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Research article

A new distance measure and corresponding TOPSIS method for interval-valued intuitionistic fuzzy sets in multi-attribute decision-making

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Abstract: Strengthening the evaluation of teaching satisfaction plays a crucial role in guiding teachers to improve their teaching quality and competence, as well as in aiding educational institutions in the formulation of effective teaching reforms and plans. The evaluation process for teaching satisfaction is usually regarded as a typical multi-attribute decision-making (MADM) process, which inherently possesses uncertainty and fuzziness due to the subjective nature of human cognition. In order to improve the subtle discrimination of evaluation information data and enhance the accuracy of the evaluation results, we have developed an integrated MADM method by combining a new distance measure and an improved TOPSIS method for interval-valued intuitionistic fuzzy sets (IvIFSs). First, a novel distance measure for IvIFSs based on triangular divergence is proposed to capture the differences between two IvIFSs, and some properties of this distance measure are investigated. Then, the superiority of this new distance measure is compared with some existing distance measures. Afterward, an improved TOPSIS method is also established based on the proposed triangular distance under the interval-valued intuitionistic fuzzy setting. Besides, to illustrate the practicality of the new method, a numerical example is presented to evaluate mathematics teaching satisfaction. Moreover, a comparative analysis that includes existing TOPSIS methods, is presented to demonstrate the superiority of the given method. The comparison outcomes show that the proposed technique can effectively discern uncertainties or subtle differences in IvIFSs, resulting in more accurate and comprehensive evaluation results for teaching satisfaction. Overall, the findings of this study emphasize the importance of incorporating the new distance measure in MADM. The proposed approach serves as a valuable tool for decision-makers to compare and evaluate alternatives effectively.

Keywords: multi-attribute decision-making; interval-valued intuitionistic fuzzy set; distance measure; TOPSIS; triangular divergence; teaching satisfaction evaluation
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1. Introduction

In recent decades, there has been a global trend of placing increased importance on the teaching quality of higher education. Teaching satisfaction, as a crucial evaluation criterion, is widely recognized as essential information for enhancing teaching quality. By evaluating teachers' teaching satisfaction, they are encouraged to identify areas for improvement, reflect on their teaching practices, address weaknesses and, ultimately, enhance the quality of education [1]. This, in turn, promotes the development and advancement of colleges and universities. Currently, there are many studies on teaching satisfaction, including the evaluation index and its influence. Previous studies [2–6] have indicated that teaching satisfaction is affected by diverse factors, such as the expertise of teachers, teaching attitude, content of courses and teaching methods, among others. Additionally, many colleges and universities have also established evaluation index systems that integrate standardized criteria and talent training programs [7]. The process of comparing and selecting a desirable option from several schemes evaluated on various dimensions or attributes is often considered as a multi-attribute decision-making (MADM) process [8].

In the specific decision-making situation of teaching satisfaction, two issues should be the focus [9]:

(1) How does a decision-maker express decision-related information appropriately?

(2) How can an ideal scheme be determined by using relevant decision-making methods?

Regarding the first issue, during the evaluation process, the meaning of each attribute representing the target is determined. For instance, when evaluating whether a teacher is fully prepared, a decision-maker may simply provide a binary opinion of "yes" or "no". However, in practical decision expressions, there is fuzziness and a large amount of uncertainties, which often involve subjective terms such as "very poor", "good" or "excellent". These terms do not always correspond to precise data. To address this issue, Zadeh [10] initially created fuzzy sets (FSs) with the concept of a membership function in 1965. By utilizing a FS, evaluation values can extend beyond the binary scale of $\{0, 1\}$ to a more flexible range of $[0, 1]$. The theory of FSs has been widely applied in engineering, business management, education and other domains.

With the development of theoretical research and practical demonstration, new problems have emerged. For instance, if 10 people decide "whether the teacher is patient with students", five evaluators may answer "yes/agreement", and four people may answer "no/disagreement". But one person may have some uncertainty in making decisions, and may even decline to answer. In this case, the hesitancy must be taken into consideration. Then, Atanassov [11] proposed the intuitionistic FS (IFS). Unlike FSs, IFSs have a membership degree, non-membership degree, hesitancy degree or intuitionistic index, which are consistent with humans' subjective habit of describing decisions with "negation", "affirmation" and "hesitation". In this case, IFSs are more suitable for describing and collecting decision-making information. However, in IFSs, the values of the membership degrees and others need to be precise numbers, which may be difficult to obtain because of the complex factors of a realistic environment and the limitations of decision-makers' cognitive abilities [8]. For example, if 0.6 represents "good" and 0.75 is "very good", a decision-maker is inclined to decide with an interval of [0.6, 0.7] rather than an exact value of 0.7. Subsequently, Atanassov and Gargov [12] further came up with the interval-valued intuitionistic fuzzy set (IvIFS) in 1989. They employ interval values rather than crisp values to express the degree of membership, non-membership and uncertainty for each element. The theory of IvIFSs has received widespread attention in theoretical research and practical application [13–16]. It has been demonstrated that the IvIFS is a more powerful tool to deal with uncertain and ambiguous information in actual environments, rather than previous theories, such as

FSs and IFSs [17, 18].

Regarding the second issue, various approaches are employed in academic research to determine the optimal scheme(s) within the framework of IvIFSs, such as the approach of determining index weights, use of a distance measure, use of the technique for order performance by similaity to ideal solution (TOPSIS), and so on.

(1) The method of determining index weight. For the case in which the weight value is unknown or uncertain, the entropy weight method has been widely recognized as an efficient technique for calculating index weights. Zhang and Jiang [19] initially established the concept of entropy and then gave a couple of formulas to compute IvIFSs' entropy, which could be used in clustering analysis and various fields. Wu and Wan [20] further advanced the entropy method with IvIFSs in supplier selection problems and computed the corresponding index weights, which made the conclusions more reliable and objective. Considering the risk preferences of experts, Zhang et al. [21] constructed a novel score function ($P - \lambda$) and then introduced the concept of average entropy for IvIFSs. They incorporated experts' risk preferences into the weighting process, enhancing the reliability of the results. Additionally, Xian et al. [14] developed a new weight approach based on the entropy measure, considering the presence of both positive and indeterminate preferences for attributes. This approach assigns unique weights to each attribute, therefore providing a comprehensive evaluation.

(2) Distance measure of IvIFSs. It is an effective tool for handling uncertain and vague information within the framework of FS theory [22–25]. IvIFSs' distance method is generalized on the basis of FSs' distance as developed in [23, 24]. Xu [22, 25] defined several distance measures, such as the (normalized) Hamming distance, (normalized) Euclidean distance and hybrid weighted distance measures. Park [26] redefined pairs of various distance measures including the (normalized) Hamming distance and (normalized) Euclidean distance, by taking the amplitude of the membership of the elements into consideration. Muharrem [27] put forward a novel distance measure, which could be utilized to compare counter intuitive examples for IvIFSs. Inspired by intervals, Liu and Jiang [17] established a new interval-valued intuitionistic distance for IvIFSs, and it preserves the entire interval information and effectively avoids information loss. Garg and Kumar [18] constructed a new exponential distance based on different connection numbers.

(3) TOPSIS method. TOPSIS is a widely employed MADM model initially introduced by Hwang and Yoon [28]. The fundamental principle of TOPSIS is to identify the ideal scheme(s) with the shortest distance to the positive ideal solution (PIS) and the longest distance to the negative ideal solution (NIS) [29]. In recent years, researchers have extended and applied the TOPSIS method for suitability with IvIFSs in various fields [30–32], such as signal processing [33], supplier selection [31, 34] and emergency rescue [35].

For instance, Qiao et al. [36] presented a TOPSIS method for IvIFSs which considers the preference information of schemes. Using the weighted TOPSIS method, Huang and Zhang [37] conducted a teaching effectiveness evaluation for higher education. Zhao [38] introduced an advanced TOPSIS method based on the conventional one to calculate the distances between schemes and the PIS/NIS; they were able to determine an ideal teaching quality. AI-Shamiri et al. [39] integrated TOPSIS and ELECTRE-I within the framework of cubic m-polar FSs to diagnose psychiatric disorders.

In light of the literature analysis and discussion provided, the motivations of this research can be summarized as follows:

(1) The evaluation process for teaching satisfaction needs to take multiple criteria into account from different perspectives, which often leads to internal ambiguity and inconsistency. Additionally, decision-makers may struggle to provide crisp values due to the inherent vagueness and uncertainty

in cognition. Because the IvIFS can effectively handle fuzziness and uncertainty by considering both membership and non-membership degrees, it can reduce the vagueness in decision-making. Hence, we utilize IvIFSs to express decision-makers' evaluation opinions.

(2) Although the research about distance measures has significantly advanced, as far as we know, the triangular divergence fails to be explored under the conditions of the interval-valued intuitionistic fuzzy (IvIF) environment. Besides, some existing measures cannot be adopted to distinguish subtle differences in data, while others involve complex and tedious calculation processes. Therefore, this research serves to introduce a novel IvIF triangular distance to enrich the information measure theory and yeild a new TOPSIS method based on it.

Based on these motivations, this research has several key contributions. First, the proposed IvIF triangular distance enhances the information measure theory by providing a new perspective. Second, an improved TOPSIS method is established by utilizing the novel triangular distance within the IvIF environment. This method enables a more accurate and reliable evaluation of teaching satisfaction. Lastly, a comprehensive framework for teaching satisfaction evaluation, as based on the novel distance measure and TOPSIS method for IvIFSs, is developed to provide decision support for education managers. This framework serves as a driving force for teachers to enhance teaching quality, and for students to improve learning efficiency.

Regarding the IvIFS environment, this study was designed to yield a new TOPSIS method through the introduction a novel distance measure for evaluating the teaching satisfaction of a college mathematics course. The structure of this paper is organized as follows. Section 2 provides an overview of the elementary concepts related to IvIFSs, including interval-valued intuitionistic fuzzy numbers (IvIFNs) and their relationships, as well as the entropy weight method. Section 3 proposes a new distance measure, and it is proved that the new distance measure satisfies the requirements for the related axiomatic properties. Additionally, several examples are illustrated to examine the superiority of the proposed distance measure in Section 4. Building upon the new distance measure, Section 5 presents an improved TOPSIS method. In Section 6, a novel decision-making approach specifically tailored to teaching quality evaluation is established, and a numerical example investigating the teaching satisfaction of mathematics courses is presented to illustrate the practical application of the proposed methodology. Furthermore, comparisons and counter intuitive examples are discussed to prove the rationality and superiority of the proposed TOPSIS method in Section 7. Finally, Section 8 concludes the study by summarizing the main findings and contributions of this research.

2. Preliminaries

In this section, related basic concepts, including the IvIFS and its properties, as well as the distance measures, are recalled briefly.

2.1. IvIFS

Definition 2.1. *[12] Suppose that Γ is a nonempty set ; an IvIFS $\widetilde{H}$ over Γ could be defined as below.*

$$\widetilde{H} = \left\{ \left\langle \tau, \left[u_{\widetilde{H}}^{L}(\tau), u_{\widetilde{H}}^{R}(\tau) \right], \left[v_{\widetilde{H}}^{L}(\tau), v_{\widetilde{H}}^{R}(\tau) \right] \right\rangle | \tau \in \Gamma \right\}, \tag{2.1}$$

where $\left[u_{\widetilde{H}}^{L}(\tau), u_{\widetilde{H}}^{R}(\tau) \right] \subseteq [0,1]$, $\left[v_{\widetilde{H}}^{L}(\tau), v_{\widetilde{H}}^{R}(\tau) \right] \subseteq [0,1]$ are the interval membership degree and non-membership degree of the element τ to $\widetilde{H}$, respectively. Besides, the formula satisfies the condition that $u_{\widetilde{H}}^{R}(\tau) + v_{\widetilde{H}}^{R}(\tau) \leq 1$ for any $\tau \in \Gamma$. The interval intuitionistic index or hesitancy degree is $\pi_{\widetilde{H}}(\tau) = \left[\pi_{\widetilde{H}}^{L}(\tau), \pi_{\widetilde{H}}^{R}(\tau) \right] = \left[1 - u_{\widetilde{H}}^{R}(\tau) - v_{\widetilde{H}}^{R}(\tau), 1 - u_{\widetilde{H}}^{L}(\tau) - v_{\widetilde{H}}^{L}(\tau) \right]$, and $\pi_{\widetilde{H}}(\tau) \subseteq [0,1]$.

For convenience, we denote the IvIFN as $\left(\left[u^L, u^R\right], \left[v^L, v^R\right]\right)$ [13].

Especially, when $u^L_{\widetilde{H}}(\tau) = u^R_{\widetilde{H}}(\tau)$ and $v^L_{\widetilde{H}}(\tau) = v^R_{\widetilde{H}}(\tau)$, an IvIFS is reduced to an IFS and an IvIFN is reduced to an intuitionistic fuzzy number.

