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Research article

An optimal fractional-order accumulative Grey Markov model with variable parameters and its application in total energy consumption

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Abstract: In this paper, we propose an optimal fractional-order accumulative Grey Markov model with variable parameters (FOGMKM (1,1)) to predict the annual total energy consumption in China and improve the accuracy of energy consumption forecasting. The new model is built upon the traditional Grey model and utilized matrix perturbation theory to study the natural and response characteristics of a system when the structural parameters change slightly. The particle swarm optimization algorithm (PSO) is used to determine the number of optimal fractional order and nonlinear parameters. An experiment is conducted to validate the high prediction accuracy of the FOGMKM (1,1) model, with mean absolute percentage error (MAPE) and root mean square error (RMSE) values of 0.51% and 1886.6, respectively, and corresponding fitting values of 0.92% and 6108.8. These results demonstrate the superior fitting performance of the FOGMKM (1,1) model when compared to other six competitive models, including GM (1,1), ARIMA, Linear, FAONGBM (1,1), FGM (1,1) and FOGM (1,1). Our study provides a scientific basis and technical references for further research in the finance as well as energy fields and can serve well for energy market benchmark research.

Keywords: FOGMKM (1,1); variable parameters; fractional grey models; energy consumption; parameter estimation and prediction; stochastic Markov process
Mathematics Subject Classification: 62M05, 62M10, 62P05, 62P20,

1. Introduction

Energy is of paramount importance for human survival, societal progress, and national economic development. However, the excessive energy consumption poses a challenge to sustainable development for the society. Consequently, the exploration of factors that can ensure a stable energy supply and constrain accelerated energy consumption has become a priority concern for countries and regions in worldwide. Over the past few decades, the rapid growth of China's economy has drawn a global attention to its energy consumption patterns.

Due to the complexity of energy consumption systems, they are influenced by a great number of factors and constraints. In response to the international situation and domestic development needs, researching total energy consumption has attracted increasing attention and extensive interest among scholars. This research has a long history in the field of statistics, which can be mostly divided into two aspects: studying the impact of single or multiple factors on total energy consumption through factor decomposition methods and researching the future trend of total energy consumption. A large number of scholars have discussed the factors that influence total energy consumption. For example, Guo [1] analyzed the characteristics of spatial differentiation of energy consumption intensity in 30 Chinese provinces in 2019. Through empirical research, Guo identified the different factors that influence energy consumption intensity in different regions. They found that technological progress and industrial structure are key factors that influence energy consumption intensity, and different factors have varying impacts on a province's energy consumption intensity. In reference [2], the researchers used the STIRPAT model to decompose the factors that influence the energy consumption of each prefecture from five aspects: scale effect, economic benefit, technical effect, structural effect, and urbanization effect. As a result, they concluded that the influence of population on energy consumption was the greatest. In the study conducted by Zhang et al. [3], the researchers considered the driving factors of energy intensity in Beijing based on the structural decomposition technology of input-output analysis. They decomposed these factors into five components, including energy input coefficient, complete demand coefficient, final demand structure coefficient, final demand and final energy consumption coefficient. Sainu et al. [4] empirically studied the temporal, dynamic and causal relationships between urbanization, energy consumption and emissions using census data from 1901 to 2011 in India, with the aim of understanding the urbanization process, including the level and rhythm of urbanization and urban growth patterns. They also reviewed the increase in energy consumption and emissions under the context of rapid urbanization. Yang et al. [5] conducted a study using the logarithmic mean Divisia index (LMDI) method to examine the impact of urbanization on the growth of renewable energy consumption. They found that the growth of renewable energy consumption can be attributed to factors such as urbanization effects, energy structure effects, energy intensity effects, economic effects and population effects. The growth of renewable energy consumption was divided into three stages: slow growth, fluctuating growth and accelerated growth. He et al. [6] investigated the sustainable development of energy in rural Chinese households, particularly the distributed use of renewable energy. They emphasized the importance of synergistic effects among different energy sources, technologies and stakeholders, as well as flexible engineering schemes and policy designs that consider temporal changes and regional differences. Chong et al. [7] utilized the LMDI decomposition method to explore the factors influencing energy consumption growth in Guangdong Province. They integrated the input-output method and established the connection between final energy utilization and major energy consumption by deriving the main energy conversion coefficients. Zhou and Feng [8] focused on the intricate relationship between environmental regulation and fossil fuel consumption. They analyzed the direct and indirect pathways through which environmental regulation affects fossil fuel consumption. Additionally, they incorporated three types of technologies (production technology, pollution prevention technology and standby technology) in their model to identify the indirect effects of environmental regulation on fossil fuel consumption.

