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Research article

An estimate for the numerical radius of the Hilbert space operators and a numerical radius inequality

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Abstract: We provide a number of sharp inequalities involving the usual operator norms of Hilbert space operators and powers of the numerical radii. Based on the traditional convexity inequalities for nonnegative real numbers and some generalize earlier numerical radius inequalities, operator. Precisely, we prove that if $\mathbf{A}_i, \mathbf{B}_i, \mathbf{X}_i \in \mathcal{B}(\mathcal{H})$ $(i = 1, 2, \cdots, n)$, $m \in \mathbb{N}$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and ϕ and ψ are non-negative functions on $[0, \infty)$ which are continuous such that $\phi(t)\psi(t) = t$ for all $t \in [0, \infty)$, then

$$w^{2r}\left(\sum_{i=1}^{n} \mathbf{X}_i \mathbf{A}_i^m \mathbf{B}_i\right) \leq \frac{n^{2r-1}}{m} \sum_{j=1}^{m} \left\| \sum_{i=1}^{n} \frac{1}{p} S_{i,j}^{pr} + \frac{1}{q} T_{i,j}^{qr} \right\| - r_0 \inf_{\|\xi\|=1} \rho(\xi),$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$, $S_{i,j} = \mathbf{X}_i \phi^2\left(\left|\mathbf{A}_i^{j*}\right|\right) \mathbf{X}_i^*$, $T_{i,j} = \left(\mathbf{A}_i^{m-j} \mathbf{B}_i\right)^* \psi^2\left(\left|\mathbf{A}_i^{j}\right|\right) \mathbf{A}_i^{m-j} \mathbf{B}_i$ and

$$\rho(\xi) = \frac{n^{2r-1}}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\left\langle S_{i,j}^{r} \xi, \xi \right\rangle^{\frac{p}{2}} - \left\langle T_{i,j}^{r} \xi, \xi \right\rangle^{\frac{q}{2}} \right)^2.$$

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1. Introduction

Let $\mathcal{H}$ be complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operator on $\mathcal{H}$. An operator $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is said to be *positive* if $\langle \mathbf{T}\xi, \xi \rangle \geq 0$ holds for all $\xi \in \mathcal{H}$. We write $\mathbf{T} \geq 0$ if $\mathbf{T}$ is positive.

The numerical radius of $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is defined by

$$w(\mathbf{T}) = \sup\{|\lambda| : \lambda \in W(\mathbf{T})\} = \sup\{|\langle \mathbf{T}\xi, \xi\rangle| : \xi \in \mathcal{H}, \|\xi\| = 1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $\mathbf{T} \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|\mathbf{T}\| \leq w(\mathbf{T}) \leq \|\mathbf{T}\|. \tag{1.1}$$

Also, if $\mathbf{T} \in \mathcal{B}(\mathcal{H})$ is normal, then $w(\mathbf{T}) = \|\mathbf{T}\|$.

An important inequality for $w(\mathbf{T})$ is the power inequality stating that $w(\mathbf{T}^n) \leq (w(\mathbf{T}))^n$ for every natural numbers n.

Several numerical radius inequalities improving the inequalities in (3.9) have been recently given in [5, 6, 12, 19–22]. For instance, Kittaneh [13, 14] proved that for any $\mathbf{A} \in \mathcal{B}(\mathcal{H})$,

$$w(\mathbf{A}) \leq \frac{1}{2}\||\mathbf{A}| + |\mathbf{A}^*|\| \leq \frac{1}{2}\left(\|\mathbf{A}\| + \left\|\mathbf{A}^2\right\|^{1/2}\right), \tag{1.2}$$

where $|\mathbf{A}| = \sqrt{\mathbf{A}^*\mathbf{A}}$ is the absolute value of $\mathbf{A}$, and

$$\frac{1}{4}\|\mathbf{A}^*\mathbf{A} + \mathbf{A}\mathbf{A}^*\| \leq w^2(\mathbf{A}) \leq \frac{1}{2}\|\mathbf{A}^*\mathbf{A} + \mathbf{A}\mathbf{A}^*\|. \tag{1.3}$$

Also, in the same paper, it was shown that

$$\|\mathbf{A} + \mathbf{B}\|^2 \leq \left\||\mathbf{A}|^2 + |\mathbf{B}|^2\right\| + \left\||\mathbf{A}^*|^2 + |\mathbf{B}^*|^2\right\|. \tag{1.4}$$

Kittaneh and El-Haddad [15] established the generalizations of inequality (1.2) and the second inequality (1.3) as follows:

$$w^r(\mathbf{A}) \leq \frac{1}{2}\left\||\mathbf{A}|^{2r\lambda} + |\mathbf{A}^*|^{2r(1-\lambda)}\right\| \tag{1.5}$$

and

$$w^{2r}(\mathbf{A}) \leq \left\|\lambda|\mathbf{A}|^{2r} + (1-\lambda)|\mathbf{A}^*|^{2r}\right\|, \tag{1.6}$$

where $0 < \lambda < 1$ and $r \geq 1$.

A general numerical radius inequality has been established by Kittaneh [14], it has been proved that if $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{T}, \mathbf{S} \in \mathcal{B}(\mathcal{H})$, then

$$w(\mathbf{ATB} + \mathbf{CSD}) \leq \frac{1}{2}\left\|\mathbf{A}|\mathbf{T}^*|^{2(1-\alpha)}\mathbf{A}^* + \mathbf{B}^*|\mathbf{T}|^{2\alpha}\mathbf{B} + \mathbf{C}|\mathbf{S}^*|^{2(1-\alpha)}\mathbf{C}^* + \mathbf{D}^*|\mathbf{S}|^{2\alpha}\mathbf{D}\right\| \tag{1.7}$$

for all $\alpha \in (0, 1)$.

Although several open problems relating to numerical radius inequalities for bounded linear operators remain unsolved, work on establishing numerical radius inequalities for a number of bounded linear operators has begun (see, for example, [10] and [19–22]). If $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\mathcal{H})$, then

$$w(\mathbf{AB}) \leq 4w(\mathbf{A})w(\mathbf{B}).$$

In the case that $\mathbf{AB} = \mathbf{BA}$, we have

$$w(\mathbf{AB}) \leq 2w(\mathbf{A})w(\mathbf{B}).$$

Moreover, if $\mathbf{A}$ and $\mathbf{B}$ are normal, then

$$w(\mathbf{AB}) \leq w(\mathbf{A})w(\mathbf{B}).$$

Recently, Dragomir [7] proved that if $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\mathcal{H})$ and $r \geq 1$, then

$$w^r(\mathbf{B}^*\mathbf{A}) \leq \frac{1}{2} \left\| |\mathbf{A}|^{2r} + |\mathbf{B}^*|^{2r} \right\|. \tag{1.8}$$

Shebrawi and Albadawi [23] discovered a fascinating numerical radius inequality, it has been shown that if $\mathbf{A}, \mathbf{X}, \mathbf{B} \in \mathcal{B}(\mathcal{H})$, then

$$w^r(\mathbf{A}^*\mathbf{X}\mathbf{B}) \leq \frac{1}{2} \left\| (\mathbf{A}^*|\mathbf{X}^*|^{2\nu}\mathbf{A})^r + (\mathbf{B}^*|\mathbf{X}|^{2(1-\nu)}\mathbf{B})^r \right\|, \ r \geq 1, 0 < \nu < 1. \tag{1.9}$$

Recently, Al-Dolat and Al-Zoubi [3], showed that if $\mathbf{A}_i, \mathbf{B}_i, \mathbf{X}_i \in \mathcal{B}(\mathcal{H})$ $(i = 1, 2, \cdots, n)$, $m \in \mathbb{N}$ and ϕ and ψ are non-negative functions on $[0, \infty)$ which are continuous such that $\phi(t)\psi(t) = t$ for all $t \in [0, \infty)$, then

$$w^r\left(\sum_{i=1}^{n} \mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\right) \leq \frac{n^{2r-1}}{2m} \sum_{j=1}^{m} \left\| \sum_{i=1}^{n} (E_{i,j})^r + (W_{i,j})^r \right\| \tag{1.10}$$

for $r \geq 1$, where $E_{i,j} = \mathbf{X}_i\phi^2\left(\left|\mathbf{A}_i^{j*}\right|\right)\mathbf{X}_i^*$ and $W_{i,j} = (\mathbf{A}_i^{m-j}\mathbf{B}_i)^*\psi^2\left(\left|\mathbf{A}_i^{j}\right|\right)\mathbf{A}_i^{m-j}\mathbf{B}_i$.

The goal of this study is to develop significant extensions of these inequalities based on the classic convexity inequalities for nonnegative real numbers and some operator inequalities. For the sum of two operators, usual operator norm inequalities and a related numerical radius inequality are also provided. In specifically, if $i = 1, 2, \cdots n \in \mathbb{N}$, $\mathbf{A}_i, \mathbf{B}_i$, and $\mathbf{X}_i$ are bounded linear operators, then we estimate the numerical radius to $\sum_{i=1}^{m} \mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i$ for some $m \in \mathbb{N}$.

2. Inequalities for sums and products of operators

Using well-known new numerical radius inequalities as an example, we constructed a general numerical radius inequality for Hilbert space operators in this section. This section is initiated with an operator for norm inequality. In fact, we provide an extra upper bound for $\|\mathbf{B}^*\mathbf{A} + \mathbf{D}^*\mathbf{C}\|$. However, the proof of the theorem depends on the next lemma.

