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Research article

Boundedness of an intrinsic square function on grand p-adic Herz-Morrey spaces

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Abstract: This research paper focuses on establishing a framework for grand Herz-Morrey spaces defined over the p-adic numbers and their associated p-adic intrinsic square function. We will define the ideas of grand p-adic Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,u),\theta}_{s(\cdot)}(\mathbb{Q}^n_p)$ and p-adic intrinsic square function. Moreover, the corresponding operator norms are estimated. Grand p-adic Herz-Morrey spaces with variable exponent is the generalization of p-adic Herz spaces. Our main goal is to obtain the boundedeness of p-adic intrinsic square function in grand p-adic Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,u),\theta}_{s(\cdot)}(\mathbb{Q}^n_p)$. The boundedness is proven by exploiting the properties of variable exponents in these function spaces.

Keywords: p-adic intrinsic square function; p-adic Lebesgue spaces with variable exponents; grand p-adic Herz-Morrey spaces
Mathematics Subject Classification: 42B35, 26D15, 46B25, 47G10

1. Introduction

During the previous decade, researchers were enticed to the p-adic analysis because it is a useful tool to study different phenomenon in biology, physics and medicine, see for instance [1–3]. For this reason, the study of operators that allows us to describe such phenomena is essential. Even more so when in the p-adic setting it is not possible to define the derivative in the classical sense. The p-adic Herz spaces are a class of function spaces defined over p-adic numbers, which are an extension of the rational numbers that arise in number theory and algebraic geometry.

Variable exponent Lebesgue spaces generalize the notion of q-integrability in the classical Lebesgue spaces, allowing the exponent to be a measurable function. In [4], authors introduced variable exponent Lebesgue spaces, where the underlying space is the field of the p-adic numbers. They proved many properties of the spaces and also studied the boundedness of the maximal operator as well as its application to convolution operators. In [5], boundedness of the fractional maximal and the fractional integral operator in the p-adic variable exponent Lebesgue spaces was obtained. As an application, they obtained unique solution for a nonhomogeneous Cauchy problem in the p-adic variable exponent Lebesgue spaces.

The p-adic Herz-Morrey spaces are the generalization of p-adic Herz spaces. The boundedness of p-adic fractional operator on p-adic Herz-Morrey spaces was proved in [6]. The boundedness of commutators of a rough p-adic fractional Hardy-type operator on Herz-type spaces obtained in [7] when the symbol functions belong to two different function spaces. The boundedness of commutators of p-adic weighted Hardy-Cesáro operator with symbols in the Lipschitz space on the weighted Morrey-Herz space was established in [8]. In [9], authors established the sharp boundedness of p-adic multilinear Hausdorff operators on the product of Lebesgue and central Morrey spaces associated with both power weights and Muckenhoupt weights. In [10], authors obtained the boundedness of the Hausdorff operator in the Triebel-Lizorkin spaces and Herz spaces with absolutely homogeneous weights and the Muckenhoupt weights on p-adic field. Moreover, the corresponding operator norms were estimated. Some applications to the Hardy, Hilbert and weighted Hardy-Cesáro operators on p-adic field were also shown.

Grand function spaces with variable exponent is active area of research (cf. [11–14]). The boundedness of variable integral operators on grand variable Herz spaces proved in [15, 16]. Grand variable Herz-Morrey spaces are the generalization of grand variable Herz spaces, see for instance [17, 18]. It is reasonable to define the grand p-adic Herz-Morrey spaces with variable exponent. Almost all results in this direction were obtained in the framework of $\mathbb{R}^n$. Motivated by these results, in this paper we introduce p-adic intrinsic square function on grand p-adic Herz-Morrey spaces with variable exponent. To the best of our knowledge, this was never considered since such kind of results have been established only in classical function spaces. Furthermore, we will discuss the boundedness of an p-adic intrinsic square function in grand p-adic Herz-Morrey spaces with variable exponent.

This work is divided as follows. Section 2 contains a quick description of the preliminary on the topic of the p-adic analysis and variable exponent Lebesgue spaces on the p-adic numbers, necessary for the development of this work. In Section 3, we define the concept of grand p-adic Herz-Morrey spaces with variable exponent and p-adic intrinsic square. Boundedness of p-adic intrinsic square function in the frame work of grand p-adic Herz-Morrey spaces with variable exponent is proved in the last section.

