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Research article

Some new results on sum index and difference index

Yuan Zhang and Haiying Wang*

School of Science, China University of Geosciences (Beijing), Beijing 100083, China

* **Correspondence:** Email: 2007012125@cugb.edu.cn.

Abstract: Let $G = (V(G), E(G))$ be a graph with a vertex set $V(G)$ and an edge set $E(G)$. For every injective vertex labeling $f : V(G) \to \mathbb{Z}$, there are two induced edge labelings denoted by $f^+ : E(G) \to \mathbb{Z}$ and $f^- : E(G) \to \mathbb{Z}$. These two edge labelings f^+ and f^- are defined by $f^+(uv) = f(u) + f(v)$ and $f^-(uv) = |f(u) - f(v)|$ for each $uv \in E(G)$ with $u, v \in V(G)$. The sum index and difference index of G are induced by the minimum ranges of f^+ and f^-, respectively. In this paper, we obtain the properties of sum and difference index labelings. We also improve the bounds on the sum indices and difference indices of regular graphs and induced subgraphs of graphs. Further, we determine the sum and difference indices of various families of graphs such as the necklace graphs and the complements of matchings, cycles and paths. Finally, we propose some conjectures and questions by comparison.

Keywords: graph labeling; degree sequence; sum index; difference index
Mathematics Subject Classification: 05C05, 05C12

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. Two edges e_1 and e_2 of G are adjacent, if they have exactly one common end vertex (otherwise, e_1 and e_2 of G are non-adjacent). If $e = \{u, v\}$ is an edge of G, then u and v are adjacent while u and e are incident. If all the vertices of G have the same degree d, then G is d-regular and its degree sequence is $\pi = (d, d, \cdots, d)$. Let $G' = (V(G'), E(G'))$, we call G' is a subgraph of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If G' contains all the edges $uv \in E(G)$ with $u,\ v \in V(G')$, then G' is an induced subgraph of G.

The complement of G denoted by $\overline{G}$, is the simple graph whose vertex set is $V(G)$ and whose edges are the pairs of non-adjacent vertices of G. A matching of a graph G is a set of pairwise non-adjacent edges in $E(G)$. A m-matching is a matching consisting of m edges denoted by mK_2. As usual, we use P_n and C_n to denote the path and cycle of order n, respectively.

Let $f : V(G) \to \mathbb{Z}$ be an injective vertex labeling of G. There are two edge labelings of G induced by f, which are defined as $f^+ : E(G) \to \mathbb{Z}$ and $f^- : E(G) \to \mathbb{Z}$. For each edge $uv \in E(G)$, two edge labelings $f^+(uv)$ and $f^-(uv)$ are defined by

$$f^+(uv) = f(u) + f(v) \text{ and } f^-(uv) = |f(u) - f(v)|.$$

The sum index $s(G)$ and difference index $d(G)$ of G are defined by the minimum ranges of f^+ and f^- of G, respectively. They were introduced by Harrington et al. in [1]. To avoid much ambiguity, we denote f^- by g^-. The injective vertex labeling corresponding to g^- is denoted by g throughout the article.

Definition 1.1. *[1] The sum index of G, denoted by $s(G)$, is the minimum positive integer k such that there exists a vertex labeling f of G satisfying $|f^+| = k$, where $|f^+|$ is the range of f. A vertex labeling f such that $|f^+| = s(G)$ is referred to as a sum index labeling of G.*

Definition 1.2. *[1] Let $g : V(G) \to \mathbb{Z}$ be a vertex labeling of G, and let $g^- : E(G) \to \mathbb{Z}$ be the induced edge labeling defined by $g^-(uv) = |g(u) - g(v)|$ for each edge $uv \in E(G)$. The difference index of G, denoted by $d(G)$, is the minimum positive integer k such that there exists a vertex labeling g of G satisfying $|g^-| = k$, where $|g^-|$ is the range of g. A vertex labeling g such that $|g^-| = d(G)$ is referred to as a difference index labeling of G.*

Degree-based indices are served as meaningful models for a broad range of applications. For example, the Randić index [2] is denoted by

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(v)}},$$

where $d_G(u)$ is the degree of the vertex $u \in V(G)$. The Randić index has great applications in modeling the properties of certain molecular structures. As another important degree-based topological index, the Sombor index [3] is used to model the higher-order interactions represented by clique structures and defined by

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

The atom-bond connectivity (ABC) index of G has proven to be a valuable predictive index in the study of the heat of function in alkanes [4, 5] and defined by

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u) d_G(v)}}.$$

In graph theory, regular graphs are one of the most important classes of graphs. Let G_0 be d-regular with order n. Then the above three indices are $R(G_0) = \frac{n}{2}$, $SO(G_0) = \frac{\sqrt{2}}{2} d^2 n$ and $ABC(G_0) = \frac{n\sqrt{2d-2}}{2}$. Moreover, the first Zagreb index [6], the variable sum exdeg index [7], the Harmonic index [8] and so on are all widely used in graphs including regular graphs.

Graph indices derived from graph labelings are also of great value in coding theory, radar, circuit design, data base management and among others. The notations of the (integral) sum labelings of graphs were introduced by F. Harary in [9]. Since that time the problems of finding the (integral) sum numbers and proving the (integral) sum graphs have been studied and discussed by scholars referring to [10, 11]. A graph G is called an (integral) sum graph if there is a bijection f from $V(G)$ to $S \subset (\mathbb{Z})\ \mathbb{N}$ such that $xy \in E(G)$ if and only if $f(x) + f(y) \in S$. The (integral) sum number of G is the minimum number of isolated vertices that must be added to G such that the resulting graph is an (integral) sum graph.