Lemma 2.1 ([4]). *Let $\xi, \zeta, \eta \in \mathcal{H}$. Then we have*

$$|\langle \eta, \xi \rangle|^2 + |\langle \eta, \zeta \rangle|^2 \leq \|\eta\|^2 \max\{\|\xi\|^2, \|\zeta\|^2\} + |\langle \xi, \zeta \rangle|. \tag{2.1}$$

Theorem 2.2. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} \|\mathbf{B}^*\mathbf{A} + \mathbf{D}\mathbf{C}^*\|^2 \ &\leq \ \frac{1}{2}\left[\left\| |\mathbf{A}^*\mathbf{B}|^2 + |\mathbf{C}\mathbf{D}^*|^2 \right\| + \left\| |\mathbf{A}^*\mathbf{B}|^2 - |\mathbf{C}\mathbf{D}^*|^2 \right\|\right] \\ &+ \ w(\mathbf{D}\mathbf{C}^*\mathbf{A}^*\mathbf{B}) + 2\,\|\mathbf{B}^*\mathbf{A}\|\,\|\mathbf{D}\mathbf{C}^*\|. \end{aligned}$$

Proof. For $\xi, \zeta \in \mathcal{H}$, we have by triangle inequality, we have

$$\begin{aligned} |\langle(\mathbf{B}^*\mathbf{A}+\mathbf{D}\mathbf{C}^*)\xi,\zeta\rangle|^2 &\leq |\langle\mathbf{B}^*\mathbf{A}\xi,\zeta\rangle|^2+|\langle\mathbf{D}\mathbf{C}^*\xi,\zeta\rangle|^2 \\ &+ 2|\langle\mathbf{B}^*\mathbf{A}\xi,\zeta\rangle\langle\mathbf{D}\mathbf{C}^*\xi,\zeta\rangle|. \end{aligned} \tag{2.2}$$

Now in inequality (2.1), for all $\xi, \zeta \in \mathcal{H}$, letting $\xi = \mathbf{A}^*\mathbf{B}\zeta$, $\zeta = \mathbf{C}\mathbf{D}^*\zeta$ and $\eta = \xi$ with $\|\xi\| = \|\zeta\| = 1$, we get

$$|\langle\xi,\mathbf{A}^*\mathbf{B}\zeta\rangle|^2+|\langle\xi,\mathbf{C}\mathbf{D}^*\zeta\rangle|^2 \leq \max\{\|\mathbf{A}^*\mathbf{B}\zeta\|^2,\|\mathbf{C}\mathbf{D}^*\zeta\|^2\}+|\langle\mathbf{A}^*\mathbf{B}\zeta,\mathbf{C}\mathbf{D}^*\zeta\rangle|.$$

Now use the fact that

$$\max\{\sigma,\tau\} = \frac{1}{2}\left[\sigma+\tau+|\sigma-\tau|\right] \quad \text{for any} \quad \sigma,\tau \in \mathbb{R},$$

we have

$$\begin{aligned} |\langle\mathbf{B}^*\mathbf{A}\xi,\zeta\rangle|^2+|\langle\mathbf{D}\mathbf{C}^*\xi,\zeta\rangle|^2 &\leq \frac{1}{2}\Big[\Big\langle\Big(|\mathbf{A}^*\mathbf{B}|^2+|\mathbf{C}\mathbf{D}^*|^2\Big)\zeta,\zeta\Big\rangle \\ &+ \Big|\Big\langle\Big(|\mathbf{A}^*\mathbf{B}|^2-|\mathbf{C}\mathbf{D}^*|^2\Big)\zeta,\zeta\Big\rangle\Big|+|\langle\mathbf{A}^*\mathbf{B}\zeta,\mathbf{C}\mathbf{D}^*\zeta\rangle|\Big]. \end{aligned} \tag{2.3}$$

Combining the inequalities (2.2) and (2.3), we have

$$\begin{aligned} |\langle(\mathbf{B}^*\mathbf{A}+\mathbf{D}\mathbf{C}^*)\xi,\zeta\rangle|^2 &\leq \frac{1}{2}\Big[\Big\langle\Big(|\mathbf{A}^*\mathbf{B}|^2+|\mathbf{C}\mathbf{D}^*|^2\Big)\zeta,\zeta\Big\rangle \\ &+ \Big|\Big\langle\Big(|\mathbf{A}^*\mathbf{B}|^2-|\mathbf{C}\mathbf{D}^*|^2\Big)\zeta,\zeta\Big\rangle\Big|+|\langle\mathbf{A}^*\mathbf{B}\zeta,\mathbf{C}\mathbf{D}^*\zeta\rangle|\Big]. \\ &+ 2\,|\langle\mathbf{B}^*\mathbf{A}\zeta,\zeta\rangle\langle\mathbf{C}\mathbf{D}^*\zeta,\zeta\rangle|\,. \end{aligned}$$

Taking the supremum over all unit vectors ξ, ζ, we obtain the desired inequality. □

In Theorem 2.2, if we let $\mathbf{A} = \mathbf{B} = \mathbf{C} = \mathbf{D} = \mathbf{S}$, we have:

Corollary 2.3. *Let* $\mathbf{S} \in \mathcal{B}(\mathcal{H})$. *Then*

$$\|\mathbf{S}^*\mathbf{S}+\mathbf{S}\mathbf{S}^*\|^2 \leq \frac{1}{2}\Big[\big\||\mathbf{S}|^4+|\mathbf{S}^*|^4\big\|+\big\||\mathbf{S}|^4-|\mathbf{S}^*|^4\big\|\Big]+w(|\mathbf{S}^*|^2|\mathbf{S}|^2)+2\,\big\||\mathbf{S}|^2\big\|\,\big\||\mathbf{S}^*|^2\big\|\,.$$

In the proof of Theorem 2.2, if we let $\xi = \zeta$, we have:

Corollary 2.4. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{B}(\mathcal{H})$. *Then*

$$\begin{aligned} w^2(\mathbf{B}^*\mathbf{A}+\mathbf{D}\mathbf{C}^*) &\leq \frac{1}{2}\Big[\big\||\mathbf{A}^*\mathbf{B}|^2+|\mathbf{C}\mathbf{D}^*|^2\big\|+\big\||\mathbf{A}^*\mathbf{B}|^2-|\mathbf{C}\mathbf{D}^*|^2\big\|\Big] \\ &+ w(\mathbf{D}\mathbf{C}^*\mathbf{A}^*\mathbf{B})+2w(\mathbf{B}^*\mathbf{A})w(\mathbf{D}\mathbf{C}^*). \end{aligned}$$

The following lemma gives a basic but useful extension for four operators of the Schwarz inequality due to Dragomir [8].

Lemma 2.5. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{B}(\mathcal{H})$. *Then for* $\xi, \zeta \in \mathcal{H}$ *we have the inequality*

$$|\langle\mathbf{D}\mathbf{C}\mathbf{B}\mathbf{A}\xi,\zeta\rangle|^2 \leq \Big\langle|\mathbf{B}\mathbf{A}|^2\xi,\xi\Big\rangle\Big\langle|(\mathbf{D}\mathbf{C})^*|^2\zeta,\zeta\Big\rangle\,.$$

The equality case holds if and only if the vectors $\mathbf{B}\mathbf{A}\xi$ *and* $\mathbf{C}^*\mathbf{D}^*\zeta$ *are linearly dependent in* $\mathcal{H}$.

The following lemma, known as the Hölder-McCarthy inequality, is a well-known conclusion derived from Jensen's inequality and the spectral theorem for positive operators (see [12]).

Lemma 2.6. *Let $\mathbf{T} \in \mathcal{B}(\mathcal{H})$, $\mathbf{T} \geq 0$ and let $\xi \in \mathcal{H}$ be any unit vector. Then we have*

(i) $\langle \mathbf{T}\xi, \xi\rangle^r \leq \langle \mathbf{T}^r\xi, \xi\rangle$ *for* $r \geq 1$.
(ii) $\langle \mathbf{T}^r\xi, \xi\rangle \leq \langle \mathbf{T}\xi, \xi\rangle^r$ *for* $0 < r \leq 1$.
(iii) If $\mathbf{T}$ *is invertible, then* $\langle \mathbf{T}\xi, \xi\rangle^r \leq \langle \mathbf{T}^r\xi, \xi\rangle$ *for all* $r < 0$.

The next result is well known in the literature as the Mond-Pečarić inequality [18].

Lemma 2.7. *If ψ is a convex function on a real interval J containing the spectrum of the self-adjoint operator $\mathbf{T}$, then for any unit vector $\xi \in \mathcal{H}$,*

$$\psi(\langle \mathbf{T}\xi, \xi\rangle) \leq \langle \psi(\mathbf{T})\xi, \xi\rangle \tag{2.4}$$

and the reverse inequality holds if ψ is concave.

The forth lemma is a direct consequence of [2, Theorem 2.3].

Lemma 2.8. *Let ψ be a non-negative non-decreasing convex function on $[0, \infty)$ and let $\mathbf{T}, \mathbf{S} \in \mathcal{B}(\mathcal{H})$ be positive operators. Then for any $0 < \mu < 1$,*

$$\|\psi(\mu\mathbf{T} + (1-\mu)\mathbf{S})\| \leq \|\mu\psi(\mathbf{T}) + (1-\mu)\psi(\mathbf{S})\|\,. \tag{2.5}$$

The above four lemmas admit the following more general result.

Theorem 2.9. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{B}(\mathcal{H})$. If ψ is a non-negative increasing convex function on $[0, \infty)$, then for any $0 < \mu < 1$,*

$$\psi(w^2(\mathbf{DCBA})) \leq \left\|\mu\psi\left(|\mathbf{BA}|^{\frac{2}{\mu}}\right) + (1-\mu)\psi\left(|(\mathbf{DC})^*|^{\frac{2}{1-\mu}}\right)\right\|. \tag{2.6}$$

In particular,

$$w^{2r}(\mathbf{DCBA}) \leq \left\|\mu|\mathbf{BA}|^{\frac{2r}{\mu}} + (1-\mu)|(\mathbf{DC})^*|^{\frac{2r}{1-\mu}}\right\| \tag{2.7}$$

for all $r \geq 1$.

Proof. For any unit vector $\xi \in \mathcal{H}$, we have

$$\begin{aligned}
|\langle \mathbf{DCBA}\xi, \xi\rangle|^2 &\leq \left\langle |\mathbf{BA}|^2\xi, \xi\right\rangle \left\langle |(\mathbf{DC})^*|^2\xi, \xi\right\rangle \quad \text{(by Lemma (2.5))}\\
&\leq \left\langle |\mathbf{BA}|^{\frac{2}{\mu}}\xi, \xi\right\rangle^{\mu} \left\langle |(\mathbf{DC})^*|^{\frac{2}{1-\mu}}\xi, \xi\right\rangle^{1-\mu}\\
&\qquad \text{(by Lemma 2.4 for concavity of } \psi(t) = t^{\mu} \text{ for } 0 < \mu < 1)\\
&\leq \mu\left\langle |\mathbf{BA}|^{\frac{2}{\mu}}\xi, \xi\right\rangle + (1-\mu)\left\langle |(\mathbf{DC})^*|^{\frac{2}{1-\mu}}\xi, \xi\right\rangle\\
&\qquad \text{(by weighted arithmetic-geometric mean inequality).}
\end{aligned}$$

Taking the supremum over $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, we infer that

$$w^2(\mathbf{DCBA}) \leq \left\|\mu|\mathbf{BA}|^{\frac{2}{\mu}} + (1-\mu)|(\mathbf{DC})^*|^{\frac{2}{1-\mu}}\right\|. \tag{2.8}$$

On account of assumptions on ψ, we can write

$$\begin{aligned}\psi(w^2(\mathbf{DCBA})) &\leq \psi\left(\left\|\mu|\mathbf{BA}|^{\frac{2}{\mu}} + (1-\mu)|(\mathbf{DC})^*|^{\frac{2}{1-\mu}}\right\|\right) \\ &\leq \left\|\mu\psi\left(|\mathbf{BA}|^{\frac{2}{\mu}}\right) + (1-\mu)\psi\left(|(\mathbf{DC})^*|^{\frac{2}{1-\mu}}\right)\right\| \quad \text{(by Lemma 2.5)}.\end{aligned}$$

The inequality (2.7) follows directly from (2.6) by taking $\psi(t) = t^r$ ($r \geq 1$). □

In the following result, we want to improve (1.9) under certain mild situations. We'll need the arithmetic-geometric mean inequality refinement [24] to do this.

