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Research article

Barbashin type characterizations for nonuniform h-dichotomy of evolution families

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Abstract: The aim of this paper is to give some Barbashin type characterizations for the nonuniform h-dichotomy of reversible evolution families in Banach spaces. Two necessary and sufficient conditions for the nonuniform h-dichotomy are pointed out using some important sets of growth functions. Additionally, as particular cases, we obtain a Barbashin type characterization for nonuniform exponential dichotomy and a necessary and sufficient condition for the nonuniform polynomial dichotomy.

Keywords: nonuniform h-dichotomy; evolution family; growth rate; Barbashin type theorem; nonuniform exponential dichotomy; nonuniform polynomial dichotomy
Mathematics Subject Classification: 34D05, 34D09

1. Introduction

In the last decades, the qualitative theory for dynamical systems on Banach spaces is intensively investigated in the literature. Various results concerning this field have witnessed considerable development. Some concepts of the qualitative behaviors were defined and improved, such as exponential (in)stability, polynomial (in)stability and h-(in)stability, based on the fact that the dynamical systems describing the process of science or engineering is extremely complex and it is difficult to determine an appropriate mathematical model. One of the most celebrated theorems in the qualitative theory of differential systems was given by Barbashin [1]. Barbashin's theorem states that the differential system

$$\begin{cases} \dot{U}(t) = U(t)A(t) \\ U(0) = I \end{cases}$$

is uniformly exponentially stable if and only if there exists $L > 0$ such that

$$\int_0^t \left\| U(t)U^{-1}(\tau) \right\| d\tau \le L$$

for all $t \ge 0$. Subsequently, this result was extended to the case of evolution families, that is, an exponentially bounded evolution family

$$\mathcal{U} = \{U(t,s)\}_{t \ge s \ge 0}$$

is uniformly exponentially stable if and only if

$$\sup_{t \ge 0} \int_0^t \| U(t,s)x \| ds < \infty$$

for all $x \in X$, where X is a Banach space. Since then, this theorem has inspired many extensions and generalizations along this line (see [2–10] and the references therein). For example, some Barbashin type conditions for uniform exponential stability of linear skew-evolution semiflows were established by Hai [2] in terms of the existence of some functionals on certain function (sequence) spaces. In [6], Dragičević formulated Barbashin type conditions for (non)uniform exponential stability for linear cocycles over both maps and flows by making use of the ergodic theory. In addition, the Barbashin type integral characterizations for uniform h-stability of evolution operators were investigated by Boruga et al. in [8]. Very recently, in [9], through the usage of Banach function (sequence) spaces, we obtained some discrete and continuous versions of Barbashin type theorem for uniform polynomial stability and respectively uniform polynomial instability of evolution families.

As a natural generalization of exponential (in)stability, exponential dichotomy is one of the most important asymptotic properties in the qualitative theory of evolution equations. To the best of our knowledge, the first study on the exponential dichotomy of differential equations was presented by Perron [11] in 1930. After the seminal work of Perron, many authors have made valuable contributions to this line of the research. For details and references, we refer the reader to [12–21].

However, there are some situations where the notion of exponential dichotomy may look as too restrictive, therefore it is important to search for more general type of dichotomic behavior. In this sense we refer to the notion of polynomial dichotomy, which was first considered in 2009 by Bento and Silva [22] for the discrete time case, Barreira and Valls [23] for the continuous time case and then it was discussed in the works of Dragičević et al. [24–26], Boruga and Megan [27]. In particular, in [27] the authors obtained two conditions of Datko type for the existence of the nonuniform polynomial dichotomy for evolution operators. In addition to the aforementioned references, we mention a recent and interesting paper by Găină et al. [28], where the authors proposed a more general notion, the so-called nonuniform h-dichotomy. Simultaneously, they established some Datko type characterizations for the nonuniform h-dichotomy of skew-evolution cocycles in Banach spaces. As is well known, in the Datko type theorems, the integration variable is the first parameter of the evolution family, while in the Barbashin type theorems, the integration variable is its second parameter. Naturally, the question arises whether Barbashin's theorem can be generalized to the case of a nonuniform h-dichotomy.