1.1. Notations

In this article we will use the following notations:

(i) For $\mu \in \mathbb{Z}$, $B^n_\mu(b)$ is the ball of radius p^μ with centre at $b = (b_1, ..., b_n) \in \mathbb{Q}^n_p$ and given as

$$B^n_\mu(b) = \left\{z \in \mathbb{Q}^n_p : \|z - b\| \leq p^\mu\right\}. \tag{1.1}$$

(ii) The corresponding sphere is

$$S^n_\mu(b) = \left\{z \in \mathbb{Q}^n_p : \|z - b\| = p^\mu\right\} = B^n_\mu(b) \setminus B^n_{\mu-1}(b). \tag{1.2}$$

(iii) $B^n_\mu(0) := B^n_\mu$ and $S^n_\mu(0) := S^n_\mu$.

(iv) We define $B^n_\mu(b)$ as $B^n_\mu(b) = B_\mu(b_1) \times ... \times B_\mu(b_n)$, where,

$$B_\mu(b_i) := \left\{z \in \mathbb{Q}^n_p : |z - b_i| \leq p^\mu\right\}, \tag{1.3}$$

is the one-dimensional ball.

(v) $S_\mu(b)$ is the sphere of $\mathbb{Q}^n_p$ with radius p^μ and centre at $b \in \mathbb{Q}^n_p$,

$$S_\mu(b) = \left\{z \in \mathbb{Q}^n_p : |z - b| = p^\mu\right\}. \tag{1.4}$$

(vi) For $b = 0$ one can write $B_\mu(0) = B_\mu$ and $S_\mu(0) = S_\mu$.

(vii) χ_μ is the characteristic function.

C is the constant and its value can vary from line to line. Note that there exists a positive Haar measure dx in $\mathbb{Q}^n_p$, since $\mathbb{Q}^n_p$ is the locally compact commutative group with respect to addition. The measure dx is normalized by $\int_{B^n_0} dx = 1$. We can easily find $\int_{B_\mu(b)} dx = p^{n\mu}$ and $\int_{S_\mu(b)} dx = p^{n\mu}(1 - p^{-n})$ for $b \in \mathbb{Q}^n_p$.

2. Preliminaries

2.1. The field of p-adic numbers

For current section authors refer to [5–7, 19–21]. Let p denotes the prime number. The $\mathbb{Q}_p$ is the field given by the completion of $\mathbb{Q}$ with respect to p-adic norm $|\cdot|_p$, defined as

$$|z|_p := \begin{cases} 0, & if\ \ z = 0, \\ p^{-\mu}, & if\ \ z = p^\mu \frac{a}{b}, \end{cases} \tag{2.1}$$

where a, b are integers co-prime to p. The integer $\mu := ord(z)(ord(0) := +\infty)$ is represented as the p-adic order of z. The norm of $|z|$ can be extended to $\mathbb{Q}^n_p$ as given by

$$\|z\|_p := \max_{1\leq i\leq n} |z_i|_p \quad for\ z = (z_1, z_2, ..., z_n) \in \mathbb{Q}^n_p, \tag{2.2}$$

and norm satisfies the strong triangular inequality

$$\|z + x\|_p \leq \max\left\{\|z\|_p, \|x\|_p\right\}, \tag{2.3}$$

with equality if $\|z\|_p \neq \|x\|_p$.

A p-adic number $z \neq 0$ can be written uniquely in formal power series expansion as,

$$z = p^\mu \sum_{j=0}^{\infty} z_j p^j, \tag{2.4}$$

where $z_j \in \{0, 1, 2, ..., p - 1\}$ and $z_0 \neq 0$.

2.2. Lebesgue space $L^q(\mathbb{Q}_p^n)$ [5]

A measurable function $g : \mathbb{Q}_p^n \to \mathbb{C}$ belongs to the Lebesgue space $L^q(\mathbb{Q}_p^n)$, $q \in [1, \infty)$, if

$$\|g\|^q_{L^q(\mathbb{Q}_p^n)} := \int_{\mathbb{Q}_p^n} |g(z)|^q dz < \infty, \quad q \in [1, \infty), \tag{2.5}$$

where

$$\int_{\mathbb{Q}_p^n} |g(z)|^q dz := \lim_{\eta \to \infty} \int_{B_\eta^n(0)} |g(z)|^q dz, \tag{2.6}$$

if the limits exists.
Now we define notion of variable exponents p-adic Lebesgue spaces. If $q : \mathbb{Q}_p^n \to [1, \infty)$ by $\mathcal{Q}(\mathbb{Q}_p^n)$. Let us denote the set of all variable exponent satisfying $q^+ < \infty$, where

(i) $q^+ := \operatorname{ess}\sup_{z \in \mathbb{Q}_p^n} q(z)$,
(ii) $q^- := \operatorname{ess}\inf_{z \in \mathbb{Q}_p^n} q(z)$.