Lemma 2.10. *Suppose that $\mu, \nu > 0$ and positive real numbers δ, Δ satisfy*

$$\min\{\mu, \nu\} \leq \delta < \Delta \leq \max\{\mu, \nu\}.$$

Then

$$\frac{\Delta + \delta}{2\sqrt{\delta\Delta}}\sqrt{\mu\nu} \leq \frac{\mu + \nu}{2}.$$

The following lemma is very useful in the proof of the next result.

Lemma 2.11. *Let ψ be a non-negative increasing convex function on $[0, \infty)$, $\psi(0) = 0$ and $\alpha \in [0, 1]$. Then $\psi(\alpha t) \leq \alpha\psi(t)$.*

Theorem 2.12. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{B}(\mathcal{H})$ and let ψ be a non-negative increasing convex function on $[0, \infty)$. If*

$$0 < |\mathbf{BA}|^2 \leq \delta < \Delta \leq |(\mathbf{DC})^*|^2$$

or

$$0 < |(\mathbf{DC})^*|^2 \leq \delta < \Delta \leq |\mathbf{BA}|^2,$$

then

$$\psi(w(\mathbf{DCBA})) \leq \frac{\sqrt{\delta\Delta}}{\delta + \Delta}\left\|\psi\left(|\mathbf{BA}|^2\right) + \psi\left(|(\mathbf{DC})^*|^2\right)\right\|. \tag{2.9}$$

Proof. It follows from Lemma 2.5 that

$$|\langle \mathbf{DCBA}\xi, \xi\rangle| \leq \sqrt{\langle |\mathbf{BA}|^2\xi, \xi\rangle\langle |(\mathbf{DC})^*|^2\xi, \xi\rangle}. \tag{2.10}$$

$$\begin{aligned} &\leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta}\left[\left\langle |\mathbf{BA}|^2\xi, \xi\right\rangle + \left\langle |(\mathbf{DC})^*|^2\xi, \xi\right\rangle\right] \qquad (2.11)\\ &= \frac{\sqrt{\Delta\delta}}{\Delta + \delta}\left\langle\left(|\mathbf{BA}|^2 + |(\mathbf{DC})^*|^2\right)\xi, \xi\right\rangle.\end{aligned}$$

Combining (2.10) and (2.11), we obtain

$$|\langle \mathbf{DCBA}\xi, \xi\rangle| \leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta}\left\langle\left(|\mathbf{BA}|^2 + |(\mathbf{DC})^*|^2\right)\xi, \xi\right\rangle. \tag{2.12}$$

Taking the supremum over $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, we infer that

$$w(\mathbf{DCBA}) \leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta} \left\| |\mathbf{BA}|^2 + |(\mathbf{DC})^*|^2 \right\|.$$

Now, since ψ is a non-negative increasing convex function, we have

$$\begin{aligned} \psi(w(\mathbf{DCBA})) &\leq \psi\left(\frac{2\sqrt{\Delta\delta}}{\Delta + \delta} \left\| \frac{|\mathbf{BA}|^2 + |(\mathbf{DC})^*|^2}{2} \right\|\right) \\ &\leq \frac{2\sqrt{\Delta\delta}}{\Delta + \delta} \psi\left(\left\| \frac{|\mathbf{BA}|^2 + |(\mathbf{DC})^*|^2}{2} \right\|\right) \\ &\qquad \left(\text{by Lemma 2.11 because } \alpha = \frac{2\sqrt{\Delta\delta}}{\Delta + \delta} \leq 1\right) \\ &\leq \frac{2\sqrt{\Delta\delta}}{\Delta + \delta} \left\| \psi\left(\frac{|\mathbf{BA}|^2 + |(\mathbf{DC})^*|^2}{2}\right) \right\| \\ &\leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta} \left\| \psi\left(|\mathbf{BA}|^2\right) + \psi\left(|(\mathbf{DC})^*|^2\right) \right\| \quad \text{(by Lemma 2.8).} \end{aligned}$$

□

As an applications of Theorem 2.12, we have:

Corollary 2.13. *Let* $\mathbf{T} \in \mathcal{B}(\mathcal{H})$*,* $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1$ *and let* ψ *be a non-negative increasing convex function on* $[0, \infty)$*. If*

$$0 < |\mathbf{T}|^{2\beta} \leq \delta < \Delta \leq |\mathbf{T}^*|^{2\alpha}$$

or

$$0 < |\mathbf{T}^*|^{2\alpha} \leq \delta < \Delta \leq |\mathbf{T}|^{2\beta},$$

then

$$\psi(w(\mathbf{T}|\mathbf{T}|^{\beta-1}\mathbf{T}|\mathbf{T}|^{\alpha-1})) \leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta} \left\| \psi\left(|\mathbf{T}|^{2\beta}\right) + \psi\left(|\mathbf{T}^*|^{2\alpha}\right) \right\|. \tag{2.13}$$

Remark 2.14. Following (2.13) we list here some particular inequalities of interest.

(i) If we let $\psi(t) = t^r$ $(r \geq 1)$, we have

$$w^r(\mathbf{DCBA}) \leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta} \left\| |\mathbf{BA}|^{2r} + |(\mathbf{DC})^*|^{2r} \right\|,$$

whenever

$$0 < |\mathbf{BA}|^2 \leq \delta < \Delta \leq |(\mathbf{DC})^*|^2 \quad \text{or} \quad 0 < |(\mathbf{DC})^*|^2 \leq \delta < \Delta \leq |\mathbf{BA}|^2.$$

(ii) Letting $\mathbf{D} = \mathbf{S}^*, \mathbf{A} = \mathbf{T}$ and let $\psi(t) = t^r$ $(r \geq 1)$, we have

$$w^r(\mathbf{T}^*\mathbf{S}) \leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta} \left\| |\mathbf{T}|^{2r} + |\mathbf{S}|^{2r} \right\|,$$

whenever

$$0 < |\mathbf{T}|^2 \leq \delta < \Delta \leq |\mathbf{S}|^2 \quad \text{or} \quad 0 < |\mathbf{S}|^2 \leq \delta < \Delta \leq |\mathbf{T}|^2.$$

(iii) Letting $\mathbf{C} = \mathbf{D} = \mathbf{B} = I$ and $\mathbf{A} = \mathbf{T}$ and let $\psi(t) = t^r$ ($r \geq 1$), we have

$$w^r(\mathbf{T}) \leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta} \left\| |\mathbf{T}|^{2r} + I \right\|,$$

whenever

$$0 < |\mathbf{T}|^2 \leq \delta < \Delta \leq MI \quad \text{or} \quad 0 < I \leq \delta < \Delta \leq |\mathbf{T}|^2.$$

We give an example to clarify part (ii) in Remark 2.14

Example 2.15. Let $\mathbf{S} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$ and $\mathbf{T} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $r = 2$. A simple calculations show that $\mathbf{S}^*\mathbf{T} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$ and so $w^2(\mathbf{S}^*\mathbf{T}) = 1$, $\left\| |\mathbf{S}|^4 + |\mathbf{T}|^4 \right\| = \frac{257}{16} = 16.0625$. If we take $\delta = 0.3$ and $\Delta = .4$, then

$$w^2(\mathbf{S}^*\mathbf{T}) = 1 \leq \frac{\sqrt{\Delta\delta}}{\Delta + \delta} \left\| |\mathbf{S}|^4 + |\mathbf{T}|^4 \right\| = 7.94.$$

If $\mathbf{A}, \mathbf{B} \in \mathcal{B}(\mathcal{H})$ are positive, the geometric mean of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A}\sharp\mathbf{B}$, is defined as

$$\mathbf{A}\sharp\mathbf{B} = \mathbf{A}^{\frac{1}{2}} \left(\mathbf{A}^{-\frac{1}{2}} \mathbf{B} \mathbf{A}^{-\frac{1}{2}} \right)^{\frac{1}{2}} \mathbf{A}^{\frac{1}{2}}.$$

For $0 \leq \nu \leq 1$, the ν-weighted geometric mean, denoted by $\mathbf{A}\sharp_\nu\mathbf{B}$, is defined as

$$\mathbf{A}\sharp_\nu\mathbf{B} = \mathbf{A}^{\frac{1}{2}} \left(\mathbf{A}^{-\frac{1}{2}} \mathbf{B} \mathbf{A}^{-\frac{1}{2}} \right)^{1-\nu} \mathbf{A}^{\frac{1}{2}}.$$

The ν-weighted geometric mean was introduced by Kubo and Ando [11], and when $\nu = \frac{1}{2}$ this is just the geometric mean. One can show that $\mathbf{A}\sharp_\nu\mathbf{B} = \mathbf{B}\sharp_{1-\nu}\mathbf{A}$ for $0 \leq \nu \leq 1$. When $\mathbf{A}$ and $\mathbf{B}$ commute, $\mathbf{A}\sharp_\nu\mathbf{B} = \mathbf{A}^\nu\mathbf{B}^{1-\nu}$. The ν-weighted arithmetic mean of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A}\nabla_\nu\mathbf{B}$, is defined as

$$\mathbf{A}\nabla_\nu\mathbf{B} = (1 - \nu)\mathbf{A} + \nu\mathbf{B}.$$

Theorem 2.16. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{B}(\mathcal{H})$*, and let* ψ *be a non-negative increasing convex function on* $[0, \infty)$*. If for given* $m', M' > 0$*,*

$$0 < m' \leq |\mathbf{BA}|^2 \leq |(\mathbf{DC})^*|^2 \leq M' \quad \text{or } 0 < m' \leq |(\mathbf{DC})^*|^2 \leq |\mathbf{BA}|^2 \leq M',$$

then

$$\psi(w(\mathbf{DCBA})) \leq \frac{1}{2\gamma} \left\| \psi\left(|\mathbf{BA}|^2\right) + \psi\left(|(\mathbf{DC})^*|^2\right) \right\|,$$

where

$$\gamma := \left(1 - \frac{1}{8}\left(1 - \frac{1}{h'}\right)^2\right)^{-1} \geq 1 \text{ with } h' = \frac{M'}{m'}.$$

To prove Theorem 2.16, we need the following result that established by Furuichi [9].