Definition 2.2. *[40] Suppose that* $\widetilde{H_k} = \left\langle \tau, \left[u^L_{\widetilde{H_k}}(\tau), u^R_{\widetilde{H_k}}(\tau)\right], \left[v^L_{\widetilde{H_k}}(\tau), v^R_{\widetilde{H_k}}(\tau)\right]\right\rangle$ $(k = 1, 2)$ *are any two IvIFSs in* Γ*, we have:*

(1) $\widetilde{H}_1 = \widetilde{H}_2$ *iff* $u^L_{\widetilde{H_1}}(\tau) = u^L_{\widetilde{H_2}}(\tau)$, $u^R_{\widetilde{H_1}}(\tau) = u^R_{\widetilde{H_2}}(\tau)$, $v^L_{\widetilde{H_1}}(\tau) = v^L_{\widetilde{H_2}}(\tau)$ *and* $v^R_{\widetilde{H_1}}(\tau) = v^R_{\widetilde{H_2}}(\tau)$ *for* $\forall \tau \in \Gamma$*;*

(2) $\widetilde{H}_1 \subseteq \widetilde{H}_2$ *iff* $u^L_{\widetilde{H_1}}(\tau) \le u^L_{\widetilde{H_2}}(\tau)$, $u^R_{\widetilde{H_1}}(\tau) \le u^R_{\widetilde{H_2}}(\tau)$, $v^L_{\widetilde{H_1}}(\tau) \ge v^L_{\widetilde{H_2}}(\tau)$ *and* $v^R_{\widetilde{H_1}}(\tau) \ge v^R_{\widetilde{H_2}}(\tau)$ *for* $\forall \tau \in \Gamma$*;*

(3) $\widetilde{H}_1 \cup \widetilde{H}_2 = \left\{\left\langle \tau, \left[\max\left(u^L_{\widetilde{H_k}}(\tau)\right), max\left(u^R_{\widetilde{H_k}}(\tau)\right)\right], \left[\min\left(v^L_{\widetilde{H_k}}(\tau)\right), min\left(v^R_{\widetilde{H_k}}(\tau)\right)\right] | \tau \in \Gamma\right\rangle\right\}$*;*

(4) $\widetilde{H}_1 \cap \widetilde{H}_2 = \left\{\left\langle \tau, \left[\min\left(u^L_{\widetilde{H_k}}(\tau)\right), min\left(u^R_{\widetilde{H_k}}(\tau)\right)\right], \left[\max\left(v^L_{\widetilde{H_k}}(\tau)\right), max\left(v^R_{\widetilde{H_k}}(\tau)\right)\right] | \tau \in \Gamma\right\rangle\right\}$*;*

(5) $\widetilde{H}^C_1 = \left\{\left\langle \tau, \left[v^L_{\widetilde{H_1}}(\tau), v^R_{\widetilde{H_1}}(\tau)\right], \left[u^L_{\widetilde{H_1}}(\tau), u^R_{\widetilde{H_1}}(\tau)\right]\right\rangle | \tau \in \Gamma\right\}$.

Definition 2.3. *[13] Assume that* $\widetilde{\beta_k} = \left(\left[u^L_{\widetilde{\beta_k}}, u^R_{\widetilde{\beta_k}}\right], \left[v^L_{\widetilde{\beta_k}}, v^R_{\widetilde{\beta_k}}\right]\right)$ *(*$k = 1, 2$*) represents any two IvIFNs; then, one has the following operational laws.*

(1) $\widetilde{\beta_1} \oplus \widetilde{\beta_2} = \left(\left[u^L_{\widetilde{\beta_1}} + u^L_{\widetilde{\beta_2}} - u^L_{\widetilde{\beta_1}} u^L_{\widetilde{\beta_2}}, u^R_{\widetilde{\beta_1}} + u^R_{\widetilde{\beta_2}} - u^R_{\widetilde{\beta_1}} u^R_{\widetilde{\beta_2}}\right], \left[v^L_{\widetilde{\beta_1}} v^L_{\widetilde{\beta_2}}, v^R_{\widetilde{\beta_1}} v^R_{\widetilde{\beta_2}}\right]\right)$*;*

(2) $\widetilde{\beta_1} \otimes \widetilde{\beta_1} = \left(\left[u^L_{\widetilde{\beta_1}} u^L_{\widetilde{\beta_2}}, u^R_{\widetilde{\beta_1}} u^R_{\widetilde{\beta_2}}\right], \left[v^L_{\widetilde{\beta_1}} + v^L_{\widetilde{\beta_2}} - v^L_{\widetilde{\beta_1}} v^L_{\widetilde{\beta_2}}, v^R_{\widetilde{\beta_1}} + v^R_{\widetilde{\beta_2}} - v^R_{\widetilde{\beta_1}} v^R_{\widetilde{\beta_2}}\right]\right)$*;*

(3) $\gamma\widetilde{\beta} = \left(\left[1 - \left(1 - u^L_{\widetilde{\beta}}\right)^\gamma, 1 - \left(1 - u^R_{\widetilde{\beta}}\right)^\gamma\right], \left[\left(v^L_{\widetilde{\beta}}\right)^\gamma, \left(v^R_{\widetilde{\beta}}\right)^\gamma\right]\right), \gamma > 0$*;*

(4) $(\widetilde{\beta})^\gamma = \left(\left[\left(u^L_{\widetilde{\beta}}\right)^\gamma, \left(u^R_{\widetilde{\beta}}\right)^\gamma\right], \left[1 - \left(1 - v^L_{\widetilde{\beta}}\right)^\gamma, 1 - \left(1 - v^R_{\widetilde{\beta}}\right)^\gamma\right]\right) \gamma > 0.$

Definition 2.4. *[13] Suppose that* $\widetilde{\beta}$ *refers to an IvIFN ; the score function and accuracy function of* $\widetilde{\beta}$ *will be given as below, respectively.*

$$f_s\left(\widetilde{\beta}\right) = \frac{u^L_{\widetilde{\beta}} + u^R_{\widetilde{\beta}} - v^L_{\widetilde{\beta}} - v^R_{\widetilde{\beta}}}{2}, -1 \le f_s\left(\widetilde{\beta}\right) \le 1. \tag{2.2}$$

$$f_a\left(\widetilde{\beta}\right) = \frac{u^L_{\widetilde{\beta}} + u^R_{\widetilde{\beta}} + v^L_{\widetilde{\beta}} + v^R_{\widetilde{\beta}}}{2}, 0 \le f_a\left(\widetilde{\beta}\right) \le 1. \tag{2.3}$$

For any two IvIFNs $\widetilde{\beta_1}$ and $\widetilde{\beta_2}$, the order relationship could be defined as follows.

(1) $\widetilde{\beta_1} > \widetilde{\beta_2}$ when $f_s\left(\widetilde{\beta_1}\right) > f_s\left(\widetilde{\beta_2}\right)$;

(2) If $f_s\left(\widetilde{\beta_1}\right) = f_s\left(\widetilde{\beta_2}\right)$, the following holds:

a. $\widetilde{\beta_1} > \widetilde{\beta_2}$ when $f_a\left(\widetilde{\beta_1}\right) > f_a\left(\widetilde{\beta_2}\right)$;

b. $\widetilde{\beta_1} = \widetilde{\beta_2}$ when $f_a\left(\widetilde{\beta_1}\right) = f_a\left(\widetilde{\beta_2}\right)$.

2.2. Entropy, aggregation operator and distances of IvIFSs

Definition 2.5. *[19] Set* Γ *as a nonempty set;* $\widetilde{H_k}$ *is a separate element in an IvIFS* $\widetilde{H}$ *in* Γ*. Then, the entropy of* $\widetilde{H_k}$ *can be given in the following form:*

$$E\left(\widetilde{H_k}\right) = 1 - \frac{1}{2n} \sum_{k=1}^{n} \left(|u^L_{\widetilde{H_k}}(\tau) - v^L_{\widetilde{H_k}}(\tau)| + |u^R_{\widetilde{H_k}}(\tau) - v^R_{\widetilde{H_k}}(\tau)|\right); \tag{2.4}$$

the weight of element $\widetilde{H_k}$ is defined as in Eq (2.5):

$$w\left(\widetilde{H_k}\right) = \frac{1 - E\left(\widetilde{H_k}\right)}{\sum_{k=1}^{n}\left(1 - E\left(\widetilde{H_k}\right)\right)}. \tag{2.5}$$

Definition 2.6. *[13] For a group of IvIFNs $\widetilde{\beta_k}$ ($k = 1, 2, \cdots, n$), the IvIF weighted arithmetic average operator (IvIFWAA) will be given by Eq (2.6).*

$$\text{IvIFWAA}\left(\widetilde{\beta_1}, \widetilde{\beta_2}, \cdots, \widetilde{\beta_n}\right) = \sum_{k=1}^{n} w_k \widetilde{\beta_k} = \left(\left[1 - \prod_{k=1}^{n}\left(1 - u^L_{\widetilde{\beta_k}}\right)^{w_k}, 1 - \prod_{k=1}^{n}\left(1 - u^R_{\widetilde{\beta_k}}\right)^{w_k}\right], \left[\prod_{k=1}^{n}\left(v^L_{\widetilde{\beta_k}}\right)^{w_k}, \prod_{k=1}^{n}\left(v^R_{\widetilde{\beta_k}}\right)^{w_k}\right]\right). \tag{2.6}$$

Here, w_k is the weight of $\widetilde{\beta_k}$, satisfying $0 \le w_k \le 1$, and $\sum_{k=1}^{n} w_k = 1$.

Definition 2.7. *[19] A mapping $d : IvIFS(\Gamma) \times IvIFS(\Gamma) \to [0, 1]$ denotes the distance measure between the IvIFSs $\widetilde{H_i}$ and $\widetilde{H_j}$ if the following conditions are satisfied:*

(1) $d\left(\widetilde{H_i}, \widetilde{H_j}\right) = 0 \Leftrightarrow \widetilde{H_i} = \widetilde{H_j}$;

(2) $d\left(\widetilde{H_i}, \widetilde{H_j}\right) = d\left(\widetilde{H_j}, \widetilde{H_i}\right)$;

(3) $0 \le d\left(\widetilde{H_i}, \widetilde{H_j}\right) \le 1$;

(4) If $\widetilde{H_i} \le \widetilde{H_j} \le \widetilde{H_k}$, then $d\left(\widetilde{H_i}, \widetilde{H_j}\right) \le d\left(\widetilde{H_i}, \widetilde{H_k}\right)$, and $d\left(\widetilde{H_j}, \widetilde{H_k}\right) \le d\left(\widetilde{H_i}, \widetilde{H_k}\right)$.

Suppose that $\widetilde{H_k} = \left\langle \tau, \left[u^L_{\widetilde{H_k}}(\tau), u^R_{\widetilde{H_k}}(\tau)\right], \left[v^L_{\widetilde{H_k}}(\tau), v^R_{\widetilde{H_k}}(\tau)\right]\right\rangle$ ($k = 1, 2$) presents any two IvIFSs in $\Gamma = \{\tau_1, \tau_2, \cdots, \tau_m\}$; some existing distance measures between $\widetilde{H_1}$ and $\widetilde{H_2}$ are stated as below, which will be used in a later discussion.

(1) Hamming distance [22]

$$d_H\left(\widetilde{H_1}, \widetilde{H_2}\right) = \frac{1}{4m}\left[\sum_{i=1}^{m}\left|u^L_{\widetilde{H_1}}(\tau_i) - u^L_{\widetilde{H_2}}(\tau_i)\right| + \left|u^R_{\widetilde{H_1}}(\tau_i) - u^R_{\widetilde{H_2}}(\tau_i)\right| + \left|v^L_{\widetilde{H_1}}(\tau_i) - v^L_{\widetilde{H_2}}(\tau_i)\right| + \left|v^R_{\widetilde{H_1}}(\tau_i) - u^R_{\widetilde{H_2}}(\tau_i)\right|\right]. \tag{2.7}$$

(2) Euclidean distance [22]

$$d_E\left(\widetilde{H_1}, \widetilde{H_2}\right) = \sqrt{\frac{1}{4m}\sum_{i=1}^{m}\left[\left(u^L_{\widetilde{H_1}}(\tau_i) - u^L_{\widetilde{H_2}}(\tau_i)\right)^2 + \left(u^R_{\widetilde{H_1}}(\tau_i) - u^R_{\widetilde{H_2}}(\tau_i)\right)^2 + \left(v^L_{\widetilde{H_1}}(\tau_i) - v^L_{\widetilde{H_2}}(\tau_i)\right)^2 + \left(v^R_{\widetilde{H_1}}(\tau_i) - u^R_{\widetilde{H_2}}(\tau_i)\right)^2\right]}. \tag{2.8}$$

(3) Hausdorff–Hamming distance [26]

$$d_{HH}\left(\widetilde{H_1}, \widetilde{H_2}\right) = \frac{1}{2m}\sum_{i=1}^{m}\left[\left|u^L_{\widetilde{H_1}}(\tau_i) - u^L_{\widetilde{H_2}}(\tau_i)\right| \vee \left|u^R_{\widetilde{H_1}}(\tau_i) - u^R_{\widetilde{H_2}}(\tau_i)\right| + \left|v^L_{\widetilde{H_1}}(\tau_i) - v^L_{\widetilde{H_2}}(\tau_i)\right| \vee \left|v^R_{\widetilde{H_1}}(\tau_i) - u^R_{\widetilde{H_2}}(\tau_i)\right|\right]. \tag{2.9}$$

(4) Hausdorff–Euclidean distance [26]

$$d_{HE}\left(\widetilde{H_1}, \widetilde{H_2}\right) = \sqrt{\frac{1}{2m}\sum_{i=1}^{m}\left[\left(\left|u^L_{\widetilde{H_1}}(\tau_i) - u^L_{\widetilde{H_2}}(\tau_i)\right| \vee \left|u^R_{\widetilde{H_1}}(\tau_i) - u^R_{\widetilde{H_2}}(\tau_i)\right|\right)^2 + \left(\left|v^L_{\widetilde{H_1}}(\tau_i) - v^L_{\widetilde{H_2}}(\tau_i)\right| \vee \left|v^R_{\widetilde{H_1}}(\tau_i) - u^R_{\widetilde{H_2}}(\tau_i)\right|\right)^2\right]}. \tag{2.10}$$

(5) Muharrem [27] created a novel distance measure for IvIFSs

$$d_p^t\left(\widetilde{H_1},\widetilde{H_2}\right)=\sqrt[p]{\begin{array}{l}\frac{1}{4m(t+1)^p}\sum_{i=1}^{m}\left\{\left|t\left(u_{\widetilde{H_1}}^L(\tau_i)-u_{\widetilde{H_2}}^L(\tau_i)\right)-\left(v_{\widetilde{H_1}}^L(\tau_i)-v_{\widetilde{H_2}}^L(\tau_i)\right)\right|^p+\left|t\left(v_{\widetilde{H_1}}^L(\tau_i)-v_{\widetilde{H_2}}^L(\tau_i)\right)-\left(u_{\widetilde{H_1}}^L(\tau_i)-u_{\widetilde{H_2}}^L(\tau_i)\right)\right|^p\right.\\ \left.+\left|t\left(u_{\widetilde{H_1}}^R(\tau_i)-u_{\widetilde{H_2}}^R(\tau_i)\right)-\left(v_{\widetilde{H_1}}^R(\tau_i)-v_{\widetilde{H_2}}^R(\tau_i)\right)\right|^p+\left|t\left(v_{\widetilde{H_1}}^R(\tau_i)-v_{\widetilde{H_2}}^R(\tau_i)\right)-\left(u_{\widetilde{H_1}}^R(\tau_i)-u_{\widetilde{H_2}}^R(\tau_i)\right)\right|^p\right\},\end{array}} \tag{2.11}$$

where $t=2,3,4,\cdots$, the parameter p represents the L_p norm and t is used to identify the uncertainty level. For the calculations, $t=2$ and $p=1$ are used in this study.