Inspired by [28], our main purpose is to obtain some Barbashin type conditions for the nonuniform h-dichotomy of reversible evolution families in Banach spaces, using some important sets of growth rates. The paper is organized as follows. In Section 2, some notations, definitions and preliminary facts will be introduced. Section 3 is devoted to establishing the Barbashin type characterizations for the nonuniform h-dichotomy of evolution families. It should be noted that the growth rates considered in the major results of this paper are different from that used in [17]. The growth rates used in this paper depends on Definition 2.7, while the growth rates used in [17] only require differentiability. Furthermore, in [17], the authors established only some Datko type conditions for the existence of nonuniform μ-dichotomy of evolution operators, and did not discuss its Barbashin type characterizations.

2. Notations and preliminaries

Throughout this paper, $X=(X,\|\cdot\|)$ is a Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all linear and bounded operators on X. Furthermore, we denote by $\mathbb{R}_+=[0,+\infty)$ and $\Delta=\{(t,s)\in\mathbb{R}_+^2 : t\ge s\}$.

Definition 2.1. ([29]) A family $\{U(t,s)\}_{t\ge s\ge 0}$ of operators in $\mathcal{B}(X)$ is called an evolution family if

(i) $U(t,t)=I$ (the identity operator on X) for every $t\ge 0$,

(ii) $U(t,\tau)U(\tau,s)=U(t,s)$ for all $t\ge\tau\ge s\ge 0$,

(iii) the map $(t,s)\mapsto U(t,s)x$ is continuous for every $x\in X$.

If the evolution family U is bijective for all $(t,s)\in\Delta$, then we say that U is reversible.

Definition 2.2. ([5]) A strongly continuous function $P:\mathbb{R}_+\to\mathcal{B}(X)$ (this means that $t\mapsto P(t)x$ is continuous for every $x\in X$) is called a projection valued function if $P^2(t)=P(t)$ for all $t\ge 0$.

Remark 2.1. If $P:\mathbb{R}_+\to\mathcal{B}(X)$ is a projection valued function, then the function $Q(t)=I-P(t)$ is also a projection valued function, which is called the complementary projection valued function of P.

Definition 2.3. A projection valued function $P:\mathbb{R}_+\to\mathcal{B}(X)$ is invariant to the evolution family $U:\Delta\to\mathcal{B}(X)$ if $U(t,s)P(s)=P(t)U(t,s)$ for all $(t,s)\in\Delta$.

If $P:\mathbb{R}_+\to\mathcal{B}(X)$ is invariant to the evolution family U, then the pair (U,P) is called a dichotomic pair.

Definition 2.4. ([30]) We say that a nondecreasing function $h:\mathbb{R}_+\to[1,\infty)$ is a growth rate if it is bijective.

In what follows, we suppose that $h:\mathbb{R}_+\to[1,\infty)$ is a growth rate.

Definition 2.5. The dichotomic pair (U,P) is called nonuniformly h-dichotomic (n.h.d.) if there exist a nondecreasing function $N:\mathbb{R}_+\to[1,\infty)$ and a constant $\nu>0$ such that:

$$(nhd_1):\ h(t)^{\nu}\|U(t,s)P(s)x\|\le N(t)h(s)^{\nu}\|P(s)x\|,$$

$$(nhd_2):\ h(t)^{\nu}\|Q(s)x\|\le N(t)h(s)^{\nu}\|U(t,s)Q(s)x\|,$$

for all $(t,s,x)\in\Delta\times X$.

It should be noted that the concept of nonuniform h-dichotomy considered in this paper is weaker than Definition 6 in [28]. In Definition 2.5 above, the function N in condition (nhd_1) depends on the first variable t of the evolution family, while in [28], the function N in condition (nhd_1) depends on the second variable s.

In the classic notions of nonuniform dichotomy, the growth function N depends on the second variable (see [20,21,27]) and roughly speaking a dichotomy means a splitting of X into Range $P(s)$ and Range $Q(s)$ such that U exhibits contraction on Range $P(s)$ and expansion on Range $Q(s)$, where P and Q are associated projection valued functions (see [20,21,26]). In Definition 2.5 the function N depends on the first variable and the contraction/expansion behaviors are not necessarily satisfied, and so, it rather describes a h-dichotomic splitting with some general growth rates.