Corollary 2.17. *Let* $0 \leq \nu \leq 1$, $-1 \leq r_1 < 0$, $0 < r_2 \leq 1$ *and let* $\mathbf{T}$ *and* $\mathbf{S}$ *be strictly positive operators satisfying (i)* $0 < m \leq \mathbf{T} \leq m' < M' \leq \mathbf{S} \leq M$ *or (ii)* $0 < m \leq \mathbf{S} \leq m' < M' \leq \mathbf{T} \leq M$ *with* $h = \frac{M}{m}$ *and* $h' = \frac{M'}{m'}$. *Then*

$$exp_{r_1}\left(\frac{\nu(1-\nu)}{2}\left(\frac{h-1}{h}\right)^2\right)\mathbf{T}\sharp_\nu\mathbf{S} \leq \mathbf{T}\nabla_\nu\mathbf{S} \leq exp_{r_2}\left(\frac{\nu(1-\nu)}{2}(h'-1)^2\right)\mathbf{T}\sharp_\nu\mathbf{S}.$$

Proof of Theorem 2.16. From Corollary 2.17, we have

$$exp_r\left(\frac{\nu(1-\nu)}{2}\left(1-\frac{1}{h'}\right)^2\right)\mathbf{T}\sharp_\nu\mathbf{S} \leq \mathbf{T}\nabla_\nu\mathbf{S}$$

for $\mathbf{T}, \mathbf{S} > 0$ with $m', M' > 0$ satisfying $0 < m' \leq \mathbf{T} \leq \mathbf{S} \leq M'$ or $0 < m' \leq \mathbf{S} \leq \mathbf{T} \leq M'$, where $exp_r(\xi) := (1 + r\xi)^{1/r}$, if $1 + r\xi > 0$, and it is undefined otherwise. Since $exp_r(\xi)$ is decreasing in $r \in [-1, 0)$, the above inequality gives a tight lower bound when $r = -1$. After all, we have the scalar inequality:

$$\gamma\sqrt{\sigma\tau} \leq \frac{\sigma+\tau}{2}$$

for $\sigma, \tau > 0$ and $m', M' > 0$ such that $0 < m' \leq \min\{\sigma, \tau\} \leq \max\{\sigma, \tau\} \leq M'$. Applying this inequality with a similar argument as in Theorem 2.12, we obtain the desired result. □

Theorem 2.18. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{B}(\mathcal{H})$ $0 < \nu < 1$ *and let* ψ *be a non-negative increasing convex function on* $[0, \infty)$. *Then*

$$\psi(w^2(\mathbf{DCBA})) \leq \left\|(1-\nu)\psi\left(|\mathbf{BA}|^{\frac{2}{1-\nu}}\right) + \nu\psi\left(|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\right\| - r\gamma(\psi) \tag{2.14}$$

where $r = \min\{\nu, 1-\nu\}$ *and*

$$\begin{aligned}\gamma(\psi) \;=\; & \inf_{\|\xi\|=1}\Big\{\psi\left(\left\langle|\mathbf{BA}|^{\frac{2}{1-\nu}}\xi,\xi\right\rangle\right) + \psi\left(\left\langle|(\mathbf{DC})^*|^{\frac{2}{\nu}}\xi,\xi\right\rangle\right) \\ & - 2\psi\left(\left\langle\left(\frac{\left\langle|\mathbf{BA}|^{\frac{2}{1-\nu}}\xi,\xi\right\rangle + \left\langle|(\mathbf{DC})^*|^{\frac{2}{\nu}}\xi,\xi\right\rangle}{2}\right)\xi,\xi\right\rangle\right)\Big\}.\end{aligned} \tag{2.15}$$

Proof. We assume $0 \leq \nu \leq \frac{1}{2}$. For each unit vector $\xi \in \mathcal{H}$,

$$\begin{aligned}& \psi\left(\left\langle\left((1-\nu)|\mathbf{BA}|^{\frac{2}{1-\nu}} + \nu|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\xi,\xi\right\rangle\right) + r\gamma(\psi) \\ =\; & \psi\left((1-\nu)\left\langle|\mathbf{BA}|^{\frac{2}{1-\nu}}\xi,\xi\right\rangle + \nu\left\langle|(\mathbf{DC})^*|^{\frac{2}{\nu}}\xi,\xi\right\rangle\right) + r\gamma(\psi) \\ =\; & \psi\left((1-2\nu)\left\langle|\mathbf{BA}|^{\frac{2}{1-\nu}}\xi,\xi\right\rangle + 2\nu\left\langle\left(\frac{|\mathbf{BA}|^{\frac{2}{1-\nu}} + |(\mathbf{DC})^*|^{\frac{2}{\nu}}}{2}\right)\xi,\xi\right\rangle\right) + r\gamma(\psi) \\ \leq\; & (1-2\nu)\psi\left(\left\langle|\mathbf{BA}|^{\frac{2}{1-\nu}}\xi,\xi\right\rangle\right) + 2\nu\psi\left(\left\langle\left(\frac{|\mathbf{BA}|^{\frac{2}{1-\nu}} + |(\mathbf{DC})^*|^{\frac{2}{\nu}}}{2}\right)\xi,\xi\right\rangle\right) + r\gamma(\psi) \\ & \text{(by convexity of } \psi).\end{aligned}$$

Hence

$$\begin{aligned}
&\psi\left(\left\langle\left((1-\nu)|\mathbf{BA}|^{\frac{2}{1-\nu}}+\nu|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\xi,\xi\right\rangle\right)+r\gamma(\psi)\\
\leq\;& (1-2\nu)\psi\left(\left\langle|\mathbf{BA}|^{\frac{2}{1-\nu}}\xi,\xi\right\rangle\right)+2\nu\psi\left(\left\langle\left(\frac{|\mathbf{BA}|^{\frac{2}{1-\nu}}+|(\mathbf{DC})^*|^{\frac{2}{\nu}}}{2}\right)\xi,\xi\right\rangle\right)\\
+\;& r\left(\psi\left(\left\langle|\mathbf{BA}|^{\frac{2}{1-\nu}}\xi,\xi\right\rangle\right)+\psi\left(\left\langle|(\mathbf{DC})^*|^{\frac{2}{\nu}}\xi,\xi\right\rangle\right)\right.\\
-\;& \left.2\psi\left(\left\langle\left(\frac{|\mathbf{BA}|^{\frac{2}{1-\nu}}+|(\mathbf{DC})^*|^{\frac{2}{\nu}}}{2}\right)\xi,\xi\right\rangle\right)\right)\\
&\quad\text{(by inequality 2.15)}\\
\leq\;& (1-\nu)\psi\left(\left\langle|\mathbf{BA}|^{\frac{2}{1-\nu}}\xi,\xi\right\rangle\right)+\nu\psi\left(\left\langle|(\mathbf{DC})^*|^{\frac{2}{\nu}}\xi,\xi\right\rangle\right)\\
\leq\;& \left\langle\left((1-\nu)\psi\left(|\mathbf{BA}|^{\frac{2}{1-\nu}}\right)+\nu\psi\left(|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\right)\xi,\xi\right\rangle\quad\text{(by Lemma 2.7)}.
\end{aligned}$$

If we apply similar arguments for $\frac{1}{2}\leq\nu\leq 1$, then we can write

$$\begin{aligned}
\psi\left(\left\langle\left((1-\nu)|\mathbf{BA}|^{\frac{2}{1-\nu}}+\nu|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\xi,\xi\right\rangle\right)\;\leq\;& \left\|\left((1-\nu)\psi\left(|\mathbf{BA}|^{\frac{2}{1-\nu}}\right)\right.\right.\\
+\;& \left.\left.\nu\psi\left(|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\right)\xi,\xi\right\|-r\gamma(\psi).
\end{aligned}$$

We know that if $\mathbf{T}\in\mathcal{B}(\mathcal{H})$ is a positive operator, then $\|\mathbf{T}\|=\sup_{\|\xi\|=1}\langle\mathbf{T}\xi,\xi\rangle$. By using this, the continuity and the increase of ψ, we have

$$\begin{aligned}
&\psi\left(\left\|(1-\nu)|\mathbf{BA}|^{\frac{2}{1-\nu}}+\nu|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right\|\right)\\
=\;& \psi\left(\sup_{\|\xi\|=1}\left\langle\left((1-\nu)|\mathbf{BA}|^{\frac{2}{1-\nu}}+\nu|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\xi,\xi\right\rangle\right)\\
=\;& \sup_{\|\xi\|=1}\psi\left(\left\langle\left((1-\nu)|\mathbf{BA}|^{\frac{2}{1-\nu}}+\nu|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\xi,\xi\right\rangle\right)\\
\leq\;& \sup_{\|\xi\|=1}\left\langle\left((1-\nu)\psi\left(|\mathbf{BA}|^{\frac{2}{1-\nu}}\right)+\nu\psi\left(|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\right)\xi,\xi\right\rangle-r\gamma(\psi)\\
=\;& \left\|(1-\nu)\psi\left(|\mathbf{BA}|^{\frac{2}{1-\nu}}\right)+\nu\psi\left(|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\right\|-r\gamma(\psi).
\end{aligned}$$

On the other hand, if $\mathbf{X}\in\mathcal{B}(\mathcal{H})$, and if ψ is a non-negative increasing function on $[0,\infty)$, then $\psi(\|\mathbf{X}\|)=\|\psi(|\mathbf{X}|)\|$.

Now from the proof of Theorem 2.9, we have

$$\begin{aligned}
\psi(w^2(\mathbf{DCBA}))\;\leq\;& \psi\left(\left\|(1-\nu)|\mathbf{BA}|^{\frac{2}{1-\nu}}+\nu|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right\|\right)\\
\leq\;& \left\|(1-\nu)\psi\left(|\mathbf{BA}|^{\frac{2}{1-\nu}}\right)+\nu\psi\left(|(\mathbf{DC})^*|^{\frac{2}{\nu}}\right)\right\|-r\gamma(\psi).
\end{aligned}$$

This completes the proof. □

Inequality (2.18) includes several numerical radius inequalities as special cases.

