int_\theta^{1-\theta} s^{N-1} ds \right)^{\frac{1}{k}} > 1,$$

here the positive constant L_2 is defined the same as in proof of Theorem 3.1.

Set

$$\Omega_3 = \{(y_1, y_2) \in X \mid \|(y_1, y_2)\| < J_3\}.$$

For any $(y_1, y_2) \in K \cap \partial\Omega_3$, according to (3.4), we can get

$$\begin{aligned} T_1(y_1, y_2)(1-\theta) &= \int_{1-\theta}^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds \right)^{\frac{1}{k}} dt \\ &\geq \int_{1-\theta}^1 \left(\frac{k}{t^{N-k}} \int_\theta^{1-\theta} \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds \right)^{\frac{1}{k}} dt \\ &= \frac{k}{2k-N} \left(\frac{N}{C_N^k} \right)^{\frac{1}{k}} \left(1 - (1-\theta)^{\frac{2k-N}{k}} \right) \left(\int_\theta^{1-\theta} s^{N-1} \lambda f(s, y_1(s), y_2(s)) ds \right)^{\frac{1}{k}} \\ &\geq \theta L_2 \left(\lambda\eta \int_\theta^{1-\theta} s^{N-1} ds \right)^{\frac{1}{k}} \cdot (\|y_1\|_\infty + \|y_2\|_\infty) \\ &\geq \|(y_1, y_2)\|, \end{aligned}$$

furtheremore, we have

$$\|T(y_1, y_2)\| = \|T_1(y_1, y_2)\|_\infty + \|T_2(y_1, y_2)\|_\infty \geq \|(y_1, y_2)\|.$$

In order to construct Ω_4, we define two new functions

$$F(r, t) = \max_{0 \leq y_1 + y_2 \leq t} f(r, y_1, y_2), \quad G(r, t) = \max_{0 \leq y_1 + y_2 \leq t} g(r, y_1, y_2),$$

it is easy to see $F(r, t)$ and $G(r, t)$ are nondecreasing about variable t. From the definitions of f^∞ and g^∞ and the expressions of F and G, it can be seen that the following two limits are valid according to $f^\infty = g^\infty = 0$,

$$\lim_{t \to \infty} \max_{r \in [0,1]} \frac{F(r, t)}{t^k} = 0, \quad \lim_{t \to \infty} \max_{r \in [0,1]} \frac{G(r, t)}{t^k} = 0,$$

therefore, there is a constant $J_4 > 2J_3$, such that

$$F(r, t) \leq \rho t^k, \quad G(r, t) \leq \rho t^k, \quad t \geq J_4, \quad 0 \leq r \leq 1,$$

for $\rho > 0$ satisfying

$$L_1 \left(\frac{\lambda\rho}{N}\right)^{\frac{1}{k}} \le \frac{1}{2}, \quad L_1 \left(\frac{\mu\rho}{N}\right)^{\frac{1}{k}} \le \frac{1}{2},$$

here positive constant L_1 is defined the same as in proof of Theorem 3.1.

Set

$$\Omega_4 = \{(y_1, y_2) \in X \mid \|(y_1, y_2)\| < J_4\}.$$

For any $(y_1, y_2) \subset K \cap \partial\Omega_4$, we have

$$\begin{aligned} T_1(y_1, y_2)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds\right)^{\frac{1}{k}} dt \\ &\le \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^1 \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds\right)^{\frac{1}{k}} dt \\ &\le \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^1 \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda F(s, J_4) ds\right)^{\frac{1}{k}} dt \\ &= \frac{k}{2k-N} \left(\frac{N}{C_N^k}\right)^{\frac{1}{k}} \left(\int_0^1 s^{N-1} \lambda F(s, J_4) ds\right)^{\frac{1}{k}} \\ &\le L_1 J_4 \left(\lambda\rho \int_0^1 s^{N-1} ds\right)^{\frac{1}{k}} \\ &\le \frac{\|(y_1, y_2)\|}{2}, \quad r \in [0, 1]. \end{aligned}$$

Therefore, we have

$$\|T_1(y_1, y_2)\|_\infty \le \frac{\|(y_1, y_2)\|}{2}. \tag{3.5}$$

Similarly, we can get

$$\|T_2(y_1, y_2)\|_\infty \le \frac{\|(y_1, y_2)\|}{2}. \tag{3.6}$$

Combining inequalities (3.5) and (3.6), for $(y_1, y_2) \in K \cap \partial\Omega_4$, one has

$$\|T(y_1, y_2)\| = \|T_1(y_1, y_2)\|_\infty + \|T_2(y_1, y_2)\|_\infty \le \|(y_1, y_2)\|.$$