(6) Liu and Jiang [17] established a new distance measure for IvIFSs:

$$d_L\left(\widetilde{H_1},\widetilde{H_2}\right)=\sqrt{\frac{1}{2}\left(D_u^2+D_v^2+D_\pi^2+D_\pi D_u+D_\pi D_v\right)} \tag{2.12}$$

where an IvIFN $\left\langle\left[u^L,u^R\right],\left[v^L,v^R\right]\right\rangle$ is converted into an interval vector $\left(\left[u^L,u^R\right],\left[v^L,v^R\right],\left[\pi^L,\pi^R\right]\right)^T$, and

$$D_u^2\left(\widetilde{H_1},\widetilde{H_2}\right)=\left(\frac{u_{\widetilde{H_1}}^L(\tau_i)+u_{\widetilde{H_1}}^R(\tau_i)}{2}-\frac{u_{\widetilde{H_2}}^L(\tau_i)+u_{\widetilde{H_2}}^R(\tau_i)}{2}\right)^2+\frac{1}{3}\left(\frac{u_{\widetilde{H_1}}^R(\tau_i)-u_{\widetilde{H_1}}^L(\tau_i)}{2}-\frac{u_{\widetilde{H_2}}^R(\tau_i)-u_{\widetilde{H_2}}^L(\tau_i)}{2}\right)^2,$$

$$D_v^2\left(\widetilde{H_1},\widetilde{H_2}\right)=\left(\frac{v_{\widetilde{H_1}}^L(\tau_i)+v_{\widetilde{H_1}}^R(\tau_i)}{2}-\frac{v_{\widetilde{H_2}}^L(\tau_i)+v_{\widetilde{H_2}}^R(\tau_i)}{2}\right)^2+\frac{1}{3}\left(\frac{v_{\widetilde{H_1}}^R(\tau_i)-v_{\widetilde{H_1}}^L(\tau_i)}{2}-\frac{v_{\widetilde{H_2}}^R(\tau_i)-v_{\widetilde{H_2}}^L(\tau_i)}{2}\right)^2,$$

$$D_\pi^2\left(\widetilde{H_1},\widetilde{H_2}\right)=\left(\frac{\pi_{\widetilde{H_1}}^L(\tau_i)+\pi_{\widetilde{H_1}}^R(\tau_i)}{2}-\frac{\pi_{\widetilde{H_2}}^L(\tau_i)+\pi_{\widetilde{H_2}}^R(\tau_i)}{2}\right)^2+\frac{1}{3}\left(\frac{\pi_{\widetilde{H_1}}^R(\tau_i)-\pi_{\widetilde{H_1}}^L(\tau_i)}{2}-\frac{\pi_{\widetilde{H_2}}^R(\tau_i)-\pi_{\widetilde{H_2}}^L(\tau_i)}{2}\right)^2,$$

and

$$D_u=\begin{cases}\sqrt{D_u^2} & \text{when } u_{\widetilde{H_1}}^L(\tau_i)+u_{\widetilde{H_1}}^R(\tau_i)\geq u_{\widetilde{H_2}}^L(\tau_i)+u_{\widetilde{H_2}}^R(\tau_i);\\ -\sqrt{D_u^2} & \text{when } u_{\widetilde{H_1}}^L(\tau_i)+u_{\widetilde{H_1}}^R(\tau_i)< u_{\widetilde{H_2}}^L(\tau_i)+u_{\widetilde{H_2}}^R(\tau_i).\end{cases}$$

$$D_v=\begin{cases}\sqrt{D_v^2} & \text{when } v_{\widetilde{H_1}}^L(\tau_i)+v_{\widetilde{H_1}}^R(\tau_i)\geq v_{\widetilde{H_2}}^L(\tau_i)+v_{\widetilde{H_2}}^R(\tau_i);\\ -\sqrt{D_v^2} & \text{when } v_{\widetilde{H_1}}^L(\tau_i)+v_{\widetilde{H_1}}^R(\tau_i)< v_{\widetilde{H_2}}^L(\tau_i)+v_{\widetilde{H_2}}^R(\tau_i).\end{cases}$$

$$D_\pi=\begin{cases}\sqrt{D_\pi^2} & \text{when } \pi_{\widetilde{H_1}}^L(\tau_i)+\pi_{\widetilde{H_1}}^R(\tau_i)\geq \pi_{\widetilde{H_2}}^L(\tau_i)+\pi_{\widetilde{H_2}}^R(\tau_i);\\ -\sqrt{D_\pi^2} & \text{when } \pi_{\widetilde{H_1}}^L(\tau_i)+\pi_{\widetilde{H_1}}^R(\tau_i)< \pi_{\widetilde{H_2}}^L(\tau_i)+\pi_{\widetilde{H_2}}^R(\tau_i).\end{cases}$$

(7) Garg and Kumar [18] defined a new exponential distance through the use of a connection set $\widetilde{H}=\{(\tau_i, r_{\widetilde{H}}(\tau_i)+s_{\widetilde{H}}(\tau_i)\,i+t_{\widetilde{H}}(\tau_i)\,j)\}$, as follows:

If $f_s\left(\widetilde{H_1}\right)\neq f_s\left(\widetilde{H_2}\right)$ and $f_a\left(\widetilde{H_1}\right)\neq f_a\left(\widetilde{H_2}\right)$, then

$$r_{\widetilde{H}}(\tau_i)=\frac{\left(u_{\widetilde{H}}^L(\tau_i)+u_{\widetilde{H}}^R(\tau_i)\right)\left(2-v_{\widetilde{H}}^L(\tau_i)-v_{\widetilde{H}}^R(\tau_i)\right)}{4};$$

$$s_{\widetilde{H}}(\tau_i)=\frac{1+\left(1-u_{\widetilde{H}}^L(\tau_i)-u_{\widetilde{H}}^R(\tau_i)\right)\left(1-v_{\widetilde{H}}^L(\tau_i)-v_{\widetilde{H}}^R(\tau_i)\right)}{2};$$

$$t_{\widetilde{H}}(\tau_i)=\frac{\left(v_{\widetilde{H}}^L(\tau_i)+v_{\widetilde{H}}^R(\tau_i)\right)\left(2-u_{\widetilde{H}}^L(\tau_i)-u_{\widetilde{H}}^R(\tau_i)\right)}{4}.$$

If $f_s\left(\widetilde{H_1}\right)=f_s\left(\widetilde{H_2}\right)$ (either $f_a\left(\widetilde{H_1}\right)\neq f_a\left(\widetilde{H_2}\right)$ or $f_a\left(\widetilde{H_1}\right)=f_a\left(\widetilde{H_2}\right)$) , one has

$$r_{\widetilde{H}}(\tau_i)=\frac{\left(u_{\widetilde{H}}^L(\tau_i)\left(1-u_{\widetilde{H}}^R(\tau_i)-v_{\widetilde{H}}^R(\tau_i)\right)+u_{\widetilde{H}}^R(\tau_i)\left(1-u_{\widetilde{H}}^L(\tau_i)-v_{\widetilde{H}}^L(\tau_i)\right)\right)\left(2-v_{\widetilde{H}}^L(\tau_i)-v_{\widetilde{H}}^R(\tau_i)\right)}{4};$$

$s_{\widetilde{H}}(\tau_i) = 1 - r_{\widetilde{H}}(\tau_i) - t_{\widetilde{H}}(\tau_i)$;

$t_{\widetilde{H}}(\tau_i) = \frac{\left(v^L_{\widetilde{H}}(\tau_i)\left(1-u^R_{\widetilde{H}}(\tau_i)-v^R_{\widetilde{H}}(\tau_i)\right)+v^R_{\widetilde{H}}(\tau_i)\left(1-u^L_{\widetilde{H}}(\tau_i)-v^L_{\widetilde{H}}(\tau_i)\right)\right)\left(2-u^L_{\widetilde{H}}(\tau_i)-u^R_{\widetilde{H}}(\tau_i)\right)}{4}$.

Thus, a new normalized exponential Hamming distance is given by Eq (2.13):

$$d^H_{\exp}\left(\widetilde{H_1}, \widetilde{H_1}\right) = \left\{1 - \exp\left[-\frac{1}{3}\sum_{i=1}^{n}\left(\left|\sqrt{r_{\widetilde{H_1}}(\tau_i)} - \sqrt{r_{\widetilde{H_2}}(\tau_i)}\right| + \left|\sqrt{s_{\widetilde{H_1}}(\tau_i)} - \sqrt{s_{\widetilde{H_2}}(\tau_i)}\right| + \left|\sqrt{t_{\widetilde{H_1}}(\tau_i)} - \sqrt{t_{\widetilde{H_2}}(\tau_i)}\right|\right)\right]\right\} \Big/ (1 - \exp(-n)), \tag{2.13}$$

and a new normalized exponential Euclidean distance is given by Eq (2.14):

$$d^E_{\exp}\left(\widetilde{H_1}, \widetilde{H_1}\right) = \left\{1 - \exp\left[-\left(\frac{1}{3}\sum_{i=1}^{n}\left(\left|\sqrt{r_{\widetilde{H_1}}(\tau_i)} - \sqrt{r_{\widetilde{H_2}}(\tau_i)}\right|^2 + \left|\sqrt{s_{\widetilde{H_1}}(\tau_i)} - \sqrt{s_{\widetilde{H_2}}(\tau_i)}\right|^2 + \left|\sqrt{t_{\widetilde{H_1}}(\tau_i)} - \sqrt{t_{\widetilde{H_2}}(\tau_i)}\right|^2\right)\right)^{1/2}\right]\right\} \Big/ \left(1 - \exp\left(-\sqrt{n}\right)\right). \tag{2.14}$$

3. A novel distance measure for IvIFSs

The distance measure plays a crucial role in distinguishing the differences among alternatives, making it a critical component in decision-making processes such as the TOPSIS method [17]. The distance measure with a higher distinguishing ability will lead to a better decision-making method, thus providing decision-makers with a definite choice. However, some existing distance measures have been established without explicit physical meaning, while others involve complex calculations. In some cases, these measures fail to adequately distinguish decision-making information, and the calculated results may even contradict theoretical requirements or intuitive feelings.

Triangular divergence, a classical measure widely applied in probability distributions, has successfully handled counter intuitive problems better than other existing distance methods [41–43]. Therefore, based upon the concept of triangular divergence and the approaches described in [43], we have created a new distance measure in the IvIFS environment. By utilizing triangular divergence, the proposed distance measure aims to overcome the limitations of existing measures and provide a more effective tool for distinguishing differences between IvIFSs.

In the following sections, we will discuss the concept of triangular divergence and its application in the new distance measure for IvIFSs.

3.1. Triangular divergence

Definition 3.1. *[42] Set* $\Psi_n = \left\{P = (p_1, p_2, \ldots, p_n) \,|\, p_i > 0, \sum_{i=1}^{n} p_i = 1\right\}, n \geq 2$ *as a set of finite discrete probability distributions. For* $\forall P, Q \in \Psi_n$*, the classical triangular divergence measure between* P *and* Q *is defined as*

$$\Delta(P, Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}. \tag{3.1}$$

The bigger the triangular divergence value, the greater the difference between the probability distributions P and Q.

With Eq (3.1), the square root of the triangular divergence could be described as follows:

$$d(P, Q) = \sqrt{\sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}}$$

where, by convention, $0/0 = 0$.

3.2. New distance measure for IvIFSs based on the triangular divergence

Definition 3.2. *Suppose that* $\widetilde{H}_k = \left\langle \tau_j, \left[u^L_{\widetilde{H}_k}(\tau), u^R_{\widetilde{H}_k}(\tau)\right], \left[v^L_{\widetilde{H}_k}(\tau), v^R_{\widetilde{H}_k}(\tau)\right]\right\rangle$ $(k = 1, 2)$ *represents any two IvIFSs in* $\Gamma = \{\tau_1, \tau_2, \cdots, \tau_m\}$*; then, the distance between* $\widetilde{H}_1$ *and* $\widetilde{H}_2$ *could be determine by using the following formula.*

$$d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right) = \sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left[\frac{\left(u^L_{\widetilde{H}_1}(\tau_j) - u^L_{\widetilde{H}_2}(\tau_j)\right)^2}{u^L_{\widetilde{H}_1}(\tau_j) + u^L_{\widetilde{H}_2}(\tau_j)} + \frac{\left(u^R_{\widetilde{H}_1}(\tau_j) - u^R_{\widetilde{H}_2}(\tau_j)\right)^2}{u^R_{\widetilde{H}_1}(\tau_j) + u^R_{\widetilde{H}_2}(\tau_j)} + \frac{\left(v^L_{\widetilde{H}_1}(\tau_j) - v^L_{\widetilde{H}_2}(\tau_j)\right)^2}{v^L_{\widetilde{H}_1}(\tau_j) + v^L_{\widetilde{H}_2}(\tau_j)} + \frac{\left(v^R_{\widetilde{H}_1}(\tau_j) - v^R_{\widetilde{H}_2}(\tau_j)\right)^2}{v^R_{\widetilde{H}_1}(\tau_j) + v^R_{\widetilde{H}_2}(\tau_j)}\right]}. \tag{3.2}$$

We denote d_{Iv} as an interval-valued intuitionistic distance measure based on triangular divergence (IvIFTD). As stated previously, the bigger the value of d_{Iv}, the greater the difference between the IvIFSs.