Corollary 2.19. *Let* $\mathbf{T} \in \mathcal{B}(\mathcal{H})$*,* $\alpha + \beta \geq 1$*,* $0 < \nu < 1$ *and let* ψ *be a non-negative increasing convex function on* $[0, \infty)$*. Then*

$$\psi(w^2(\mathbf{T}|\mathbf{T}|^{\beta-1}\mathbf{T}|\mathbf{T}|^{\alpha-1})) \leq \left\|(1-\nu)\psi\left(|\mathbf{T}|^{\frac{2\beta}{1-\nu}}\right) + \nu\psi\left(|\mathbf{T}^*|^{\frac{2\alpha}{\nu}}\right)\right\| - r\gamma(\psi) \tag{2.16}$$

where r *and* $\gamma(\psi)$ *as in Theorem 2.18.*

Proof. Let $\mathbf{T} = U|\mathbf{T}|$ be the polar decomposition of the operator $\mathbf{T}$, where U is partial isometry and the kernel $\ker(U) = N(|\mathbf{T}|)$. If we take $\mathbf{D} = U, \mathbf{C} = |\mathbf{T}|^\beta, \mathbf{B} = U$ and $\mathbf{A} = |\mathbf{T}|^\alpha$, we have

$$\mathbf{DCBA} = \mathbf{T}|\mathbf{T}|^{\beta-1}\mathbf{T}|\mathbf{T}|^{\alpha-1}, |\mathbf{BA}|^2 = |\mathbf{T}|^{2\alpha} \quad \text{and} \quad |(\mathbf{DC})^*|^2 = |\mathbf{T}^*|^{2\beta}.$$

So, the result follows by Theorem 2.18. □

Corollary 2.20. *Let* $\mathbf{T} \in \mathcal{B}(\mathcal{H})$*,* $\alpha, \beta \geq 0$ *such that* $\alpha + \beta \geq 2$*,* $0 < \nu < 1$ *and let* ψ *be a non-negative increasing convex function on* $[0, \infty)$*. Then*

$$\psi(w^2(\mathbf{T}^*|\mathbf{T}^*|^{\alpha+\beta-2}\mathbf{T})) \leq \left\|(1-\nu)\psi\left(|\mathbf{T}|^{\frac{2\beta}{1-\nu}}\right) + \nu\psi\left(|\mathbf{T}|^{\frac{2\alpha}{\nu}}\right)\right\| - r\gamma(\psi) \tag{2.17}$$

where r *and* $\gamma(\psi)$ *as in Theorem 2.18.*

Proof. Let $\mathbf{T}^* = U|\mathbf{T}^*|$ be the polar decomposition of the operator $\mathbf{T}^*$, where U is partial isometry and the kernel $\ker(U) = N(|\mathbf{T}|)$. Then $\mathbf{T} = |\mathbf{T}^*|U^*$. If we take $\mathbf{D} = U, \mathbf{C} = |\mathbf{T}^*|^\beta, \mathbf{B} = |\mathbf{T}^*|^\alpha$ and $\mathbf{A} = U^*$, we have

$$\mathbf{DCBA} = U|\mathbf{T}^*|^\beta|\mathbf{T}^*|^\alpha U^* = \mathbf{T}^*|\mathbf{T}^*|^{\alpha+\beta-2}\mathbf{T}, |\mathbf{BA}|^2 = |\mathbf{T}|^{2\alpha} \text{ and } |(\mathbf{DC})^*|^2 = |\mathbf{T}|^{2\beta}.$$

So, the result follows by Theorem 2.18. □

Corollary 2.21. *Let* $\mathbf{T} \in \mathcal{B}(\mathcal{H})$*,* $\alpha, \beta \geq 0$*,* $0 < \nu < 1$ *and let* ψ *be a non-negative increasing convex function on* $[0, \infty)$*. Then*

$$\psi(w^2(|\mathbf{T}|^\alpha\mathbf{T}^2|\mathbf{S}|^\beta)) \leq \left\|(1-\nu)\psi\left(|\mathbf{T}|^{\frac{2\beta+2}{1-\nu}}\right) + \nu\psi\left(|\mathbf{T}^*|\mathbf{T}|^\alpha|^{\frac{2}{\nu}}\right)\right\| - r\gamma(\psi) \tag{2.18}$$

where r *and* $\gamma(\psi)$ *as in Theorem 2.18.*

Proof. In Theorem 2.18, if we let $\mathbf{D} = |\mathbf{T}|^\alpha, \mathbf{C} = \mathbf{T}, \mathbf{B} = \mathbf{T}$ and $\mathbf{A} = |\mathbf{T}|^\beta$, then

$$\begin{aligned} |\mathbf{BA}|^2 &= \mathbf{A}^*|\mathbf{B}|^2\mathbf{A} = |\mathbf{T}|^\beta|\mathbf{T}|^2|\mathbf{T}|^\alpha = |\mathbf{T}|^{2\beta+2} \\ |(\mathbf{DC})^*|^2 &= \mathbf{D}|\mathbf{C}^*|^2\mathbf{D}^* = |\mathbf{T}|^\alpha|\mathbf{T}^*|^2|\mathbf{T}|^\alpha = |\mathbf{T}|^\alpha\mathbf{T}\mathbf{T}^*|\mathbf{T}|^\alpha \\ &= |\mathbf{T}|^\alpha\mathbf{T}(|\mathbf{T}|^\alpha\mathbf{T})^* = |(|\mathbf{T}|^\alpha\mathbf{T})^*|^2 = |\mathbf{T}^*|\mathbf{T}|^\alpha|^2, \end{aligned}$$

so the result: □

Inequalities for numerical radius and operator norm have now been given, although in the context of superquadratic functions. Remember that a function $\psi : [0, \infty) \longrightarrow \mathbb{R}$ is termed superquadratic if there exists a constant $C_x \in \mathbb{R}$ such that

$$\psi(t) \geq \psi(\xi) + C_\xi(t - \xi) + \psi(|t - \xi|) \tag{2.19}$$

for all $t \geq 0$. If $-\psi$ is superquadratic, we say ψ is subquadratic. As a result, for a superquadratic function, ψ must be above its tangent line plus a translation of ψ. Superquadratic functions appear to be stronger than convex functions at first glance, however they may be deemed weaker if ψ has negative values. If ψ is superquadratic and non-negative, then ψ is increasing and convex, and if C_ξ is equal to (2.19), then $C_\xi \geq 0$ [1].

Theorem 2.22. *Let $\mathbf{A} \in \mathcal{B}(\mathcal{H})$ and let ψ be a non-negative superquadratic function. Then*

$$\psi(w(\mathbf{A})) \leq \|\psi(|\mathbf{A}|)\| - \inf_{\|\xi\|=1} \left\|\psi(||\mathbf{A}| - \|\mathbf{A}\||)^{\frac{1}{2}}\right\|^2. \tag{2.20}$$

Proof. Letting $\xi = \|\mathbf{A}\|$ in the inequality (2.19), we get

$$\psi(t) \geq \psi(\|\mathbf{A}\|) + C_{\|\mathbf{A}\|}(t - \|\mathbf{A}\|) + \psi(|t - \|\mathbf{A}\||). \tag{2.21}$$

By applying functional calculus for the operator $|\mathbf{A}|$ in (2.21) we get

$$\psi(|\mathbf{A}|) \geq \psi(\|\mathbf{A}\|) + C_{\|\mathbf{A}\|}(|\mathbf{A}| - \|\mathbf{A}\|) + \psi(||\mathbf{A}| - \|\mathbf{A}\||). \tag{2.22}$$

Hence,

$$\langle\psi(|\mathbf{A}|)\xi, \xi\rangle \geq \psi(\|\mathbf{A}\|) + C_{\|\mathbf{A}\|}(\langle|\mathbf{A}|\xi, \xi\rangle - \|\mathbf{A}\|) + \langle\psi(||\mathbf{A}| - \|\mathbf{A}\||)\xi, \xi\rangle.$$

Consequently,

$$\langle\psi(|\mathbf{A}|)\xi, \xi\rangle \geq \psi(\|\mathbf{A}\|) + C_{\|\mathbf{A}\|}(\langle|\mathbf{A}|\xi, \xi\rangle - \|\mathbf{A}\|) + \left\|\psi(||\mathbf{A}| - \|\mathbf{A}\||)^{\frac{1}{2}}x\right\|^2 \tag{2.23}$$

for every unit vector $\xi \in \mathcal{H}$.

Now, by taking supremum over $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ in (2.23), and using the fact $w(|\mathbf{A}|) = \|\mathbf{A}\| \geq w(\mathbf{A})$, and ψ is increasing, we deduce the desired inequality (2.20). □

Applying Theorem 2.22 to the superquadratic function $\psi(t) = t^r$ $(r \geq 2)$, we reach the following corollary:

Corollary 2.23. *Let $A \in \mathcal{B}(\mathcal{H})$. Then for any $r \geq 2$,*

$$w^r(\mathbf{A}) \leq \|\mathbf{A}\|^r - \inf_{\|\xi\|=1} \left\|||\mathbf{A}| - \|\mathbf{A}\||^{\frac{r}{2}}\right\|^2.$$

In particular

$$w(\mathbf{A}) \leq \sqrt{\|\mathbf{A}\|^2 - \inf_{\|\xi\|=1} \|||\mathbf{A}| - \|\mathbf{A}\|||^2} \leq \|\mathbf{A}\|.$$

3. Further refinements of numerical radius inequalities

In this section, We provide various inequalities involving power numerical radii and the usual operator norms of Hilbert space operators. In particular, if $\mathbf{A}_i, \mathbf{B}_i$ and $\mathbf{X}_i$ are bounded linear operators $(i = 1, 2, \cdots n \in \mathbb{N})$, then we estimate the numerical radius to $\sum_{i=1}^m \mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i$ for some $m \in \mathbb{N}$.

The following lemma is a straightforward application of Jensen's inequality about the convexity or concavity of certain power functions. Schlömilch's inequality for the weighted means of non-negative real numbers is a specific example of this inequality.