On the other hand, many scholars have conducted extensive research on energy consumption trends. For instance, Wang and Zhang [9] adopted the trend method to predict China's total energy consumption and compared it with the linear regression method. The empirical results showed that the trend prediction method has a smaller prediction error than the linear regression method. Paul et al. [10] used a nonlinear autoregressive neural network (NARNET) to predict the energy consumption of four campuses of a South African university. They filtered the data based on three-year daily energy consumption data and applied the Singular Spectrum Analysis (SSA) technique. Moreover, Emmanuel and Chandana [11] employed various parametric and non-parametric prediction techniques to analyze and evaluate energy consumption forecasts of electricity, petroleum, coal and renewable energy sources in Sri Lanka. Wang et al. [12] proposed a clustering model based on set pair analysis (SPA), and used Fisher's optimal partition method to perform cluster analysis on the annual dynamic relative index (DRI) of historical energy consumption. Christopher [13] studied the interaction between climate variability and residential electricity consumption in a state in the southeast United States. He conducted a survey of residential electricity consumers to gain a better understanding of how to promote positive attitudes and behavior related to energy efficiency (EE) in households. After taking into account changes in various sectors and identifying possible deviation limits, their aim was to generate long-term forecasts of global energy consumption and projections of total energy consumption at the global level [14]. Wu et al. [15] proposed a new Grey system model with fractional order accumulation, which can better reflect the priority of new information as the accumulation order number becomes smaller in the in-sample model. Since then, considerable theoretical and application research has been developed in this area. Fan et al. [16] first proposed a hybrid prediction model, the GM-S-SIGM-GA model, to forecast China's natural gas demand from 2011 to 2017, which constructed a grey model (GM (1,1)) and an adaptive intelligent grey model (SIGM). Ma and Liu [17] discussed a novel time-delayed polynomial grey prediction model (TDPGM (1, 1)) based on the grey system theory. They compared its performance in forecasting the natural gas consumption of China with commonly used prediction models. Based on statistical data of natural gas consumption in China from 1995 to 2011, Zhang and Zhou [18] used the Boltzmann model and a third-order polynomial curve model to fit the historical data. Their aim was to explore past variation tendencies and identify a suitable model for forecasting. The fitting results demonstrated that a combination model based on the Boltzmann model and a third-order polynomial curve model showed excellent fit. To maximize the utilization of existing grey system models, ensemble learning was employed to develop a new strategy for constructing forecasting models for electricity supply in China [19]. Two numerical validation cases were conducted to validate the proposed method in comparison with other well-known models. Additionally, a novel time-delayed fractional grey model has been developed to forecast natural gas consumption, considering the time-delayed effects [20]. Theoretical analysis showed that it had a more general formulation, was unbiased and had greater flexibility than existing models. Wu et al. [21] investigated natural gas consumption in the United States, Germany, the United Kingdom, China and Japan using a new Grey Bernoulli model. They also derived analytical formulations for the time response function, restored values and linear parameter estimation. Chen et al. [22] established a novel Fractional Hausdorff

Discrete Grey Model (FHDGM (1,1)) to forecast renewable energy consumption for the years 2021 to 2023 in three regions: the Asia Pacific, Europe, and the world. By improving the grey action quantity of the traditional grey model with an exponential time term, She et al. [23] proposed a novel power-driven grey model to forecast the total residential and thermal energy consumption in China as reference data for decision makers. Zhang et al. [24] constructed the Fractional Order Cumulative Multivariate Grey Model with Equal-Dimensional Recursive Optimization (EFMGM (1,2)) to predict the trend of carbon emissions from fossil energy consumption and constant-price GDP in China and calculated the carbon emissions from fossil energy consumption. Li et al. [25] provided two time series models, namely, the improved grey model (IGM (0,n)) and the Optimized fractional Grey model (OFGM (1,1)), to forecast waste water discharge and energy consumption in China. He et al. [26] established a novel structure adaptive new information priority discrete grey prediction model to forecast long term renewable energy generation, and the disturbance analysis shows that it is suitable for small sample modeling. The grey model was systematically studied based on the new definitions of the conformable fractional accumulation and difference to analysis the carbon dioxide emissions of BRICS [27]. Wang et al. [28] proposed a novel structural adaptive grey model FCSAGM (p,1) with Caputo fractional derivative and a new Caputo fractional order accumulation generation operator to predict China's total energy consumption, China's total primary energy production, China's thermal power generation and China's hydropower generation. A time-delayed power effect with high flexibility was considered to develop a new grey system model, which can be more efficient in dealing with small and complex time series and shares a more general formulation [29]. Wang et al. [30] proposed a novel fractional structural adaptive grey Chebyshev polynomial Bernoulli model to forecast China's renewable energy. They used Monte Carlo simulation and probability density analysis to illustrate the robustness and accuracy of the proposed model. Based on the above research results, many scholars have made a great deal of efforts to the grey prediction model, and it is easy to generate random error among these models in fact. To capture the nonlinear trend in annual energy consumption data of China and obtain an appreciate prediction accuracy, we propose a FOGMKM (1,1) model.