Based on the above, graph labelings of graphs attracted lots of attention of researchers. For example, the vertex-magic total labelings were introduced by MacDougall, Miller, Slamin and Wallis in [12]. Ponraj and Parthipan posited pair sum labelings in [13]. Harrington and Wong used super totient numbers to label vertices of graphs and found that the restricted super totient indices of graphs were equal to their sum indices in [14]. In this regard, Harrington, Henninger-Voss et al. defined the sum indices and difference indices of graphs and lower bounds on these two indices were also determined in [1]. Further, both the sum and difference indices of several graphs were also obtained such as the complete graphs, complete bipartite graphs, caterpillars, cycles, wheels and rectangular grids in [1]. Besides, Haslegrave improved bounds on these two indices in [15].

Up to now, some useful results about the sum indices and difference indices of graphs are shown as follows.

Property 1.1. *[14] The sum index is greater than or equal to the maximum degree of G,*

$$s(G) \geq \Delta(G).$$

Property 1.2. *[1] The sum index is greater than or equal to the chromatic index of G,*

$$s(G) \geq \chi'(G).$$

Property 1.3. *[1] Let $\delta(G)$ be the minimum degree of G, and recall that $\chi'(G)$ is the chromatic index of G. Then we have*

$$d(G) \geq \max\left\{\left\lceil \frac{\chi'(G)}{2} \right\rceil, \delta(G)\right\}.$$

Property 1.4. *[15] For any graph G, we have*

$$s(G) \geq max_{k\geq 1}(\delta_k(G) + \delta_{k+1}(G) - k) \text{ and } d(G) \geq max_{k\geq 1}(\delta_{2k}(G) + 1 - k).$$

In this paper, we continue to study the sum indices and difference indices of graphs. Firstly, we improve some bounds on the sum and difference indices in Section 2. In particular, we find relationships about these two indices between a graph G and its induced subgraph G'. For regular graphs, we make slight generalizations of Haslegrave's bounds on the sum and difference indices in [15]. Secondly, we obtain the sum and difference indices of some graphs in Section 3 such as the necklace graphs, the complements of matchings, cycles and paths. Finally, we compare the sum indices with (integral) sum numbers and analyze the sum and difference indices of regular graphs. Combined with conclusions of the paper, we also put forward some conjectures and open questions.

2. Properties and bounds of sum indices and difference indices of graphs

In this section, we explore the properties of sum index labelings and difference index labelings of graphs. We also improve the bounds on the sum and difference indices of regular graphs and induced subgraphs of graphs. In particular, we make slight improvements to Property 1.4 such that we can directly determine lower bounds on the sum indices and difference indices of regular graphs.

Theorem 2.1. *If f is a sum index labeling of G, then kf is also a sum index labeling of G for any non-zero integer. The result holds for a difference index labeling g as well.*

Proof. If f is a sum index labeling of G, then $|f^+| = s(G)$. Denote $s(G) = s$ and $f^+ = \{\alpha_1, \alpha_2, \cdots, \alpha_s\}$, where $\alpha_1, \alpha_2, \cdots, \alpha_s$ are integers. According to the definition of $s(G)$, one has

$$(kf)^+ (v_i v_j) = kf(v_i) + kf(v_j) = k\left(f(v_i) + f(v_j)\right) = kf^+(v_i v_j).$$

Hence we observe that

$$\left|(kf)^+\right| = |\{k\alpha_1, k\alpha_2, \cdots, k\alpha_s\}| = s(G).$$

Thus kf is also a sum index labeling of G. □

Remark 2.1. *For a graph G, the sum index of G is unique, while the number of corresponding sum index labelings is infinite.*

Theorem 2.2. *Let G' be an induced subgraph of a graph G. Then $s(G') \leq s(G)$ and $d(G') \leq d(G)$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $V(G') = \left\{v_{i_1}, v_{i_2}, v_{i_3}, \cdots, v_{i_j}\right\}$, where $V(G') \subset V(G)$ for $i \in \mathbb{Z}$ and $j \leq n$. For any vertex v_{i_k} in G', we can always find a vertex v_l in G such that $v_{i_k} = v_l$ for $1 \leq k \leq j$ and $1 \leq l \leq n$. If f is a sum index labeling and g is a difference index labeling of G, then $|f^+| = s(G)$ and $|g^-| = d(G)$. For each vertex labeling f' and g' of G', let $f'(v_{i_k}) = f(v_l)$ and $g'(v_{i_k}) = g(v_l)$. Thus we have

$$s(G') \leq s(G) \text{ and } d(G') \leq d(G).$$

□

Theorem 2.3. *Let G be d-regular with order n. Then $s(G) \geq 2d - 1$ and $d(G) \geq d$.*

Proof. Let $f : V(G) \to \mathbb{Z}$ be an injective vertex labeling of G. Denote $f : V(G) \to \{a_1, a_1, \cdots, a_n\}$, where $a_1, a_1, \cdots, a_n$ are integers such that $a_1 < a_2 < \cdots < a_n$. It follows that

$$a_1 + a_2 < a_1 + a_3 < \cdots < a_1 + a_n < a_2 + a_n < a_3 + a_n < \cdots < a_{n-1} + a_n.$$

Note that G is a d-regular graph. Then the sum of the two numbers above has at most $2(n - 1 - d)$ elements which do not belong to f^+. Thus, we obtain

$$s(G) \geq (2n - 3) - 2(n - 1 - d) = 2d - 1.$$

Next, we will show that $d(G) \geq d$. Let $g = f$ be an injective vertex labeling of G. Without loss of generality, we assume that the vertex labeled a_1 is adjacent to vertices labeled a_2, a_3, $\cdots$, a_{d+1}. By utilizing the definition of $d(G)$, we have $d(G) \geq d$. □

Remark 2.2. *Theorem 2.3 is viewed as slight generalizations of Haslegrave's bounds on the sum and difference indices of graphs in [15] (see Property 1.4).*

3. Sum indices and difference indices of some graphs

In this section, we discuss the sum indices and difference indices of some graphs such as the necklace graphs, the complements of matchings, cycles and paths. At the beginning, we introduce a difinition about the edge labeling of a graph that will be needed in the remaining part of the paper.