Lemma 3.1. *Let* $\sigma, \tau > 0$ *and* $0 \le \alpha \le 1$. *Then*

$$\sigma^{\alpha}\tau^{1-\alpha} \le \alpha\sigma + (1-\alpha)\tau \le (\alpha\sigma^{r} + (1-\alpha)\tau^{r})^{\frac{1}{r}} \quad for \quad r \ge 1. \tag{3.1}$$

The following result was established by Kittaneh and Manasrah [16], which is a refinement of the scalar Young inequality.

Lemma 3.2. *Let* $\sigma, \tau > 0$, *and* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\sigma\tau + r_0(\sigma^{\frac{p}{2}} - \tau^{\frac{q}{2}})^2 \le \frac{\sigma^p}{p} + \frac{\tau^q}{q}, \tag{3.2}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$.

Manasrah and Kittaneh have generalized (3.2) in [17], as follows:

Lemma 3.3. *Let* $\sigma, \tau > 0$, *and* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for* $m = 1, 2, \cdots$, *we have*

$$(\sigma^{\frac{1}{p}}\tau^{\frac{1}{q}})^m + r_0^m(\sigma^{\frac{m}{2}} - \tau^{\frac{m}{2}})^2 \le \left(\frac{\sigma^r}{p} + \frac{\tau^r}{q}\right)^{\frac{m}{r}}, \quad r \ge 1 \tag{3.3}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$. *In particular, if* $p = q = 2$, *then*

$$(\sqrt{\sigma\tau})^m + \frac{1}{2^m}(\sigma^{\frac{m}{2}} - \tau^{\frac{m}{2}})^2 \le 2^{-\frac{m}{r}}(\sigma^r + \tau^r)^{\frac{m}{r}}. \tag{3.4}$$

For $m = 1$, *and* $p = q = 2$, *we have*

$$\sqrt{\sigma\tau} + \frac{1}{2}(\sqrt{\sigma} - \sqrt{\tau})^2 \le 2^{-\frac{1}{r}}(\sigma^r + \tau^r)^{\frac{1}{r}}. \tag{3.5}$$

The convexity of the function $\psi(t) = t^r$, $r \ge 1$ leads to the following lemma, which deals with positive real numbers.

Lemma 3.4. *Let* $\sigma_i, i = 1, \cdots, n$ *be positive real numbers. Then*

$$\left(\sum_{i=1}^{n}\sigma_i\right)^r \le n^{r-1}\sum_{i=1}^{n}\sigma_i^r \quad for \ r \ge 1. \tag{3.6}$$

Theorem 3.5. *Let* $\mathbf{A}_i, \mathbf{C}_i, D_i \in \mathcal{B}(\mathcal{H})$, $(i = 1, \cdots, n)$, $m \in \mathbb{N}$. *Then*

$$w^r\left(\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\right) \le \frac{n^{r-1}}{2m}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n}\left(|\mathbf{C}_i^j\mathbf{A}_i|^{2r} + |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2r}\right)\right\| \tag{3.7}$$

for all $r \ge 1$.

Proof. Let $\xi \in \mathcal{H}$ be any unit vector. Then by Lemma 2.5, Lemma 3.1 and Lemma 3.4, we obtain that

$$\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\xi, \xi\right\rangle\right|^r = \frac{1}{m}\sum_{j=1}^{m}\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^{m-j}\mathbf{C}_i^j\mathbf{A}_i\xi, \xi\right\rangle\right|^r$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left|\left\langle \mathbf{D}_i\mathbf{C}_i^{m-j}\mathbf{C}_i^{j}\mathbf{A}_i\xi,\xi\right\rangle\right|\right)^r.$$

This implies that

$$\begin{aligned}
\left|\left\langle \sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^{m}\mathbf{A}_i\xi,\xi\right\rangle\right|^r &\leq \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left|\left\langle \mathbf{D}_i\mathbf{C}_i^{m-j}\mathbf{C}_i^{j}\mathbf{A}_i\xi,\xi\right\rangle\right|^r \\
&\leq \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle |\mathbf{C}_i^{j}\mathbf{A}_i|^2\xi,\xi\right\rangle^{\frac{r}{2}}\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^2\xi,\xi\right\rangle^{\frac{r}{2}} \\
&\leq \frac{n^{r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle |\mathbf{C}_i^{j}\mathbf{A}_i|^{2r}\xi,\xi\right\rangle^{\frac{1}{2}}\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2r}\xi,\xi\right\rangle^{\frac{1}{2}} \\
&\leq \frac{n^{r-1}}{2m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle \left(|\mathbf{C}_i^{j}\mathbf{A}_i|^{2r}+|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2r}\right)\xi,\xi\right\rangle.
\end{aligned}$$

Taking the supremum over all unit vectors $\xi \in \mathcal{H}$, we get the result. □

For $\mathbf{D}_i = \mathbf{A}_i = I$ in inequality (3.9), we have:

Corollary 3.6. *Let* $\mathbf{C}_i \in \mathcal{B}(\mathcal{H})$*,* $(i = 1, \cdots, n)$*,* $m \in \mathbb{N}$*. Then*

$$w^r\left(\sum_{i=1}^{n}\mathbf{C}_i^{m}\right) \leq \frac{n^{r-1}}{2m}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n}\left(|\mathbf{C}_i^{j}|^{2r}+|(\mathbf{C}_i^{m-j})^*|^{2r}\right)\right\| \tag{3.8}$$

for all $r \geq 1$*.*

The following is an example of how Corollary 3.6 may be used. It entails a numerical radius inequality for operator powers.

Corollary 3.7. *Let* $\mathbf{C} \in \mathcal{B}(\mathcal{H})$ *and* $m \in \mathbb{N}$*. Then for all* $r \geq 1$*, we have*

$$w^r(\mathbf{C}^m) \leq \frac{1}{2m}\sum_{j=1}^{m}\left\||\mathbf{C}^{j}|^{2r}+|(\mathbf{C}^{m-j})^*|^{2r}\right\|.$$

Theorem 3.8. *Let* $\mathbf{A}_i, \mathbf{C}_i, \mathbf{D}_i \in \mathcal{B}(\mathcal{H})$*,* $(i = 1, \cdots, n)$*,* $m \in \mathbb{N}$ *and* $0 \leq \alpha \leq 1$*. Then*

$$w\left(\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^{m}\mathbf{A}_i\right) \leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|\alpha|\mathbf{C}_i^{j}\mathbf{A}_i|^{\frac{2r}{\alpha}}+(1-\alpha)|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{\frac{2r}{1-\alpha}}\right\|^{\frac{1}{2r}} \tag{3.9}$$

for all $r \geq 1$*.*

Proof. Let $\xi \in \mathcal{H}$ be any unit vector. Then by Lemmas 2.5, 3.1 and 3.4, we obtain

$$\left|\left\langle \sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^{m}\mathbf{A}_i\xi,\xi\right\rangle\right| = \frac{1}{m}\sum_{j=1}^{m}\left|\left\langle \sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^{m-j}\mathbf{C}_i^{j}\mathbf{A}_i\xi,\xi\right\rangle\right|$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle |\mathbf{C}_i^j\mathbf{A}_i|^2\xi,\xi\right\rangle^{\frac{1}{2}}\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^2\xi,\xi\right\rangle^{\frac{1}{2}}.$$

Hence,

$$\begin{aligned}\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\xi,\xi\right\rangle\right| &\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle |\mathbf{C}_i^j\mathbf{A}_i|^2\xi,\xi\right\rangle\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^2\xi,\xi\right\rangle\right)^{\frac{1}{2}}\\
&\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle |\mathbf{C}_i^j\mathbf{A}_i|^{\frac{2}{\alpha}}\xi,\xi\right\rangle^{\alpha}\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{\frac{2}{1-\alpha}}\xi,\xi\right\rangle^{1-\alpha}\right)^{\frac{1}{2}}\\
&\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\alpha\left\langle |\mathbf{C}_i^j\mathbf{A}_i|^{\frac{2}{\alpha}}\xi,\xi\right\rangle^{r}+(1-\alpha)\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{\frac{2}{1-\alpha}}\xi,\xi\right\rangle^{r}\right)^{\frac{1}{2r}}\\
&\leq \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle\left(\alpha|\mathbf{C}_i^j\mathbf{A}_i|^{\frac{2r}{\alpha}}+(1-\alpha)|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{\frac{2r}{1-\alpha}}\right)\xi,\xi\right\rangle^{\frac{1}{2r}}.\end{aligned}$$

Taking the supremum over all unit vectors $\xi\in\mathcal{H}$, we deduce the desired result. □

Theorem 3.9. *Let* $\mathbf{A}_i,\mathbf{C}_i,\mathbf{D}_i\in\mathcal{B}(\mathcal{H})$*,* $(i=1,\cdots,n)$*,* $m\in\mathbb{N}$ *and* $p,q>1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$*. Then*

$$w^2\left(\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\right)\leq\frac{1}{m}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n}\frac{1}{p}|\mathbf{C}_i^j\mathbf{A}_i|^{2p}+\frac{1}{q}|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2q}\right\|-r_0\inf_{\|\xi\|=1}\psi(\xi),\tag{3.10}$$

where $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$ *and*

$$\psi(\xi)=\frac{n}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle ||\mathbf{C}_i^j\mathbf{A}_i|^2\xi,\xi\right\rangle^{\frac{p}{2}}-\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|\xi,\xi\right\rangle^{\frac{q}{2}}\right)^2.$$

Proof. Let $\xi\in\mathcal{H}$ be any unit vector. Then by Lemmas 2.5, 3.2 and 3.4, we obtain

$$\begin{aligned}\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\xi,\xi\right\rangle\right|^2 &= \frac{1}{m}\sum_{j=1}^{m}\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^{m-j}\mathbf{C}_i^j\mathbf{A}_i\xi,\xi\right\rangle\right|^2\\
&\leq \frac{n}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left|\left\langle\mathbf{D}_i\mathbf{C}_i^{m-j}\mathbf{C}_i^j\mathbf{A}_i\xi,\xi\right\rangle\right|^2\\
&\leq \frac{n}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle |\mathbf{C}_i^j\mathbf{A}_i|^2\xi,\xi\right\rangle\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^2\xi,\xi\right\rangle\\
&\leq \frac{n}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\frac{1}{p}\left\langle |\mathbf{C}_i^j\mathbf{A}_i|^{2p}\xi,\xi\right\rangle+\frac{1}{q}\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2q}\xi,\xi\right\rangle\right)\\
&\quad-\frac{nr_0}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle |\mathbf{C}_i^j\mathbf{A}_i|^2\xi,\xi\right\rangle^{\frac{p}{2}}-\left\langle |(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^2\xi,\xi\right\rangle^{\frac{q}{2}}\right)^2.\end{aligned}$$