Remark 2.2. In Definition 2.5, if we consider
(i) $N(t) = N$ (a constant), then we obtain the property of uniform h-dichotomy (u.h.d.),
(ii) $h(t) = e^t$, then the concept of nonuniform exponential dichotomy is obtained,
(iii) $h(t) = t + 1$, then the concept of nonuniform polynomial dichotomy is obtained.

Example 2.1. (Dichotomic pair which is n.h.d. and is not u.h.d.).

Let $h : \mathbb{R}_+ \to [1, \infty)$ be a growth rate with $\lim\limits_{t\to\infty} 2^t h(t)^{\nu-1} = \infty$, where $\nu \in (0,1)$. Let $X = \mathbb{R}^2$ with the norm $\|x\| = |x_1| + |x_2|$, where $x = (x_1, x_2) \in X$. The evolution family $U : \Delta \to \mathcal{B}(X)$ is defined by

$$U(t,s)x = \left(2^{t-s} \frac{h(s)}{h(t)} x_1, 2^{s-t} \frac{h(t)}{h(s)} x_2 \right)$$

for all $(t,s,x) \in \Delta \times X$. Let us consider the projection families $P, Q : \mathbb{R}_+ \to \mathcal{B}(X)$ defined by $P(t)x = (x_1, 0)$ and $Q(t)x = (0, x_2)$, for all $t \geq 0$ and all $(x_1, x_2) \in X$. We have that

$$\|U(t,s)P(s)x\| = 2^{t-s} \frac{h(s)}{h(t)} |x_1| \leq 2^t \frac{h(s)}{h(t)} \|P(s)x\|$$

and

$$2^t \|U(t,s)Q(s)x\| \geq 2^{t-s} \|U(t,s)Q(s)x\| = \frac{h(t)}{h(s)} |x_2| = \frac{h(t)}{h(s)} \|Q(s)x\|$$

for all $(t,s,x) \in \Delta \times X$. Thus Definition 2.5 is satisfied for $N(t) = 2^t$ and $\nu = 1$. It results that (U, P) is n.h.d.

If we suppose that (U, P) is u.h.d., then there exist two constants $N \geq 1$ and $\nu > 0$ such that

$$2^{t-s} \left(\frac{h(t)}{h(s)} \right)^{\nu-1} \leq N$$

for all $(t,s) \in \Delta$. In particular, for $s = 0$ and $t \to \infty$, we obtain a contradiction and thus (U, P) is not u.h.d.

Remark 2.3. The dichotomic pair (U, P) is nonuniformly h-dichotomic if and only if there are a nondecreasing function $N : \mathbb{R}_+ \to [1, \infty)$ and a constant $\nu > 0$ such that:

$$(nhd_1'): h(t)^{\nu} \|U(t,t_0)P(t_0)x\| \leq N(t) h(s)^{\nu} \|U(s,t_0)P(t_0)x\|,$$

$$(nhd_2'):\ h(t)^{\nu}\left\|U(s,t_0)Q(t_0)x\right\| \le N(t)h(s)^{\nu}\left\|U(t,t_0)Q(t_0)x\right\|,$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$.

Remark 2.4. Let U be a reversible evolution family. Then the dichotomic pair (U,P) is nonuniformly h-dichotomic if and only if there are a nondecreasing function $N:\mathbb{R}_+ \to [1,\infty)$ and a constant $\nu > 0$ such that:

$$(nhd_1''):\ h(t)^{\nu}\left\|P(t)x\right\| \le N(t)h(s)^{\nu}\left\|U(t,s)^{-1}P(t)x\right\|,$$

$$(nhd_2''):\ h(t)^{\nu}\left\|U(t,s)^{-1}Q(t)x\right\| \le N(t)h(s)^{\nu}\left\|Q(t)x\right\|,$$

for all $(t,s,x) \in \Delta \times X$.