Although the Grey prediction model excels in handling short-term data in time series, it struggles to accurately make long-term predictions, especially when capturing nonlinear relationships within large datasets. To overcome these limitations and enhance prediction accuracy, we propose using the FOGM (1,1) model to capture nonlinear trends and improve the accuracy of predicting annual energy consumption data in China. The major contributions of this study can be summarized as follows:

1) The FOGMKM (1,1) model is established based on the Particle Swarm Optimization (PSO) algorithm. The optimal order and nonlinear parameters of the FOGMKM (1,1) model are determined by minimizing the mean relative errors.

2) Concrete expressions for the estimated and predicted values of the FOGMKM (1,1) model are constructed based on the Markov transition probability matrix and state division.

3) The proposed model's validity is verified through numerical examples and applied to forecast China's annual energy consumption. Six comparative models are analysed with the proposed model. The validity and robustness of each model are expressed by the statistics R^2 and F.

The paper is organized as follows: Section 2 constructs the FOGMKM (1,1) model using the FOGM (1,1) model and the PSO algorithm. It also illustrates the steps to modify the forecast using the Markov model. Section 3 displays the application of the model, presenting the results of estimation and prediction using the FOGM (1,1) model and Markov model for modifying energy consumption. Finally, Section 4 concludes the paper and discusses its implications.

2. The FOGMKM (1,1) model

2.1. The evolution of Grey model and Markov Chain

The fractional grey model has received significant attention in the field of grey systems in recent years. The classical grey model is based on 1-AGO. Wu et al. [15] proposed a fractional grey model based on FAGO, followed by Ma et al. [31], who established the discrete FAGO model. Wu et al. [32] constructed the FAGO grey Bernoulli models. Mao et al. [33–34] introduced fractional derivatives based on FAGO. Kang et al. [35] proposed a variable-order fractional grey model. Xie et al. [36] developed a generalized fractional grey model by introducing a generalized fractional derivative that conforms to the memory effect. While the fractional grey model can appropriately reflect the new information priority principle and memory characteristics of the system, Markov correction can improve its accuracy.

Andrey Markov, a world-renowned Russian mathematician, studied and proposed a general model to solve changes in natural laws using mathematical methods and models in 1906. This model later became known as the Markov chain. It is a random process without after-effects, meaning that the current state is known, and the subsequent state is only related to the current state and not to the previous state. Markov theory is an essential part of stochastic processes and has a wide range of applications in many fields, such as operations research, biology and physics. Markov processes are found in various aspects of real life, such as the Brownian motion of the liquid in a cup. Markov chains, which have countable states and homogeneous time, are well-known examples of Markov processes [37]. In the energy field, the combination of grey prediction models and Markov processes has been applied. D'Amico et al. [38] used a second-order semi-Markov chain model to solve the problem of wind energy production, considering both state and duration. Ren and Gu [39] constructed a Markov chain model to predict the transition of the primary energy structure. They applied the GM (1,1) model and linear regression model to forecast the total energy consumption in 2020 and 2030 based on wind speed. Liu [37] employed the Grey Markov model in an empirical analysis to study the relationship between renewable energy consumption and economic growth.

2.2. The FOGM (1,1) model

We construct FOGM (1, 1) model through the followings.

Let $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$ be the non-negative sequence, we use to set up the fractional accumulation operator $X^{(r)} = (x^{(r)}(1), x^{(r)}(2), \cdots, x^{(r)}(n))$.

Similar to the definition of the non-negative sequence, we define the cumulative decreasing sequence with order r is $X^{(-r)} = (x^{(-r)}(1), x^{(-r)}(2), \cdots, x^{(-r)}(n))$, where

$$x^{(-r)}(k) = \sum_{i=1}^{k} (-1)^{i} \frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)} x^{(0)}(k-i), k = 1,2,\cdots, n. \tag{1}$$

Let

$$\frac{\mathrm{d}x^{(r)}(t)}{\mathrm{dt^u}} + ax^{(r)}(t) = b \tag{2}$$

be the whitening equation of the FOGM (1,1) model. We define

$$x^{(r)}(k) - x^{(r)}(k-1) + az^{(r)}(k) = b(k^u - (k-1)^u) \tag{3}$$

as the basic form of the FOGM (1,1) model,
where

$$z^{(r)}(k) = 0.5(uk^{u-1}x^{(r)}(\mathrm{k}) + u(k-1)^{u-1}x^{(r)}(k-1)). \tag{4}$$

Next, we will give out the prove process for the Eq (3).
Proof. Take the definite integral from k-1 to k of both sides of Eq (2).