Let $q \in \mathcal{Q}(\mathbb{Q}_p^n)$, $L^{q(\cdot)}(\mathbb{Q}_p^n)$ denotes the space of measurable functions $g : \mathbb{Q}_p^n \to \mathbb{R}$ such that

$$\|g\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \inf \left\{ \gamma > 0 : \eta_{q(\cdot)}(\frac{g}{\gamma}) \leq 1 \right\} < \infty, \tag{2.7}$$

where $\eta_{q(\cdot)}(g) := \int_{\mathbb{Q}_p^n} |g(z)|^{q(z)} dz$. For variable exponent Lebesgue space, as a result

$$\|g\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \eta_{q(\cdot)}(g) + 1, \tag{2.8}$$

$$\eta_{q(\cdot)}(g) \leq \left(1 + \|g\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{q^+}, \tag{2.9}$$

$$\|g\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \||g|\|^{\frac{1}{s}}_{L^{q(\cdot)/s}(\mathbb{Q}_p^n)}, \; s \in (0, q^-]. \tag{2.10}$$

Lemma 2.1. *[5] [Hölder's inequality] Assume that $f \in L^{q(\cdot)}(\mathbb{Q}_p^n)$ and $g \in L^{q'(\cdot)}(\mathbb{Q}_p^n)$ then we have*

$$\int_{\mathbb{Q}_p^n} |f(z)g(z)| dz \leq \|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \|g\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)},$$

where q and q' are conjugate exponents defined as $1 = \frac{1}{q(z)} + \frac{1}{q'(z)}$ for every $z \in \mathbb{Q}_p^n$.

Let $q \in \mathcal{Q}(\mathbb{Q}_p^n)$, then we say that $q \in W_0(\mathbb{Q}_p^n)$ if there exists a constant C such that

$$\gamma \left(q^-(B_\gamma^n(y) - q^+(B_\gamma^n(y))\right) \leq C, \tag{2.11}$$

for all $y \in \mathbb{Q}_p^n$. Let $q \in \mathcal{Q}(\mathbb{Q}_p^n)$, then we say that $q \in W^\infty(\mathbb{Q}_p^n)$ if there exists a constant C such that

$$|q(z_1) - q(z_2)| \leq \frac{C}{\log_p(p + \min\{\|z_2\|_p, \|z_1\|_p\})}, \tag{2.12}$$

for $z_1, z_2 \in \mathbb{Q}_p^n$. We define class $W_0^\infty(\mathbb{Q}_p^n)$ as $W_0^\infty(\mathbb{Q}_p^n) := W^\infty(\mathbb{Q}_p^n) \cap W_0(\mathbb{Q}_p^n)$.

Lemma 2.2. *[5] Let $r \in W_0^{\infty}(\Omega_p^n)$, then*

$$\|\chi_{B_\mu^n(z)}(\cdot)\|_{L^{r(\cdot)}(\Omega_p^n)} \leq C p^{\mu n/r(z,\mu)}, \tag{2.13}$$

where

$$r(z,\mu) = \begin{cases} r(z), & \mu < 0, \\ r(\infty), & \mu \geq 0. \end{cases} \tag{2.14}$$

3. Main results

Now we will define main definitions and prove main results.

3.1. Grand p-adic Herz-Morrey spaces with variable exponent

Next we define grand p-adic Herz-Morrey spaces with variable exponent.

Definition 3.1. [7] Let $0 < q < \infty$, $\alpha \in \mathbb{R}$ and $0 < s < \infty$. The homogeneous p-adic Herz space $\dot{K}_r^{\alpha,q}(\mathbb{Q}_p^n)$ is defined by

$$\dot{K}_r^{\alpha,q}(\mathbb{Q}_p^n) = \left\{ g \in L^r(\mathbb{Q}_p^n) : \|g\|_{\dot{K}_r^{\alpha,q}(\mathbb{Q}_p^n)} < \infty \right\}, \tag{3.1}$$

where

$$\|g\|_{\dot{K}_r^{\alpha,q}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|p^{k\alpha} g \chi_k\|_{L^r(\mathbb{Q}_p^n)}^q \right)^{\frac{1}{q}}.$$