Definition 2.6. We say that the dichotomic pair (U,P) has a nonuniform h-growth (n.h.g.) if there exist a nondecreasing function $M:\mathbb{R}_+ \to [1,\infty)$ and a constant $\omega > 0$ such that:

$$(nhg_1):\ h(s)^{\omega}\left\|U(t,s)P(s)x\right\| \le M(t)h(t)^{\omega}\left\|P(s)x\right\|,$$

$$(nhg_2):\ h(s)^{\omega}\left\|Q(s)x\right\| \le M(t)h(t)^{\omega}\left\|U(t,s)Q(s)x\right\|,$$

for all $(t,s,x) \in \Delta \times X$.

Remark 2.5. As particular cases of Definition 2.6, we give the following:
(i) For $M(t) = M$, we say that the pair (U,P) has a uniform h-growth;
(ii) For $h(t) = e^t$, we say that the pair (U,P) has a nonuniform exponential growth;
(iii) For $h(t) = t+1$, we say that the pair (U,P) has a nonuniform polynomial growth.

Remark 2.6. If the pair (U,P) is n.h.d., then it has n.h.g. However, the converse is not necessarily valid.

Example 2.2. (Dichotomic pair which has n.h.g. and is not n.h.d.).

Let X, P, Q be defined as in Example 2.1. Let us consider the evolution family

$$U:\Delta \to \mathcal{B}(X),\quad U(t,s)x = \left(\frac{N(t)h(t)}{N(s)h(s)}x_1, \frac{N(s)h(s)}{N(t)h(t)}x_2\right),$$

where $N:\mathbb{R}_+ \to [1,\infty)$ is given by Definition 2.5. It is easy to check that (U,P) satisfies Definition 2.6 for $M(t) = N(t)$ and $\omega = 1$. Thus, (U,P) has n.h.g.

If we suppose that (U,P) is n.h.d., then from Definition 2.5, we have

$$\left(\frac{h(t)}{h(s)}\right)^{\nu+1} \le N(s)$$

for all $(t,s) \in \Delta$. In particular, for $s = 0$ and $t \to \infty$, we obtain a contradiction and thus (U,P) is not n.h.d.

Remark 2.7. The dichotomic pair (U,P) has a nonuniform *h*-growth if and only if there are a nondecreasing function $M:\mathbb{R}_+\to[1,\infty)$ and a constant $\omega>0$ such that:

$$(nhg_1'):\ h(s)^{\omega}\left\|U(t,t_0)P(t_0)x\right\|\le M(t)h(t)^{\omega}\left\|U(s,t_0)P(t_0)x\right\|,$$

$$(nhg_2'):\ h(s)^{\omega}\left\|U(s,t_0)Q(t_0)x\right\|\le M(t)h(t)^{\omega}\left\|U(t,t_0)Q(t_0)x\right\|,$$

for all $t\ge s\ge t_0\ge 0$ and $x\in X$.

Remark 2.8. Let U be a reversible evolution family. Then the dichotomic pair (U,P) has a nonuniform *h*-growth if and only if there are a nondecreasing function $M:\mathbb{R}_+\to[1,\infty)$ and a constant $\omega>0$ such that:

$$(nhg_1''):\ h(s)^{\omega}\left\|P(t)x\right\|\le M(t)h(t)^{\omega}\left\|U(t,s)^{-1}P(t)x\right\|,$$

$$(nhg_2''):\ h(s)^{\omega}\left\|U(t,s)^{-1}Q(t)x\right\|\le M(t)h(t)^{\omega}\left\|Q(t)x\right\|,$$

for all $(t,s,x)\in\Delta\times X$.

Definition 2.7. ([8]) We introduce the following classes of growth rates, which are very helpful for us to prove the main results:
(i) $\mathcal{H}_0$ is the set of all growth rates $h:\mathbb{R}_+\to[1,\infty)$ with $h(t)\ge t+1$, for all $t\ge 0$.
(ii) $\mathcal{H}$ is the set of all growth rates $h:\mathbb{R}_+\to[1,\infty)$ with the property that there exists $H>1$ such that

$$h(t+1)\le Hh(t) \tag{2.1}$$

for all $t\ge 0$.
(iii) $\mathcal{H}_1$ is the set of all growth rates $h:\mathbb{R}_+\to[1,\infty)$ with the property that there exists $H_1>1$ such that

