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Research article

Extended Perron complements of M-matrices

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Abstract: This paper aims to consider the extended Perron complements for the collection of M-matrices. We first exhibit the connection between the extended Perron complements of M-matrices and nonnegative matrices. Moreover, we present some common inequalities involving extended Perron complements, Schur complements, and principal submatrices of irreducible M-matrices by utilizing the properties of M-matrices. We also discuss the monotonicity of the extended Perron complements and minimum eigenvalue. For the collection of M-matrices, we demonstrate that all (extended) Perron complements are M-matrices. Especially, we deduce that M-matrices and their Perron complements share the same minimum eigenvalue. Finally, a simple example is presented to illustrate our findings.

Keywords: M-matrix; Z-matrix; minimum eigenvalue; extended Perron complement; Schur complement
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1. Introduction

In this paper, we deal with n-square real matrices. We denote the set $\{1, 2, \cdots, n\}$ by $\langle n \rangle$. The expression $A \geqslant 0$ implies that all elements of the matrix A are nonnegative. We write $B \geqslant A$ if and only if $B - A \geqslant 0$. According to the Perron-Frobenius theorem ([1], Theorem 1.1, p.26), if $A \geqslant 0$, then the spectral radius, denoted by $\rho(A)$, is an eigenvalue of the matrix A and dominates all other eigenvalues in modulus.

The n-square real matrix $K = \left(k_{ij}\right)$ is defined to be a Z-matrix provided $k_{ij} \leq 0,\ i \neq j,\ i, j \in \langle n \rangle$. The collection of Z-matrices appears in many mathematical and physics problems. Let K be a Z-matrix and define

$$q(K) = \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma(K)\},$$

where $\sigma(K)$ denotes the spectrum of the matrix K. Then, $q(K) \in \sigma(K)$ and $q(K)$ is called the minimum eigenvalue of K ([2], Problem 16 in Section 2.5, p.129–130). From the definition of Z-

matrices, we can write a Z-matrix K as follows:

$$K = sI - A,\ A \geq 0.$$

Here, I is the n-square identity matrix, and s is a real number. Clearly, $q(K) = s - \rho(A)$.

A Z-matrix K is further defined to be a nonsingular M-matrix provided

$$K = sI - A,\ A \geq 0,\ s > \rho(A). \tag{1.1}$$

For an M-matrix K, $q(K)$ is also the eigenvalue of K with minimum modulus. Moreover, if K is an irreducible M-matrix, there must exist a positive vector u such that $Ku = q(K)u$. The set of M-matrices is one of the most famous subclasses of Z-matrices. These matrices are of considerable importance in the research on the convergence of iterative processes in linear and nonlinear equation systems, as well as the research on nonnegative solutions to such systems. In addition, there has been considerable interest in inverse M-matrices, that is, nonsingular matrices whose inverses are M-matrices. Undoubtedly, inverse M-matrices inherit significant structural properties from M-matrices. M-matrices and inverse M-matrices have important applications in computational mathematics, physics, intelligence science, biology, and other fields. Furthermore, the research on submatrices and related matrices of M-matrices is an important study direction. Some interesting results proved in references [3–8] inspired our research on the collection of M-matrices.

We now give some definitions and preliminary results that will be used in this study. Unless otherwise specified, we assume that β is an increasing sequence of integers chosen from $\langle n \rangle$ and $\alpha = \langle n \rangle \setminus \beta$. We denote a submatrix of A by $A[\alpha,\beta]$, whose rows depend on α and columns depend on β. In particular, if $\alpha = \beta$, $A[\beta,\beta]$ is abbreviated as $A[\beta]$ for convenience.

Suppose that $\varnothing \neq \beta \subset \langle n \rangle$. For the nonsingular submatrix $A[\beta]$ of A, the expression

$$A/A[\beta] = A[\alpha] - A[\alpha,\beta](A[\beta])^{-1}A[\beta,\alpha]$$

is defined to be the Schur complement concerning $A[\beta]$.

The Schur complement is a powerful tool in the study of matrix theory. Utilizing the Schur complement is a classical approach when dealing with matrix problems, such as deriving matrix inequalities, deducing determinants, and dealing with large-scale matrix computations. Significant research has been conducted on Schur complements of some special matrices since the late 1960s.