Definition 3.1. *For two edge labelings f^+ and g^- of G, the contributing elements of f^+ are the elements first appearing in f^+. For each edge $v_iv_j \in E(G)$, let $f'^+\left(v_iv_j\right)$ be a contributing set of f^+, which is a set of all contributing elements of f^+. Define $\left|f'^+\left(v_iv_j\right)\right|$ as contributions corresponding to $f'^+\left(v_iv_j\right)$ of f^+. If $f'^+\left(v_iv_j\right) = \emptyset$, then $\left|f'^+\left(v_iv_j\right)\right| = 0$. The definitions of $g'^-\left(v_iv_j\right)$ and $\left|g'^-\left(v_iv_j\right)\right|$ are similar to $f'^+\left(v_iv_j\right)$ and $\left|f'^+\left(v_iv_j\right)\right|$, respectively.*

According to Definition 3.1, it is not difficult to find that

$$\sum_{i,j}\left|f'^+\left(v_iv_j\right)\right| = \left|f^+\right| \text{ and } \sum_{i,j}\left|g'^-\left(v_iv_j\right)\right| = \left|g^-\right|.$$

Remark 3.1. *[1] For the matching mK_2 with $m \geq 1$, we have $s(mK_2) = 1$ and $d(mK_2) = 1$. The inverse problem is also true.*

Remark 3.2. *[1] For the cycle C_n with $n \geq 3$, we have $s(C_n) = 3$ and $d(C_n) = 2$.*

Theorem 3.1. *For the necklace graph, or briefly by N_{ek} for $k \geq 1$, we have $s(N_{ek}) = 5$ and $d(N_{ek}) = 3$.*

Proof. Note that the degree sequence of N_{ek} is $\pi = (3, 3, \cdots, 3)$, and hence $s(N_{ek}) \geq 5$ by Theorem 2.3. In what follows, we just find a vertex labeling f such that $|f^+| = 5$.

Let $f : V(N_{ek}) \to \mathbb{Z}$ be an injective labeling, where $V(N_{ek}) = \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k, c_1, c_2\}$. For $1 \leq i \leq k$, we label the vertices in $V(N_{ek})$ according to the following scheme.

$$f(a_i) = \begin{cases} i, & \text{if } i \text{ is odd,} \\ -i, & \text{if } i \text{ is even,} \end{cases}$$

$$f(b_i) = -f(a_i),$$

$$f\left(c_j\right) = \begin{cases} 0, & \text{if } j = 1, \\ k+1, & \text{if } j = 2, \end{cases}$$

where $c_1a_1 \in E(N_{ek})$, $c_1b_1 \in E(N_{ek})$, $c_1c_2 \in E(N_{ek})$, $c_2a_k \in E(N_{ek})$, $c_2b_k \in E(N_{ek})$, $a_1b_1 \in E(N_{ek})$, $a_1a_2 \in E(N_{ek})$ and $b_1b_2 \in E(N_{ek})$. For $2 \leq i \leq k-1$, we observe that $a_{i-1}a_i \in E(N_{ek})$, $a_ia_{i+1} \in E(N_{ek})$, $a_ib_i \in E(N_{ek})$, $b_{i-1}b_i \in E(N_{ek})$ and $b_ib_{i+1} \in E(N_{ek})$ (see Figure 1).

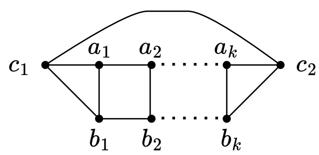

Figure 1. A necklace graph N_{ek}.

It is clear to see that under the vertex labeling f we have

$$s(N_{ek}) \leq |\{0, 1, -1, k+1, 2k+1\}| = 5.$$

On the other hand, we will show that $d(N_{ek}) = 3$. Since we know $d(N_{ek}) \geq 3$ by Theorem 2.3, it suffices to find a vertex labeling g such that $|g^-| = 3$.

Let $g : V(N_{ek}) \to \mathbb{Z}$ be an injective vertex labeling of N_{ek}. For $1 \leq i \leq k$, we consider the vertex labeling g defined such that

$$g(a_i) = 2i - 1,\ g(b_i) = 2i,$$

$$g\left(c_j\right) = \begin{cases} 0, & \text{if } j = 1, \\ 2k+1, & \text{if } j = 2. \end{cases}$$

It is clear to see that

$$d(N_{ek}) \leq |\{1, 2, 2k+1\}| = 3.$$

We illustrate the vertex labelings f and g of N_{e4}, respectively (see Figure 2). □

Figure 2. A sum index labeling f and a difference index labeling g of N_{e4}.

Remark 3.3. *[1, 15] The prism graph is also 3-regular, which satisfies* $s(\Pi_n) = 5$ *and* $d(\Pi_n) = 3$.