We get

$$\int_{k-1}^{k} \frac{\mathrm{d}x^{(r)}(t)}{\mathrm{dt}} + a\int_{k-1}^{k} ut^{u-1}x^{(r)}(t)dt = b\int_{k-1}^{k} ut^{u-1}dt.$$

Assume that

$$x^{(r)}(k) - x^{(r)}(k-1) + az^{(r)}(k) = b(k^u - (k-1)^u),$$

we know

$$x^{(r)}(k) - x^{(r)}(k-1) + az^{(r)}(k) = b(k^u - (k-1)^u).$$

Assume that $X^{(0)}, X^{(r)}, Z^{(r)}$ are defined as in (1). Suppose the parameters column be $\theta = [a, b, c]^T$, set

$$Y = \begin{bmatrix} x^{(r)}(2) - x^{(r)}(1) \\ x^{(r)}(3) - x^{(r)}(2) \\ \vdots \\ x^{(r)}(n) - x^{(r)}(n-1) \end{bmatrix}, B = \begin{bmatrix} -z^{(r)}(2) & 2^u - 1^u \\ -z^{(r)}(3) & 3^u - 2^u \\ \vdots & \vdots \\ -z^{(r)}(n) & n^u - (n-1)^u \end{bmatrix}. \tag{5}$$

Then, we apply the least squares estimation, the parameters of the FOGM(1,1) model $x^{(r)}(k) - x^{(r)}(k-1) + az^{(r)}(k) = b(k^u - (k-1)^u)$ can be estimated as $\theta = (B^T B)^{-1} B^T Y$.

Assume B, Y are defined as in (4), taking $\hat{x}^{(r)}(1) = x^{(0)}(1)$, then the solution of the time response function $\frac{\mathrm{d}x^{(r)}(t)}{\mathrm{dt^u}} + ax^{(r)}(t) = b$ (which also known as the whitening equation) is

$$\hat{x}^{(r)}(\mathrm{t}) = e^{a(1-t^u)}x^{(0)}(1) - \frac{b}{a}e^{a(1-t^u)} + \frac{b}{a}. \tag{6}$$

And the time response sequence of the equation

$$x^{(r)}(k) - x^{(r)}(k-1) + az^{(r)}(k) = b(k^u - (k-1)^u),$$

which contained in FOGM (1,1) model is

$$\hat{x}^{(r)}(k) = e^{a(1-k^{u})}x^{(0)}(1) - \frac{b}{a}e^{a(1-k^{u})} + \frac{b}{a}. \tag{7}$$

The reduced value of the Eq (7) is

$$\hat{x}^{(0)}(k) = \begin{cases} \hat{x}^{(\mathrm{r})}(k),\ k = 1 \\ \sum_{i=1}^{k}(-1)^{i}\dfrac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)}\hat{x}^{(\mathrm{r})}(k-i),\ k = 2,\cdots,n \end{cases}. \tag{8}$$

Proof. According to the derivative formula:

$$\frac{\mathrm{d}t^{\mathrm{u}}}{\mathrm{d}t} = ut^{u-1},$$

we have,

$$\frac{\mathrm{d}x^{(r)}(t)}{\mathrm{d}t} = \frac{\mathrm{d}x^{(r)}(t)}{\mathrm{d}t^{\mathrm{u}}}\frac{\mathrm{d}t^{\mathrm{u}}}{\mathrm{d}t} = \frac{\mathrm{d}x^{(r)}(t)}{\mathrm{d}t^{\mathrm{u}}}\mathrm{u}t^{u-1}. \tag{9}$$

Then, the Eq (2) can be written as

$$\frac{\mathrm{d}x^{(r)}(t)}{\mathrm{d}t} + a\mathrm{u}t^{u-1}x^{(r)}(t) = b\mathrm{u}t^{u-1}. \tag{10}$$

Multiply the $e^{at^{u}}$ on the two sides of Eq (10),

$$e^{at^{u}}[\frac{\mathrm{d}x^{(r)}(t)}{\mathrm{d}t} + a\mathrm{u}t^{u-1}x^{(r)}(t)] = e^{at^{u}}b\mathrm{u}t^{u-1}, \tag{11}$$

$$\frac{\mathrm{d}e^{at^{u}}x^{(r)}(t)}{\mathrm{d}t} = e^{at^{u}}b\mathrm{u}t^{u-1}. \tag{12}$$

Take the definite integral on both sides of the above equation on the interval [1, t], we get

$$\int_{1}^{t}\frac{\mathrm{d}e^{at^{u}}\mathrm{x}^{(r)}(t)}{\mathrm{d}t} = \int_{1}^{t}e^{at^{u}}b\mathrm{u}t^{u-1}, \tag{13}$$

$$e^{at^{u}}x^{(r)}(t)\Big|_{1}^{t} = \frac{b}{a}e^{at^{u}}\Big|_{1}^{t}, \tag{14}$$

$$e^{at^{u}}x^{(r)}(t) - e^{a}x^{(r)}(1) = \frac{b}{a}(e^{at^{u}} - e^{a}). \tag{15}$$

We obtain the time response function as following

$$x^{(r)}(t) = e^{a(1-t^{u})}x^{(r)}(1) - \frac{b}{a}e^{a(1-t^{u})} + \frac{b}{a}. \tag{16}$$

And the time response sequence is

$$\hat{x}^{(r)}(k) = e^{a(1-k^{u})}x^{(0)}(1) - \frac{b}{a}e^{a(1-k^{u})} + \frac{b}{a}. \tag{17}$$

Make the cumulative decrease of order r to the time response sequence, we obtain the final prediction value as follow:

$$\hat{x}^{(0)}(k) = \begin{cases} \hat{x}^{(r)}(k), k = 1 \\ \sum_{i=1}^{k} (-1)^{i} \frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)} \hat{x}^{(r)}(k-i), k = 2, \cdots, n \end{cases}. \tag{18}$$

This completes the proof.