For a nonnegative irreducible matrix A, the notion of Perron complement concerning $A[\beta]$ was introduced by Meyer [5] as follows:

$$P(A/A[\beta]) = A[\alpha] + A[\alpha,\beta](\rho(A)I - A[\beta])^{-1}A[\beta,\alpha]. \tag{1.2}$$

Because A is irreducible, we can easily get $\rho(A) > \rho(A[\beta])$. Therefore, $\rho(A)I - A[\beta]$ is an invertible M-matrix. Meyer thoroughly explored the properties of $P(A/A[\beta])$ and derived that the matrix $P(A/A[\beta])$ inherits the nonnegativity and irreducibility of the matrix A. Meanwhile, A and its Perron complement $P(A/A[\beta])$ possess the same spectral radius. Extensive studies have been conducted on the Perron complements of matrices with special structures, such as Z-matrices [6], inverse N_0-matrices [7], and inverse M-matrices [8], and notable results have been achieved [9–11].

Neumann [8] made an improvement on (1.2) by substituting t for $\rho(A)$ and proposed the notion of extended Perron complement relative to $A[\beta]$ at t as follows:

$$P_t(A/A[\beta]) = A[\alpha] + A[\alpha,\beta](tI - A[\beta])^{-1}A[\beta,\alpha],\ t \geq \rho(A). \tag{1.3}$$

Clearly, the above expression is well-defined because $t \geq \rho(A) > \rho(A[\beta])$.

To obtain tight bounds of $\rho(A)$, the concept of generalized Perron complement $P_t(A/A[\beta])$ for $t > \rho(A[\beta])$ was first proposed in [12]. Inspired by the definitions of Meyer and Neumann, we present the following notions. Let $\varnothing \neq \beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle \backslash \beta$. For an irreducible M-matrix K of order n , we define the extended Perron complement relative to $K[\beta]$ at t as follows:

$$Q_t(K/K[\beta]) = K[\alpha] + K[\alpha,\beta](tI - K[\beta])^{-1}K[\beta,\alpha],\ t \leq q(K). \tag{1.4}$$

Remark 1.1. Set $K = sI - A,\ A \geq 0$, and $s > \rho(A)$. For any $\varnothing \neq \beta \subset \langle n\rangle$, clearly, $q(K) < q(K[\beta])$ because K is irreducible. Therefore, we have $t \leq q(K) < q(K[\beta])$. This shows that

$$s - t \geq s - q(K) > s - q(K[\beta]) = \rho(sI - K[\beta]).$$

As $sI - K[\beta] \geq 0$, we conclude from (1.1) that $(s-t)I - (sI - K[\beta]) = K[\beta] - tI$ is an invertible M-matrix and $(K[\beta] - tI)^{-1} \geq 0$. Meanwhile,

$$K[\alpha] + K[\alpha,\beta](tI - K[\beta])^{-1}K[\beta,\alpha] = K[\alpha] - K[\alpha,\beta](K[\beta] - tI)^{-1}K[\beta,\alpha].$$

Therefore, the definition of $Q_t(K/K[\beta])$ in (1.4) is valid when $t \leq q(K)$.

If we set t=$q(K)$ in (1.4), we will obtain the Perron complement relative to $K[\beta]$:

$$Q(K/K[\beta]) = K[\alpha] + K[\alpha,\beta](q(K)I - K[\beta])^{-1}K[\beta,\alpha]. \tag{1.5}$$

This paper focuses on studying the properties of extended Perron complements concerning irreducible M-matrices. The content of the paper is arranged as follows. For the collection of irreducible M-matrices, we establish our concepts and notations of (extended) Perron complement in Section 1.

In Section 2, utilizing the special structure of M-matrices, we establish the connection between the (extended) Perron complements of M-matrices and nonnegative matrices.

In Section 3, some general inequalities involving extended Perron complements, Schur complements, and submatrices of irreducible M-matrices are presented by employing the properties of M-matrices. Additionally, we discuss the monotonicity of the extended Perron complements and minimum eigenvalue.

At the end of this paper, we prove that the collection of M-matrices is closed on any (extended) Perron complementation. We also demonstrate that an M-matrix and its Perron complement share the same minimum eigenvalue.

2. Relationship between extended Perron complements of M-matrices and nonnegative matrices

For convenience and clarity, we employ the following notations. We will simply denote Perron and Schur complements by the index sets rather than principal submatrices. For example, $Q_t(K/K[\beta])$ is abbreviated as $Q_t(K/\beta)$, and $K/K[\beta]$ is denoted by K/β .