Theorem 3.2. *The complement of a matching* mK_2*, or briefly by* $\overline{mK_2}$*, is* $(n-2)$*-regular for* $m \geq 4$*. Then*

$$s(\overline{mK_2}) = 2n - 5 \text{ and } d(\overline{mK_2}) = n - 2.$$

Proof. Throughout this proof, since the complement of a matching mK_2 is $(n-2)$-regular. The degree sequence of $\overline{mK_2}$ is $\pi = (n-2, n-2, \cdots, n-2)$. According to the properties of degree sequence, we know $\sum_{v \in V(G)} d_G(v) = 2|E|$. It implies that n is even.

Firstly, we discuss the sum index of $\overline{mK_2}$. By Theorem 2.3, we have $s(\overline{mK_2}) \geq 2n - 5$. Then we just find a vertex labeling f such that $|f^+| = 2n - 5$. For $1 \leq i \leq n$, let $f : V(\overline{mK_2}) \to \mathbb{Z}$ be an injective labeling such that

$$f(v_i) = \begin{cases} -\frac{i-1}{2}, & \text{if } i \text{ is odd}, \\ \frac{i}{2}, & \text{if } i \text{ is even}, \end{cases}$$

where $v_j v_{j+1} \notin E(\overline{mK_2})$ and $v_{n+1} = v_1$ for $2 \leq j \leq n$ and j is even.

According to the labeling f and Definition 3.1, we observe that the following three equations hold.

$$\left|f'^{+}(v_1 v_i)\right| = \left|\left\{1, 2, \cdots, \frac{n}{2} - 1, -1, -2, \cdots, -\left(\frac{n}{2} - 1\right)\right\}\right| = n - 2,$$

$$\left|f'^{+}(v_n v_i)\right| = \left|\left\{\frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, \frac{n}{2} + \left(\frac{n}{2} - 1\right)\right\}\right| = \frac{n}{2} - 1,$$

$$\left|f'^{+}(v_{n-1}v_i)\right| = \left|\left\{-\left(\frac{n}{2}-1+1\right), -\left(\frac{n}{2}-1+2\right), \cdots, -\left(\frac{n}{2}-1+\frac{n}{2}-2\right)\right\}\right| = \frac{n}{2}-2.$$

Except the above contributions corresponding three contributing sets, the sum of other contributions of f^+ is equal to 0. Therefore we know

$$s(\overline{mK_2}) \leq \left|f^+\right| = (n-2) + \left(\frac{n}{2}-1\right) + \left(\frac{n}{2}-2\right) = 2n-5.$$

Next, we consider the difference index of $\overline{mK_2}$. Note that $d(\overline{mK_2}) \geq n-2$ by Theorem 2.3. It remains to show that $d(\overline{mK_2}) \leq n-2$. Let $g : V(\overline{mK_2}) \to \mathbb{Z}$ be an injective labeling, where $g(v_i) = f(v_i)$ for $1 \leq i \leq n$. We assume that all vertices of $\overline{mK_2}$ satisfy $v_j v_{j+1} \notin E(\overline{mK_2})$, when j is odd and $1 \leq j \leq n-1$.

By Definition 3.1, it holds that

$$\left|g'^{-}(v_1 v_i)\right| = \left|\left\{1, 2, \cdots, \frac{n}{2}\right\}\right| = \frac{n}{2},$$
$$\left|g'^{-}(v_n v_i)\right| = \left|\left\{\frac{n}{2}+1, \frac{n}{2}+2, \cdots, \frac{n}{2}+\left(\frac{n}{2}-2\right)\right\}\right| = \frac{n}{2}-2.$$

The sum of other contributions of g^- is equal to 0, if the contributions corresponding two contributing sets above are excluded. Thus we have

$$d(\overline{mK_2}) \leq \left|g^-\right| = \frac{n}{2} + \left(\frac{n}{2}-2\right) = n-2.$$

We illustrate the vertex labelings f and g of $\overline{6K_2}$, respectively (see Figure 3). □

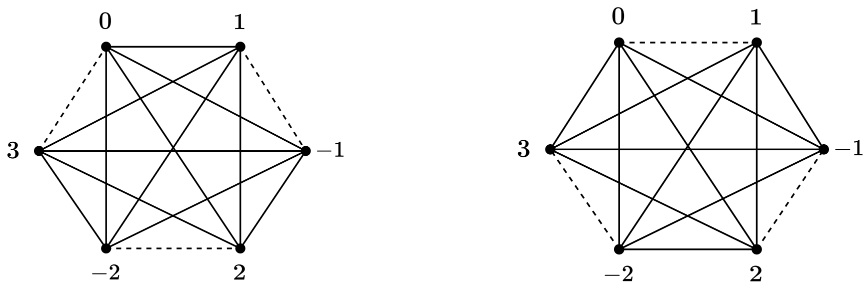

Figure 3. A sum index labeling f and a difference index labeling g of $\overline{6K_2}$.

Next, we achieve the following theorems of the complements of a cycle C_n and a path P_n relating to sum and difference indices, respectively.

Theorem 3.3. *The complement of a cycle C_n, or briefly by $\overline{C_n}$, is $(n-3)$-regular for $n \geq 6$. Then*

$$s(\overline{C_n}) = 2n-7 \text{ and } d(\overline{C_n}) = n-3.$$

Proof. To begin with, we consider the sum index of $\overline{C_n}$. By Theorem 2.3, we have $s(\overline{C_n}) \geq 2n-7$. It remains to show that $s(\overline{C_n}) \leq 2n-7$ by defining a vertex labeling f such that $|f^+| = 2n-7$. For $1 \leq i \leq n$, let $f : V(\overline{C_n}) \to \mathbb{Z}$ be an injective labeling such that

$$f(v_i) = \begin{cases} -\frac{i-1}{2}, & \text{if } i \text{ is odd,} \\ \frac{i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

In what follows, we distinguish two cases according to the parity of n.