Theorem 3.1. *Set* $d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right)$ *in Eq (3.2) as the distance measure between two IvIFSs; then, the following properties are satisfied:*

(1) $d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right) = 0 \Leftrightarrow \widetilde{H}_1 = \widetilde{H}_2$*;*

(2) $d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right) = d_{Iv}\left(\widetilde{H}_2, \widetilde{H}_1\right)$*;*

(3) $0 \le d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right) \le 1$*;*

(4) If $\widetilde{H}_1 \le \widetilde{H}_2 \le \widetilde{H}_3$*, then one has* $d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right) \le d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_3\right)$ *and* $d_{Iv}\left(\widetilde{H}_2, \widetilde{H}_2\right) \le d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_3\right)$.

Proof. (1) $d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right) = 0 \Leftrightarrow \widetilde{H}_1 = \widetilde{H}_2$.

Necessity:

For any $\tau_j \in \Gamma$, if $d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right) = 0$, one has

$$d_{Iv}\left(\widetilde{H}_1, \widetilde{H}_2\right) = \sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left[\frac{\left(u^L_{\widetilde{H}_1}(\tau_j) - u^L_{\widetilde{H}_2}(\tau_j)\right)^2}{u^L_{\widetilde{H}_1}(\tau_j) + u^L_{\widetilde{H}_2}(\tau_j)} + \frac{\left(u^R_{\widetilde{H}_1}(\tau_j) - u^R_{\widetilde{H}_2}(\tau_j)\right)^2}{u^R_{\widetilde{H}_1}(\tau_j) + u^R_{\widetilde{H}_2}(\tau_j)} + \frac{\left(v^L_{\widetilde{H}_1}(\tau_j) - v^L_{\widetilde{H}_2}(\tau_j)\right)^2}{v^L_{\widetilde{H}_1}(\tau_j) + v^L_{\widetilde{H}_2}(\tau_j)} + \frac{\left(v^R_{\widetilde{H}_1}(\tau_j) - v^R_{\widetilde{H}_2}(\tau_j)\right)^2}{v^R_{\widetilde{H}_1}(\tau_j) + v^R_{\widetilde{H}_2}(\tau_j)}\right]} = 0.$$

Then we have

$$\frac{\left(u^L_{\widetilde{H}_1}(\tau_j) - u^L_{\widetilde{H}_2}(\tau_j)\right)^2}{u^L_{\widetilde{H}_1}(\tau_j) + u^L_{\widetilde{H}_2}(\tau_j)} = \frac{\left(u^R_{\widetilde{H}_1}(\tau_j) - u^R_{\widetilde{H}_2}(\tau_j)\right)^2}{u^R_{\widetilde{H}_1}(\tau_j) + u^R_{\widetilde{H}_2}(\tau_j)} = \frac{\left(v^L_{\widetilde{H}_1}(\tau_j) - v^L_{\widetilde{H}_2}(\tau_j)\right)^2}{v^L_{\widetilde{H}_1}(\tau_j) + v^L_{\widetilde{H}_2}(\tau_j)} = \frac{\left(v^L_{\widetilde{H}_1}(\tau_j) - v^L_{\widetilde{H}_2}(\tau_j)\right)^2}{v^L_{\widetilde{H}_1}(\tau_j) + v^L_{\widetilde{H}_2}(\tau_j)} = 0,$$

that is

$$\left(u^L_{\widetilde{H}_1}(\tau_j) - u^L_{\widetilde{H}_2}(\tau_j)\right)^2 = \left(u^R_{\widetilde{H}_1}(\tau_j) - u^R_{\widetilde{H}_2}(\tau_j)\right)^2 = \left(v^L_{\widetilde{H}_1}(\tau_j) - v^L_{\widetilde{H}_2}(\tau_j)\right)^2 = \left(v^L_{\widetilde{H}_1}(\tau_j) - v^L_{\widetilde{H}_2}(\tau_j)\right)^2 = 0.$$

According to Definition 2.1, one has

$$0 \le u^L_{\widetilde{H}_1}, u^R_{\widetilde{H}_1}, v^L_{\widetilde{H}_1}, v^R_{\widetilde{H}_1}, u^L_{\widetilde{H}_2}, u^R_{\widetilde{H}_2}, v^L_{\widetilde{H}_2}, v^R_{\widetilde{H}_2} \le 1;$$

hence, we have

$$u^L_{\widetilde{H}_1}(\tau_j) = u^L_{\widetilde{H}_2}(\tau_j), u^R_{\widetilde{H}_1}(\tau_j) = u^R_{\widetilde{H}_2}(\tau_j), v^L_{\widetilde{H}_1}(\tau_j) = v^L_{\widetilde{H}_2}(\tau_j), v^L_{\widetilde{H}_1}(\tau_j) = v^L_{\widetilde{H}_2}(\tau_j).$$

Therefore, $\widetilde{H_1} = \widetilde{H_2}$ is deduced.

Sufficiency:

When $\widetilde{H_1} = \widetilde{H_2}$, one has

$$u^L_{\widetilde{H_1}}\left(\tau_j\right) = u^L_{\widetilde{H_2}}\left(\tau_j\right), u^R_{\widetilde{H_1}}\left(\tau_j\right) = u^R_{\widetilde{H_2}}\left(\tau_j\right), v^L_{\widetilde{H_1}}\left(\tau_j\right) = v^L_{\widetilde{H_2}}\left(\tau_j\right), v^L_{\widetilde{H_1}}\left(\tau_j\right) = v^L_{\widetilde{H_2}}\left(\tau_j\right).$$

Then, we can obtain

$$d_{Iv}\left(\widetilde{H_1}, \widetilde{H_2}\right) = \sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left[\frac{\left(u^L_{\widetilde{H_1}}\left(\tau_j\right) - u^L_{\widetilde{H_2}}\left(\tau_j\right)\right)^2}{u^L_{\widetilde{H_1}}\left(\tau_j\right) + u^L_{\widetilde{H_2}}\left(\tau_j\right)} + \frac{\left(u^R_{\widetilde{H_1}}\left(\tau_j\right) - u^R_{\widetilde{H_2}}\left(\tau_j\right)\right)^2}{u^R_{\widetilde{H_1}}\left(\tau_j\right) + u^R_{\widetilde{H_2}}\left(\tau_j\right)} + \frac{\left(v^L_{\widetilde{H_1}}\left(\tau_j\right) - v^L_{\widetilde{H_2}}\left(\tau_j\right)\right)^2}{v^L_{\widetilde{H_1}}\left(\tau_j\right) + v^L_{\widetilde{H_2}}\left(\tau_j\right)} + \frac{\left(v^R_{\widetilde{H_1}}\left(\tau_j\right) - v^R_{\widetilde{H_2}}\left(\tau_j\right)\right)^2}{v^R_{\widetilde{H_1}}\left(\tau_j\right) + v^R_{\widetilde{H_2}}\left(\tau_j\right)}\right]} = 0.$$

□

Proof. (2) $d_{Iv}\left(\widetilde{H_1}, \widetilde{H_2}\right) = d_{Iv}\left(\widetilde{H_2}, \widetilde{H_1}\right)$.

$$\begin{aligned}
d_{Iv}\left(\widetilde{H_1}, \widetilde{H_2}\right) &= \sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left[\frac{\left(u^L_{\widetilde{H_1}}(\tau_j) - u^L_{\widetilde{H_2}}(\tau_j)\right)^2}{u^L_{\widetilde{H_1}}(\tau_j) + u^L_{\widetilde{H_2}}(\tau_j)} + \frac{\left(u^R_{\widetilde{H_1}}(\tau_j) - u^R_{\widetilde{H_2}}(\tau_j)\right)^2}{u^R_{\widetilde{H_1}}(\tau_j) + u^R_{\widetilde{H_2}}(\tau_j)} + \frac{\left(v^L_{\widetilde{H_1}}(\tau_j) - v^L_{\widetilde{H_2}}(\tau_j)\right)^2}{v^L_{\widetilde{H_1}}(\tau_j) + v^L_{\widetilde{H_2}}(\tau_j)} + \frac{\left(v^R_{\widetilde{H_1}}(\tau_j) - v^R_{\widetilde{H_2}}(\tau_j)\right)^2}{v^R_{\widetilde{H_1}}(\tau_j) + v^R_{\widetilde{H_2}}(\tau_j)}\right]} \\
&= \sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left[\frac{\left(u^L_{\widetilde{H_2}}(\tau_j) - u^L_{\widetilde{H_1}}(\tau_j)\right)^2}{u^L_{\widetilde{H_2}}(\tau_j) + u^L_{\widetilde{H_1}}(\tau_j)} + \frac{\left(u^R_{\widetilde{H_2}}(\tau_j) - u^R_{\widetilde{H_1}}(\tau_j)\right)^2}{u^R_{\widetilde{H_2}}(\tau_j) + u^R_{\widetilde{H_1}}(\tau_j)} + \frac{\left(v^L_{\widetilde{H_2}}(\tau_j) - v^L_{\widetilde{H_1}}(\tau_j)\right)^2}{v^L_{\widetilde{H_2}}(\tau_j) + v^L_{\widetilde{H_1}}(\tau_j)} + \frac{\left(v^R_{\widetilde{H_2}}(\tau_j) - v^R_{\widetilde{H_1}}(\tau_j)\right)^2}{v^R_{\widetilde{H_2}}(\tau_j) + v^R_{\widetilde{H_1}}(\tau_j)}\right]} \\
&= d_{Iv}\left(\widetilde{H_2}, \widetilde{H_1}\right).
\end{aligned}$$

□

Proof. (3) $0 \leq d_{Iv}\left(\widetilde{H_1}, \widetilde{H_2}\right) \leq 1$.

Clearly, $0 \leq d_{Iv}\left(\widetilde{\beta_i}, \widetilde{\beta_j}\right)$ holds.

According to Definition 2.1, one has

$$0 \leq u^L_{\widetilde{H_1}} + v^L_{\widetilde{H_1}} \leq u^R_{\widetilde{H_1}} + v^R_{\widetilde{H_1}} \leq 1, 0 \leq u^L_{\widetilde{H_2}} + v^L_{\widetilde{H_2}} \leq u^R_{\widetilde{H_2}} + v^R_{\widetilde{H_2}} \leq 1. \tag{3.3}$$

So the following inequalities hold:

$$\left(u^L_{\widetilde{H_1}} - u^L_{\widetilde{H_2}}\right)^2 \leq \left(u^L_{\widetilde{H_1}} + u^L_{\widetilde{H_2}}\right)^2, \left(u^R_{\widetilde{H_1}} - u^R_{\widetilde{H_2}}\right)^2 \leq \left(u^R_{\widetilde{H_1}} + u^R_{\widetilde{H_2}}\right)^2, \left(v^L_{\widetilde{H_1}} - v^L_{\widetilde{H_2}}\right)^2 \leq \left(v^L_{\widetilde{H_1}} + v^L_{\widetilde{H_2}}\right)^2, \left(v^R_{\widetilde{H_1}} - v^R_{\widetilde{H_2}}\right)^2 \leq \left(v^R_{\widetilde{H_1}} + v^R_{\widetilde{H_2}}\right)^2;$$

then, one has

$$\begin{aligned}
d_{Iv}\left(\widetilde{H_1}, \widetilde{H_2}\right) &= \sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left[\frac{\left(u^L_{\widetilde{H_1}}(\tau_j) - u^L_{\widetilde{H_2}}(\tau_j)\right)^2}{u^L_{\widetilde{H_1}}(\tau_j) + u^L_{\widetilde{H_2}}(\tau_j)} + \frac{\left(u^R_{\widetilde{H_1}}(\tau_j) - u^R_{\widetilde{H_2}}(\tau_j)\right)^2}{u^R_{\widetilde{H_1}}(\tau_j) + u^R_{\widetilde{H_2}}(\tau_j)} + \frac{\left(v^L_{\widetilde{H_1}}(\tau_j) - v^L_{\widetilde{H_2}}(\tau_j)\right)^2}{v^L_{\widetilde{H_1}}(\tau_j) + v^L_{\widetilde{H_2}}(\tau_j)} + \frac{\left(v^R_{\widetilde{H_1}}(\tau_j) - v^R_{\widetilde{H_2}}(\tau_j)\right)^2}{v^R_{\widetilde{H_1}}(\tau_j) + v^R_{\widetilde{H_2}}(\tau_j)}\right]} \\
&\leq \sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left[\frac{\left(u^L_{\widetilde{H_1}}(\tau_j) + u^L_{\widetilde{H_2}}(\tau_j)\right)^2}{u^L_{\widetilde{H_1}}(\tau_j) + u^L_{\widetilde{H_2}}(\tau_j)} + \frac{\left(u^R_{\widetilde{H_1}}(\tau_j) + u^R_{\widetilde{H_2}}(\tau_j)\right)^2}{u^R_{\widetilde{H_1}}(\tau_j) + u^R_{\widetilde{H_2}}(\tau_j)} + \frac{\left(v^L_{\widetilde{H_1}}(\tau_j) + v^L_{\widetilde{H_2}}(\tau_j)\right)^2}{v^L_{\widetilde{H_1}}(\tau_j) + v^L_{\widetilde{H_2}}(\tau_j)} + \frac{\left(v^R_{\widetilde{H_1}}(\tau_j) + v^R_{\widetilde{H_2}}(\tau_j)\right)^2}{v^R_{\widetilde{H_1}}(\tau_j) + v^R_{\widetilde{H_2}}(\tau_j)}\right]} \\
&= \sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left(u^L_{\widetilde{H_1}}\left(\tau_j\right) + u^L_{\widetilde{H_2}}\left(\tau_j\right) + u^R_{\widetilde{H_1}}\left(\tau_j\right) + u^R_{\widetilde{H_2}}\left(\tau_j\right) + v^L_{\widetilde{H_1}}\left(\tau_j\right) + v^L_{\widetilde{H_2}}\left(\tau_j\right) + v^R_{\widetilde{H_1}}\left(\tau_j\right) + v^R_{\widetilde{H_2}}\left(\tau_j\right)\right)} \\
&\leq \sqrt{\frac{1}{4m}\sum_{j=1}^{m}(4)} \\
&= 1.
\end{aligned}$$

Consequently, the formula $0 \leq d_{Iv}\left(\widetilde{\beta_i}, \widetilde{\beta_j}\right) \leq 1$ is proved. □

Proof. (4) If $\widetilde{H_1} \leq \widetilde{H_2} \leq \widetilde{H_3}$, then $d_{Iv}\left(\widetilde{H_1}, \widetilde{H_2}\right) \leq d_{Iv}\left(\widetilde{H_1}, \widetilde{H_3}\right)$ and $d_{Iv}\left(\widetilde{H_2}, \widetilde{H_2}\right) \leq d_{Iv}\left(\widetilde{H_1}, \widetilde{H_3}\right)$.