This implies that

$$\begin{aligned}\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\xi,\xi\right\rangle\right|^2 &\leq \frac{n}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle\left(\frac{1}{p}|\mathbf{C}_i^j\mathbf{A}_i|^{2p}+\frac{1}{q}|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2q}\right)\xi,\xi\right\rangle\\ &- \frac{nr_0}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle|\mathbf{C}_i^j\mathbf{A}_i|^2\xi,\xi\right\rangle^{\frac{p}{2}}-\left\langle|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^2\xi,\xi\right\rangle^{\frac{q}{2}}\right)^2.\end{aligned}$$

Taking the supremum over all unit vectors $\xi\in\mathcal{H}$, we deduce the desired result. □

Theorem 3.10. *Let $\mathbf{A}_i,\mathbf{C}_i,\mathbf{D}_i\in\mathcal{B}(\mathcal{H})$, $(i=1,\cdots,n)$, $m\in\mathbb{N}$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $k=1,2,\cdots$. Then*

$$w^{2k}\left(\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\right)\leq\frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|\frac{1}{p}|\mathbf{C}_i^j\mathbf{A}_i|^{2rp}+\frac{1}{q}|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2qr}\right\|^{\frac{k}{r}}-r_0^k\inf_{\|\xi\|=1}\eta(\xi),\tag{3.11}$$

where $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$ and

$$\eta(\xi)=\frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle|\mathbf{C}_i^j\mathbf{A}_i|^{2p}\xi,\xi\right\rangle^{\frac{k}{2}}-\left\langle|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2q}\xi,\xi\right\rangle^{\frac{k}{2}}\right)^2$$

for all $r\geq1$.

Proof. Let $\xi\in\mathcal{H}$ be any unit vector. Then by Lemmas 2.5, 3.3 and 3.4, we obtain

$$\begin{aligned}\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\xi,\xi\right\rangle\right|^{2k} &= \frac{1}{m}\sum_{j=1}^{m}\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^{m-j}\mathbf{C}_i^j\mathbf{A}_i\xi,\xi\right\rangle\right|^{2k}\\ &\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left|\left\langle\mathbf{D}_i\mathbf{C}_i^{m-j}\mathbf{C}_i^j\mathbf{A}_i\xi,\xi\right\rangle\right|^{2k}.\end{aligned}$$

This implies that

$$\begin{aligned}\left|\left\langle\sum_{i=1}^{n}\mathbf{D}_i\mathbf{C}_i^m\mathbf{A}_i\xi,\xi\right\rangle\right|^{2k} &\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle|\mathbf{C}_i^j\mathbf{A}_i|^2\xi,\xi\right\rangle\left\langle|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^2\xi,\xi\right\rangle\right)^k\\ &\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle|\mathbf{C}_i^j\mathbf{A}_i|^{\frac{2p}{p}}\xi,\xi\right\rangle\left\langle|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{\frac{2q}{q}}\xi,\xi\right\rangle\right)^k\\ &\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle|\mathbf{C}_i^j\mathbf{A}_i|^{2p}\xi,\xi\right\rangle^{\frac{1}{p}}\left\langle|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2q}\xi,\xi\right\rangle^{\frac{1}{q}}\right)^k\\ &\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle\left(\frac{1}{p}|\mathbf{C}_i^j\mathbf{A}_i|^{2rp}+\frac{1}{q}|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2qr}\right)\xi,\xi\right\rangle^{\frac{k}{r}}\\ &- \frac{n^{2k-1}r_0^k}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle|\mathbf{C}_i^j\mathbf{A}_i|^{2p}\xi,\xi\right\rangle^{\frac{k}{2}}-\left\langle|(\mathbf{D}_i\mathbf{C}_i^{m-j})^*|^{2q}\xi,\xi\right\rangle^{\frac{k}{2}}\right)^2.\end{aligned}$$

Taking the supremum over all unit vectors $\xi\in\mathcal{H}$, we deduce the desired result. □

For $k = 1$, and $p = q = 2$, we have:

Corollary 3.11. *Let* $\mathbf{A}_i, \mathbf{C}_i, \mathbf{D}_i \in \mathcal{B}(\mathcal{H})$, $(i = 1, \cdots, n)$, $m \in \mathbb{N}$. *Then*

$$w^2\left(\sum_{i=1}^{n} \mathbf{D}_i \mathbf{C}_i^m \mathbf{A}_i\right) \leq \frac{n2^{-\frac{1}{r}}}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left\| |\mathbf{C}_i^j \mathbf{A}_i|^{4r} + |(\mathbf{D}_i \mathbf{C}_i^{m-j})^*|^{4r} \right\|^{\frac{1}{r}} - \frac{1}{2} \inf_{\|\xi\|=1} \eta(\xi), \tag{3.12}$$

where

$$\eta(\xi) = \frac{n}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\left\langle |\mathbf{C}_i^j \mathbf{A}_i|^4 \xi, \xi \right\rangle^{\frac{1}{2}} - \left\langle |(\mathbf{D}_i \mathbf{C}_i^{m-j})^*|^4 \xi, \xi \right\rangle^{\frac{1}{2}} \right)^2$$

for all $r \geq 1$.

The following lemma is an extended variant of the mixed Schwarz inequality, which has been shown by Kittaneh [12] and is highly relevant in the following results.

Lemma 3.12. *Let* $\mathbf{A} \in \mathcal{B}(\mathcal{H})$, *and* ψ *and* ϕ *be non-negative functions on* $[0, \infty)$ *which are continuous such that* $\psi(t)\phi(t) = t$ *for all* $t \in [0, \infty)$. *Then*

$$|\langle \mathbf{A}\xi, \zeta \rangle| \leq \|\psi(|\mathbf{A}|)\xi\| \, \|\phi(|\mathbf{A}^*|)\zeta\|, \tag{3.13}$$

for all $\xi, \zeta \in \mathcal{H}$.

The next results give improvements of the inequality (1.10).

Theorem 3.13. *Let* $\mathbf{A}_i, \mathbf{B}_i, \mathbf{X}_i \in \mathcal{B}(\mathcal{H})$, $(i = 1, \cdots, n)$, $m \in \mathbb{N}$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and let* ψ *and* ϕ *be as in Lemma 3.12. Then for all* $r \geq 1$, *we have*

$$w^{2r}\left(\sum_{i=1}^{n} \mathbf{X}_i \mathbf{A}_i^m \mathbf{B}_i\right) \leq \frac{n^{2r-1}}{m} \sum_{j=1}^{m} \left\| \sum_{i=1}^{n} \frac{1}{p} S_{i,j}^{pr} + \frac{1}{q} T_{i,j}^{qr} \right\| - r_0 \inf_{\|\xi\|=1} \rho(\xi), \tag{3.14}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$, $S_{i,j} = \mathbf{X}_i \psi^2\left(\left|\mathbf{A}_i^{j*}\right|\right) \mathbf{X}_i^*$, $T_{i,j} = \left(\mathbf{A}_i^{m-j} \mathbf{B}_i\right)^* \phi^2\left(\left|\mathbf{A}_i^j\right|\right) \mathbf{A}_i^{m-j} \mathbf{B}_i$ *and*

$$\rho(\xi) = \frac{n^{2r-1}}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left(\left\langle S_{i,j}^r \xi, \xi \right\rangle^{\frac{p}{2}} - \left\langle T_{i,j}^r \xi, \xi \right\rangle^{\frac{q}{2}} \right)^2.$$

Proof. Let $\xi \in \mathcal{H}$ be any unit vector. Then by Lemma 3.3, Lemma 3.4 and Lemma 3.12, we obtain

$$\begin{aligned} \left| \sum_{i=1}^{n} \langle \mathbf{X}_i \mathbf{A}_i^m \mathbf{B}_i \xi, \xi \rangle \right|^{2r} &= \frac{1}{m} \sum_{j=1}^{m} \left| \sum_{i=1}^{n} \left\langle \mathbf{X}_i \mathbf{A}_i^{m-j} \mathbf{A}_i^j \mathbf{B}_i \xi, \xi \right\rangle \right|^{2r} \\ &\leq \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \left| \left\langle \mathbf{X}_i \mathbf{A}_i^{m-j} \mathbf{A}_i^j \mathbf{B}_i \xi, \xi \right\rangle \right| \right)^{2r} \\ &\leq \frac{n^{2r-1}}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left| \left\langle \mathbf{X}_i \mathbf{A}_i^{m-j} \mathbf{A}_i^j \mathbf{B}_i \xi, \xi \right\rangle \right|^{2r} \end{aligned}$$

and so

$$\begin{aligned}
\left|\sum_{i=1}^{n}\langle \mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\xi,\xi\rangle\right|^{2r} &\le \frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left|\left\langle \mathbf{A}_i^{j*}\mathbf{X}_i^{*}x,\mathbf{A}_i^{m-j}\mathbf{B}_i x\right\rangle\right|^{2r}\\
&\le \frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|\psi\left(\left|\mathbf{A}_i^{j*}\right|\right)\mathbf{X}_i^{*}x\right\|^{2r}\left\|\phi\left(\left|\mathbf{A}_i^{j}\right|\right)\mathbf{A}_i^{m-j}\mathbf{B}_i x\right\|^{2r}\\
&\le \frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle S_{i,j}\xi,\xi\right\rangle^{r}\left\langle T_{i,j}\xi,\xi\right\rangle^{r}\\
&\le \frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle S_{i,j}^{r}\xi,\xi\right\rangle\left\langle T_{i,j}^{r}\xi,\xi\right\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\sum_{i=1}^{n}\langle \mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\xi,\xi\rangle\right|^{2r} &\le \frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle\left(\frac{1}{p}S_{i,j}^{pr}+\frac{1}{q}T_{i,j}^{qr}\right)\xi,\xi\right\rangle\\
&- r_0\frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle S_{i,j}^{r}\xi,\xi\right\rangle^{\frac{p}{2}}-\left\langle T_{i,j}^{r}\xi,\xi\right\rangle^{\frac{q}{2}}\right)^{2}.
\end{aligned}$$

Taking the supremum over all unit vectors $\xi\in\mathcal{H}$, we deduce the desired result. □

Inequality (3.17) includes several numerical radius inequalities as special cases. Samples of inequalities are demonstrated in what follows, for $\psi(t)=t^{\lambda}$ and $\phi(t)=t^{1-\lambda}$, $\lambda\in(0,1)$ in inequality (3.17).