$$h(h(t))\le H_1h(t) \tag{2.2}$$

for all $t\ge 0$.
(iv) $\mathcal{H}_2$ is the set of all growth rates $h:\mathbb{R}_+\to[1,\infty)$ with the property that for all $\alpha>0$, there exists $H_2>1$ such that

$$\int_0^t h(s)^{\alpha}ds\le H_2h(t)^{\alpha} \tag{2.3}$$

for all $t\ge 0$.
(v) $\mathcal{H}_3$ is the set of all growth rates $h:\mathbb{R}_+\to[1,\infty)$ with the property that for all $\alpha>0$, there exists $H_3>1$ such that

$$\int_0^t h(s)^{\alpha-1}ds\le H_3h(t)^{\alpha} \tag{2.4}$$

for all $t\ge 0$.

3. The main results

In this section, we will extend the Barbashin type results from [8] on uniform h-stability of evolution operators to the case of nonuniform h-dichotomy of evolution families. Throughout this section, we suppose that U is reversible and $Q(t)$ is the complementary projection valued function of $P(t)$.

We start with a Barbashin type characterization for the nonuniform h-dichotomy, using growth rates in $\mathcal{H}\cap\mathcal{H}_2$.

Theorem 3.1. Let $h\in\mathcal{H}\cap\mathcal{H}_2$ and (U,P) has a nonuniform h-growth. Then, the pair (U,P) is nonuniformly h-dichotomic if and only if there are a nondecreasing function $B:\mathbb{R}_+\to[1,\infty)$ and a constant $b>0$ such that:

$$(nhD_1):\int_0^t \frac{d\tau}{h(\tau)^b\left\|U(t,\tau)^{-1}P(t)x\right\|}\le\frac{B(t)}{h(t)^b\left\|P(t)x\right\|},$$

for all $(t,x)\in\mathbb{R}_+\times X$ with $P(t)x\ne 0$;

$$(nhD_2):\int_0^t \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^b}d\tau\le\frac{B(t)}{h(t)^b}\left\|Q(t)x\right\|,$$

for all $(t,x)\in\mathbb{R}_+\times X$.

Proof. Necessity. If (U,P) is n.h.d., then, by Remark 2.4, there are a nondecreasing function $N:\mathbb{R}_+\to[1,\infty)$ and $v>0$ such that the relations (nhd_1'') and (nhd_2'') are satisfied. We consider $b\in(0,v)$.

By (nhd_1'') and (2.3) we have

$$\int_0^t \frac{d\tau}{h(\tau)^b\left\|U(t,\tau)^{-1}P(t)x\right\|}\le\int_0^t N(t)\left(\frac{h(\tau)}{h(t)}\right)^v\frac{1}{h(\tau)^b\left\|P(t)x\right\|}d\tau=\frac{N(t)}{h(t)^v\left\|P(t)x\right\|}\int_0^t h(\tau)^{v-b}d\tau$$
$$\le\frac{N(t)}{h(t)^v\left\|P(t)x\right\|}\cdot H_2h(t)^{v-b}=\frac{B(t)}{h(t)^b\left\|P(t)x\right\|}$$

for all $(t,x)\in\mathbb{R}_+\times X$ with $P(t)x\ne 0$, where $B(t)=N(t)H_2$.

Analogously, by (nhd_2'') and (2.3) we have

$$\int_0^t \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^b}d\tau\le\int_0^t N(t)\left(\frac{h(\tau)}{h(t)}\right)^v\frac{\left\|Q(t)x\right\|}{h(\tau)^b}d\tau=\frac{N(t)\left\|Q(t)x\right\|}{h(t)^v}\int_0^t h(\tau)^{v-b}d\tau$$
$$\le\frac{N(t)\left\|Q(t)x\right\|}{h(t)^v}\cdot H_2h(t)^{v-b}=\frac{B(t)}{h(t)^b}\left\|Q(t)x\right\|$$

for all $(t,x)\in\mathbb{R}_+\times X$.

Sufficiency. Let $(t,s)\in\Delta$. First, we prove that (nhD_1) implies (nhd_1'').