Equation (1.1) implies that there exists a close connection between M-matrices and nonnegative matrices. The following theorem shows that this close relationship can be carried over to their extended Perron complements.

Theorem 2.1. Suppose K is an $n \times n$ irreducible M-matrix such that $K = sI - A,\ A \geq 0$, and $s > \rho(A)$. For any $\varnothing \neq \beta \subset \langle n \rangle$, the following identity holds:

$$Q_t(K/\beta) + P_{s-t}(A/\beta) = sI,\ t \leq q(K). \tag{2.1}$$

Proof. For any $\varnothing \neq \beta \subset \langle n \rangle$, we suppose that K is symmetrically permuted to the following block matrix:

$$K = \begin{pmatrix} K[\alpha] & K[\alpha,\beta] \\ K[\beta,\alpha] & K[\beta] \end{pmatrix} = \begin{pmatrix} B & C \\ D & E \end{pmatrix}.$$

Therefore, we have

$$A = sI - K = \begin{pmatrix} sI - B & -C \\ -D & sI - E \end{pmatrix} \geq 0. \tag{2.2}$$

For any $t \leq q(K)$, we obtain $s - t \geq s - q(K) = \rho(A)$. According to (2.2), we get the extended Perron complement with respect to $A[\beta]$ at $s - t$ as follows:

$$\begin{aligned} P_{s-t}(A/\beta) &= (sI - B) + (-C)\left[(s-t)I - (sI - E)\right]^{-1}(-D) \\ &= (sI - B) + C(E - tI)^{-1}D \\ &= (sI - B) - C(tI - E)^{-1}D \\ &= sI - \left[B + C(tI - E)^{-1}D\right] \\ &= sI - Q_t(K/\beta). \end{aligned} \tag{2.3}$$

From (2.3), we can immediately obtain our conclusion.

If we set $t = q(K)$ in (2.1), we will obtain the following consequence.

Corollary 2.1. Suppose K is an $n \times n$ irreducible M-matrix such that $K = sI - A,\ A \geq 0$, and $s > \rho(A)$. For any $\varnothing \neq \beta \subset \langle n \rangle$, the following identity holds:

$$Q(K/\beta) + P(A/\beta) = sI.$$

3. Some basic properties of the extended Perron complement of M-matrices

In the following, we discuss some essential properties of the extended Perron complement of M-matrices. We will establish some general inequalities involving extended Perron complements, Schur complements, and principal submatrices of irreducible M-matrices. The monotonicity of the extended Perron complements and minimum eigenvalue are also demonstrated. First, we introduce some basic lemmas that are helpful in the proof of our conclusions.

Lemma 3.1. [1] If nonsingular M-matrices E and F satisfy $F \leq E$, then $0 \leq E^{-1} \leq F^{-1}$.

Lemma 3.2. [12] Suppose the matrix A is irreducible and nonnegative. Then, the generalized Perron complement $P_t(A/\beta)$ for $t > \rho(A[\beta])$ is irreducible and nonnegative. In addition, as a function with respect to t, the spectral radius $\rho[P_t(A/\beta)]$ is strictly decreasing on $(\rho(A[\beta]), +\infty)$.

Theorem 3.1. Suppose K is an $n \times n$ irreducible M-matrix. For any $\varnothing \neq \beta \subset \langle n \rangle$, the following orderings hold:
(1) $Q_t(K/\beta) \leq K/\beta \leq K[\alpha]$ for $0 \leq t \leq q(K)$;
(2) $K/\beta \leq Q_t(K/\beta) \leq K[\alpha]$ for $t \leq 0$.
Moreover, we have

$$\lim_{t \to -\infty} Q_t(K/\beta) = K[\alpha]. \tag{3.1}$$

Proof. For any $\varnothing \neq \beta \subset \langle n \rangle$ and $\alpha = \langle n \rangle \setminus \beta$, we assume that K is symmetrically permuted to the block matrix:

$$K = \begin{pmatrix} K[\alpha] & K[\alpha, \beta] \\ K[\beta, \alpha] & K[\beta] \end{pmatrix} = \begin{pmatrix} B & C \\ D & E \end{pmatrix}. \tag{3.2}$$

For any $t \leq q(K)$, we have

$$\begin{aligned} Q_t(K/\beta) &= B + C(tI - E)^{-1} D \\ &= B - C(E - tI)^{-1} D \end{aligned}$$

and

$$K/\beta = B - CE^{-1} D.$$

To obtain our conclusions, we discuss two cases.