Definition 3.2. Let $\alpha \in L^{\infty}(\mathbb{Q}_p^n)$, $u \in [1,\infty)$, $s : \mathbb{Q}_p^n \to [1,\infty)$, $\theta > 0$ and $0 \leq \lambda < \infty$. A homogeneous grand p-adic Herz-Morrey spaces with variable exponent $M\dot{K}_{s(\cdot)}^{\alpha,u),\theta}(\mathbb{Q}_p^n)$ is defined as

$$M\dot{K}_{\lambda,s(\cdot)}^{\alpha,u),\theta}(\mathbb{Q}_p^n) = \left\{ g \in L_{\text{loc}}^{s(\cdot)}(\mathbb{Q}_p^n \setminus \{0\}) : \|g\|_{M\dot{K}_{\lambda,s(\cdot)}^{\alpha,u),\theta}(\mathbb{Q}_p^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}_{\lambda,s(\cdot)}^{\alpha,u),\theta}(\mathbb{Q}_p^n)} = \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^{\theta} \sum_{k=-\infty}^{k_0} p^{\alpha k u(1+\epsilon)} \|g\chi_k\|_{L^{s(\cdot)}}^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}}.$$

Non-homogeneous grand p-adic Herz-Morrey spaces can be defined in a similar way. Now we will define intrinsic square function $H_\beta f(z_1)$.

Definition 3.3. Let $z_1 \in \mathbb{Q}_p^n$, we define a set,

$$\Gamma(z_1) :\equiv \{(z_2,t) \in \mathbb{Q}_{p^+}^{n+1} : |z_1 - z_2| < t\},$$

where $\mathbb{Q}_{p^+}^{n+1} = \mathbb{Q}_p^n \times (0,\infty)$. Let $0 < \beta \leq 1$ be a constant. The set C_β consists of all functions ϕ defined on $\mathbb{Q}_p^n$ such that

(i) $\operatorname{supp} \phi \subset \{|z_1| \leq 1\}$,
(ii) $\int_{\mathbb{Q}_p^n} \phi(z_1) dz_1 = 0$,
(iii) $|\phi(z_1) - \phi(z_1')| \leq |z_1 - z_1'|^{\beta}$ for $z_1, z_1' \in \mathbb{Q}_p^n$.

For every $(z_2, t) \in \mathbb{Q}_{p^+}^{n+1}$ we write $\phi_t(z_2) = t^{-n}\phi(z_2/t)$. Then we define a maximal function for $f \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$.
$A_\beta f(z_2, t) := \sup_{\phi\in C_\beta} |f * \phi_t(z_2)|$, where $(z_2, t) \in \mathbb{Q}_{p^+}^{n+1}$. Using above, we define the intrinsic square function with order β by

$$H_\beta f(z_1) := \left(\iint_{\Gamma(z_1)} A_\beta f(z_2, t)^2 \frac{dz_2 dt}{t^{n+1}} \right)^{1/2}.$$

3.2. Boundedness of p-adic intrinsic square function

In this section, we show that p-adic intrinsic square function is bounded on $M\dot{K}^{\alpha,r}_{q(\cdot)}(\mathbb{Q}_p^n)$.

Theorem 3.4. *Let* $1 \leq u < \infty$, $\alpha, q(\cdot) \in W_0^\infty(\mathbb{Q}_p^n)$, *q is defined as* $1/q(z_1) := 1/q'(z_1) - 1$ *and* $0 \leq \lambda < 0$. *And* α *is satisfying the following conditions*

i) $-n/q(z_1) < \alpha < n/q'(z_1)$,
ii) $-n/q(\infty) < \alpha < n/q'(\infty)$.

Then p-adic intrinsic square function with order β *is bounded on* $M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)$.

Proof. Let $f \in M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)$, and $f(z_1) = \Sigma_{l=-\infty}^{\infty} f(z_1)\chi_l(z_1) = \Sigma_{l=-\infty}^{\infty} f_l(z_1)$. We will find the estimate when k_0 is positive, since the negative case can be treated similarly. We have

$$\begin{aligned}
\|H_\beta f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)} &= \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{k_0} p^{k\alpha u} \|\chi_k H_\beta f\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&\leq \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{k_0} p^{k\alpha u(1+\epsilon)} \left(\sum_{l=-\infty}^{\infty} \|\chi_k H_\beta f(\chi_l)\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)\right)^{\frac{1}{u(1+\epsilon)}} \\
&\leq \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{k_0} p^{k\alpha u(1+\epsilon)} \left(\sum_{l=-\infty}^{k} \|\chi_k H_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&+ \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=-\infty}^{k_0} p^{k\alpha u(1+\epsilon)} \left(\sum_{l=k+1}^{\infty} \|\chi_k H_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&=: E_1 + E_2.
\end{aligned}$$