2.3. PSO algorithm

In 1995, Kennedy and Eberhart [40] first proposed the Particle Swarm Optimization (PSO) algorithm, which is a swarm intelligence optimization algorithm in the field of computational intelligence, along with ant colony algorithm and fish swarm algorithm. The algorithm is easy to understand, requires less parameter adjustment, is easily programmable and has strong stability, among other benefits. PSO is a stochastic global optimization technique that finds optimal regions in complex search spaces through interactions between particles. The PSO algorithm is suitable for high-dimensional optimization problems with multiple local optimal solutions and low requirements for result accuracy. It has a wide range of applications in neural network training and function optimization.

In this section, we mainly apply the PSO algorithm to minimize the mean relative error. We then search for the optimal Weibull parameter and the optimal non-linear parameter of the FOGM (1,1) model as follows:

$$\min f(u, r) = \frac{1}{n-1} \sum_{k=2}^{n} \left| \frac{\hat{x}^{(0)}(k) - x^{(0)}(k)}{x^{(0)}(k)} \right|. \tag{19}$$

2.4. The Markov model

A Markov chain is a random process with no aftereffects, where the future state depends only on the current state, which is known as the Markov property. Most sample sequences exhibit the Markov property, which can be utilized to forecast the future states by considering different initial states and the state transition probabilities of the targets, thereby enhancing the accuracy of prediction models [41]. In one study, a BP neural network and Markov model were constructed to predict the degree of opioid-related flooding in the United States [42]. Researchers have also explored quantum Markov chains (QMCs) on graphs and trees, which are associated with many significant models that arise from quantum statistical mechanics and quantum information [43].

2.4.1. Partition of state space

According to the Markov chain, the data sequence is divided into multiple different states $t_1, t_2, \cdots, t_n$, which are represented by $E_1, E_2, \cdots, E_n$, and the state transition only occurs at equal-countable moments. State interval is

$$E_i = [Q_{i1}, Q_{i2}], \ (i = 1,2, \cdots j), \tag{20}$$

where, Q_{i1}, Q_{i2} respectively represent the lower and the upper limits of relative errors in the state interval, and j represents the number of states divided.

2.4.2. State transition probability matrix

The transfer probability of Markov chain from state E_i to state E_j through k steps is denoted by $p_{ij}(k)$,

$$p_{ij}(k) = \frac{m_{ij}(k)}{M_i}, \tag{21}$$

where, M_i represents the total number of occurrences of state E_i, $m_{ij}(k)$ represents the number of state E_i transferring to state E_j by k steps, and m represents the number of states divided. The matrix composed of transition probability is called transition probability matrix, which can reflect the potential rule of one-step or multi-step state transition and predict the unknown state according to the known state. The transition probability matrix is non-negative and the sum of the elements in each row is 1. Its matrix form is as follow:

$$P(1) = \begin{bmatrix} p_{11}(1) & p_{12}(1) & \cdots & p_{1m}(1) \\ p_{21}(1) & p_{22}(1) & \cdots & p_{2m}(1) \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1}(1) & p_{m2}(1) & \cdots & p_{mm}(1) \end{bmatrix}. \tag{22}$$

2.5. Modify the forecast

According to the sequence from near to far, we select j groups of data nearest to the predicted data, and determine the step number t as $1,2, \cdots, j$. Then, the row vectors of the t steps state transition matrix corresponding to each data are taken to form a new matrix. We determine the most likely state of the predicted value by summing the column vectors in the new matrix. The predicted value of the curve fitting model is then adjusted through the predicted state interval obtained, and the midpoint of the interval is selected for calculation. The formula is as follows:

$$\hat{u}_{FOGMKM}(k) = \frac{\hat{u}_{FOGM}(k)}{1 + \frac{1}{2}(Q_{i1} + Q_{i2})}. \tag{23}$$

3. Application of the model

3.1. *Estimation and prediction of the model FOGM (1,1)*

Importing the statistical yearbook data of China into R software, and combing the PSO algorithm with (2.2), we get the optimal fractional order $r = 0.8$ and the Weibull parameters $u = 1.26$ of the accumulating generation operator respectively.