When $\widetilde{H_1} \leq \widetilde{H_2} \leq \widetilde{H_3}$, we have

$$u^L_{\widetilde{H_1}} \leq u^L_{\widetilde{H_2}} \leq^L_{\widetilde{H_3}}, u^R_{\widetilde{H_1}} \leq u^R_{\widetilde{H_2}} \leq u^R_{\widetilde{H_3}}, v^L_{\widetilde{H_3}} \leq v^L_{\widetilde{H_2}} \leq v^L_{\widetilde{H_1}}, v^R_{\widetilde{H_3}} \leq v^R_{\widetilde{H_2}} \leq v^R_{\widetilde{H_1}}. \tag{3.4}$$

For $0 \leq \eta_k \leq 1$ $(k = 1, 2, 3, 4)$ and $0 \leq \eta_1 + \eta_3 \leq 1$, $0 \leq \eta_2 + \eta_4 \leq 1$, a function $g(x_1, x_2, x_3, x_4)$ could be established as below:

$$g(x_1, x_2, x_3, x_4) = \sum_{k=1}^{4} \frac{(x_k - \eta_k)^2}{x_k + \eta_k}, x_k \in [0, 1]; \tag{3.5}$$

then, the partial derivation of the function $g(x_1, x_2, x_3, x_4)$ in terms of x_i will be calculated as follows:

$$\frac{\partial g}{\partial x_k} = \frac{(x_k - \eta_k)(x_k + 3\eta_k)}{(x_k + \eta_k)^2}; \tag{3.6}$$

from the partial derivation function of Eq (3.6), one has

$$\begin{cases} \frac{\partial g}{\partial x_k} \geq 0, & 0 \leq \eta_k \leq x_k \leq 1, \\ \frac{\partial g}{\partial x_k} < 0, & 0 \leq x_k < \eta_k \leq 1. \end{cases} \tag{3.7}$$

Therefore, when $x_k \geq \eta_k$, $g(x_1, x_2, x_3, x_4)$ is a monotonically increasing function for x_k, and when $x_k \leq \eta_k$, $g(x_1, x_2, x_3, x_4)$ is a monotonically decreasing function for x_k.

Let $\eta_1 = u^L_{\widetilde{H_1}}$, $\eta_2 = u^R_{\widetilde{H_1}}$, $\eta_3 = v^L_{\widetilde{H_1}}$ and $\eta_4 = v^R_{\widetilde{H_1}}$.

When $\widetilde{H_1} \leq \widetilde{H_2} \leq \widetilde{H_3}$, we have

$$\eta_1 = u^L_{\widetilde{H_1}} \leq u^L_{\widetilde{H_2}} \leq u^L_{\widetilde{H_3}}, \eta_2 = u^R_{\widetilde{H_1}} \leq u^R_{\widetilde{H_2}} \leq u^R_{\widetilde{H_3}}, v^L_{\widetilde{H_3}} \leq v^L_{\widetilde{H_2}} \leq v^L_{\widetilde{H_1}} = \eta_3, v^R_{\widetilde{H_3}} \leq v^R_{\widetilde{H_2}} \leq v^R_{\widetilde{H_1}} = \eta_4.$$

Because $g(x_1, x_2, x_3, x_4)$ is monotonically increasing when $x_1 \geq \eta_1$, if $u^L_{\widetilde{H_3}} \geq u^L_{\widetilde{H_2}}$, one has

$$g\left(u^L_{\widetilde{H_3}}, u^R_{\widetilde{H_3}}, v^L_{\widetilde{H_3}}, v^R_{\widetilde{H_3}}\right) \geq g\left(u^L_{\widetilde{H_2}}, u^R_{\widetilde{H_3}}, v^L_{\widetilde{H_3}}, v^R_{\widetilde{H_3}}\right); \tag{3.8}$$

similarly, because $g(x_1, x_2, x_3, x_4)$ is monotonically increasing when $x_2 \geq \eta_2$, if $u^R_{\widetilde{H_3}} \geq u^R_{\widetilde{H_2}}$, one obtains

$$g\left(u^L_{\widetilde{H_2}}, u^R_{\widetilde{H_3}}, v^L_{\widetilde{H_3}}, v^R_{\widetilde{H_3}}\right) \geq g\left(u^L_{\widetilde{H_2}}, u^R_{\widetilde{H_2}}, v^L_{\widetilde{H_3}}, v^R_{\widetilde{H_3}}\right); \tag{3.9}$$

meanwhile, because $g(x_1, x_2, x_3, x_4)$ is monotonically decreasing when $x_3 \leq \eta_3$, if $v^L_{\widetilde{H_3}} \leq v^L_{\widetilde{H_2}}$, one has

$$g\left(u^L_{\widetilde{H_2}}, u^R_{\widetilde{H_2}}, v^L_{\widetilde{H_3}}, v^R_{\widetilde{H_3}}\right) \geq g\left(u^L_{\widetilde{H_2}}, u^R_{\widetilde{H_2}}, v^L_{\widetilde{H_2}}, v^R_{\widetilde{H_3}}\right); \tag{3.10}$$

besides, because $g(x_1, x_2, x_3, x_4)$ is monotonically decreasing when $x_4 \leq \eta_4$, if $v^R_{\widetilde{H_3}} \leq v^R_{\widetilde{H_2}}$, one has

$$g\left(u^L_{\widetilde{H_2}}, u^R_{\widetilde{H_2}}, v^L_{\widetilde{H_2}}, v^R_{\widetilde{H_3}}\right) \geq g\left(u^L_{\widetilde{H_2}}, u^R_{\widetilde{H_2}}, v^L_{\widetilde{H_2}}, v^R_{\widetilde{H_2}}\right). \tag{3.11}$$

Combining Eqs (3.8)–(3.11), one has

$$g\left(u^L_{\widetilde{H_3}}, u^R_{\widetilde{H_3}}, v^L_{\widetilde{H_3}}, v^R_{\widetilde{H_3}}\right) \geq g\left(u^L_{\widetilde{H_2}}, u^R_{\widetilde{H_2}}, v^L_{\widetilde{H_2}}, v^R_{\widetilde{H_2}}\right), \tag{3.12}$$

that is,

$$\frac{\left(u^L_{\widetilde{H_2}}-u^L_{\widetilde{H_1}}\right)^2}{u^L_{\widetilde{H_2}}+u^L_{\widetilde{H_1}}}+\frac{\left(u^R_{\widetilde{H_2}}-u^R_{\widetilde{H_1}}\right)^2}{u^R_{\widetilde{H_2}}+u^R_{\widetilde{H_1}}}+\frac{\left(v^L_{\widetilde{H_2}}-v^L_{\widetilde{H_1}}\right)^2}{v^L_{\widetilde{H_2}}+v^L_{\widetilde{H_1}}}+\frac{\left(v^R_{\widetilde{H_2}}-v^R_{\widetilde{H_1}}\right)^2}{v^R_{\widetilde{H_2}}+v^R_{\widetilde{H_1}}}\le\frac{\left(u^L_{\widetilde{H_3}}-u^L_{\widetilde{H_1}}\right)^2}{u^L_{\widetilde{H_3}}+u^L_{\widetilde{H_1}}}+\frac{\left(u^R_{\widetilde{H_3}}-u^R_{\widetilde{H_1}}\right)^2}{u^R_{\widetilde{H_3}}+u^R_{\widetilde{H_1}}}+\frac{\left(v^L_{\widetilde{H_3}}-v^L_{\widetilde{H_1}}\right)^2}{v^L_{\widetilde{H_3}}+v^L_{\widetilde{H_1}}}+\frac{\left(v^R_{\widetilde{H_3}}-v^R_{\widetilde{H_1}}\right)^2}{v^R_{\widetilde{H_3}}+v^R_{\widetilde{H_1}}}. \quad (3.13)$$

Consequently, we have

$$\begin{aligned}
d_{Iv}\left(\widetilde{H_1},\widetilde{H_2}\right)&=\sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left(\frac{\left(u^L_{\widetilde{H_2}}(\tau_j)-u^L_{\widetilde{H_1}}(\tau_j)\right)^2}{u^L_{\widetilde{H_2}}(\tau_j)+u^L_{\widetilde{H_1}}(\tau_j)}+\frac{\left(u^R_{\widetilde{H_2}}(\tau_j)-u^R_{\widetilde{H_1}}(\tau_j)\right)^2}{u^R_{\widetilde{H_2}}(\tau_j)+u^R_{\widetilde{H_1}}(\tau_j)}+\frac{\left(v^L_{\widetilde{H_2}}(\tau_j)-v^L_{\widetilde{H_1}}(\tau_j)\right)^2}{v^L_{\widetilde{H_2}}(\tau_j)+v^L_{\widetilde{H_1}}(\tau_j)}+\frac{\left(v^R_{\widetilde{H_2}}(\tau_j)-v^R_{\widetilde{H_1}}(\tau_j)\right)^2}{v^R_{\widetilde{H_2}}(\tau_j)+v^R_{\widetilde{H_1}}(\tau_j)}\right)}\\
&\le\sqrt{\frac{1}{4m}\sum_{j=1}^{m}\left(\frac{\left(u^L_{\widetilde{H_3}}(\tau_j)-u^L_{\widetilde{H_1}}(\tau_j)\right)^2}{u^L_{\widetilde{H_3}}(\tau_j)+u^L_{\widetilde{H_1}}(\tau_j)}+\frac{\left(u^R_{\widetilde{H_3}}(\tau_j)-u^R_{\widetilde{H_1}}(\tau_j)\right)^2}{u^R_{\widetilde{H_3}}(\tau_j)+u^R_{\widetilde{H_1}}(\tau_j)}+\frac{\left(v^L_{\widetilde{H_3}}(\tau_j)-v^L_{\widetilde{H_1}}(\tau_j)\right)^2}{v^L_{\widetilde{H_3}}(\tau_j)+v^L_{\widetilde{H_1}}(\tau_j)}+\frac{\left(v^R_{\widetilde{H_3}}(\tau_j)-v^R_{\widetilde{H_1}}(\tau_j)\right)^2}{v^R_{\widetilde{H_3}}(\tau_j)+v^R_{\widetilde{H_1}}(\tau_j)}\right)}\\
&=d_{Iv}\left(\widetilde{H_1},\widetilde{H_3}\right).
\end{aligned}$$

Hence, $d_{Iv}\left(\widetilde{H_1},\widetilde{H_2}\right)\le d_{Iv}\left(\widetilde{H_1},\widetilde{H_3}\right)$ is proved.

Similarly, $d_{Iv}\left(\widetilde{H_2},\widetilde{H_3}\right)\le d_{Iv}\left(\widetilde{H_1},\widetilde{H_3}\right)$ could be proved, too. □

Definition 3.3. *Specifically, for any two IvIFNs $\widetilde{\beta_i}$ and $\widetilde{\beta_j}$, the distance measure between $\widetilde{\beta_i}$ and $\widetilde{\beta_j}$ could be given by Eq (3.14).*

$$d_{Iv}\left(\widetilde{\beta_i},\widetilde{\beta_j}\right)=\sqrt{\frac{1}{4}\left[\frac{\left(u^L_{\widetilde{\beta_i}}-u^L_{\widetilde{\beta_j}}\right)^2}{u^L_{\widetilde{\beta_i}}+u^L_{\widetilde{\beta_j}}}+\frac{\left(u^R_{\widetilde{\beta_i}}-u^R_{\widetilde{\beta_j}}\right)^2}{u^R_{\widetilde{\beta_i}}+u^R_{\widetilde{\beta_j}}}+\frac{\left(v^L_{\widetilde{\beta_i}}-v^L_{\widetilde{\beta_j}}\right)^2}{v^L_{\widetilde{\beta_i}}+v^L_{\widetilde{\beta_j}}}+\frac{\left(v^R_{\widetilde{\beta_i}}-v^R_{\widetilde{\beta_j}}\right)^2}{v^R_{\widetilde{\beta_i}}+v^R_{\widetilde{\beta_j}}}\right]}. \quad (3.14)$$

Example 3.1. *There are three IvIFSs , $\widetilde{H_1}=\{\langle[0,0],[1,1]\rangle\}$, $\widetilde{H_2}=\{\langle[0.35,0.55],[0.25,0.35]\rangle\}$ and $\widetilde{H_3}=\{\langle[1,1],[0,0]\rangle\}$. According to Definition 2.2, it holds that $\widetilde{H_1}\subseteq\widetilde{H_2}\subseteq\widetilde{H_3}$.*

With the proposed IvIFTD distance measure, one has the following:

$d_{Iv}\left(\widetilde{H_1},\widetilde{H_2}\right)=\sqrt{\frac{1}{4}\times\left(\frac{0.35^2}{0.35}+\frac{0.55^2}{0.55}+\frac{(1-0.25)^2}{1+0.25}+\frac{(1-0.35)^2}{1+0.35}\right)}=0.6448,$

$d_{Iv}\left(\widetilde{H_2},\widetilde{H_3}\right)=\sqrt{\frac{1}{4}\times\left(\frac{(0.35-1)^2}{0.35+1}+\frac{(0.55-1)^2}{0.55+1}+\frac{0.25^2}{0.25}+\frac{0.35^2}{0.35}\right)}=0.5108,$

$d_{Iv}\left(\widetilde{H_1},\widetilde{H_3}\right)=\sqrt{\frac{1}{4}\times\left(\frac{1^2}{1}+\frac{1^2}{1}+\frac{1^2}{1}+\frac{1^2}{1}\right)}=1.$

Hence, we have that $d_{Iv}\left(\widetilde{H_1},\widetilde{H_2}\right)\le d_{Iv}\left(\widetilde{H_1},\widetilde{H_3}\right)$ and $d_{Iv}\left(\widetilde{H_2},\widetilde{H_3}\right)\le d_{Iv}\left(\widetilde{H_1},\widetilde{H_3}\right)$.