By part (2) of Lemma 2.2, system (1.1) has at least one radial solution. □

Let

$$L_3 = \frac{k}{2k-N} \left(\frac{1}{C_N^k}\right)^{\frac{1}{k}}, \quad L_4 = \frac{k}{2k-N} \left(\int_\theta^{1-\theta} s^{N-1} ds\right)^{\frac{1}{k}} \left(\frac{N}{C_N^k}\right)^{\frac{1}{k}} \left(1 - (1-\theta)^{\frac{2k-N}{k}}\right).$$

Theorem 3.3. *Assume (H) holds, f^0, g^0, f_∞, $g_\infty \in (0, \infty)$, $f_\infty > f^0$ and $g_\infty > g^0$. Then, for each $\lambda \in \left(\frac{1}{L_4^k f_\infty \theta^k}, \frac{1}{L_3^k f^0 2^k}\right)$ and $\mu \in \left(\frac{1}{L_4^k g_\infty \theta^k}, \frac{1}{L_3^k g^0 2^k}\right)$, the k-Hessian system (1.1) has at least one radial solution.*

Proof. Let λ and μ be given in (3.7). Now, let $\varepsilon > 0$ be chosen, such that

$$\frac{1}{L_4^k(f_\infty - \varepsilon)\theta^k} \leq \lambda \leq \frac{1}{L_3^k(f^0 + \varepsilon)2^k} \tag{3.7}$$

and

$$\frac{1}{L_4^k(g_\infty - \varepsilon)\theta^k} \leq \mu \leq \frac{1}{L_3^k(g^0 + \varepsilon)2^k}. \tag{3.8}$$

Since $f^0, g^0 \in (0, \infty)$, for $\varepsilon > 0$, there exists $J_5 > 0$, such that

$$f(r, y_1, y_2) \leq (f^0 + \varepsilon)(y_1 + y_2)^k, \quad g(r, y_1, y_2) \leq (g^0 + \varepsilon)(y_1 + y_2)^k, \tag{3.9}$$

for $r \in [0, 1]$, $0 < y_1 + y_2 \leq J_5$. Therefore, choosing $(y_1, y_2) \in K$ with $\|(y_1, y_2)\| = J_5$, one gets

$$\begin{aligned}
T_1(y_1, y_2)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds \right)^{\frac{1}{k}} dt \\
&\leq \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^1 \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds \right)^{\frac{1}{k}} dt \\
&\leq \frac{k}{2k - N} \left(\frac{N}{C_N^k} \right)^{\frac{1}{k}} \left(\int_0^1 s^{N-1} \lambda (f^0 + \varepsilon)(y_1(s) + y_2(s))^k ds \right)^{\frac{1}{k}} \\
&\leq \lambda^{\frac{1}{k}} \frac{k}{2k - N} \left(\frac{N}{C_N^k} \right)^{\frac{1}{k}} (\int_0^1 s^{N-1} ds)^{\frac{1}{k}} (f^0 + \varepsilon)^{\frac{1}{k}} \cdot \|(y_1, y_2)\| \\
&= \lambda^{\frac{1}{k}} \frac{k}{2k - N} \left(\frac{1}{C_N^k} \right)^{\frac{1}{k}} (f^0 + \varepsilon)^{\frac{1}{k}} \cdot \|(y_1, y_2)\| \\
&= \lambda^{\frac{1}{k}} L_3 (f^0 + \varepsilon)^{\frac{1}{k}} \cdot \|(y_1, y_2)\| \\
&\leq \frac{\|(y_1, y_2)\|}{2}, \ r \in [0, 1].
\end{aligned}$$

Therefore, we have

$$\|T_1(y_1, y_2)\|_\infty \leq \frac{\|(y_1, y_2)\|}{2}. \tag{3.10}$$

Similarly, we can get

$$\|T_2(y_1, y_2)\|_\infty \leq \frac{\|(y_1, y_2)\|}{2}. \tag{3.11}$$

Combining (3.11) and (3.12), we can get

$$\|T(y_1, y_2)\| = \|T_1(y_1, y_2)\|_\infty + \|T_2(y_1, y_2)\|_\infty \leq \|(y_1, y_2)\|.$$

Therefore, if we set

$$\Omega_5 = \{(y_1, y_2) \in X \mid \|(y_1, y_2)\| < J_5\},$$

then, for $(y_1, y_2) \in K \cap \partial\Omega_5$,

$$\|T(y_1, y_2)\| \leq \|(y_1, y_2)\|.$$

In view of $f_\infty \in (0,\infty)$, for $\varepsilon > 0$, there is $\bar{J} > 0$, such that

$$f(r,y_1,y_2) \geq (f_\infty - \varepsilon)(y_1+y_2)^k, \quad r \in [\theta, 1-\theta], \ y_1+y_2 \geq \bar{J}.$$