Applying the accumulation with order $r = 0.8$ on Eq (18), we obtain the estimate value as $k = 1,2,3,\cdots,16$ and the prediction value as $k = 17,18,19,20$. Using five models as the comparison model, the estimated equations of GM, ARIMA, FGM, linear and FAONGBM are respective as follows:

$$\hat{x}(k+1) = e^{0.05609k}(x^{(0)}(1) + 3640701) - 3640701, \tag{24}$$

$$\hat{x}_k = 306476.05 + 1.9071\hat{x}_{k-1} - 0.9359\hat{x}_{k-2} + \varepsilon_k, \tag{25}$$

$$\hat{x}^{(0.34)}(k) = -3731054e^{-0.02k} + 3886601, \tag{26}$$

$$\hat{x}(k) = 151161.65 + 120004.394k, \tag{27}$$

$$\hat{x}(k) = ((x(16)^{0.01} - 1.1403)e^{-0.0146(k-16)} + 1.1403)^{100}. \tag{28}$$

At the same time, we get the estimated equation of FOGM,

$$\hat{x}(k) = ((x(1) - 398980004)e^{0.00002486(1-k^{1.26})} + 398980004. \tag{29}$$

The prediction results are listed in Tables 1 and 3.

As shown in Table 1, the results of the FOGM (1,1) model are closer to the actual values, with a smaller relative error than other models. When predicting the test data and estimating the training data, the FOGM (1,1) model also yields the smallest RMSE and MAPE values.

Table 1. Comparison of GM (1,1), ARIMA, Linear and FOGM (1,1) results.

Year	Raw	FOGM (1,1)		GM (1,1)		Linear		ARIMA	
		Predicted value	Relative error	Predicted value	Relative error	Predicted value	Relative error	Predicted value	Relative error
2001	155547	155547.0	0	155547.0	0	171166.0	10.0413%	155547	0
2002	169577	169503.3	-0.0435%	219031.7	29.1636%	191170.4	12.7337%	169577	0
2003	197083	204829.8	3.9307%	231669.1	17.5490%	211174.8	7.1502%	184599.2	-6.3343%
2004	230281	235149.4	2.1141%	245035.7	6.4073%	231179.2	0.3900%	221546.4	-3.7930%
2005	261369	261758.9	0.1492%	259173.5	-0.8400%	251183.6	-3.8969%	259146.8	-0.8502%
2006	286467	285668.7	-0.2787%	274127.0	-4.3077%	271188.0	-5.3336%	289715.1	1.1339%
2007	311442	307529.2	-1.2563%	289943.3	-6.9030%	291192.4	-6.5019%	311615.9	0.0558%

Continued on next page

Year	Raw	FOGM (1,1)		GM (1,1)		Linear		ARIMA	
		Predicted value	Relative error	Predicted value	Relative error	Predicted value	Relative error	Predicted value	Relative error
2008	320611	327774.6	2.2344%	306672.1	-4.3476%	311196.8	-2.9363%	336061.3	4.8190%
2009	336126	346708.9	3.1485%	324366.1	-3.4987%	331201.2	-1.4652%	336359.8	0.0696%
2010	360648	364553.3	1.0829%	343081.0	-4.8710%	351205.6	-2.6182%	353579.3	-1.9600%
2011	387043	381474.7	-1.4387%	362875.7	-6.2441%	371210.0	-4.0908%	381408.8	-1.4557%
2012	402138	397602.3	-1.1279%	383812.5	-4.5570%	391214.4	-2.7164%	407866.1	1.4244%
2013	416913	413038.2	-0.9294%	405957.3	-2.6278%	411218.8	-1.3658%	416558.2	-0.0851%
2014	428334	427865.2	-0.1094%	429379.8	0.2442%	431223.2	0.6745%	429666.2	0.3110%
2015	434113	442151.0	1.8516%	454153.6	4.6164%	451227.6	3.9424%	437891.3	0.8704%
2016	441492	455952.1	3.2753%	480356.9	8.8031%	471232.0	6.7362%	436115.7	-1.2178%
		RMSE	6219.4	RMSE	22463	RMSE	14678	RMSE	6354.3
		MAPE	1.44%	MAPE	1.79%	MAPE	4.54%	MAPE	0.44%
2017	455827	469316.0	2.9592%	508072.0	11.4616%	491235.6	7.7680%	443604.56	-2.6814%
2018	471925	482283.2	2.1949%	537386.2	13.8711%	511240.0	8.3308%	441291.58	-6.4912%
2019	487488	494888.5	1.5181%	568391.7	16.5960%	531244.4	8.9759%	434706.89	-10.8272%
2020	498000	507162.1	1.8398%	601186.1	20.7201%	551248.8	10.6925%	423727.57	-14.9141%
		RMSE	10344	RMSE	77794	RMSE	43444	RMSE	48451
		MAPE	2.13%	MAPE	15.66%	MAPE	8.94%	MAPE	8.73%

3.2. Markov model modifying energy consumption

3.2.1. State transition probability matrix

In general, the state interval is partitioned according to the relative error of the FOGM model. From Table 1, we observe that the minimum and maximum relative errors of the first 16 fitted data points using this model are -1.44% and 3.941%, respectively. Therefore, based on the equal spacing rule, the state interval is divided into four partitions as follows: $E_1(-1.44\%,-0.095\%]$, $E_2(-0.095\%,1.251\%]$, $E_3(1.251\%,2.596\%]$, $E_4(2.596\%,3.941\%]$.