Case 1. $0 \leq t \leq q(K)$

Because E is a principal submatrix of K, E is also an M-matrix. Clearly, $q(K) < q(E)$ because K is irreducible. Therefore, we obtain $0 \leq t < q(E)$, which guarantees that $E - tI$ is a nonsingular M-matrix. This important fact has also been mentioned in Remark 1.1. Based on Lemma 3.1, we obtain $(E - tI)^{-1} \geq E^{-1} \geq 0$. Meanwhile, for M-matrix K to be a Z-matrix, we must have that $C \leq 0$ and $D \leq 0$. Therefore, we obtain

$$C(E - tI)^{-1} D \geq CE^{-1} D \geq 0.$$

That is,

$$B - C(E - tI)^{-1} D \leq B - CE^{-1} D \leq B.$$

The above inequality implies that $Q_t(K/\beta) \leq K/\beta \leq B = K[\alpha]$.

Case 2. $t \leq 0$

Clearly, $E - tI \geq E$ and $E - tI, E$ are nonsingular M-matrices. From Lemma 3.1, we obtain $E^{-1} \geq (E - tI)^{-1} \geq 0$. As $C \leq 0$ and $D \leq 0$, we obtain

$$CE^{-1} D \geq C(E - tI)^{-1} D \geq 0.$$

We further obtain

$$B - CE^{-1} D \leq B - C(E - tI)^{-1} D \leq B.$$

This means that $K/\beta \leq Q_t(K/\beta) \leq B = K[\alpha]$.

Next, we discuss the limit of $Q_t(K/\beta)$ when $t \to -\infty$. If $t \neq 0$, we have

$$(tI - E)^{-1} = \frac{1}{t}\left(I - \frac{1}{t} E\right)^{-1}$$

$$= \frac{1}{t}\left(I + \frac{1}{t}E + \frac{1}{t^2}E^2 + \cdots\right).$$

Clearly, $\lim\limits_{t\to-\infty}(tI - E)^{-1} = 0$. Therefore, we have

$$\begin{aligned}\lim_{t\to-\infty} Q_t(K/\beta) &= \lim_{t\to-\infty}\left[B + C(tI - E)^{-1}D\right]\\ &= B + \lim_{t\to-\infty} C(tI - E)^{-1}D\\ &= B.\end{aligned}$$

Thus, the proof of Theorem 3.1 is completed.

Theorem 3.2. Suppose K is an $n \times n$ irreducible M-matrix. For any $\emptyset \neq \beta \subset \langle n\rangle$, the extended Perron complement $Q_t(K/\beta)$ is entry-wise decreasing concerning t on $(-\infty, q(K)]$.

Proof. Let K be partitioned according to (3.2). Suppose $t_1 \leq t_2 \leq q(K)$. We have

$$Q_{t_1}(K/\beta) = B + C(t_1 I - E)^{-1}D$$

and

$$Q_{t_2}(K/\beta) = B + C(t_2 I - E)^{-1}D.$$

Therefore,

$$\begin{aligned}Q_{t_1}(K/\beta) - Q_{t_2}(K/\beta) &= C(t_1 I - E)^{-1}D - C(t_2 I - E)^{-1}D\\ &= C(E - t_2 I)^{-1}D - C(E - t_1 I)^{-1}D\\ &= C\left[(E - t_2 I)^{-1} - (E - t_1 I)^{-1}\right]D.\end{aligned}$$

According to Remark 1.1, we observe that $E - t_1 I$ and $E - t_2 I$ are both nonsingular M-matrices when $t_1 \leq t_2 \leq q(K)$. Moreover, $E - t_1 I \geq E - t_2 I$. From Lemma 3.1, we have

$$(E - t_2 I)^{-1} \geq (E - t_1 I)^{-1}.$$

Because K is an M-matrix, it must first be a Z-matrix, resulting in $C \leq 0, D \leq 0$. Hence, we deduce that

$$C\left[(E - t_2 I)^{-1} - (E - t_1 I)^{-1}\right]D \geq 0.$$

This means $Q_{t_1}(K/\beta) \geq Q_{t_2}(K/\beta)$. Therefore, as a function concerning t, the extended Perron complement $Q_t(K/\beta)$ is entry-wise decreasing on $(-\infty, q(K)]$.