For E_1, we take $k \in \mathbb{Z}$, $l \leq k$, $z_1 \in S_k^n$ and $(z_2, t) \in \Gamma(z_1)$. For $\phi \in C_\beta$ we have

$$\begin{aligned}
|f(\chi_l) * \phi_t(z_2)| &= \left| \int_{S_l^n} \phi_t(z_2) f(x) dx \right| \\
&\leq C t^{-n} \int_{\{x\in S_l^n : |z_2 - x| < t\}} |f(x)| dx.
\end{aligned}$$

For $x \in S_l^n$ with $|z_2 - x| < t$ implies that

$$t = \frac{1}{2}(t + t) > \frac{1}{2}(|z_1 - z_2| + |z_2 - x|) \geq \frac{1}{4}|z_1 - x|.$$

As a result

$$
\begin{aligned}
&|S_{\beta}(f\chi_l)(z_1)| \\
&= \left(\iint_{\Gamma(z_1)} \left(\sup_{\phi \in C_{\beta}} |f\chi_l * \phi_t(z_2)|^2 \frac{dz_2 dt}{t^{n+1}} \right)^2 \right)^{1/2} \\
&\le C \left(\int_{\frac{|z_1 - x|}{4}}^{\infty} \int_{\{z_2 : |z_1 - z_2| < t\}} \left(\frac{1}{t^n} \int_{\{x \in S_l^n : |z_2 - x| < t\}} |f(x)| dx \right)^2 \frac{dz_2 dt}{t^{n+1}} \right)^{1/2} \\
&\le C \left(\int_{S_l^n} |f(x)| dx \right) \left(\int_{\frac{|z_1 - x|}{4}}^{\infty} \left(\int_{\{z_2 : |z_1 - z_2| < t\}} dz_2 \right) \frac{dt}{t^{3n+1}} \right)^{1/2} \\
&= C \left(\int_{S_l^n} |f(x)| dx \right) \left(\int_{\frac{|z_1 - x|}{4}}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2} \\
&= C \left(\int_{S_l^n} |f(x)| dx \right) |z_1 - x|^{-n}.
\end{aligned}
$$

By using Hölder's inequality,

$$
\begin{aligned}
S_{\beta}(f\chi_l)(z_1)| &\le C p^{-kn} \left(\int_{S_l^n} |f(x)| dx \right) \\
&\le C p^{-kn} \|f\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n))} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n))}.
\end{aligned}
$$

Splitting E_1 by the virtue of Minkowski's inequality we get

$$
\begin{aligned}
E_1 \le & \sup_{\epsilon > 0} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\epsilon^{\theta} \sum_{k=-\infty}^{-1} p^{k\alpha u(1+\epsilon)} \left(\sum_{l=-\infty}^{k} \|\chi_k H_{\beta}(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
& + \sup_{\epsilon > 0} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\epsilon^{\theta} \sum_{k=0}^{k_0} p^{k\alpha u(1+\epsilon)} \left(\sum_{l=-\infty}^{k} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
& := E_{11} + E_{12}.
\end{aligned}
$$

By virtue of Lemma 2.2

$$
p^{-kn} \|\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \|\chi_l\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)} \le C p^{-kn} p^{\frac{kn}{q(z_1)}} p^{\frac{ln}{q'(z_1)}} \le C p^{\frac{(l-k)n}{q'(z_1)}}. \tag{3.2}
$$

By applying above estimates to E_{11} we obtain

$$
E_{11} \le \sup_{\epsilon > 0} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\epsilon^{\theta} \sum_{k=-\infty}^{-1} p^{k\alpha u(1+\epsilon)} \left(\sum_{l=-\infty}^{k} \|\chi_k T(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}}
$$

$$\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left[\epsilon^{\theta}\sum_{k=-\infty}^{-1}p^{k\alpha u(1+\epsilon)}\left(\sum_{l=-\infty}^{k}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{-kn}\right.\right.$$
$$\left.\left.\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right]^{\frac{1}{u(1+\epsilon)}}.$$

Let $b=\frac{n}{q'(z_1)}-\alpha$,

$$E_{11}\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left[\epsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k}p^{\alpha l}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{b(l-k)}\right)^{u(1+\epsilon)}\right]^{\frac{1}{u(1+\epsilon)}}. \tag{3.3}$$