Combining these state partitions with the probability of the current state transferring to the next state, we obtain the state transition matrix of steps one to four.

$$P(1) = \begin{bmatrix} 0.66 & 0 & 0.34 & 0 \\ 0 & 1 & 0 & 0 \\ 0.4 & 0 & 0.2 & 0.4 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad P(2) = \begin{bmatrix} 0.5716 & 0 & 0.2924 & 0.1360 \\ 0 & 1 & 0 & 0 \\ 0.3440 & 0 & 0.5760 & 0.0800 \\ 0.4000 & 0 & 0.2000 & 0.4000 \end{bmatrix},$$

$$P(3)=\begin{bmatrix}0.4942 & 0 & 0.3888 & 0.1170\\ 0 & 1 & 0 & 0\\ 0.4574 & 0 & 0.3122 & 0.2304\\ 0.3440 & 0 & 0.5760 & 0.0800\end{bmatrix},\ P(4)=\begin{bmatrix}0.4817 & 0 & 0.3628 & 0.1555\\ 0 & 1 & 0 & 0\\ 0.4268 & 0 & 0.4484 & 0.1249\\ 0.4574 & 0 & 0.3122 & 0.2304\end{bmatrix}.$$

3.2.2. China's energy consumption forecast

Constructing a new state transition matrix using the most recent sets of data, we get the state of 2017, which is listed in Table 2.

Table 2. The prediction status of 2017.

Year	Initial status	Transferring steps	P_{ij}	E_1	E_2	E_3	E_4
2016	4	1	P_{14}	0	0	1	0
2015	3	2	P_{23}	0.3400	0	0.5760	0.0800
2014	1	3	P_{31}	0.4942	0	0.3888	0.1170
2013	1	4	P_{41}	0.4817	0	0.3628	0.1555
Total				1.3519	0	2.3276	0.3525

In Table 2, the value of P_{ij} comes from the value of row j of matrix $P(i)$ in the article. The result shows that the most likely state of China's energy consumption in 2017 is E_3, because E_3 has the largest value in the total. The predicted value of FOGM model in 2017 is 469316, and the predicted value of Markov model is 460459 according to Formula (23). In accordance with the same method, the predicted values of Markov model in 2018–2020 can be obtained, and the specific results are shown in Table 3

In Table 3, the results predicted by the FOGMKM (1,1) model are closer to the actual values, and the relative error is smaller than that of the FOGM (1,1) model. Using the FOGMKM (1,1) model to forecast the data from 2017–2020 and estimate the data from 2001–2016, we can obtain the smallest value of the RMSE as well as the MAPE.

Table 3. The comparison results among FOGM (1,1), FAONGBM (1,1), FGM (1,1) and FOGMKM (1,1).

Year	Raw	FOGM (1,1)			FAONGBM (1,1)		FGM (1,1)		FOGMKM (1,1)	
		Predicted value	Relative error	Station value	Predicted value	Relative error	Predicted value	Relative error	Predicted value	Relative error
2001	155547	155547.0	0	2	144368.7	-7.1864%	155547.0	0.0000%	156750.0	0.7734%
2002	169577	169503.3	-0.0435%	2	170997.4	0.8376%	176057.2	3.8214%	168529.2	-0.618%
2003	197083	204829.8	3.9307%	4	197791.6	0.3595%	205410.7	4.2255%	198346.8	0.6413%
2004	230281	235149.4	2.1141%	3	224205.6	-2.6383%	233281.4	1.3029%	230711.6	0.0187%
2005	261369	261758.9	0.1492%	3	249792.8	-4.4291%	258994.1	-0.9086%	256818.9	-1.741%
2006	286467	285668.7	-0.2787%	1	274212.5	-4.2778%	282681.6	-1.3214%	287878.1	0.4926%
2007	311442	307529.2	-1.2563%	1	297226.4	-4.5644%	304568.1	-2.2071%	309907.7	-0.493%
2008	320611	327774.6	2.2344%	3	318689.1	-0.5994%	324861.7	1.3258%	321588.8	0.305%
2009	336126	346708.9	3.1485%	4	338535.3	0.7168%	343738.7	2.2648%	335735.3	-0.116%
2010	360648	364553.3	1.0829%	3	356767.3	-1.0760%	361346.6	0.1937%	357673.4	-0.825%
2011	387043	381474.7	-1.4387%	1	373444.4	-3.5135%	377808.7	-2.3859%	384425.1	-0.676%
2012	402138	397602.3	-1.1279%	1	388676.5	-3.3475%	393229.6	-2.2153%	400677.4	-0.363%
2013	416913	413038.2	-0.9294%	1	402630.3	-3.4258%	407698.4	-2.2102%	416232.7	-0.163%
2014	428334	427865.2	-0.1094%	1	415563.1	-2.9815%	421292.1	-1.6440%	431174.4	0.6631%
2015	434113	442151.0	1.8516%	3	427968.5	-1.4154%	434077.9	-0.0081%	433806.7	0.0710%
2016	441492	455952.1	3.2753%	4	441492.0	0.0000%	446114.7	1.0471%	441520.9	0.0065%
		RMSE	6219.4		RMSE	9507.2	RMSE	6044.5	RMSE	1886.6
		MAPE	1.44%		MAPE	2.59%	MAPE	1.69%	MAPE	0.51%
2017	455827	469316.0	2.9592%	4	441120.7	-3.2263%	457455.0	0.3572%	460459.0	1.0161%
2018	471925	482283.2	2.1949%	3	449616.0	-4.7272%	468145.4	-0.8009%	473181.5	0.2662%
2019	487488	494888.5	1.5181%	3	458055.0	-6.0377%	478227.8	-1.8996%	498716.1	2.3032%
2020	498000	507162.1	1.8398%	3	468138.4	-5.9963%	487740.1	-2.0602%	497590.9	-0.0821%
		RMSE	10344		RMSE	24860	RMSE	7210.3	RMSE	6108.8
		MAPE	2.13%		MAPE	5.00%	MAPE	1.28%	MAPE	0.92%