Example 3.2. *Suppose that there are two IvIFSs $\widetilde{H_4}$ and $\widetilde{H_5}$, as follows:*

$\widetilde{H_4}=\{\langle x_1,[0.55,0.67],[0.1,0.28]\rangle,\langle x_2,[0.26,0.37],[0.55,0.63]\rangle\}.$

$\widetilde{H_5}=\{\langle x_1,[0.7,0.8],[0.15,0.2]\rangle,\langle x_2,[0.61,0.71],[0,0.1]\rangle\}.$

The distance measure for $\left(\widetilde{H_4},\widetilde{H_5}\right)$ will be calculated by using the proposed distance measure, as follows:

$$\begin{aligned}
d_{Iv}\left(\widetilde{H_4},\widetilde{H_5}\right)&=\sqrt{\frac{1}{4\times2}\left(\frac{(0.55-0.7)^2}{0.55+0.67}+\frac{(0.67-0.8)^2}{0.67+0.8}+\frac{(0.1-0.15)^2}{0.1+0.15}+\frac{(0.28-0.2)^2}{0.28+0.2}+\frac{(0.26-0.61)^2}{0.26+0.61}+\frac{(0.37-0.71)^2}{0.37+0.71}+\frac{(0.55)^2}{0.55}+\frac{(0.63-0.1)^2}{0.63+0.1}\right)}\\
&=0.39.
\end{aligned}$$

4. The superiority of the IvIFTD distance measure

To demonstrate the superiority of the IvIFTD distance measure over some previous measures, some examples are presented below.

Example 4.1. *Assume that there are three IvIFSs, as below:*

$\widetilde{H_6} = \{\langle [0.2, 0.3], [0.3, 0.4] \rangle\}, \widetilde{H_7} = \{\langle [0.25, 0.35], [0.35, 0.45] \rangle\}, \widetilde{H_8} = \{\langle [0.25, 0.35], [0.25, 0.35] \rangle\}.$

We note that $\widetilde{H_7} \neq \widetilde{H_8}$*, so the distance measure between* $\left(\widetilde{H_6}, \widetilde{H_7}\right)$ *and* $\left(\widetilde{H_6}, \widetilde{H_8}\right)$ *should be different.*

Table 1. Comparison of distance measures for Example 4.1.

Pair	d_H	d_E	d_{HH}	d_{HE}	d_p^t	d_L	$d_{\exp}^H$	$d_{\exp}^E$	d_{Iv}
$\left(\widetilde{H_6}, \widetilde{H_7}\right)$	**0.05**	**0.05**	**0.05**	**0.05**	0.0167	**0.05**5	0.0387	0.0233	0.0636
$\left(\widetilde{H_6}, \widetilde{H_8}\right)$	**0.05**	**0.05**	**0.05**	**0.05**	0.05	**0.05**	0.0582	0.0402	0.0657

Table 1 lists the values for different distance methods. It shows that the results in bold, with the Hamming distance, Euclidean distance and some other existing distance methods yielding the same results between $\left(\widetilde{H_6}, \widetilde{H_7}\right)$ *and* $\left(\widetilde{H_6}, \widetilde{H_8}\right)$*. However, the values calculated by using the exponential distance [18] and our proposed distance method are consistent with the intuitive experience and theoretical requirements, i.e.,* $d_{\exp}^H\left(\widetilde{H_6}, \widetilde{H_7}\right) < d_{\exp}^H\left(\widetilde{H_6}, \widetilde{H_8}\right)$*,* $d_{\exp}^E\left(\widetilde{H_6}, \widetilde{H_7}\right) < d_{\exp}^E\left(\widetilde{H_6}, \widetilde{H_8}\right)$ *and* $d_{Iv}\left(\widetilde{H_6}, \widetilde{H_7}\right) < d_{Iv}\left(\widetilde{H_6}, \widetilde{H_8}\right)$*.*

Example 4.2. *We discuss the distance measure for two IvIFNs* $\widetilde{H_9} = \{\langle [0, 0], [0, 0] \rangle\}$ *and* $\widetilde{H_{10}} = \{\langle [0.5, 0.5], [0.5, 0.5] \rangle\}$*.*

Obviously, $\widetilde{H_9} \neq \widetilde{H_{10}}$*. Therefore, the distance between* $\widetilde{H_9}$ *and* $\widetilde{H_{10}}$ *should not be 0. However, using the exponential distance measure described in [18], we have that* $d_{\exp}^H\left(\widetilde{H_9}, \widetilde{H_{10}}\right) = d_{\exp}^E\left(\widetilde{H_9}, \widetilde{H_{10}}\right) = 0$ *as shown in Table 2, which demonstrates that the exponential distance method is limited in this example. Alternatively, the result is 0.7071 with the proposed IvIFTD distance measure, which is in line with an actual intuitive experience.*

Table 2. Distance measures for Example 4.2.

Pair	d_H	d_E	d_{HH}	d_{HE}	d_p^t	d_L	$d_{\exp}^H$	$d_{\exp}^E$	d_{Iv}
$\left(\widetilde{H_9}, \widetilde{H_{10}}\right)$	0.5	0.5	0.5	0.5	0.1667	0.5	**0**	**0**	0.7071

Example 4.3. *In the case of three IvIFSs* $\widetilde{H_{11}}, \widetilde{H_{12}}$ *and* $\widetilde{H_{13}}$*, we have*

$\widetilde{H_{11}} = \{\langle [0.3, 0.4], [0.2, 0.3] \rangle\}, \widetilde{H_{12}} = \{\langle [0.5, 0.55], [0.35, 0.4] \rangle\}, \widetilde{H_{13}} = \{\langle [0.5, 0.55], [0.2, 0.3] \rangle\}.$

In the case of the distance measures for the pair of IvIFSs $\left(\widetilde{H_{11}}, \widetilde{H_{12}}\right)$ *and* $\left(\widetilde{H_{11}}, \widetilde{H_{13}}\right)$*, the distance between the pair* $\left(\widetilde{H_{11}}, \widetilde{H_{12}}\right)$ *should be larger than the distance between* $\left(\widetilde{H_{11}}, \widetilde{H_{13}}\right)$ *from an intuitive perspective.*

However, regarding the results for the different distance measures in Table 3, the distance value for d_p^t *[27] is* $d_p^t\left(\widetilde{H_{12}}, \widetilde{H_{12}}\right) = 0.05 < d_p^t\left(\widetilde{H_{11}}, \widetilde{H_{13}}\right) = 0.0875$*, and those for* $d_{\exp}^H$ *and* $d_{\exp}^H$ *[18] are, respectively,* $d_{\exp}^H\left(\widetilde{H_{11}}, \widetilde{H_{12}}\right) = 0.0696 < d_{\exp}^H\left(\widetilde{H_{11}}, \widetilde{H_{13}}\right) = 0.1186$ *and* $d_{\exp}^E\left(\widetilde{H_{11}}, \widetilde{H_{12}}\right) = 0.0438 <$

$d^E_{\exp}\left(\widetilde{H_{11}}, \widetilde{H_{13}}\right) = 0.0734$. *These results are counter intuitive and different from those for the other distance measures, i.e.,* d_H, d_E *[22],* d_{HH}, d_{HE} *[26] and* d_L *[17] , as well as the new IvIFTD distance measure. Therefore, the existing distance methods using* d^t_p, $d^H_{\exp}$ *and* $d^H_{\exp}$ *are invalid in this example. Alternatively,* d_H, d_E, d_{HH}, d_{HE}, d_L *and our proposed new IvIFTD can work well in this situation.*

Table 3. Distance measures for Example 4.3.

Pair	d_H	d_E	d_{HH}	d_{HE}	d^t_p	d_L	$d^H_{\exp}$	$d^E_{\exp}$	d_{Iv}
$\left(\widetilde{H_{11}}, \widetilde{H_{12}}\right)$	0.15	0.1541	0.175	0.1768	**0.05**	0.1527	**0.0696**	**0.0438**	0.1795
$\left(\widetilde{H_{11}}, \widetilde{H_{13}}\right)$	0.0875	0.125	0.1	0.1414	**0.0875**	0.1242	**0.1186**	**0.0734**	0.1357

Example 4.4. *Mr. X needs to choose a product from an alternative house set, that is* $\{p_i|i = 1, 2, \cdots, 6\}$, *from five of the same weighted attributes* $\{a_1, a_2, \cdots, a_5\}$. *Relevant decision-making information in the IvIFS is supplied as shown in the following matrix, and it is assumed that the ideal alternative is* p_0. *The following is proposed to determine the best choice by adopting the proposed IvIFTD distance measure :*

$$M_{6\times5} = \begin{bmatrix} ([0.7,0.8],[0.1,0.2]) & ([0.82,0.84],[0.05,0.15]) & ([0.52,0.72],[0.18,0.25]) & ([0.55,0.6],[0.3,0.35]) & ([0.7,0.8],[0.1,0.2]) \\ ([0.85,0.9],[0.05,0.1]) & ([0.7,0.74],[0.17,0.25]) & ([0.1,0.23],[0.6,0.7]) & ([0.15,0.25],[0.2,0.3]) & ([0.05,0.1],[0.65,0.8]) \\ ([0.5,0.7],[0.2,0.3]) & ([0.86,0.9],[0.04,0.1]) & ([0.6,0.7],[0.2,0.28]) & ([0.2,0.3],[0.5,0.6]) & ([0.65,0.8],[0.15,0.2]) \\ ([0.4,0.6],[0.3,0.4]) & ([0.52,0.64],[0.23,0.35]) & ([0.72,0.78],[0.11,0.21]) & ([0.3,0.5],[0.4,0.5]) & ([0.8,0.9],[0.05,0.1]) \\ ([0.6,0.8],[0.15,0.2]) & ([0.3,0.35],[0.5,0.65]) & ([0.58,0.68],[0.18,0.3]) & ([0.68,0.77],[0.1,0.2]) & ([0.72,0.85],[0.1,0.15]) \\ ([0.3,0.5],[0.3,0.45]) & ([0.5,0.68],[0.25,0.3]) & ([0.33,0.43],[0.5,0.55]) & ([0.62,0.65],[0.15,0.35]) & ([0.84,0.93],[0.04,0.07]) \end{bmatrix}.$$

Table 4. The ideal solution for Example 4.4.

	a_1	a_2	a_3	a_4	a_5	a_6
p_0	$([1,1],[0,0])$	$([1,1],[0,0])$	$([1,1],[0,0])$	$([1,1],[0,0])$	$([1,1],[0,0])$	$([1,1],[0,0])$

Table 5. The distance measure for Example 4.4.

Pair	(p_1, p_0)	(p_2, p_0)	(p_3, p_0)	(p_4, p_0)	(p_5, p_0)	(p_6, p_0)
d_{Iv}	0.3437	0.2908	0.4125	0.4178	0.4031	0.446

The proposed IvIFTD distance measure yielded the distance values presented in Table 5. According to the results for the IvIFTD distance, the ranking order is $p_2 \succ p_1 \succ p_5 \succ p_3 \succ p_4 \succ p_6$. *Hence,* h_2 *is accessed as the best choice. On the one hand, the ranking order is the same as that in [17], which proved the rationality of the proposed IvIFTD distance measure. On the other hand, it is unlike the original ranking in [44]:* $p_2 \succ p_1 = p_5 \succ p_3 = p_4 = p_6$, *which also demonstrates the superiority of the proposed IvIFTD distance measure especially in terms of discriminating information with subtle differences, such as* p_1, p_5 *or* p_3, p_4, p_6.

From the above examples, unlike some of the existing distance measures that failed to work for the IvIFSs, the proposed IvIFTD distance measure can effectively reflect the differences among IvIFSs. Therefore, the new IvIFTD distance measure is rational and superior to some existing distance methods.

5. An improved TOPSIS method

According to the new IvIFSTD distance measure for IvIFSs, an improved TOPSIS method is established correspondingly. The specific implementation process for TOPSIS is outlined as follows.

Step 1. Set the biggest IvIFN $\widetilde{\beta}^{+}$ as the PIS and the smallest $\widetilde{\beta}^{-}$ as the NIS. Then, for any IvIFN $\widetilde{\beta}_k$, the distance between $\widetilde{\beta}_k$ and $\widetilde{\beta}^{+}$ ($\widetilde{\beta}^{-}$) will be $d_{Iv}\left(\widetilde{\beta}_k, \widetilde{\beta}^{+}\right)$ ($d_{Iv}\left(\widetilde{\beta}_k, \widetilde{\beta}^{-}\right)$).

Step 2. Calculate the relative closeness of the scheme $\widetilde{\beta}_k$ with respect to $\widetilde{\beta}^{+}$, which could be given by the following expression :

$$\rho_k = \frac{d_{Iv}\left(\widetilde{\beta}_k, \widetilde{\beta}^{-}\right)}{d_{Iv}\left(\widetilde{\beta}_k, \widetilde{\beta}^{+}\right) + d_{Iv}\left(\widetilde{\beta}_k, \widetilde{\beta}^{-}\right)}. \tag{5.1}$$

Step 3. Rank the schemes based on the values of ρ_k. The larger the value of ρ_k, the better the scheme performs. Therefore, the decision-maker can select the optimal scheme based on the ranking results.

6. Decision-making method and application of the proposed TOPSIS method

As mentioned in the introduction, teaching quality in higher education is a hot topic for educational administrators, teachers and students alike. To further observe the teaching quality in higher education, a relevant decision-making method and application was employed by using the proposed TOPSIS method. This analysis serves to provide valuable information and evaluation regarding teaching quality in higher education.

6.1. Establish a new decision-making framework for teaching satisfaction

With the proposed TOPSIS method and related knowledge in Section 2, a new MADM method for teaching satisfaction evaluation was constructed as shown in Figure 1.