Let $J_6 = \max\{2J_5, \frac{1}{\theta}\bar{J}\}$ and set

$$\Omega_6 = \{(y_1,y_2) \in X \mid \|(y_1,y_2)\| < J_6\}.$$

If $(y_1,y_2) \in K$ with $\|(y_1,y_2)\| = J_6$, then

$$\min_{r\in[\theta,1-\theta]}(y_1(r)+y_2(r)) \geq \theta(\|y_1\|_\infty + \|y_2\|_\infty) \geq \bar{J}.$$

From the above inequality and (3.10), we have

$$\begin{aligned}
T_1(y_1,y_2)(1-\theta) &= \int_{1-\theta}^1 \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}\lambda f(s,y_1(s),y_2(s))ds\right)^{\frac{1}{k}} dt \\
&\geq \int_{1-\theta}^1 \left(\frac{k}{t^{N-k}}\int_\theta^{1-\theta} \frac{s^{N-1}}{C_{N-1}^{k-1}}\lambda f(s,y_1(s),y_2(s))ds\right)^{\frac{1}{k}} dt \\
&\geq \frac{k}{2k-N}\left(\frac{N}{C_N^k}\right)^{\frac{1}{k}}\left(1-(1-\theta)^{\frac{2k-N}{k}}\right)\left(\int_\theta^{1-\theta} s^{N-1}\lambda(f_\infty-\varepsilon)(y_1(s)+y_2(s))^k ds\right)^{\frac{1}{k}} \\
&\geq \lambda^{\frac{1}{k}}\frac{k}{2k-N}\left(\frac{N}{C_N^k}\right)^{\frac{1}{k}}\left(1-(1-\theta)^{\frac{2k-N}{k}}\right)\left(\int_\theta^{1-\theta} s^{N-1}ds\right)^{\frac{1}{k}}(f_\infty-\varepsilon)^{\frac{1}{k}}\theta\cdot\|(y_1,y_2)\| \\
&= \lambda^{\frac{1}{k}}L_4(f_\infty-\varepsilon)^{\frac{1}{k}}\theta\cdot\|(y_1,y_2)\| \\
&\geq \|(y_1,y_2)\|.
\end{aligned}$$

Furthermore, $\|T(y_1,y_2)\| = \|T_1(y_1,y_2)\|_\infty + \|T_2(y_1,y_2)\|_\infty \geq \|(y_1,y_2)\|$. System (1.1) has at least one radial solution for the given λ and μ due to Lemma 2.2.

Similarly, we can also prove $\|T(y_1,y_2)\| \geq \|(y_1,y_2)\|$ if $g_\infty \in (0,\infty)$. This completes the proof. □

Theorem 3.4. *Assume (H) holds, f_0, g_0, f^∞, $g^\infty \in (0,\infty)$, $f_0 > f^\infty$ and $g_0 > g^\infty$. Then, for each $\lambda \in \left(\frac{1}{L_4^k f_0\theta^k}, \frac{1}{L_3^k f^\infty 2^k}\right)$ and $\mu \in \left(\frac{1}{L_4^k g_0\theta^k}, \frac{1}{L_3^k g^\infty 2^k}\right)$, the k-Hessian system (1.1) has at least one radial solution.*

Proof. Let λ and μ be given in (3.13). Now, let $\varepsilon > 0$ be chosen, such that

$$\frac{1}{L_4^k(f_0-\varepsilon)\theta^k} \leq \lambda \leq \frac{1}{L_3^k(f^\infty+\varepsilon)2^k} \tag{3.12}$$

and

$$\frac{1}{L_4^k(g_0-\varepsilon)\theta^k} \leq \mu \leq \frac{1}{L_3^k(g^\infty+\varepsilon)2^k}. \tag{3.13}$$