Corollary 3.14. *Let $\mathbf{A}_i,\mathbf{B}_i,\mathbf{X}_i\in\mathcal{B}(\mathcal{H})$, $(i=1,\cdots,n)$, $m\in\mathbb{N}$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and let ψ and g be as in Lemma 3.12. Then for all $r\ge 1$, we have*

$$w^{2r}\left(\sum_{i=1}^{n}\mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\right)\le\frac{n^{2r-1}}{m}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n}\frac{1}{p}S_{i,j}^{pr}+\frac{1}{q}T_{i,j}^{qr}\right\|-r_0\inf_{\|\xi\|=1}\rho(\xi),\tag{3.15}$$

where $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$, $S_{i,j}=\mathbf{X}_i\left|\mathbf{A}_i^{j*}\right|^{2\lambda}\mathbf{X}_i^{*}$, $T_{i,j}=\left(\mathbf{A}_i^{m-j}\mathbf{B}_i\right)^{*}\left|\mathbf{A}_i^{j}\right|^{2(1-\lambda)}\mathbf{A}_i^{m-j}\mathbf{B}_i$ and

$$\rho(\xi)=\frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle S_{i,j}^{r}\xi,\xi\right\rangle^{\frac{p}{2}}-\left\langle T_{i,j}^{r}\xi,\xi\right\rangle^{\frac{q}{2}}\right)^{2}.$$

For $\mathbf{X}_i=\mathbf{B}_i=I$ in inequality (3.17) we get the following numerical radius inequality.

Corollary 3.15. *Let $\mathbf{A}_i,\in\mathcal{B}(\mathcal{H})$, $(i=1,\cdots,n)$, $m\in\mathbb{N}$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and let ψ and g be as in Lemma 3.12. Then for all $r\ge 1$, we have*

$$w^{2r}\left(\sum_{i=1}^{n}\mathbf{A}_i^m\right)\le\frac{n^{2r-1}}{m}\sum_{j=1}^{m}\left\|\sum_{i=1}^{n}\frac{1}{p}S_{i,j}^{pr}+\frac{1}{q}T_{i,j}^{qr}\right\|-r_0\inf_{\|\xi\|=1}\rho(\xi),\tag{3.16}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$, $S_{i,j} = \psi^2\left(\left|\left(\mathbf{A}_i^j\right)^*\right|\right)$, $T_{i,j} = \left(\mathbf{A}_i^{m-j}\right)^* \phi^2\left(\left|\mathbf{A}_i^j\right|\right)\mathbf{A}_i^{m-j}$ *and*

$$\rho(\xi) = \frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle S_{i,j}^{r}\xi,\xi\right\rangle^{\frac{p}{2}} - \left\langle T_{i,j}^{r}\xi,\xi\right\rangle^{\frac{q}{2}}\right)^2.$$

An application of Corollary 3.15 can be seen in the following result. It involves a numerical radius inequality for the powers of operator.

Corollary 3.16. *Let* $\mathbf{A} \in \mathcal{B}(\mathcal{H})$, $m \in \mathbb{N}$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and let* $\psi(t) = t^{\lambda}$ *and* $\phi(t) = t^{1-\lambda}$. *Then for all* $r \geq 1$, *we have*

$$w^{2r}\left(\mathbf{A}^m\right) \leq \frac{1}{m}\sum_{j=1}^{m}\left\|\frac{1}{p}S_j^{pr} + \frac{1}{q}T_j^{qr}\right\| - r_0 \inf_{\|\xi\|=1}\rho(\xi), \tag{3.17}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$, $S_j = \left|\left(A^j\right)^*\right|^{2\lambda}$, $T_j = \left(\mathbf{A}^{m-j}\right)^*\left|\mathbf{A}^j\right|^{2(1-\lambda)}\mathbf{A}^{m-j}$ *and*

$$\rho(\xi) = \frac{1}{m}\sum_{j=1}^{m}\left(\left\langle S_j^{r}\xi,\xi\right\rangle^{\frac{p}{2}} - \left\langle T_j^{r}\xi,\xi\right\rangle^{\frac{q}{2}}\right)^2.$$

Theorem 3.17. *Let* $\mathbf{A}_i, \mathbf{B}_i, \mathbf{X}_i \in \mathcal{B}(\mathcal{H})$, $(i = 1, \cdots, n)$, $m, k \in \mathbb{N}$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and let* ψ *and* ϕ *be as in Lemma 3.12. Then for all* $r \geq 1$, *we have*

$$w^{2k}\left(\sum_{i=1}^{n}\mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\right) \leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|\frac{1}{p}S_{i,j}^{pr} + \frac{1}{q}T_{i,j}^{qr}\right\|^{\frac{k}{r}} - r_0^k \inf_{\|\xi\|=1}\omega(\xi), \tag{3.18}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$, $S_{i,j} = \mathbf{X}_i\psi^2\left(\left|\mathbf{A}_i^{j*}\right|\right)\mathbf{X}_i^*$, $T_{i,j} = \left(\mathbf{A}_i^{m-j}\mathbf{B}_i\right)^*\phi^2\left(\left|\mathbf{A}_i^j\right|\right)\mathbf{A}_i^{m-j}\mathbf{B}_i$ *and*

$$\omega(\xi) = \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle S_{i,j}^{p}\xi,\xi\right\rangle^{\frac{k}{2}} - \left\langle T_{i,j}^{q}\xi,\xi\right\rangle^{\frac{k}{2}}\right)^2.$$

Proof. Let $\xi \in \mathcal{H}$ be any unit vector. Then by Lemmas 3.3, 3.4 and 3.12, we obtain

$$\begin{aligned}
\left|\sum_{i=1}^{n}\left\langle\mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\xi,\xi\right\rangle\right|^{2k} &= \frac{1}{m}\sum_{j=1}^{m}\left|\sum_{i=1}^{n}\left\langle\mathbf{X}_i\mathbf{A}_i^{m-j}\mathbf{A}_i^j\mathbf{B}_i\xi,\xi\right\rangle\right|^{2k} \\
&\leq \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left|\left\langle\mathbf{X}_i\mathbf{A}_i^{m-j}\mathbf{A}_i^j\mathbf{B}_i\xi,\xi\right\rangle\right|\right)^{2k} \\
&\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left|\left\langle\mathbf{X}_i\mathbf{A}_i^{m-j}\mathbf{A}_i^j\mathbf{B}_i\xi,\xi\right\rangle\right|^{2k} \\
&\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left|\left\langle\mathbf{A}_i^{j*}\mathbf{X}_i^*x,\mathbf{A}_i^{m-j}\mathbf{B}_ix\right\rangle\right|^{2k}
\end{aligned}$$

this implies that

$$\begin{aligned}\left|\sum_{i=1}^{n}\langle \mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\xi,\xi\rangle\right|^{2k} &\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|\psi\left(\left|\mathbf{A}_i^{j*}\right|\right)\mathbf{X}_i^* x\right\|^{2k}\left\|\phi\left(\left|\mathbf{A}_i^{j}\right|\right)\mathbf{A}_i^{m-j}\mathbf{B}_i x\right\|^{2k}\\ &\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle S_{i,j}^{p}\xi,\xi\right\rangle^{\frac{1}{p}}\left\langle T_{i,j}^{q}\xi,\xi\right\rangle^{\frac{1}{q}}\right)^{k}\\ &\leq \frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle \frac{1}{p}S_{i,j}^{pr}\xi,\xi\right\rangle+\frac{1}{q}\left\langle T_{i,j}^{qr}\xi,\xi\right\rangle\right)^{\frac{k}{r}}\\ &- r_0^k\frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle S_{i,j}^{p}\xi,\xi\right\rangle^{\frac{k}{2}}-\left\langle T_{i,j}^{q}\xi,\xi\right\rangle^{\frac{k}{2}}\right)^{2}.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\sum_{i=1}^{n}\langle \mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\xi,\xi\rangle\right|^{2k} &\leq \frac{n^{2r-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\langle\left(\frac{1}{p}S_{i,j}^{pr}+\frac{1}{q}T_{i,j}^{qr}\right)\xi,\xi\right\rangle^{\frac{k}{r}}\\ &- r_0^k\frac{n^{2k-1}}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle S_{i,j}^{p}\xi,\xi\right\rangle^{\frac{k}{2}}-\left\langle T_{i,j}^{q}\xi,\xi\right\rangle^{\frac{k}{2}}\right)^{2}.\end{aligned}$$

Taking the supremum over all unit vectors $\xi \in \mathcal{H}$, we deduce the desired result. □

If we take $k = 1$ and $p = q$, we have:

Corollary 3.18. *Let* $\mathbf{A}_i, \mathbf{B}_i, \mathbf{X}_i \in \mathcal{B}(\mathcal{H})$, $(i = 1, \cdots, n)$, $m \in \mathbb{N}$, *and let* ψ *and* ϕ *be as in Lemma 3.12. Then for all* $r \geq 1$, *we have*

$$w^2\left(\sum_{i=1}^{n}\mathbf{X}_i\mathbf{A}_i^m\mathbf{B}_i\right)\leq \frac{n}{m2^{\frac{1}{r}}}\sum_{j=1}^{m}\sum_{i=1}^{n}\left\|S_{i,j}^{2r}+T_{i,j}^{2r}\right\|^{\frac{1}{r}}-\frac{1}{2}\inf_{\|\xi\|=1}\omega(\xi), \tag{3.19}$$

where $S_{i,j} = \mathbf{X}_i\psi^2\left(\left|\mathbf{A}_i^{j*}\right|\right)\mathbf{X}_i^*$, $T_{i,j} = \left(\mathbf{A}_i^{m-j}\mathbf{B}_i\right)^*\phi^2\left(\left|\mathbf{A}_i^{j}\right|\right)\mathbf{A}_i^{m-j}\mathbf{B}_i$ *and*

$$\omega(\xi) = \frac{n}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\left(\left\langle S_{i,j}^{2}\xi,\xi\right\rangle^{\frac{1}{2}}-\left\langle T_{i,j}^{2}\xi,\xi\right\rangle^{\frac{1}{2}}\right)^{2}.$$

4. Conclusions

In this work, we have derived a series of precise inequalities involving the standard operator norms of Hilbert space operators and powers of the numerical radii. These inequalities build upon traditional convexity inequalities for nonnegative real numbers and extend earlier numerical radius inequalities in the context of operator theory.

As part of future work, further investigations could explore the practical implications and applications of these inequalities in the context of operator theory and related areas of mathematics and physics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interests.

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