Case 1. If n is even.

In this case, we assume that $v_1v_2 \notin E(\overline{C_n})$ and $v_{n-1}v_n \notin E(\overline{C_n})$. For each $1 \leq m \leq \frac{n-2}{2}$, all vertices of $\overline{C_n}$ satisfy $v_{2m}v_{2m+2} \notin E(\overline{C_n})$ and $v_{2m-1}v_{2m+1} \notin E(\overline{C_n})$. Then Definition 3.1 implies that

$$\begin{aligned}
\left|f'^{+}(v_1v_i)\right| &= \left|\left\{2, 3, \cdots, \frac{n}{2}, -2, -3, \cdots, -(\frac{n}{2}-1)\right\}\right| = n-3, \\
\left|f'^{+}(v_nv_i)\right| &= \left|\left\{\frac{n}{2}+1, \frac{n}{2}+2, \cdots, \frac{n}{2}+\left(\frac{n}{2}-2\right)\right\}\right| = \frac{n}{2}-2, \\
\left|f'^{+}(v_{n-1}v_i)\right| &= \left|\left\{0, -\left(\frac{n}{2}-1+1\right), -\left(\frac{n}{2}-1+2\right), \cdots, -\left(\frac{n}{2}-1+\frac{n}{2}-3\right)\right\}\right| = \frac{n}{2}-2.
\end{aligned}$$

Except the above contributions corresponding three contributing sets, the sum of other contributions of f^+ is equal to 0. Therefore we obtain

$$s(\overline{C_n}) \leq \left|f^+\right| = (n-3) + \left(\frac{n}{2}-2\right) + \left(\frac{n}{2}-2\right) = 2n-7.$$

Case 2. If n is odd.

If n is odd, we assume that $v_1v_2 \notin E(\overline{C_n})$, $v_1v_3 \notin E(\overline{C_n})$ and $v_{n-1}v_n \notin E(\overline{C_n})$. For each $1 \leq m \leq \frac{n-3}{2}$, all vertices of $\overline{C_n}$ satisfy $v_{2m}v_{2m+2} \notin E(\overline{C_n})$ and $v_{2m+1}v_{2m+3} \notin E(\overline{C_n})$. Further, Definition 3.1 implies that

$$\begin{aligned}
\left|f'^{+}(v_1v_i)\right| &= \left|\left\{2, 3, \cdots, \frac{n}{2}, -2, -3, \cdots, -(\frac{n}{2}-1)\right\}\right| = n-3, \\
\left|f'^{+}(v_{n-1}v_i)\right| &= \left|\left\{\frac{n-1}{2}+1, \frac{n-1}{2}+2, \cdots, \frac{n-1}{2}+\frac{n-5}{2}\right\}\right| = \frac{n-5}{2}, \\
\left|f'^{+}(v_nv_i)\right| &= \left|\left\{-\left(\frac{n-1}{2}+1\right), -\left(\frac{n-1}{2}+2\right), \cdots, -\left(\frac{n-1}{2}+\frac{n-5}{2}\right)\right\}\right| = \frac{n-5}{2}.
\end{aligned}$$

The sum of other contributions of f^+ is equal to 1, if the contributions above are excluded. Thus based on the definition of the sum index, we obtain

$$s(\overline{C_n}) \leq \left|f^+\right| = (n-3) + \left(\frac{n-5}{2}\right) + \left(\frac{n-5}{2}\right) + 1 = 2n-7.$$

We illustrate the sum index labelings f of $\overline{C_7}$ and $\overline{C_8}$, respectively (see Figure 4).

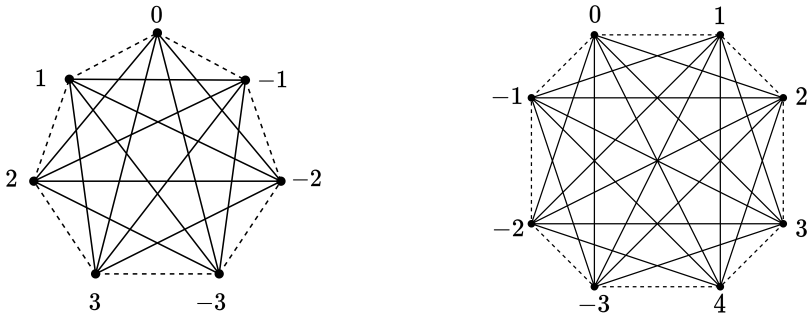

Figure 4. Sum index labelings f of $\overline{C_7}$ and $\overline{C_8}$.

On the other hand, we consider the difference index of $\overline{C_n}$. Note that $d(\overline{C_n}) \geq n-3$ by Theorem 2.3. Then it remains to show that $d(\overline{C_n}) \leq n-3$. For $1 \leq i \leq n$, let $g : V(\overline{C_n}) \to \mathbb{Z}$ be an injective labeling such that $g(v_i) = f(v_i)$. Below, we distinguish two cases according to the parity of n.

Case 1. If n is even.

If n is even, then we assume that $v_1v_2 \notin E(\overline{C_n})$ and $v_{n-1}v_n \notin E(\overline{C_n})$. And for each $1 \leq m \leq \frac{n-2}{2}$, all vertices of $\overline{C_n}$ satisfy $v_{2m}v_{2m+2} \notin E(\overline{C_n})$ and $v_{2m-1}v_{2m+1} \notin E(\overline{C_n})$.