For $f_0 \in (0,\infty)$, an analogous estimate holds for $g_0 \in (0,\infty)$, there exists positive constant J_7, for $r \in [\theta, 1-\theta]$, $0 < y_1+y_2 \leq J_7$, such that

$$f(r,y_1,y_2) \geq (f_0-\varepsilon)(y_1+y_2)^k.$$

Therefore, choosing $(y_1, y_2) \in K$ with $\|(y_1, y_2)\| = J_7$, one gets from (3.14)

$$\begin{aligned}
T_1(y_1, y_2)(1-\theta) &= \int_{1-\theta}^{1} \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds \right)^{\frac{1}{k}} dt \\
&\geq \frac{k}{2k-N} \left(\frac{N}{C_N^k} \right)^{\frac{1}{k}} \left(\int_{\theta}^{1-\theta} s^{N-1} \lambda (f_0 - \varepsilon)(y_1(s) + y_2(s))^k ds \right)^{\frac{1}{k}} \\
&\geq \theta \lambda^{\frac{1}{k}} \frac{k}{2k-N} \left(\frac{N}{C_N^k} \right)^{\frac{1}{k}} \left(1 - (1-\theta)^{\frac{2k-N}{k}} \right) \left(\int_{\theta}^{1-\theta} s^{N-1} ds \right)^{\frac{1}{k}} (f_0 - \varepsilon)^{\frac{1}{k}} \cdot \|(y_1, y_2)\| \\
&= \theta L_4 \lambda^{\frac{1}{k}} (f_0 - \varepsilon)^{\frac{1}{k}} \cdot \|(y_1, y_2)\| \\
&\geq \|(y_1, y_2)\|.
\end{aligned}$$

Thus, if we set

$$\Omega_7 = \{ (y_1, y_2) \in X \mid \|(y_1, y_2)\| < J_7 \},$$

then, for $(y_1, y_2) \in K \cap \partial\Omega_7$, we have

$$\|T(y_1, y_2)\| = \|T_1(y_1, y_2)\|_\infty + \|T_2(y_1, y_2)\|_\infty \geq \|(y_1, y_2)\|.$$

In order to construct Ω_8, for $t \geq 0$, we define two new functions

$$F(r, t) = \max_{0 \leq y_1 + y_2 \leq t} f(r, y_1, y_2), \quad G(r, t) = \max_{0 \leq y_1 + y_2 \leq t} g(r, y_1, y_2).$$

It is easy to see $F(r, t)$ and $G(r, t)$ are nondecreasing about variable t. From the definitions of f^∞ and g^∞ and the expressions of F and G, it can be seen that the following two limits are valid according to $f^\infty \in (0, \infty)$ and $g^\infty \in (0, \infty)$,

$$\lim_{t \to \infty} \max_{r \in [0,1]} \frac{F(r, t)}{t^k} = f^\infty, \quad \lim_{t \to \infty} \max_{r \in [0,1]} \frac{G(r, t)}{t^k} = g^\infty.$$

Therefore, there is a constant $J_8 > 2J_7$, such that

$$F(r, t) \leq (f^\infty + \varepsilon) t^k, \ \ G(r, t) \leq (g^\infty + \varepsilon) t^k, \qquad t \geq J_8, \ \ 0 \leq r \leq 1,$$

where $\varepsilon > 0$ satisfying

$$L_3 \lambda^{\frac{1}{k}} (f^\infty + \varepsilon)^{\frac{1}{k}} \leq \frac{1}{2}, \ \ L_4 \mu^{\frac{1}{k}} (g^\infty + \varepsilon)^{\frac{1}{k}} \leq \frac{1}{2}.$$

Set

$$\Omega_8 = \{ (y_1, y_2) \in X \mid \|(y_1, y_2)\| < J_8 \},$$

for any $(y_1, y_2) \subset K \cap \partial\Omega_8$, we have

$$\begin{aligned}
T_1(y_1, y_2)(r) &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds \right)^{\frac{1}{k}} dt \\
&\leq \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^1 \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda f(s, y_1(s), y_2(s)) ds \right)^{\frac{1}{k}} dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^1 \frac{s^{N-1}}{C_{N-1}^{k-1}} \lambda F(s, J_8) ds \right)^{\frac{1}{k}} dt \\
&= \frac{k}{2k-N} \left(\frac{N}{C_N^k} \right)^{\frac{1}{k}} \left(\int_0^1 s^{N-1} \lambda F(s, J_8) ds \right)^{\frac{1}{k}} \\
&\leq J_8 L_3 \lambda^{\frac{1}{k}} (f^\infty + \varepsilon)^{\frac{1}{k}} \\
&\leq \frac{\|(y_1, y_2)\|}{2}, \ r \in [0, 1].
\end{aligned}
$$

Therefore, we have

$$
\|T_1(y_1, y_2)\|_\infty \leq \frac{\|(y_1, y_2)\|}{2}. \tag{3.14}
$$

Similarly, we can get

$$
\|T_2(y_1, y_2)\|_\infty \leq \frac{\|(y_1, y_2)\|}{2}. \tag{3.15}
$$

Combining (3.16) and (3.17), for $(y_1, y_2) \in K \cap \partial\Omega_8$, we can get

$$
\|T(y_1, y_2)\| = \|T_1(y_1, y_2)\|_\infty + \|T_2(y_1, y_2)\|_\infty \leq \|(y_1, y_2)\|.
$$