Now by applying Fubini's theorem for series, the Hölder's inequality and the fact $p^{-u(1+\epsilon)}<p^{-u}$ we get,

$$\begin{aligned}
E_{11}\leq& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left[\epsilon^{\theta}\sum_{k=-\infty}^{-1}\left(\sum_{l=-\infty}^{k}p^{\alpha u(1+\epsilon)l}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{bu(1+\epsilon)(l-k)/2}\right.\right.\\
&\left.\left.\times\sum_{l=-\infty}^{k}p^{b(u(1+\epsilon)'(l-k)/2}\right)^{\frac{u}{(u)'}}\right]^{\frac{1}{u(1+\epsilon)}}\\
=&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}\sum_{l=-\infty}^{k}p^{\alpha u(1+\epsilon)l}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{bu(1+\epsilon)(l-k)/2}\right)^{\frac{1}{u(1+\epsilon)}}\\
=&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{l=-\infty}^{-1}p^{\alpha u(1+\epsilon)l}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\sum_{k=l}^{-1}p^{bu(1+\epsilon)(l-k)/2}\right)^{\frac{1}{u(1+\epsilon)}}\\
<&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{l=-\infty}^{-1}p^{\alpha u(1+\epsilon)l}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\sum_{k=l}^{-1}p^{bp(l-k)/2}\right)^{\frac{1}{u(1+\epsilon)}}\\
\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{l=-\infty}^{-1}p^{\alpha u(1+\epsilon)l}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{\frac{1}{u(1+\epsilon)}}\\
=&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{l=-\infty}^{k_0}p^{\alpha u(1+\epsilon)l}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{\frac{1}{u(1+\epsilon)}}\\
\leq&C\|f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Now for E_{12} using Minkowski's inequality

$$\begin{aligned}
E_{12}\leq&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}p^{k\alpha u(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_kH_{\beta}(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}p^{k\alpha u(1+\epsilon)}\left(\sum_{l=0}^{k}\|\chi_kH_{\beta}(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
:=&A_1+A_2.
\end{aligned}$$

We can find the estimate of A_2 by using similar arguments applied for the case of E_{11}. To get that estimate we will simply replace $q'(z_1)$ with $q'(\infty)$, and use the fact $\frac{n}{q'(\infty)} - \alpha > 0$.
For the estimate of A_1 we will use Lemma 2.2 to obtain

$$\begin{aligned} p^{k(-n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)} &\leq Cp^{k(-n)}p^{\frac{kn}{q(\infty)}}p^{\frac{ln}{q'(z_1)}} \\ &\leq Cp^{\frac{-kn}{q'(\infty)}}p^{\frac{ln}{q'(z_1)}}. \end{aligned}$$

As $\alpha - \frac{n}{q'(\infty)} < 0$, estimate for A_1 is given as

$$\begin{aligned} A_1 &\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}p^{k\alpha u(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_k H_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}} \\ &\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\epsilon^\theta\sum_{k=0}^{k_0}p^{k\alpha u(1+\epsilon)}\times\left(\sum_{l=-\infty}^{-1}p^{\frac{-kn}{q(\infty)}}p^{\frac{ln}{q'(z_1)}}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right]^{\frac{1}{u(1+\epsilon)}} \\ &\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\sum_{k=0}^{k_0}p^{(k\alpha-kn/q'(\infty))u(1+\epsilon)}\times\left(\sum_{l=-\infty}^{-1}p^{\frac{ln}{q'(z_1)}}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right]^{\frac{1}{u(1+\epsilon)}} \\ &\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\left(\sum_{l=-\infty}^{-1}p^{\frac{ln}{q'(z_1)}}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}} \\ &\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\left(\sum_{l=-\infty}^{-1}p^{\frac{ln}{q'(z_1)}-\alpha l}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{\alpha l}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}. \end{aligned}$$

Now by applying the fact that $\frac{n}{q'(z_1)} - \alpha > 0$, and the Hölder's inequality implies that

$$\begin{aligned} A_1 &\leq \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\left(\sum_{l=-\infty}^{-1}p^{\frac{ln}{q'(z_1)}-\alpha l}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{\alpha l}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}} \\ &\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\epsilon^\theta\sum_{l=-\infty}^{-1}p^{\alpha l u(1+\epsilon)}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right. \\ &\qquad\left.\times\left(\sum_{l=-\infty}^{-1}p^{(\frac{ln}{q'(z_1)}-\alpha l)(u)'}\right)^{\frac{u(1+\epsilon)}{(u(1+\epsilon))'}}\right]^{\frac{1}{u(1+\epsilon)}} \\ &\leq C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^\theta\sum_{l=-\infty}^{k_0}p^{\alpha l u(1+\epsilon)}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{\frac{1}{u(1+\epsilon)}} \\ &\leq C\|f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)}. \end{aligned}$$