In addition, Definition 3.1 enables us to ensure that

$$\left|g'^{-}(v_1v_i)\right| = \left|\left\{2, 3, \cdots, \frac{n}{2}\right\}\right| = \frac{n}{2} - 1,$$
$$\left|g'^{-}(v_nv_i)\right| = \left|\left\{\frac{n}{2}+1, \frac{n}{2}+2, \cdots, \frac{n}{2}+\left(\frac{n}{2}-2\right)\right\}\right| = \frac{n}{2} - 2.$$

The sum of other contributions of g^{-} is equal to 0, if the contributions corresponding two contributing sets above are excluded. Thus, we have

$$d(\overline{C_n}) \leq \left|g^{-}\right| = \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 2\right) = n - 3.$$

Case 2. If n is odd.

If n is odd, then we assume that $v_1v_2 \notin E(\overline{C_n})$, $v_1v_3 \notin E(\overline{C_n})$, $v_{n-5}v_{n-2} \notin E(\overline{C_n})$, $v_{n-4}v_{n-3} \notin E(\overline{C_n})$, $v_{n-3}v_n \notin E(\overline{C_n})$, $v_{n-2}v_{n-1} \notin E(\overline{C_n})$ and $v_{n-1}v_n \notin E(\overline{C_n})$. And for each $1 \leq m \leq \frac{n-7}{2}$, all vertices of $\overline{C_n}$ satisfy $v_{2m}v_{2m+2} \notin E(\overline{C_n})$ and $v_{2m+1}v_{2m+3} \notin E(\overline{C_n})$. By Definition 3.1, it holds that

$$\left|g'^{-}(v_1v_i)\right| = \left|\left\{2, 3, \cdots, \frac{n-1}{2}\right\}\right| = \frac{n-1}{2} - 1,$$
$$\left|g'^{-}(v_nv_i)\right| = \left|\left\{1, \frac{n-1}{2}+1, \frac{n-1}{2}+2, \cdots, \frac{n-1}{2}+\left(\frac{n-1}{2}-2\right)\right\}\right| = \frac{n-1}{2} - 1.$$

Except the above contributions corresponding two contributing sets, the sum of other contributions of g^{-} is equal to 0. It is clear to see that under the vertex labeling g we have

$$d(\overline{C_n}) \leq \left|g^{-}\right| = \frac{n-1}{2} - 1 + \frac{n-1}{2} - 1 = n - 3.$$

We illustrate difference index labelings g of $\overline{C_7}$ and $\overline{C_8}$, respectively (see Figure 5).

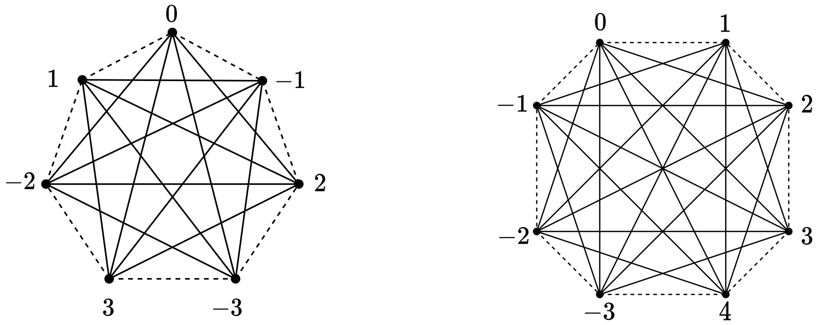

Figure 5. Difference index labelings g of $\overline{C_7}$ and $\overline{C_8}$.

□

Theorem 3.4. *Let $\overline{P_n}$ be the complement of a path P_n for $n \geq 6$. Then*

$$s(\overline{P_n}) = 2n - 6 \text{ and } d(\overline{P_n}) = n - 3.$$

Proof. In this proof, let $\overline{P_n}$ be the complement of a path P_n for $n \geq 6$. Then its degree sequence is $\pi = (n-2, n-2, n-3, n-3, \cdots, n-3)$. Without loss of generality, we assume that $d(v_1) = d(v_2) = n-2$, $d(v_i) = n-3$ for $3 \leq i \leq n$.

Firstly, we discuss the sum index of $\overline{P_n}$. Property 1.4 implies that $s(\overline{P_n}) \geq 2n-6$. Then it remains to show that $s(\overline{C_n}) \leq 2n-7$ by defining a vertex labeling f such that $|f^+| = 2n-6$. Let $f : V(\overline{C_n}) \to \mathbb{Z}$ be an injective labeling of $\overline{C_n}$ such that

$$f(v_i) = \begin{cases} -\frac{i-1}{2}, & \text{if } i \text{ is odd,} \\ \frac{i}{2}, & \text{if } i \text{ is even,} \end{cases}$$

for $1 \leq i \leq n$. Below, we distinguish two cases according to the parity of n.

Case 1. If n is even.

If n is even, then we assume that $v_1v_2 \notin E(\overline{P_n})$ and $v_{n-1}v_n \notin E(\overline{P_n})$. And for each $1 \leq m \leq \frac{n-2}{2}$, all vertices of $\overline{P_n}$ satisfy $v_{2m}v_{2m+2} \notin E(\overline{P_n})$ and $v_{2m-1}v_{2m+1} \notin E(\overline{P_n})$.