System (1.1) has at least one radial solution for the given λ and μ due to Lemma 2.2. □

4. Examples

Example 4.1. *Consider the following 3-Hessian system*

$$
\begin{cases}
S_3(\sigma(D^2 z_1)) = \lambda(|x| - z_1)(-z_1 - z_2)^5, \ \text{in } \Omega, \\
S_3(\sigma(D^2 z_2)) = \mu(|x| - z_1)\sin^4(-z_1 - z_2), \ \text{in } \Omega, \\
z_1 = z_2 = 0, \ \ \text{on } \partial\Omega,
\end{cases} \tag{4.1}
$$

where $\Omega = \{x \in \mathbb{R}^4 : |x| < 1\}$. Similar to the transformation method in Section 2, the corresponding nonlinear terms in (4.1) have the following form

$$
f(r, y_1, y_2) = (r + y_1)(y_1 + y_2)^5, \ \ g(r, y_1, y_2) = (r + y_1)\sin^4(y_1 + y_2).
$$

Therefore, the nonlinear terms f and g are continuous functions. It is not difficult to check that $f^0 = g^0 = 0$ and $f_\infty = \infty$. Then, all the conditions of Theorem 3.1 are fulfilled, the conclusion follows, that is, for any positive parameters λ and μ, (4.1) has at least one radial solution.

Example 4.2. *Consider the following 4-Hessian system*

$$
\begin{cases}
S_4(\sigma(D^2 z_1)) = \lambda(|x| - z_1 - z_2), \ \text{in } \Omega, \\
S_4(\sigma(D^2 z_2)) = \mu \sin^6(|x| - z_1 - z_2), \ \text{in } \Omega, \\
z_1 = z_2 = 0, \ \ \text{on } \partial\Omega,
\end{cases} \tag{4.2}
$$

where $\Omega = \{x \in \mathbb{R}^5 : |x| < 1\}$. Similar to the transformation method in Section 2, the corresponding nonlinear terms of (4.2) have the following form

$$
f(r, y_1, y_2) = r + y_1 + y_2, \ \ g(r, y_1, y_2) = \sin^6(r + y_1 + y_2).
$$

So, the nonlinear terms f and g are continuous functions. It is not difficult to check that $f_0 = \infty$ and $f^\infty = g^\infty = 0$. Therefore, all the conditions of Theorem 3.2 are satisfied, the conclusion follows, that is, for any positive parameters λ and μ, (4.2) has at least one radial solution.

Example 4.3. *Consider the following 3-Hessian system*

$$\begin{cases} S_3(\sigma(D^2 z_1)) = \lambda f(|x|, -z_1, -z_2), \text{ in } \Omega, \\ S_3(\sigma(D^2 z_2)) = \mu g(|x|, -z_1, -z_2), \text{ in } \Omega, \\ z_1 = z_2 = 0, \text{ on } \partial\Omega, \end{cases} \tag{4.3}$$

where $\Omega = \{x \in \mathbb{R}^4 : |x| < 1\}$, the nonlinearities f and g are given by

$$\begin{aligned} & f(|x|, -z_1, -z_2) \\ = & \begin{cases} (-z_1 - z_2)^3 \arctan(|x| - z_1 - z_2), & 0 \le -z_1 - z_2 < 90, \\ 90^3 \arctan(|x| - z_1 - z_2)[(91^3 \times 4 - 1)(-z_1 - z_2 - 90) + 1], & 90 \le -z_1 - z_2 < 91, \\ 90^3 \cdot 4(-z_1 - z_2)^3 \arctan(|x| - z_1 - z_2), & -z_1 - z_2 \ge 91, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & g(|x|, -z_1, -z_2) \\ = & \begin{cases} (-z_1 - z_2)^3 \sin^2(|x| - z_1 - z_2), & 0 \le -z_1 - z_2 < 30, \\ 30^3 \sin^2(|x| - z_1 - z_2) + 62^3 \times 10^6(-z_1 - z_2 - 30) \arctan(|x| - z_1 - z_2), & 30 \le -z_1 - z_2 < 31, \\ 30^3 \sin^2(|x| - z_1 - z_2) + 8 \times 10^6 \arctan(|x| - z_1 - z_2)(-z_1 - z_2)^3, & -z_1 - z_2 \ge 31, \end{cases} \end{aligned}$$

respectively.