For E_2, take $k \in \mathbb{Z}$, $l \geq k+1$, $z_1 \in S_k^n$ and $(z_2, t) \in \Gamma(z_1)$. For $\phi \in C_\beta$ implies that

$$|f(\chi_l) * \phi_t(z_2)| = \left|\int_{S_l^n}\phi_t(z_2)f(x)dx\right|$$

$$\leq Ct^{-n}\int_{\{x\in S_l^n:|z_2-x|<t\}}|f(x)|dx.$$

For $x\in S_l^n$ with $|z_2-x|<t$ we obtain

$$t=\frac{1}{2}(t+t)>\frac{1}{2}(|z_1-z_2|+|z_2-x|)\geq\frac{1}{4}|z_1-x|\geq\frac{1}{4}|x-z_1|.$$

As a result

$$\begin{aligned}
&|S_\beta(f\chi_l)(z_1)|\\
&=\left(\iint_{\Gamma(z_1)}\left(\sup_{\phi\in C_\beta}|f\chi_l*\phi_t(z_2)|^2\frac{dz_2dt}{t^{n+1}}\right)^2\right)^{1/2}\\
&\leq C\left(\int_{\frac{|x-z_1|}{4}}^{\infty}\int_{\{z_2:|z_1-z_2|<t\}}\left(\frac{1}{t^n}\int_{\{x\in S_l^n:|z_2-x|<t\}}|f(x)|dx\right)^2\frac{dz_2dt}{t^{n+1}}\right)^{1/2}\\
&\leq C\left(\int_{S_l^n}|f(x)|dx\right)\left(\int_{\frac{|x-z_1|}{4}}^{\infty}\left(\int_{\{z_2:|z_1-z_2|<t\}}dz_2\right)\frac{dt}{t^{3n+1}}\right)^{1/2}\\
&=C\left(\int_{S_l^n}|f(x)|dx\right)\left(\int_{\frac{|x-z_1|}{4}}^{\infty}\frac{dt}{t^{2n+1}}\right)^{1/2}\\
&=C\left(\int_{S_l^n}|f(x)|dx\right)|x-z_1|^{-n}.
\end{aligned}$$

Splitting E_2 by the virtue of Minkowski's inequality to obtain

$$\begin{aligned}
E_2\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{k_0}p^{k\alpha u(1+\epsilon)}\left(\sum_{l=k+1}^{\infty}\|\chi_kH_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
\leq&C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{-1}p^{k\alpha u(1+\epsilon)}\left(\sum_{l=k+1}^{\infty}\|\chi_kH_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&+C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\epsilon^\theta\sum_{k=0}^{k_0}p^{k\alpha u(1+\epsilon)}\left(\sum_{l=k+1}^{\infty}\|\chi_kH_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
:=&E_{21}+E_{22}.
\end{aligned}$$

For E_{22} Lemma 2.2 yields

$$p^{-ln}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\leq Cp^{-ln}p^{\frac{kn}{q(\infty)}}p^{\frac{ln}{q_1'(\infty)}}\leq Cp^{\frac{(k-l)n}{q(\infty)}},\tag{3.4}$$

these estimates yields

$$
\begin{aligned}
E_{22} &\le \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0} p^{k\alpha u(1+\epsilon)}\left(\sum_{l=k+1}^{\infty}\|\chi_k H_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg[\epsilon^{\theta}\sum_{k=0}^{k_0} p^{k\alpha u(1+\epsilon)}\Bigg(\sum_{l=k+1}^{\infty}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{-ln}\cdot\\
&\qquad\qquad \|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\Bigg)^{u(1+\epsilon)}\Bigg]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}\left(\sum_{l=k+1}^{\infty}p^{\alpha l}\|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{d(k-l)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}},
\end{aligned}
$$

where $d=\frac{n}{q(\infty)}+\alpha>0$. Then we use Hölder's theorem for series and the fact $p^{-u(1+\epsilon)}<p^{-u}$ to obtain