Case 1. If $t\ge s+1$, then by (nhg_1''), (2.1) and (nhD_1), we have

$$\frac{1}{h(s)^b \left\| U(t,s)^{-1}P(t)x \right\|} = \int_s^{s+1} \frac{d\tau}{h(s)^b \left\| U(\tau,s)^{-1}U(t,\tau)^{-1}P(t)x \right\|}$$

$$\leq \int_s^{s+1} M(\tau)\left(\frac{h(\tau)}{h(s)}\right)^{\omega} \frac{1}{h(s)^b \left\| U(t,\tau)^{-1}P(t)x \right\|} d\tau$$

$$= \int_s^{s+1} M(\tau)\left(\frac{h(\tau)}{h(s)}\right)^{\omega+b} \frac{1}{h(\tau)^b \left\| U(t,\tau)^{-1}P(t)x \right\|} d\tau$$

$$\leq M(s+1)\int_s^{s+1} \left(\frac{h(s+1)}{h(s)}\right)^{\omega+b} \frac{1}{h(\tau)^b \left\| U(t,\tau)^{-1}P(t)x \right\|} d\tau$$

$$\leq M(t)H^{\omega+b}\int_0^{t} \frac{1}{h(\tau)^b \left\| U(t,\tau)^{-1}P(t)x \right\|} d\tau$$

$$\leq \frac{M(t)B(t)H^{\omega+b}}{h(t)^b \left\| P(t)x \right\|}.$$

Thus we get

$$h(t)^b \left\| P(t)x \right\| \leq M(t)B(t)H^{\omega+b} h(s)^b \left\| U(t,s)^{-1}P(t)x \right\| \tag{3.1}$$

for all $t \geq s+1$.

Case 2. If $t \in [s, s+1)$, then by (nhg_1'') and (2.1), we have that

$$\frac{1}{h(s)^b \left\| U(t,s)^{-1}P(t)x \right\|} \leq M(t)\left(\frac{h(t)}{h(s)}\right)^{\omega+b} \frac{1}{h(t)^b \left\| P(t)x \right\|}$$

$$\leq M(t)\left(\frac{h(s+1)}{h(s)}\right)^{\omega+b} \frac{1}{h(t)^b \left\| P(t)x \right\|} \leq \frac{M(t)H^{\omega+b}}{h(t)^b \left\| P(t)x \right\|}$$

and therefore

$$h(t)^b \left\| P(t)x \right\| \leq M(t)H^{\omega+b} h(s)^b \left\| U(t,s)^{-1}P(t)x \right\| \tag{3.2}$$

for all $t \in [s, s+1)$.

Based on (3.1) and (3.2) we obtained that there exist $N(t) = M(t)B(t)H^{\omega+b}$ and $\nu = b$ such that (nhd_1'') holds for all $(t,s,x) \in \Delta \times X$.

Now we prove that (nhD_2) implies (nhd_2'').

Case 1'. If $t \geq s+1$, then by (nhg_2''), (2.1) and (nhD_2), we have

$$h(t)^b \left\| U(t,s)^{-1}Q(t)x \right\| = h(t)^b \int_s^{s+1} \left\| U(\tau,s)^{-1}U(t,\tau)^{-1}Q(t)x \right\| d\tau$$

$$\leq h(t)^b \int_s^{s+1} M(\tau)\left(\frac{h(\tau)}{h(s)}\right)^{\omega} \left\| U(t,\tau)^{-1}Q(t)x \right\| d\tau$$

$$= h(t)^b \int_s^{s+1} M(\tau)\left(\frac{h(\tau)}{h(s)}\right)^{\omega+b} \frac{h(s)^b \left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^b} d\tau$$

$$\le M(s+1)h(t)^b h(s)^b \int_s^{s+1}\left(\frac{h(s+1)}{h(s)}\right)^{\omega+b} \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^b} d\tau$$

$$\le M(t)h(t)^b h(s)^b H^{\omega+b}\int_s^{s+1} \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^b} d\tau$$

$$\le M(t)h(t)^b h(s)^b H^{\omega+b}\int_0^{t} \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^b} d\tau$$

$$\le M(t)B(t)H^{\omega+b}h(s)^b \left\|Q(t)x\right\|.$$

Case 2’. If $t \in [s, s+1)$, then by (nhg_2'') and (2.1), we have

$$h(t)^b \left\|U(t,s)^{-1}Q(t)x\right\| \le M(t)\left(\frac{h(t)}{h(s)}\right)^{\omega+b} h(s)^b \left\|Q(t)x\right\|$$

$$\le M(t)\left(\frac{h(s+1)}{h(s)}\right)^{\omega+b} h(s)^b \left\|Q(t)x\right\| \le M(t)H^{\omega+b}h(s)^b \left\|Q(t)x\right\|.$$