In fact, similar to the transformation method in Section 2, the corresponding nonlinear terms in (4.3) have the following form

$$f(r, y_1, y_2) = \begin{cases} (y_1 + y_2)^3 \arctan(r + y_1 + y_2), & 0 \le y_1 + y_2 < 90, \\ 90^3 \arctan(r + y_1 + y_2)[(91^3 \times 4 - 1)(y_1 + y_2 - 90) + 1], & 90 \le y_1 + y_2 < 91, \\ 90^3 \cdot 4(y_1 + y_2)^3 \arctan(r + y_1 + y_2), & y_1 + y_2 \ge 91, \end{cases}$$

and

$$g(r, y_1, y_2) = \begin{cases} (y_1 + y_2)^3 \sin^2(r + y_1 + y_2), & 0 \le y_1 + y_2 < 30, \\ 30^3 \sin^2(r + y_1 + y_2) + 62^3 \times 10^6(y_1 + y_2 - 30) \arctan(r + y_1 + y_2), & 30 \le y_1 + y_2 < 31, \\ 30^3 \sin^2(r + y_1 + y_2) + 8 \times 10^6 \arctan(r + y_1 + y_2)(y_1 + y_2)^3, & y_1 + y_2 \ge 31. \end{cases}$$

From the expressions of f and g, we can calculate

$$f^0 = \arctan 1, \quad f_\infty = 2 \cdot 90^3 \pi, \quad g^0 = \sin^2 1, \quad g_\infty = 4 \cdot 10^6 \pi,$$

so, $f_\infty > f^0$ and $g_\infty > g^0$. Then all the conditions of Theorem 3.3 hold.

Furthermore, for fixed $\theta = \frac{1}{4}$,

$$L_3 = \frac{3}{2} \cdot \left(\frac{1}{4}\right)^{\frac{1}{3}}, \quad L_4 = \frac{3}{2}\left(\frac{5}{128}\right)^{\frac{1}{3}}\left(1 - \left(\frac{3}{4}\right)^{\frac{2}{3}}\right).$$

Therefore, it follows from Theorem 3.3 that (4.3) has at least one radial solution for any $\lambda \in (\frac{1}{50}, \frac{37}{200})$ and $\mu \in (\frac{3}{400}, 10)$.

Example 4.4. *Consider the following 3-Hessian system*

$$\begin{cases} S_3(\sigma(D^2 z_1)) = \lambda f(|x|, -z_1, -z_2), \text{ in } \Omega, \\ S_3(\sigma(D^2 z_2)) = \mu g(|x|, -z_1, -z_2), \text{ in } \Omega, \\ z_1 = z_2 = 0, \text{ on } \partial\Omega, \end{cases} \tag{4.4}$$

where $\Omega = \{x \in \mathbb{R}^4 : |x| < 1\}$, *the nonlinearities* f *and* g *are given by*

$$\begin{aligned} & f(|x|, -z_1, -z_2) \\ =& \begin{cases} (-z_1 - z_2)^3(\cos(|x| - z_1 - z_2) + 200), & 0 \le -z_1 - z_2 < 30, \\ 30^3(\cos(|x| - z_1 - z_2) + 200) + (-z_1 - z_2 - 30)\arctan(|x| - z_1 - z_2), & 30 \le -z_1 - z_2 < 31, \\ 30^3(\cos(|x| - z_1 - z_2) + 200) + \arctan(|x| - z_1 - z_2)(-z_1 - z_2)^3, & -z_1 - z_2 \ge 31, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & g(|x|, -z_1, -z_2) \\ =& \begin{cases} (-z_1 - z_2)^3(\sin(|x| - z_1 - z_2) + 200), & 0 \le -z_1 - z_2 < 30, \\ 30^3(\sin(|x| - z_1 - z_2) + 200) + (-z_1 - z_2 - 30)\arctan(|x| - z_1 - z_2), & 30 \le -z_1 - z_2 < 31, \\ 30^3(\sin(|x| - z_1 - z_2) + 200) + \arctan(|x| - z_1 - z_2)(-z_1 - z_2)^3, & -z_1 - z_2 \ge 31, \end{cases} \end{aligned}$$

respectively.