$$
\begin{aligned}
E_{22} \le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg[\epsilon^{\theta}\sum_{k=0}^{k_0}\left(\sum_{l=k+1}^{\infty}p^{\alpha l u(1+\epsilon)}\|f\chi_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{du(1+\epsilon)(k-l)/2}\right)\\
&\qquad\times\left(\sum_{l=k+1}^{\infty}p^{d(u(1+\epsilon))'(k-l)/2}\right)^{\frac{u(1+\epsilon)}{(u(1+\epsilon))'}}\Bigg]^{\frac{1}{u(1+\epsilon)}}\\
\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}\sum_{l=k+1}^{\infty}\sum_{j=-\infty}^{l}p^{\alpha j u(1+\epsilon)}\|f\chi_j\|^{u(1+\epsilon)}_{L^{q(\cdot)}(\mathbb{Q}_p^n)}p^{du(1+\epsilon)(k-l)/2}\right)^{\frac{1}{u(1+\epsilon)}}\\
\le& C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=0}^{k_0}\sum_{l=k+1}^{\infty}p^{du(1+\epsilon)(k-l)/2}\right)^{\frac{1}{u(1+\epsilon)}}\|f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)}\\
\le& C\|f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Now for E_{21}, again applying Minkowski's inequality we get

$$
\begin{aligned}
E_{21} \le& \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1} p^{k\alpha u(1+\epsilon)}\left(\sum_{l=k+1}^{-1}\|\chi_k H_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1} p^{k\alpha u(1+\epsilon)}\left(\sum_{l=0}^{\infty}\|\chi_k H_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
:=& B_1+B_2.
\end{aligned}
$$

We can easily find the estimate for B_1 by using same arguments as used in E_{22}. We will simply replace $q(\infty)$ with $q(z_1)$, and use the fact $\frac{n}{q(z_1)}+\alpha>0$. For B_2 apply Lemma 2.2 to get

$$
p^{-ln}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)} \le Cp^{-ln}p^{\frac{kn}{q(z_1)}}p^{\frac{ln}{q'(\infty)}} \le Cp^{\frac{kn}{q(z_1)}}p^{\frac{-ln}{q(\infty)}}, \tag{3.5}
$$

$$
\begin{aligned}
B_2 \leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=\infty}^{-1} p^{k\alpha u(1+\epsilon)} \left(\sum_{l=0}^{\infty} \|\chi_k H_\beta(f\chi_l)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
\leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=\infty}^{-1} p^{k\alpha u(1+\epsilon)} \times \left(\sum_{l=0}^{\infty} p^{-ln} p^{\frac{kn}{q(z_1)}} p^{\frac{ln}{q(\infty)}} \|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
\leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=\infty}^{-1} p^{k\alpha u(1+\epsilon)} \times \left(\sum_{l=0}^{\infty} p^{\frac{kn}{q(z_1)}} p^{\frac{-ln}{q(\infty)}} \|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
\leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \sum_{k=\infty}^{-1} p^{k(\alpha+n)/q(z_1)u(1+\epsilon)} \times \left(\sum_{l=0}^{\infty} p^{\frac{-ln}{q(\infty)}} \|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
\leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \left(\sum_{l=0}^{\infty} p^{\frac{-ln}{q(\infty)}} \|f\chi_l\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
\leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\epsilon^\theta \left(\sum_{l=0}^{\infty} \sum_{j=-\infty}^{l} p^{\alpha j} \|f\chi_j\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} p^{l(nq(\infty)+\alpha)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} .
\end{aligned}
$$

As $\frac{n}{q(\infty)} + \alpha > 0$, the Hölder's inequality implies

$$
\begin{aligned}
B_2 \leq & C \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\left(\sum_{l=0}^{\infty} p^{l(nq(\infty)+\alpha)} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \|f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)} \\
\leq & C\|f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Combining the estimates for E_1 and E_2 yields

$$
\|H_\beta f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)} \leq C\|f\|_{M\dot{K}^{\alpha,u),\theta}_{\lambda,q(\cdot)}(\mathbb{Q}_p^n)},
$$

which completes the proof. □

4. Conclusions

In this paper we defined the concept of grand p-adic Herz-Morrey spaces with variable exponents. Additionally, we introduced the concept of a p-adic intrinsic square function, which characterizes some important aspect of functions in these spaces. We estimate the operator norms associated with the p-adic intrinsic square function within the grand p-adic Herz-Morrey spaces. The primary focus of the research was to establish that the p-adic intrinsic square function is bounded when considered within the grand p-adic Herz-Morrey spaces with variable exponents.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

It is declared that the authors do not have any competing interests.

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