By Definition 3.1, we note that

$$\begin{aligned} \left|f'^+(v_1v_i)\right| &= \left|\left\{1, 2, 3, \cdots, \frac{n}{2}, -2, -3, \cdots, -(\frac{n}{2}-1)\right\}\right| = n-2, \\ \left|f'^+(v_nv_i)\right| &= \left|\left\{\frac{n}{2}+1, \frac{n}{2}+2, \cdots, \frac{n}{2}+\left(\frac{n}{2}-2\right)\right\}\right| = \frac{n}{2}-2, \\ \left|f'^+(v_{n-1}v_i)\right| &= \left|\left\{0, -\left(\frac{n}{2}-1+1\right), -\left(\frac{n}{2}-1+2\right), \cdots, -\left(\frac{n}{2}-1+\frac{n}{2}-3\right)\right\}\right| = \frac{n}{2}-2. \end{aligned}$$

We find that except the above contributions corresponding three contributing sets, the sum of other contributions of f^+ is equal to 0. Thus we obtain

$$s(\overline{P_n}) \leq \left|f^+\right| = (n-2) + \left(\frac{n}{2}-2\right) + \left(\frac{n}{2}-2\right) = 2n-6.$$

Case 2. If n is odd.

If n is odd, then we assume that $v_1v_2 \in E(\overline{P_n})$, $v_1v_3 \notin E(\overline{P_n})$ and $v_{n-1}v_n \notin E(\overline{P_n})$. And for each $1 \le m \le \frac{n-3}{2}$, all vertices of $\overline{P_n}$ satisfy $v_{2m}v_{2m+2} \notin E(\overline{P_n})$ and $v_{2m+1}v_{2m+3} \notin E(\overline{P_n})$.

By Definition 3.1, it holds that

$$\begin{aligned}
\left|f'^{+}(v_1v_i)\right| &= \left|\left\{1, 2, 3, \cdots, \frac{n-1}{2}, -2, -3, \cdots, -\left(\frac{n-1}{2}\right)\right\}\right| = n-2, \\
\left|f'^{+}(v_{n-1}v_i)\right| &= \left|\left\{\frac{n-1}{2}+1, \frac{n-1}{2}+2, \cdots, \frac{n-1}{2}+\left(\frac{n-5}{2}\right)\right\}\right| = \frac{n-5}{2}, \\
\left|f'^{+}(v_nv_i)\right| &= \left|\left\{-\left(\frac{n-1}{2}+1\right), -\left(\frac{n-1}{2}+2\right), \cdots, -\left(\frac{n-1}{2}+\frac{n-5}{2}\right)\right\}\right| = \frac{n-5}{2}.
\end{aligned}$$

We claim that except the above contributions corresponding three contributing sets, the sum of other contributions of f^+ is equal to 1.

As a result, we have

$$s(\overline{P_n}) \le \left|f^+\right| = (n-2) + \left(\frac{n-5}{2}\right) + \left(\frac{n-5}{2}\right) + 1 = 2n-6.$$

We illustrate sum index labelings f of $\overline{P_7}$ and $\overline{P_8}$, respectively (see Figure 6).

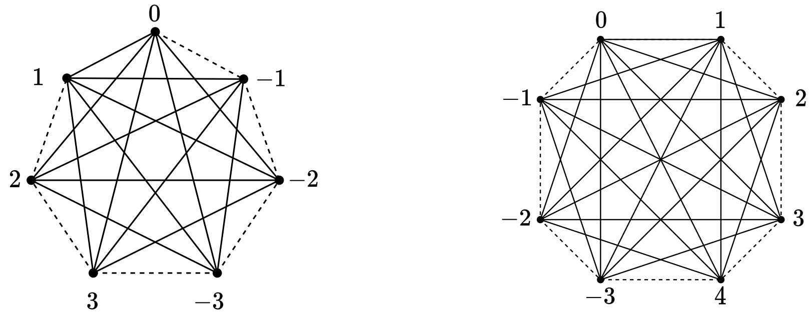

Figure 6. Sum index labelings f of $\overline{P_7}$ and $\overline{P_8}$.

Next, we consider the difference index of $\overline{P_n}$. We have $d(\overline{P_n}) \ge n-3$ by Property 1.4, which means that we just find a vertex labeling g such that $|g^-| = n-3$ in the following. We distinguish two cases according to the parity of n.

Case 1. If n is even.

Let $g : V(\overline{P_n}) \to \mathbb{Z}$ be an injective labeling of $\overline{P_n}$ such that

$$g(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ -\frac{i}{2}, & \text{if } i \text{ is even,} \end{cases}$$

for $1 \le i \le n$.

In this case, we assume that $v_1v_2 \in E(\overline{P_n})$, $v_{n-5}v_{n-2} \notin E(\overline{P_n})$, $v_{n-4}v_{n-3} \notin E(\overline{P_n})$, $v_{n-3}v_n \notin E(\overline{P_n})$, $v_{n-2}v_{n-1} \notin E(\overline{P_n})$ and $v_{n-1}v_n \notin E(\overline{P_n})$. And for each $1 \le m \le \frac{n-6}{2}$, all vertices of $\overline{P_n}$ satisfy $v_{2m}v_{2m+2} \notin E(\overline{P_n})$ and $v_{2m-1}v_{2m+1} \notin E(\overline{P_n})$.

According to Definition 3.1, we have

$$\left|g'^{-}(v_1v_i)\right| = \left|\left\{2, 3, \cdots, \frac{n}{2}+1\right\}\right| = \frac{n}{2},$$
$$\left|g'^{-}(v_nv_i)\right| = \left|\left\{\frac{n}{2}+2, \frac{n}{2}+3, \cdots, \frac{n}{2}+\left(\frac{n}{2}-2\right)\right\}\right| = \frac{n}{2}-3.$$

Since the sum of other contributions of g^- is equal to 0 excepting the above contributions.

As a result, it holds that

$$d(\overline{P_n}) \le \left|g^-\right| = \frac{n}{2} + \left(\frac{n}{2}-3\right) = n-3.$$

Case 2. If n is odd.