Theorem 3.3. Suppose K is an $n \times n$ irreducible M-matrix. For any $\emptyset \neq \beta \subset \langle n\rangle$, the minimum eigenvalue $q[Q_t(K/\beta)]$ is strictly decreasing concerning t on $(-\infty, q(K)]$.

Proof. Let $K = sI - A$, $A \geq 0$, and $s > \rho(A)$. Because $t \leq q(K)$ and A is irreducible and nonnegative, we have $s - t \geq s - q(K) = \rho(A) > \rho(A[\beta])$. According to Lemma 3.2, it can easily be seen that $P_{s-t}(A/\beta)$ is an irreducible nonnegative matrix. Moreover, as $s - t$ is strictly decreasing concerning t, from Lemma 3.2, we deduce that $\rho[P_{s-t}(A/\beta)]$ is strictly increasing concerning t on

$(\rho(A[\beta]), +\infty)$. In particular, $\rho[P_{s-t}(A/\beta)]$ is strictly increasing concerning t on $[\rho(A), +\infty)$ because $[\rho(A), +\infty) \subset (\rho(A[\beta]), +\infty)$. Meanwhile, by Theorem 2.1, we obtain

$$Q_t(K/\beta) = sI - P_{s-t}(A/\beta), \ t \leq q(K).$$

As $P_{s-t}(A/\beta)$ is nonnegative, $Q_t(K/\beta)$ is a Z-matrix, and we obtain

$$q[Q_t(K/\beta)] = s - \rho[P_{s-t}(A/\beta)].$$

Therefore, $q[Q_t(K/\beta)]$ is strictly decreasing concerning t on $(-\infty, q(K)]$.

4. Closure of the (extended) Perron complements of M-matrices

As one of the most well-known subclasses of the Z-matrices, M-matrices possess many features and properties. For example, all the Schur complements and principal submatrices of nonsingular M-matrices are nonsingular M-matrices. We highlight this to prove that the class of M-matrices is closed on any (extended) Perron complementation, that is, all (extended) Perron complements of irreducible M-matrices are irreducible M-matrices. Before proving this, we must recall the following conclusions.

Lemma 4.1. [5] Suppose the matrix A is irreducible and nonnegative. For any $\varnothing \neq \beta \subset \langle n \rangle$,
(1) $P(A/\beta)$ is irreducible and nonnegative and
(2) $\rho[P(A/\beta)] = \rho(A)$.

Lemma 4.2. [1] Suppose the matrix A is irreducible and nonnegative. Then, A has a positive eigenvector v corresponding to $\rho(A)$.

Lemma 4.3. [2] Suppose K is a Z-matrix. Then, K is a nonsingular M-matrix if and only if $K^{-1} \geq 0$.

Lemma 4.4. [13] Suppose the m-square matrix P and k-square matrix L are nonsingular. Let M be an $m \times k$ matrix and N be an $k \times m$ matrix. Then, $P + MLN$ is invertible if and only if $L^{-1} + NP^{-1}M$ is invertible and the following equation holds:

$$(P + MLN)^{-1} = P^{-1} - P^{-1}M\left(L^{-1} + NP^{-1}M\right)^{-1}NP^{-1}.$$

The above-mentioned result is called the Woodbury formula.

Theorem 4.1. Suppose K is an $n \times n$ irreducible M-matrix. For any $\varnothing \neq \beta \subset \langle n \rangle$, the Perron complement $Q(K/\beta)$ is an irreducible M-matrix such that $q[Q(K/\beta)] = q(K)$.

Proof. Set $K = sI - A$, $A \geq 0$, and $s > \rho(A)$. As irreducibility is independent of the main diagonal elements, we find that A is irreducible. In terms of Lemma 4.1, we know $P(A/\beta)$ is nonnegative and irreducible. Moreover, $s > \rho(A) = \rho[P(A/\beta)]$. According to Corollary 2.1, we have

$$Q(K/\beta) = sI - P(A/\beta). \tag{4.1}$$

Thus, we conclude from (1.1) and (4.1) that $Q(K/\beta)$ is an irreducible M-matrix.