As a matter of fact, similar to the transformation method in Section 2, the corresponding nonlinear terms in (4.4) have the following form

$$f(r, y_1, y_2) = \begin{cases} (y_1 + y_2)^3(\cos(r + y_1 + y_2) + 200), & 0 \le y_1 + y_2 < 30, \\ 30^3(\cos(r + y_1 + y_2) + 200) + 31(y_1 + y_2 - 30)\arctan(r + y_1 + y_2), & 30 \le y_1 + y_2 < 31, \\ 30^3(\cos(r + y_1 + y_2) + 200) + \arctan(r + y_1 + y_2)(y_1 + y_2)^3, & y_1 + y_2 \ge 31, \end{cases}$$

and

$$g(r, y_1, y_2) = \begin{cases} (y_1 + y_2)^3(\sin(r + y_1 + y_2) + 200), & 0 \le y_1 + y_2 < 30, \\ 30^3(\sin(r + y_1 + y_2) + 200) + 31(y_1 + y_2 - 30)\arctan(r + y_1 + y_2), & 30 \le y_1 + y_2 < 31, \\ 30^3(\sin(r + y_1 + y_2) + 200) + \arctan(r + y_1 + y_2)(y_1 + y_2)^3, & y_1 + y_2 \ge 31. \end{cases}$$

From the expressions of f and g, we can calculate

$$f_0 = g_0 = 200, \quad f^\infty = g^\infty = \frac{\pi}{2},$$

so, $f_0 > f^\infty$ and $g_0 > g^\infty$. Then all the conditions of Theorem 3.4 are fulfilled.

Furthermore, for fixed $\theta = \frac{1}{4}$,

$$L_3 = \frac{3}{2} \cdot \left(\frac{1}{4}\right)^{\frac{1}{3}}, \quad L_4 = \frac{3}{2}\left(\frac{5}{128}\right)^{\frac{1}{3}}\left(1 - \left(\frac{3}{4}\right)^{\frac{2}{3}}\right).$$

Therefore, it follows from Theorem 3.4 that (4.4) has at least one radial solution for any $\lambda \in (\frac{1}{100}, \frac{21}{250})$ and $\mu \in (\frac{1}{100}, \frac{21}{250})$.

5. Conclusions

In this paper, a class of k-Hessian system with parameter is concerned. we are interested in the range of parameters λ and μ on which there exists at least one radial solution. By using the Guo-Krasnosel'skii fixed point theorem, we find that there is a close relation between the range of parameters and the growth of nonlinearities f and g at the origin and infinity. The results obtained in this paper generalize and develop some of the known results, such as, parts of the results in [21, 26] from two sides. The one is that the more general form of nonlinearities f and g is investigated here. The other is that there are no existence results for k-Hessian system under the conditions that nonlinearities f and g satisfy k-superlinear, k-sublinear and k-asymptotically linear conditions at the origin and infinity. Finally, we also illustrate the obtained results with four examples.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. L. A. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations(III): Functions of the eigenvalues of the Hessian, *Acta. Math.*, **155** (1985), 261–301. http://doi.org/10.1007/bf02392544
2. X. J. Wang, The k-Hessian equation, In: *Geometric analysis and PDEs*, Heidelberg: Springer, **1977** (2009), 177–252. http://doi.org/10.1007/978-3-642-01674-5_5
3. Z. Zhang, K. Wang, Existence and non-existence of solutions for a class of Monge-Ampère equations, *J. Differential Equations*, **246** (2009), 2849–2875. https://doi.org/10.1016/j.jde.2009.01.004
4. J. Bao, H. Li, L. Zhang, Monge-Ampère equation on exterior domains, *Calc. Var. Partial Differential Equations*, **52** (2015), 39–63. https://doi.org/10.1007/s00526-013-0704-7
5. D. P. Covei, Solutions with radial symmetry for a semilinear elliptic system with weights, *Appl. Math. Lett.*, **76** (2018), 187–194. https://doi.org/10.1016/j.aml.2017.09.003
6. Z. J. Zhang, Optimal global and boundary asymptotic behavior of large solutions to the Monge-Ampère equation, *J. Funct. Anal.*, **278** (2020), 108512. https://doi.org/10.1016/j.jfa.2020.108512

7. H. Wang, Convex solutions of systems arising from Monge-Ampère equations, *Electron. J. Qual. Theory Differ. Equ., Spec. Ed. I*, **26** (2009), 1–8. https://doi.org/10.14232/ejqtde.2009.4.26

8. B. Guan, The Dirichlet problem for Hessian equations on Riemannian manifolds, *Calc. Var. Partial Differential Equations*, **8** (1999), 45–69. https://doi.org/10.1007/s005260050116

9. G. W. Dai, Bifurcation and admissible solutions for the Hessian equation, *J. Funct. Anal.*, **273** (2017), 3200–3240. https://doi.org/10.1016/j.jfa.2017.08.001

10. S. Chang, M. J. Gursky, P. C. Yang, An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature, *Ann. Math.*, **155** (2004), 709–787. https://doi.org/10.2307/3062131