Let $g : V(\overline{P_n}) \to \mathbb{Z}$ be an injective labeling. For $1 \le i \le n$, we define g of $\overline{P_n}$ as

$$g(v_i) = \begin{cases} -\frac{i-1}{2}, & \text{if } i \text{ is odd,} \\ \frac{i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

In this case, we assume that $v_1v_2 \in E(\overline{P_n})$, $v_1v_3 \notin E(\overline{P_n})$, $v_{n-5}v_{n-2} \notin E(\overline{P_n})$, $v_{n-4}v_{n-3} \notin E(\overline{P_n})$, $v_{n-3}v_n \notin E(\overline{P_n})$, $v_{n-2}v_{n-1} \notin E(\overline{P_n})$ and $v_{n-1}v_n \notin E(\overline{P_n})$. Note that all vertices of $\overline{P_n}$ satisfy $v_{2m}v_{2m+2} \notin E(\overline{P_n})$ and $v_{2m+1}v_{2m+3} \notin E(\overline{P_n})$ for each $1 \le m \le \frac{n-7}{2}$.

Hence, according to Definition 3.1, it follows that

$$\left|g'^{-}(v_1v_i)\right| = \left|\left\{1, 2, 3, \cdots, \frac{n-1}{2}\right\}\right| = \frac{n-1}{2},$$
$$\left|g'^{-}(v_nv_i)\right| = \left|\left\{\frac{n-1}{2}+1, \frac{n-1}{2}+2, \cdots, \frac{n-1}{2}+\left(\frac{n-5}{2}\right)\right\}\right| = \frac{n-5}{2}.$$

The sum of other contributions of g^- is equal to 0 excepting the above contributions corresponding two contributing sets.

As a result, we have

$$d(\overline{P_n}) \le \left|g^-\right| = \frac{n-1}{2} + \frac{n-5}{2} = n-3.$$

We illustrate difference index labelings g of $\overline{P_7}$ and $\overline{P_8}$, respectively (see Figure 7). □

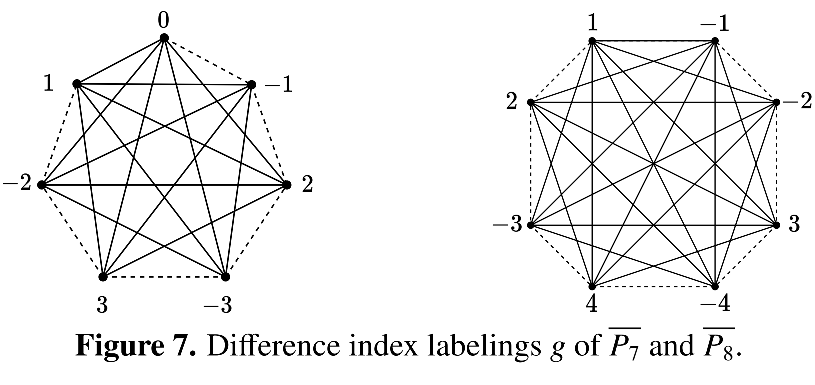

Figure 7. Difference index labelings g of $\overline{P_7}$ and $\overline{P_8}$.

4. Conclusions

In this article, we first obtain the properties of sum index labelings and difference index labelings. We also improve the sharp lower bounds on sum and difference indices of generalized regular graphs. In this regard, the sum indices and difference indices of the necklace graphs and the complements of matchings, cycles and paths are determined. Finally, we compare the sum indices with the (integral) sum numbers of several types of graphs and find that there are certain relationships in this section (see Table 1).

According to the sum indices and difference indices of regular graphs in this paper, the values of two invariants of a part of known regular graphs are listed in Table 2. Preliminary investigations on regular graphs lead us to the following conjecture.

Conjecture 4.1. *If G is d-regular, then $d(G) = d$.*

Nowadays, the problems of the (integral) sum numbers have been studied comprehensively, while the problems of the sum and difference indices of graphs constitute challenges for researchers in graph labeling theory. Based on Table 1, we find that there exist certain relationships between the sum indices and the (integral) sum numbers of graphs. Of course, the current manuscript defines a starting point for the study of the sum and difference indices of regular graphs. However, a much deeper study is required to determine these two invariants of more graph families including regular graphs. Thus it would be interesting to find solutions to the questions below.

Table 1. Comparing sum index with (integral) sum number.

	sum index	sum number	integral sum number
matchings	1	1	0
cycles	3	$3(n = 4)$	0
complements of cycles	$2n - 7(n \geq 6)$	$2n - 7(n \geq 7)$	$2n - 7(n \geq 7)$
complements of matchings	$2n - 5(n \geq 4)$	$2n - 5(n \geq 7)$	$2n - 5(n \geq 7)$
complete graphs	$2n - 3\ (n \geq 2)$	$2n - 3(n \geq 4)$	$2n - 3(n \geq 4)$

Problem 4.1. *What are the specific relationships between the (integral) sum numbers and sum indices, or in what graphs do these relationships exist referring to Table 1?*

Problem 4.2. *As we all know, the generalized Peterson graphs have been extensively investigated as many nice structural and algorithmic properties. Determine the sum and difference indices of generalized Peterson graphs, which are 3-regular.*

Problem 4.3. *The torus grid graph is the graph formed from the Cartesian product $C_m \times C_n$ of the cycle graphs C_m and C_n. Try to determine the sum and difference indices of torus grid graphs, which are 4-regular.*

Table 2. The sum index and difference index of regular graphs.

	sum index	difference index
matchings	1	1
cycles	3	2
necklace graphs	5	3
prism graphs	5	3
complements of cycles	$2n - 7$	$n - 3$
complements of matchings	$2n - 5$	$n - 2$
complete graphs	$2n - 3$	$n - 1$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest that could affect the publication of this paper.

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