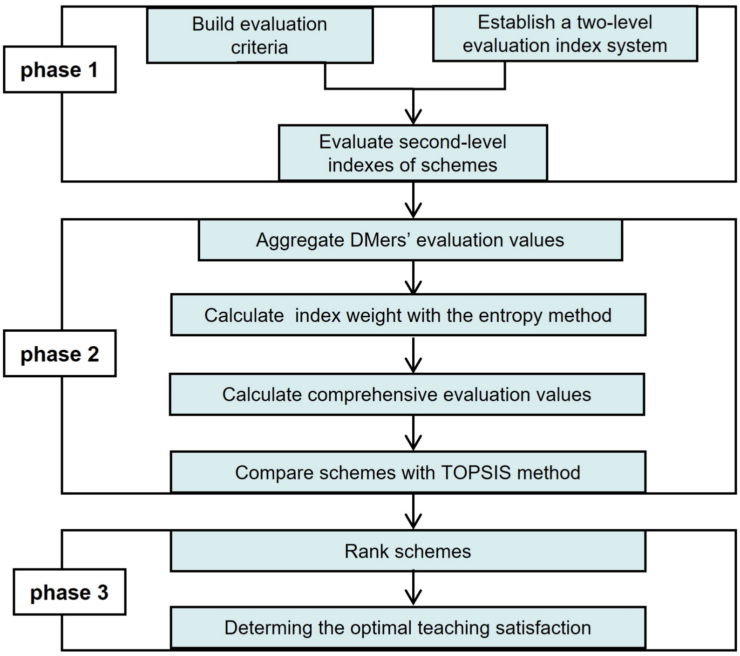

Figure 1. The decision-making flowchart for teaching satisfaction.

6.2. Illustrative examples

Example 6.1. *L University needs to determine the teaching satisfaction for four mathematics teaching courses. Assume that there are four experts with relevant knowledge and rich experience. Let* $S = \{S_1, S_2, S_3, S_4\}$ *be a scheme set to be evaluated,* C_j $(j = 1, 2, \cdots m)$ *-the first level criteria for this teaching satisfaction evaluation system and* C_j *consist of second level attributes, i.e.,* $C_j = \left\{a_{j1}, a_{j2}, \cdots, a_{jl}, \cdots a_{jn}\right\}$. *The characteristics of* a_{jl} *are expressed as IvIFNs, that is,* $a_{jl} = \left(\left[u^L_{a_{jl}}, u^R_{a_{jl}}\right], \left[v^L_{a_{jl}}, v^R_{a_{jl}}\right]\right)$. *All expert weights are assumed to be equal to each other, but the criteria weight* (w_{C_j}) *and attribute weight* ($w_{a_{jl}}$) *are unknown and need to be determined.*

6.2.1. Preparation of MADM problem

Based on the existing theoretical research and practical evaluation environment, an index system was constructed as shown in Table 6.

Table 6. Teaching satisfaction evaluation system.

First level: Criterion (C_j)	Second level: Attribute(a_{jl})
C_1: Teaching attitude	a_{11}: Fully prepared, a_{12}: Rigorous attitude, a_{13}: Respectful/patient with students, a_{14}: Manage the classroom,
C_2: Teaching content	a_{21}: Correct concept/knowledge, a_{22}: Explain clearly, a_{23}: Highlight key and difficult points, a_{24}: Connection between mathematics and social life,
C_3: Teaching method	a_{31}: Various methods, a_{32}: Teaching material, a_{33}: Participation and interaction, a_{34}: Focus on inspiration,
C_4: Teaching effect	a_{41}: Understanding /mastering of knowledge, a_{42}: Achievement of teaching objective, a_{43}: Stress the cultivation of comprehensive quality.

Table 7. Evaluation reference criteria.

Linguistic variables	IvIFNs
Especially good (EG)	$([0.9, 0.95], [0.005, 0.01])$
Very good (VG)	$([0.778, 0.9], [0.01, 0.05])$
Good (G)	$([0.667, 0.778], [0.1, 0.172])$
Relatively good (RG)	$([0.556, 0.6675], [0.2, 0.283])$
Medium (M)	$([0.445, 0.556], [0.3, 0.394])$
Relatively bad (RB)	$([0.334, 0.445], [0.4, 0.505])$
Bad (B)	$([0.223, 0.334], [0.5, 0.616])$
Very bad (VB)	$([0.1, 0.223], [0.6, 0.72])$
Especially bad (EB)	$([0, 0.1], [0.72, 0.9])$

In this study, four experts were invited to evaluate the courses by using IvIFNs. To facilitate the decision-making process for experts, we established evaluation reference criteria by using linguistic variables, which are presented as shown in Table 7.

The decision-making data from four experts are shown in Tables 8–11.

Table 8. The initial decision-making data by Expert$_1$.

Scheme	a_{11}	a_{12}	a_{13}	a_{14}	a_{21}	a_{22}	a_{23}	a_{24}	a_{31}	a_{32}	a_{33}	a_{34}	a_{41}	a_{42}	a_{43}
S_1	G	RG	G	VG	EG	G	G	VG	G	VG	G	VG	G	VG	RG
S_2	RG	VG	VG	RG	G	VG	M	G	RG	G	VG	M	RG	G	VG
S_3	G	VG	RB	M	VG	G	VG	G	VG	RG	G	G	VG	G	G
S_4	VG	G	VG	G	RG	MG	G	RG	G	G	RG	G	MG	RG	G

Table 9. The initial decision-making data by Expert$_2$.

Scheme	a_{11}	a_{12}	a_{13}	a_{14}	a_{21}	a_{22}	a_{23}	a_{24}	a_{31}	a_{32}	a_{33}	a_{34}	a_{41}	a_{42}	a_{43}
S_1	VG	RG	EG	VG	G	G	RG	G	G	M	G	M	G	G	M
S_2	G	G	VG	RG	G	VG	G	VG	G	VG	VG	RG	G	VG	RG
S_3	M	RG	G	G	VG	M	G	G	RG	G	RG	G	VG	G	G
S_4	G	G	VG	VG	RG	G	VG	RG	VG	VG	VG	M	VG	G	G

Table 10. The initial decision-making data by Expert$_3$.

Scheme	a_{11}	a_{12}	a_{13}	a_{14}	a_{21}	a_{22}	a_{23}	a_{24}	a_{31}	a_{32}	a_{33}	a_{34}	a_{41}	a_{42}	a_{43}
S_1	RG	G	G	RG	VG	M	EG	G	MG	G	G	G	RG	RG	RG
S_2	G	RG	M	G	G	RG	G	M	G	VG	M	VG	EB	G	VG
S_3	G	G	VG	VG	G	G	G	VG	RG	G	G	RG	G	G	EG
S_4	VG	VG	G	VG	M	VG	VG	RG	G	VG	VG	G	VG	VG	G

Table 11. The initial decision-making data by Expert$_4$.

Scheme	a_{11}	a_{12}	a_{13}	a_{14}	a_{21}	a_{22}	a_{23}	a_{24}	a_{31}	a_{32}	a_{33}	a_{34}	a_{41}	a_{42}	a_{43}
S_1	G	G	RG	M	VG	G	G	G	VG	G	VG	VG	G	MG	VG
S_2	RG	RG	G	VG	RG	RG	RG	RG	G	RG	RG	G	M	RG	G
S_3	VG	G	VG	G	G	VG	EG	G	EG	VG	G	VG	VG	G	RG
S_4	G	G	M	G	VG	G	M	VG	G	M	VG	G	RG	G	RB

6.2.2. Decision-making method and implementation process

After obtaining the evaluation data, the decision-making procedure generally proceeds as follows.

Step 1. Aggregating experts' evaluation values for the attribute into one value by using Eq (2.6);

Step 2. Computing the weight of attributes with by using the entropy method via Eqs (2.4) and (2.5);

Step 3. Aggregating attributes' values into corresponding criteria by using Eq (2.6);

Step 4. Calculating the weight by using the entropy method via Eqs (2.4) and (2.5);

Step 5. Obtaining a comprehensive evaluation value for (S_i) by using Eq (2.6);

Step 6. Computing the distance between the scheme and the PIS (NIS) by using Eq (3.14);

Step 7. Computing the relative closeness value for scheme (S_i) by using Eq (5.1);

Step 8. Ranking schemes.

Step 1. Taking attribute a_{11} as an example, the evaluation values from four experts were aggregated to obtain one value by using Eq (2.6). Here, the weights of experts are the same, i.e., 0.25:

$$\left(\left[1-(1-0.667)^{0.25}(1-0.778)^{0.25}(1-0.556)^{0.25}(1-0.667)^{0.25}, 1-(1-0.778)^{0.25}(1-0.9)^{0.25}(1-0.667)^{0.25}(1-0.778)^{0.25}\right],\right.$$
$$\left.\left[0.1^{0.25}0.01^{0.25}0.2^{0.25}0.1^{0.25}, 0.172^{0.25}0.05^{0.25}0.283^{0.25}0.172^{0.25}\right]\right)$$
$$= ([0.6767, 0.7987], [0.0669, 0.1430]).$$

Similarly, the aggregation values for all attributes can be obtained as shown in Table 12.

Table 12. Aggregation values for attributes.

	S_1	S_2	S_3	S_4
a_{11}	([0.677, 0.799], [0.067, 0.143])	([0.616, 0.728], [0.141, 0.221])	([0.658, 0.784], [0.074, 0.155])	([0.728, 0.851], [0.032, 0.093])
a_{12}	([0.616, 0.728], [0.141, 0.221])	([0.653, 0.777], [0.08, 0.162])	([0.677, 0.799], [0.067, 0.143])	([0.699, 0.818], [0.056, 0.126])
a_{13}	([0.735, 0.831], [0.056, 0.096])	([0.691, 0.823], [0.042, 0.114])	([0.677, 0.813], [0.045, 0.122])	([0.691, 0.823], [0.042, 0.114])
a_{14}	([0.668, 0.804], [0.05, 0.129])	([0.653, 0.777], [0.08, 0.162])	([0.658, 0.784], [0.074, 0.155])	([0.728, 0.851], [0.032, 0.927])
a_{21}	([0.799, 0.897], [0.150, 0.0456])	([0.642, 0.754], [0.119, 0.195])	([0.728, 0.851], [0.032, 0.093])	([0.605, 0.735], [0.106, 0.199])
a_{22}	([0.622, 0.736], [0.132, 0.212])	([0.686, 0.818], [0.045, 0.119])	([0.658, 0.784], [0.074, 0.155])	([0.677, 0.799], [0.074, 0.143])
a_{23}	([0.735, 0.831], [0.056, 0.088])	([0.593, 0.708], [0.157, 0.24])	([0.777, 0.875], [0.027, 0.062])	([0.691, 0.823], [0.042, 0.1])
a_{24}	([0.699, 0.818], [0.056, 0.126])	([0.633, 0.761], [0.088, 0.164])	([0.699, 0.818], [0.056, 0.126])	([0.627, 0.754], [0.095, 0.184])
a_{31}	([0.658, 0.784], [0.074, 0.145])	([0.642, 0.754], [0.119, 0.195])	([0.743, 0.847], [0.038, 0.08])	([0.699, 0.818], [0.056, 0.126])
a_{32}	([0.658, 0.784], [0.074, 0.145])	([0.708, 0.835], [0.038, 0.105])	([0.677, 0.798], [0.067, 0.143])	([0.691, 0.823], [0.042, 0.114])
a_{33}	([0.699, 0.818], [0.056, 0.126])	([0.668, 0.804], [0.05, 0.129])	([0.642, 0.754], [0.119, 0.195])	([0.708, 0.835], [0.038, 0.106])
a_{34}	([0.691, 0.823], [0.042, 0.105])	([0.633, 0.761], [0.088, 0.176])	([0.677, 0.799], [0.067, 0.143])	([0.699, 0.818], [0.056, 0.126])
a_{41}	([0.642, 0.754], [0.119, 0.195])	([0.517, 0.633], [0.221, 0.314])	([0.754, 0.878], [0.017, 0.068])	([0.583, 0.715], [0.116, 0.217])
a_{42}	([0.633, 0.761], [0.088, 0.176])	([0.677, 0.799], [0.067, 0.144])	([0.667, 0.778], [0.1, 0.172])	([0.671, 0.835], [0.038, 0.105])
a_{43}	([0.605, 0.735], [0.105, 0.199])	([0.708, 0.835], [0.038, 0.105])	([0.735, 0.831], [0.056, 0.096])	([0.604, 0.721], [0.141, 0.225])

Step 2. By using the entropy weight method as given by Eqs (2.4) and (2.5), the attribute weight ($w_{a_{jl}}$) at the second level could be obtained as shown in Table 13.

Table 13. Attribute weights.

Criterion	Attribute	Entropy	Weight ($w_{a_{jl}}$)
C_1	a_{11}	0.386	0.242
	a_{12}	0.404	0.235
	a_{13}	0.318	0.269
	a_{14}	0.365	0.254
C_2	a_{211}	0.349	0.258
	a_{22}	0.397	0.239
	a_{23}	0.342	0.26
	a_{24}	0.386	0.243
C_3	a_{31}	0.361	0.249
	a_{32}	0.344	0.255
	a_{33}	0.361	0.249
	a_{34}	0.363	0.248
C_4	a_{41}	0.299	0.302
	a_{42}	0.178	0.354
	a_{43}	0.199	0.345

Step 3. Using the weights from Step 2, repeat the aggregation method with the corresponding criteria at the first level. Then, we obtain the values listed in Table 14.

Table 14. Aggregation values for criteria.