11. Y. C. Shiu, T. Y. Shing, On the regularity of the solution of the n-dimensional Minkowski problem, *Commun. Pure Appl. Math.*, **29** (1976), 495–516. https://doi.org/10.1002/cpa.3160290504

12. X. J. Chen, Q. Tu, N. Xiang, A class of Hessian quotient equations in Euclidean space, *J. Differential Equations*, **269** (2020), 11172–11194. https://doi.org/10.1016/j.jde.2020.08.048

13. Z. J. Zhang, S. Zhou, Existence of entire positive k-convex radial solutions to Hessian equations and systems with weights, *Appl. Math. Lett.*, **50** (2015), 48–55. https://doi.org/10.1016/j.aml.2015.05.018

14. D. Covei, A necessary and a sufficient condition for the existence of the positive radial solutions to Hessian equations and systems with weights, *Acta Math. Sci.*, **37** (2017), 47–57. https://doi.org/10.1016/S0252-9602(16)30114-X

15. G. T. Wang, Z. D. Yang, L. H. Zhang, D. Baleanu, Radial solutions of a nonlinear k-Hessian system involving a nonlinear operator, *Commun. Nonlinear Sci.*, **91** (2020), 105396. https://doi.org/10.1016/j.cnsns.2020.105396

16. X. G. Zhang, H. Tain, Y. H. Wu, B. Wiwatanapataphee, The radial solution for an eigenvalue problem of singular augmented Hessian equation, *Appl. Math. Lett.*, **134** (2022), 108330. https://doi.org/10.1016/j.aml.2022.108330

17. X. G. Zhang, P. T. Xu, Y. H. Wu, The eigenvalue problem of a singular k -Hessian equation, *Appl. Math. Lett.*, **124** (2021), 107666. https://doi.org/10.1016/j.aml.2021.107666

18. M. Q. Feng, New results of coupled system of k-Hessian equations, *Appl. Math. Lett.*, **94** (2019), 196–203. https://doi.org/10.1016/j.aml.2019.03.008

19. X. M. Zhang, Analysis of nontrivial radial solutions for singular superlinear k-Hessian equations, *Appl. Math. Lett.*, **106** (2020), 106409. https://doi.org/10.1016/j.aml.2020.106409

20. J. X. He, X. G. Zhang, L. S. Liu, Y. H. Wu, Existence and nonexistence of radial solutions of Dirichlet problem for a class of general k-Hessian equations, *Nonlinear Anal. Model.*, **23** (2018), 475–492. https://doi.org/10.15388/NA.2018.4.2

21. Z. D. Yang, Z. B. Bai, Existence and multiplicity of radial solutions for a k-Hessian system, *J. Math. Anal. Appl.*, **512** (2022), 126159. https://doi.org/10.1016/j.jmaa.2022.126159

22. A. V. Lair, A. Mohammed, Large solutions to semi-linear elliptic systems with variable exponents, *J. Math. Anal. Appl.*, **420** (2014), 1478–1499. https://doi.org/10.1016/j.jmaa.2014.06.068

23. Z. Zhang, Large solutions to the Monge-Ampère equations with nonlinear gradient terms: Existence and boundary behavior, *J. Differential Equations*, **264** (2018), 263–296. https://doi.org/10.1016/j.jde.2017.09.010

24. X. G. Zhang, J. F. Xu, J. Q. Jiang, Y. H. Wu, Y. J. Cui, The convergence analysis and uniqueness of blow-up solutions for a Dirichlet problem of the general k-Hessian equations, *Appl. Math. Lett.*, **102** (2020), 106124. https://doi.org/10.1016/j.aml.2019.106124

25. X. G. Zhang, P. Chen, Y. H. Wu, B. Wiwatanapataphee, A necessary and sufficient condition for the existence of entire large solutions to a k-Hessian system, *Appl. Math. Lett.*, **145** (2023), 108745. https://doi.org/10.1016/j.aml.2023.108745

26. Z. B. Bai, Z. D. Yang, Existence of k-convex solutions for the k-Hessian equation, *Mediterr. J. Math.*, **20** (2023), 150. https://doi.org/10.1007/s00009-023-02364-8

27. X. H. Ji, J. G. Bao, Necessary and sufficient conditions on solvability for Hessian inequalities, *Proc. Amer. Math. Soc.*, **138** (2010), 175–188. http://doi.org/10.1090/S0002-9939-09-10032-1

28. D. J. Guo, V. Lakshmikantham, Positive fixed point theory, In: *Nonlinear problems in abstract cones*, New York: Academic Press, 1988. https://doi.org/10.1016/B978-0-12-293475-9.50005-4