In the following, we prove that the irreducible M-matrix K and its Perron complement $Q(K/\beta)$ share the same minimum eigenvalue. From Lemma 4.2, for the nonnegative irreducible matrix $P(A/\beta)$, there must exist a positive vector v satisfying $P(A/\beta)v = \rho[P(A/\beta)]v$. As $\rho(A) = \rho[P(A/\beta)]$, we get

$$P(A/\beta)v = \rho(A)v.$$

Therefore, according to (4.1), we have

$$\begin{aligned} Q(K/\beta)v &= [sI - P(A/\beta)]v \\ &= sv - P(A/\beta)v \\ &= sv - \rho(A)v \\ &= [s - \rho(A)]v \\ &= q(K)v. \end{aligned}$$

This shows that $q(K)$ is an eigenvalue of the matrix $Q(K/\beta)$ with a corresponding positive eigenvector v. According to the theory of M-matrices, the minimum eigenvalue is the only eigenvalue that can have a positive eigenvector. Thus, we infer that $q(K)$ is indeed the minimum eigenvalue of the matrix $Q(K/\beta)$. This completes the proof.

Remark 4.1. We can obtain $q[Q(K/\beta)] = q(K)$ more straightforwardly. According to (4.1), we get

$$q[Q(K/\beta)] = s - \rho[P(A/\beta)] = s - \rho(A) = q(K).$$

At the end of this paper, we demonstrate that the class of M-matrices is closed on any extended Perron complementation.

Theorem 4.2. Suppose K is an $n \times n$ irreducible M-matrix. For any $\emptyset \neq \beta \subset \langle n \rangle$ and $t \leq q(K)$, the extended Perron complement $Q_t(K/\beta)$ is an irreducible M-matrix.

Proof. Set $K = sI - A$, $A \geq 0$, and $s > \rho(A)$. In proving Theorem 3.3, we found that $P_{s-t}(A/\beta)$ is irreducible and nonnegative. In addition, Theorem 2.1 states that

$$Q_t(K/\beta) = sI - P_{s-t}(A/\beta), t \leq q(K).$$

Therefore, $Q_t(K/\beta)$ is an irreducible Z-matrix. Next, we focus on considering the inverse of $Q_t(K/\beta)$ and proving $[Q_t(K/\beta)]^{-1} \geq 0$. Let K be partitioned according to (3.2). For any $t \leq q(K)$, we have

$$Q_t(K/\beta) = B + C(tI - E)^{-1}D. \tag{4.2}$$

As is established, all the Schur complements of a nonsingular M-matrix are nonsingular M-matrices. Therefore, we know that K/α is a nonsingular M-matrix. We now discuss two cases when $t \leq q(K)$:
Case 1. If $t \leq 0$, then $K/\alpha - tI$ is again a nonsingular M-matrix.
Case 2. If $0 < t \leq q(K)$, by Theorem 3.3, we have

$$q[Q_0(K/\alpha)] > q[Q_t(K/\alpha)] \geq q\left[Q_{q(K)}(K/\alpha)\right]. \tag{4.3}$$

In terms of the partitioned form of (3.2), we compute the extended Perron complement with respect to $K[\alpha]$ as follows:

$$Q_t(K/\alpha) = E + D(tI - B)^{-1}C. \tag{4.4}$$

Setting $t = 0$ and $t = q(K)$ in (4.4), we obtain

$$Q_0(K/\alpha) = E - DB^{-1}C = K/\alpha \tag{4.5}$$

and

$$Q_{q(K)}(K/\alpha) = E + D[q(K)I - B]^{-1}C = Q(K/\alpha), \tag{4.6}$$

respectively. Therefore, from (4.3), (4.5), and (4.6), we infer that

$$q(K/\alpha) > q[Q_t(K/\alpha)] \geq q[Q(K/\alpha)].$$

Meanwhile, according to Theorem 4.1, for any $\varnothing \neq \alpha \subset \langle n \rangle$, we have $q[Q(K/\alpha)] = q(K)$. Thus, we immediately get the following result:

$$q(K/\alpha) > q[Q_t(K/\alpha)] \geq q(K).$$

As $0 < t \leq q(K)$, we obtain $0 < t < q(K/\alpha)$. This ensures that $K/\alpha - tI$ is an invertible M-matrix and $(K/\alpha - tI)^{-1} \geq 0$.