The results in Figure 1 show that the curve of FOGMKM (1,1) model is closer to the true values than that of FOGM (1,1) model.

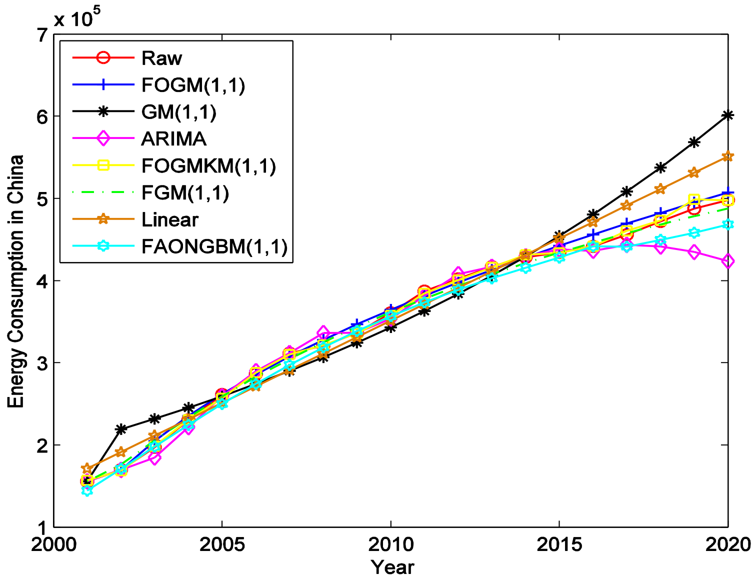

Figure 1. The forecast results of different models for China's energy consumption.

Table 4. Statistical values of various models.

Statistics	FOGMKM (1,1)	FAONGBM (1,1)	Linear	GM (1,1)	FGM (1,1)	FOGM (1,1)	ARIMA
R^2	0.9991	0.9805	0.9589	0.8951	0.9962	0.9954	0.9468
F	19547	925.4	438.01	171.62	4692.3	3937.4	338.53

Table 4 shows that the values of F statistics we calculated are all greater than the critical value 4.41 ($F_{0.05}(1,18) = 4.41$). The value of $F_{FOGMKM} = 19547$ is the maximum of these models. The goodness of fit of the FOGMKM (1,1) model is R^2=0.9991, which has a larger goodness of fit value than that of other models.

4. Conclusions

We establish a new optimal fractional-order accumulative grey Markov model (FOGMKM) with variable parameters. The appropriate state is determined using a Markov transition matrix, and a particle swarm optimization algorithm is used to find the optimal order and nonlinear parameters of the accumulative generation operator. The FOGMKM model is compared with the FOGM model, and it was found that the fitting effect and estimation accuracy of the FOGMKM model are better than those of the other six competitive models, including the FOGM model, GM, linear model, FGM, FAONGBM and ARIMA model. Finally, we apply the new model to predict the total energy consumption in China from 2017 to 2020. The results show that compared with the FOGM model, the new model has more accurate and effective prediction and evaluation.

As the combination of the optimal weighted Markov model with variable parameters and the optimal fractional-order accumulative grey model has proven to be an effective method for improving prediction accuracy, we plan to further investigate the multi-variable optimal fractional-order accumulative grey Markov model with variable parameters and inverse problems for fractional equations [44–49]. We aim to use some frontier optimization algorithms, such as the ant lion optimizer, grey wolf optimizer and whale optimizer, to search for optimal parameters in our future work, hoping to achieve even more significant progress.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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