	$\mathbf{S}_1$	$\mathbf{S}_2$	$\mathbf{S}_3$	$\mathbf{S}_4$
C_1	$([0.679, 0.795], [0.071, 0.139])$	$([0.655, 0.78], [0.077, 0.159])$	$([0.668, 0.795], [0.063, 0.143])$	$([0.712, 0.837], [0.039, 0.106])$
C_2	$([0.723, 0.832], [0.049, 0.01])$	$([0.639, 0.762], [0.094, 0.175])$	$([0.721, 0.837], [0.043, 0.102])$	$([0.652, 0.781], [0.074, 0.151])$
C_3	$([0.677, 0.803], [0.06, 0.129])$	$([0.664, 0.792], [0.066, 0.147])$	$([0.664, 0.792], [0.066, 0.147])$	$([0.699, 0.824], [0.047, 0.118])$
C_4	$([0.626, 0.75], [0.102, 0.189])$	$([0.648, 0.775], [0.078, 0.163])$	$([0.719, 0.831], [0.049, 0.106])$	$([0.623, 0.767], [0.083, 0.167])$

Step 4. Repeat the entropy weight method; the criteria weight (w_{C_j}) at the first level will be obtained as shown in Table 15.

Table 15. Criterion weight.

Criterion	Entropy	Weight (w_{C_j})
C_1	0.3594	0.2529
C_2	0.3553	0.2546
C_3	0.3524	0.2557
C_4	0.4003	0.2368

Step 5. With the criterion weight, we could get the integrated evaluation values shown in Table 16 by using the IvIFWAA operator.

Table 16. Integrated evaluation values.

Scheme	Integrated evaluation values ($\widetilde{\beta}_i$)
C_1	$([0.6789, 0.7977], [0.0673, 0.1349])$
C_2	$([0.6517, 0.7774], [0.0783, 0.1606])$
C_3	$([0.699, 0.8170], [0.0545, 0.1201])$
C_4	$([0.6742, 0.8045], [0.0577, 0.133])$

Steps 6–8. Suppose that $\widetilde{\beta}^+ = ([1, 1], [0, 0])$, $\widetilde{\beta}^- = ([0, 0], [1, 1])$ are, respectively, the PIS and NIS for the IvIFS in this example. Then, we could obtain the relative closeness degree for all schemes by using our proposed TOPSIS method. Then, all courses' teaching satisfaction is as ranked in Table 17.

Table 17. The closeness of different schemes.

Scheme	$d_{Iv}\left(\widetilde{\beta}_i, \widetilde{\beta}^+\right) / d_{Iv}\left(\widetilde{\beta}_i, \widetilde{\beta}^-\right)$	ρ_{Iv}	Ranking
$\mathbf{S}_1$	0.2676/0.8589	0.7625	3
$\mathbf{S}_2$	0.2917/0.8402	0.7423	4
$\mathbf{S}_3$	0.2482/0.8739	0.7788	1
$\mathbf{S}_4$	0.2623/0.8634	0.7628	2

In this evaluation, the teaching course S_3 had the highest satisfaction degree. On the contrary, S_2 was evaluated with the lowest satisfaction degree. That is, $S_3 \succ S_4 \succ S_1 \succ S_2$.

7. Comparative analysis

To provide a more objective comparison of the proposed IvIF-TOPSIS method and existing methods, we adopt an example of teaching quality evaluation under the conditions of the IvIFS environment originally presented by Zhao [38]. This example will serve to illustrate the comparative process and showcase the effectiveness of the IvIF-TOPSIS method.

Example 7.1. *We evaluate five schools' teaching quality by using IvIFSs. Five alternatives* $\mathrm{A}_i\,(i = 1, 2, 3, 4, 5)$ *need to be evaluated based on four attributes* $\mathrm{G}_j\,(j = 1, 2, 3, 4)$*. The weight vector for the four attributes is* $w_j = (0.15, 0.35, 0.395, 0.105)$*, and the decision matrix is*

$$M_{5\times4} = \begin{bmatrix} ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) & ([0.3,0.4],[0.4,0.5]) & ([0.5,0.6],[0.1,0.3]) \\ ([0.5,0.6],[0.2,0.3]) & ([0.6,0.7],[0.2,0.3]) & ([0.5,0.6],[0.3,0.4]) & ([0.4,0.7],[0.1,0.2]) \\ ([0.3,0.5],[0.3,0.4]) & ([0.1,0.3],[0.5,0.6]) & ([0.2,0.5],[0.4,0.5]) & ([0.2,0.3],[0.4,0.6]) \\ ([0.2,0.5],[0.3,0.4]) & ([0.4,0.7],[0.1,0.2]) & ([0.4,0.5],[0.3,0.5]) & ([0.5,0.8],[0.1,0.2]) \\ ([0.3,0.4],[0.1,0.3]) & ([0.7,0.8],[0.1,0.2]) & ([0.5,0.6],[0.2,0.4]) & ([0.6,0.7],[0.1,0.2]) \end{bmatrix}.$$

In this evaluation, the PIS and NIS are, respectively,

$\widetilde{s}^{+} = [([0.5, 0.6], [0.1, 0.3]), ([0.7, 0.8], [0.1, 0.2]), ([0.5, 0.6], [0.2, 0.4]), ([0.6, 0.8], [0.1, 0.2])]$,

$\widetilde{s}^{-} = [([0.2, 0.4], [0.3, 0.4]), ([0.1, 0.3], [0.5, 0.6]), ([0.2, 0.4], [0.4, 0.5]), ([0.2, 0.3], [0.4, 0.6])]$.

Then, we used our proposed IvIFTD distance measure to decide which alternative is better, as shown in Table 18.

Table 18. Conclusions for Example 7.1.

Scheme	$d_{Iv}(A_i, \widetilde{s}^{+}) / d_{Iv}(A_i, \widetilde{s}^{-})$	ρ_{Iv}	Ranking
$\mathbf{A}_1$	0.2013/0.2035	0.5027	4
$\mathbf{A}_2$	0.0985/0.3128	0.7605	2
$\mathbf{A}_3$	0.3572/0.0341	0.0870	5
$\mathbf{A}_4$	0.1394/0.27	0.6595	3
$\mathbf{A}_5$	0.0268/0.3578	0.9304	1

According to the results presented in Table 18, the ranking result is $\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_1 \succ \mathrm{A}_3$*. This ranking order is aligned with the original order reported in [38].*

We also performed a comparison with other methods, including the score function described by Xu [13], similarity function described by Wang [45], classical TOPSIS based on Hamming distance as described by Hu and Xu [30], Euclidean distance described by Qiao et al [36], M-TOPSIS method described by Aikhuele and Turan [34], correlation coefficient method described by Jun [46] and a new TOPSIS based on exponential distance by using connections, as described by Garg and Kumar [18]. These different methods were applied to the given data; the corresponding results are presented in Table 19. It is worth noting that, except for some minor differences observed with the score function [13], all of the ranking results for the five alternatives remained the same as that for the improved IvIF-TOPSIS method.

Table 19. Ranking results for different methods for Example 7.1.

Methods	V_{A_1}	V_{A_2}	V_{A_3}	V_{A_4}	V_{A_5}	Ranking
Score function [13]	0.0832	0.306	0.1808	0.2123	0.3838	$\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_3 \succ \mathrm{A}_1$
Similarity function [45]	0.5377	0.6345	0.427	0.5843	0.668	$\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_1 \succ \mathrm{A}_3$
Hu's TOPSIS [30]	0.4815	0.788	0.0888	0.6387	0.9215	$\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_1 \succ \mathrm{A}_3$
Qiao's TOPSIS [36]	0.4693	0.822	0.0548	0.6742	0.9444	$\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_1 \succ \mathrm{A}_3$
Aikhuele's M-TOPSIS [34]	0.13	0.0293	0.2295	0.0773	0	$\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_1 \succ \mathrm{A}_3$
Correlation coefficient method [46]	0.7486	0.8884	0.5044	0.8178	0.9053	$\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_1 \succ \mathrm{A}_3$
Garg and Kumar's TOPSIS [18]	0.4367	0.7762	0.067	0.6234	0.9259	$\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_1 \succ \mathrm{A}_3$
Our Proposed TOPSIS	0.5027	0.7605	0.0870	0.6595	0.9304	$\mathrm{A}_5 \succ \mathrm{A}_2 \succ \mathrm{A}_4 \succ \mathrm{A}_1 \succ \mathrm{A}_3$

Example 7.2. *Suppose that there are four courses* B_1, B_2, B_3 *and* B_4 *that need to be evaluated; its comprehensive IvIFN values are* $\widetilde{\gamma_{B_1}} = ([0.15, 0.25], [0.25, 0.35])$, $\widetilde{\gamma_{B_2}} = ([0.2, 0.3], [0.15, 0.25])$, $\widetilde{\gamma_{B_3}} = ([0.25, 0.35], [0.2, 0.3])$ *and* $\widetilde{\gamma_{B_4}} = ([0.352, 0.43], [0.095, 0.123])$ *respectively.*

According to Definition 2.4, the score function values are $f_s(\widetilde{\gamma_{B_1}}) = -0.1$, $f_s(\widetilde{\gamma_{B_2}}) = 0.05$, $f_s(\widetilde{\gamma_{B_3}}) = 0.05$ *and* $f_s(\widetilde{\gamma_{B_4}}) = 0.282$ *and the accuracy function values are* $f_a(\widetilde{\gamma_{B_2}}) = 0.45$ *and* $f_a(\widetilde{\gamma_{B_3}}) = 0.55$. *Thus, one has* $\widetilde{\gamma_{B_2}} \prec \widetilde{\gamma_{B_3}} \prec \widetilde{\gamma_{B_4}}$.

Besides, we set the PIS as $\widetilde{\gamma^+} = \left(\left[\max\left\{u^L_{B_i}\right\}, \max\left\{u^R_{B_i}\right\}\right], \left[\min\left\{v^L_{B_i}\right\}, \min\left\{v^R_{B_i}\right\}\right]\right)$ *and the NIS as* $\widetilde{\gamma^-} = \left(\left[\min\left\{u^L_{B_i}\right\}, \min\left\{u^R_{B_i}\right\}\right], \left[\max\left\{v^L_{B_i}\right\}, \max\left\{v^R_{B_i}\right\}\right]\right)$. *Hence, we can get the PIS* ($\widetilde{\gamma^+}$) *and NIS* ($\widetilde{\gamma^-}$) *as* $\widetilde{\gamma_{B_4}}$ *and* $\widetilde{\gamma_{B_1}}$, *respectively. Then , the relative closeness for the four courses was calculated by using Eq (5.1), and all schemes are ranked in Table 20.*

Table 20. Comparison with different TOPSIS methods for Example 7.2.

TOPSIS Methods	The relative closeness degree	Ranking
Hu and Xu's [30], Qiao's [36]	$\rho_H(B_2) = \rho_H(B_3) = 0.3927, \rho_H(B_1) = 0, \rho_H(B_4) = 1$	$B_4 > B_2 = B_3 > B_1$
Wang's [45], Zhou's [47]	$\rho_E(B_2) = \rho_E(B_3) = 0.3940, \rho_E(B_1) = 0, \rho_E(B_4) = 1$	$B_4 > B_2 = B_3 > B_1$
This study	$\rho_{Iv}(B_1) = 0, \rho_{Iv}(B_2) = 0.3995, \rho_{Iv}(B_3) = 0.3792, \rho_{Iv}(B_4) = 1$	$B_4 > B_2 > B_3 > B_1$

As observed, certain traditional TOPSIS methods are unable to effectively compare the four courses due to the identical relative closeness values. Specifically, $\rho_H(B_2) = \rho_H(B_3) = 0.3927$ *and* $\rho_E(B_2) = \rho_E(B_3) = 0.3940$. *On the contrary, the proposed IvIF-TOPSIS method yielded* $\rho_{Iv}(B_2) = 0.3995$ *and* $\rho_{Iv}(B_3) = 0.3792$, *indicating a noticeable distinction. Consequently, the IvIF-TOPSIS method allows for a comparison between course* B_2 *and* B_3, *with* B_2 *being superior to* B_3.

Based on the aforementioned comparisons, it is evident that the proposed IvIF-TOPSIS method is not only applicable to decision-making problems, but it also demonstrates a superior ability to rank schemes with subtle differences. Therefore, the improved IvIF-TOPSIS method is proven to be advantageous for decision-making problems.

8. Conclusions and future research

Teaching satisfaction evaluation plays an essential role in enhancing teaching quality in higher education. However, due to human limitations in terms of knowledge, cognitive uncertainty, and thinking habits, IvIFSs are often utilized to address MADM issues. In this domain, two vital aspects have arisen: how to objectively determine evaluation index weights and how to compare schemes with a decision-making method. To address these problems, we created a new distance measure based on the triangular divergence and demonstrated that the IvIFTD distance measure meets the requirements for the properties of the distance metric. Compared to some existing distance methods without explicit physical meaning or with complex calculations, the proposed IvIFTD distance measure is more in line with humans' intuitive experience and theoretical requirements. Additionally, it proves superior in the area of distinguishing subtle differences between different IvIFSs.

Based on the IvIFTD distance measure, an improved TOPSIS method has been proposed. This method was subsequently applied for the establishment of an MADM approach for teaching satisfaction evaluation. An example was conducted to illustrate the decision-making process, and it included problem construction, calculation of comprehensive evaluation values, ranking and a selection of schemes using the IvIFTD distance measure and TOPSIS method. Comparative analyses have been presented to validate the rationality and superiority of the proposed method. The outcomes demonstrate that the improved TOPSIS method, based on the new distance measure, effectively handles uncertainty and subtle differences in actual evaluation problems involving different IvIFSs or IvIFNs. This advantage allows for the utilization of diverse evaluation values, providing more comprehensive decision-making information for teaching satisfaction evaluation.

However, our study also has limitations that need to be addressed in future work. First, the proposed method does not consider the subjective weight of the evaluation criteria, thus overlooking the subjective preferences of decision-makers in the criteria. Additionally, the teaching satisfaction evaluation index system can be further improved by incorporating other innovative criteria. Moreover, considering a group decision-making approach for teaching satisfaction evaluation may be a more viable method to achieve objective evaluations.

In future studies, we will aim to construct a more comprehensive teaching satisfaction evaluation index system from multiple perspectives through expert investigation and consultation. We will also extend subjective weight methods such as the best-worst method, full-consistency method, and step-wise weight assessment ratio analysis to the IvIFS environment to incorporate objective criterion importance. Furthermore, the construction of group decision-making methods, considering multi-granularity linguistic information, consensus processes, and behavioral decision theory are crucial for MADM problems.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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