Moreover, we have that $C \leq 0$ and $D \leq 0$ because K is a Z-matrix. As a principal submatrix of K, B is also an M-matrix and $B^{-1} \geq 0$. By the Woodbury formula, we have

$$\begin{aligned}
[Q_t(K/\beta)]^{-1} &= \left[B + C(tI - E)^{-1}D\right]^{-1} \\
&= B^{-1} - B^{-1}C\left(tI - E + DB^{-1}C\right)^{-1}DB^{-1} \\
&= B^{-1} - B^{-1}C\left[tI - \left(E - DB^{-1}C\right)\right]^{-1}DB^{-1}.
\end{aligned}$$

From (4.5), we get

$$\begin{aligned}
[Q_t(K/\beta)]^{-1} &= B^{-1} - B^{-1}C(tI - K/\alpha)^{-1}DB^{-1} \\
&= B^{-1} + B^{-1}C(K/\alpha - tI)^{-1}DB^{-1} \\
&\geq 0.
\end{aligned}$$

Combined with the previous analysis, $Q_t(K/\beta)$ is an irreducible Z-matrix, while the inverse of $Q_t(K/\beta)$ is nonnegative. Thus, we conclude from Lemma 4.3 that $Q_t(K/\beta)$ is an irreducible M-matrix. The proof of Theorem 4.2 is completed.

5. Numerical example

In this section, we present the following example to verify our results. Let

$$A = \begin{pmatrix} 2 & 1 & 3 & 3 \\ 1 & 2 & 2 & 1 \\ 4 & 1 & 2 & 3 \\ 2 & 2 & 1 & 4 \end{pmatrix}.$$

We have $\rho(A) = 8.70$. As $s > \rho(A)$, we may set $s = 14$. Thus, we obtain

$$K = sI - A = \begin{pmatrix} 12 & -1 & -3 & -3 \\ -1 & 12 & -2 & -1 \\ -4 & -1 & 12 & -3 \\ -2 & -2 & -1 & 10 \end{pmatrix}$$

and $q(K) = 5.30$. Set $t_1 = 5$, $t_2 = q(K) = 5.30$, and $t_3 = -10$. For $\alpha = \{1, 2, 3\}$, $\beta = \{4\}$, we have

$$K[\alpha] = \begin{pmatrix} 12 & -1 & -3 \\ -1 & 12 & -2 \\ -4 & -1 & 12 \end{pmatrix},\ K/\beta = \begin{pmatrix} 11.40 & -1.60 & -3.30 \\ -1.20 & 11.80 & -2.10 \\ -4.60 & -1.60 & 11.70 \end{pmatrix},$$

$$P_{s-t_1}(A/\beta) = \begin{pmatrix} 3.20 & 2.20 & 3.60 \\ 1.40 & 2.40 & 2.20 \\ 5.20 & 2.20 & 2.60 \end{pmatrix},\ P(A/\beta) = \begin{pmatrix} 3.28 & 2.28 & 3.64 \\ 1.43 & 2.43 & 2.21 \\ 5.28 & 2.28 & 2.64 \end{pmatrix},$$

$$Q_{t_1}(K/\beta) = \begin{pmatrix} 10.80 & -2.20 & -3.60 \\ -1.40 & 11.60 & -2.20 \\ -5.20 & -2.20 & 11.40 \end{pmatrix},\ Q_{t_2}(K/\beta) = Q(K/\beta) = \begin{pmatrix} 10.72 & -2.28 & -3.64 \\ -1.43 & 11.57 & -2.21 \\ -5.28 & -2.28 & 11.36 \end{pmatrix},$$

$$Q_{t_3}(K/\beta) = \begin{pmatrix} 11.70 & -1.30 & -3.15 \\ -1.10 & 11.90 & -2.05 \\ -4.30 & -1.30 & 11.85 \end{pmatrix}.$$

A direct calculation by MATLAB yields that

$$q[Q_{t_1}(K/\beta)] = 5.45,\ q[Q_{t_2}(K/\beta)] = q[Q(K/\beta)] = 5.30,\ q[Q_{t_3}(K/\beta)] = 7.20.$$

According to the above data, it is easy to see that

$$Q_{t_1}(K/\beta) + P_{s-t_1}(A/\beta) = sI,\ Q(K/\beta) + P(A/\beta) = sI.$$

These are consistent with Theorem 2.1 and Corollary 2.1.

For $0 \le t_1 = 5 \le q(K)$, we have

$$Q_{t_1}(K/\beta) = \begin{pmatrix} 10.80 & -2.20 & -3.60 \\ -1.40 & 11.60 & -2.20 \\ -5.20 & -2.20 & 11.40 \end{pmatrix} \le K/\beta = \begin{pmatrix} 11.40 & -1.60 & -3.30 \\ -1.20 & 11.80 & -2.10 \\ -4.60 & -1.60 & 11.70 \end{pmatrix} \le K[\alpha] = \begin{pmatrix} 12 & -1 & -3 \\ -1 & 12 & -2 \\ -4 & -1 & 12 \end{pmatrix}.$$

For $t_3 = -10 \le 0$, we have

$$K/\beta = \begin{pmatrix} 11.40 & -1.60 & -3.30 \\ -1.20 & 11.80 & -2.10 \\ -4.60 & -1.60 & 11.70 \end{pmatrix} \le Q_{t_3}(K/\beta) = \begin{pmatrix} 11.70 & -1.30 & -3.15 \\ -1.10 & 11.90 & -2.05 \\ -4.30 & -1.30 & 11.85 \end{pmatrix} \le K[\alpha] = \begin{pmatrix} 12 & -1 & -3 \\ -1 & 12 & -2 \\ -4 & -1 & 12 \end{pmatrix}.$$

For $t \le q(K)$, we have

$$\begin{aligned} Q_t(K/\beta) &= \begin{pmatrix} 12 & -1 & -3 \\ -1 & 12 & -2 \\ -4 & -1 & 12 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ -3 \end{pmatrix} (t-10)^{-1} \begin{pmatrix} -2 & -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 12 & -1 & -3 \\ -1 & 12 & -2 \\ -4 & -1 & 12 \end{pmatrix} + \frac{1}{t-10} \begin{pmatrix} 6 & 6 & 3 \\ 2 & 2 & 1 \\ 6 & 6 & 3 \end{pmatrix}. \end{aligned}$$

It is obvious that

$$\lim_{t\to-\infty} Q_t\left(K/\beta\right) = \begin{pmatrix} 12 & -1 & -3 \\ -1 & 12 & -2 \\ -4 & -1 & 12 \end{pmatrix} = K\left[\alpha\right].$$

These are consistent with Theorem 3.1.

Moreover, when $t_1 = 5 < t_2 = 5.30 \leq q\left(K\right)$, we have

$$Q_{t_1}\left(K/\beta\right) = \begin{pmatrix} 10.80 & -2.20 & -3.60 \\ -1.40 & 11.60 & -2.20 \\ -5.20 & -2.20 & 11.40 \end{pmatrix} > Q_{t_2}\left(K/\beta\right) = \begin{pmatrix} 10.72 & -2.28 & -3.64 \\ -1.43 & 11.57 & -2.21 \\ -5.28 & -2.28 & 11.36 \end{pmatrix}$$

and

$$q\left[Q_{t_1}\left(K/\beta\right)\right] = 5.45 > q\left[Q_{t_2}\left(K/\beta\right)\right] = 5.30.$$

These are consistent with Theorems 3.2 and 3.3.

Finally, we can easily verify that $Q_{t_1}\left(K/\beta\right)$, $Q\left(K/\beta\right)$, and $Q_{t_3}\left(K/\beta\right)$ are all irreducible M-matrices. These are complied with Theorems 4.1 and 4.2. The above-calculated data verify the correctness of the conclusions presented in this paper.

6. Conclusions

In this paper, we have discussed various properties of the extended Perron complements of irreducible M-matrices, including the connection between the extended Perron complements of M-matrices and nonnegative matrices.

Furthermore, we presented a comparison among extended Perron complements, Schur complements, and principal submatrices of irreducible M-matrices and rigorously proved the monotonicity of the extended Perron complements and minimum eigenvalue.

Notably, for the collection of irreducible M-matrices, we demonstrated that all the (extended) Perron complements are irreducible M-matrices. Particularly, we deduced that M-matrices and their Perron complements share the same minimum eigenvalue. Subsequently, a simple example was presented to illustrate and explain our results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare that they have no competing interests.

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