Combining Case 1’ with Case 2’, we obtained that there exist $N(t) = M(t)B(t)H^{\omega+b}$ and $\nu = b$ such that (nhd_2'') holds for all $(t,s,x) \in \Delta \times X$.

Finally, by virtue of Remark 2.4, we conclude that the pair (U,P) is n.h.d.

As a direct consequence of Theorem 3.1, we obtain the following corollary, which is a version of Barbashin’s theorem for the case of the nonuniform exponential dichotomy concept.

Corollary 3.1. We suppose that the pair (U,P) has a nonuniform exponential growth. Then, it is nonuniformly exponentially dichotomic if and only if there are a nondecreasing function $B:\mathbb{R}_+ \to [1,\infty)$ and a constant $b > 0$ such that:

$$(neD_1): \int_0^t \frac{d\tau}{e^{b\tau}\left\|U(t,\tau)^{-1}P(t)x\right\|} \le \frac{B(t)}{e^{bt}\left\|P(t)x\right\|}$$

for all $(t,x) \in \mathbb{R}_+ \times X$ with $P(t)x \ne 0$;

$$(neD_2): \int_0^t \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{e^{b\tau}} d\tau \le \frac{B(t)}{e^{bt}}\left\|Q(t)x\right\|$$

for all $(t,x) \in \mathbb{R}_+ \times X$.

Proof. It follows immediately from Theorem 3.1 for $h(t) = e^t$.

Since the polynomial growth rate $h(t) = t+1 \notin \mathcal{H}_2$, Theorem 3.1 does not include the particular case of nonuniform polynomial dichotomy. In order to make the conclusion include the case of nonuniform polynomial dichotomy, we present another characterization of Barbashin type for the nonuniform h-dichotomy.

Theorem 3.2. Let $h \in \mathcal{H}_0 \cap \mathcal{H}_1 \cap \mathcal{H}_3$ and (U,P) has a nonuniform h-growth. Then, the pair (U,P) is nonuniformly h-dichotomic if and only if there are a nondecreasing function $B:\mathbb{R}_+ \to [1,\infty)$ and a constant $b>0$ such that:

$$(nhD_1'): \int_0^t \frac{d\tau}{h(\tau)^{b+1}\left\|U(t,\tau)^{-1}P(t)x\right\|} \le \frac{B(t)}{h(t)^b\left\|P(t)x\right\|}$$

for all $(t,x)\in \mathbb{R}_+\times X$ with $P(t)x\neq 0$;

$$(nhD_2'): \int_0^t \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^{b+1}}d\tau \le \frac{B(t)}{h(t)^b}\left\|Q(t)x\right\|$$

for all $(t,x)\in \mathbb{R}_+\times X$.

Proof. Necessity. If (U,P) is n.h.d., then by Remark 2.4, there are a nondecreasing function $N:\mathbb{R}_+ \to [1,\infty)$ and $\nu>0$ such that the relations (nhd_1'') and (nhd_2'') are satisfied. Let $b\in(0,\nu)$.

By (nhd_1'') and (2.4) we have

$$\int_0^t \frac{d\tau}{h(\tau)^{b+1}\left\|U(t,\tau)^{-1}P(t)x\right\|} \le \int_0^t N(t)\left(\frac{h(\tau)}{h(t)}\right)^{\nu}\frac{1}{h(\tau)^{b+1}\left\|P(t)x\right\|}d\tau = \frac{N(t)}{h(t)^{\nu}\left\|P(t)x\right\|}\int_0^t h(\tau)^{\nu-b-1}d\tau$$

$$\le \frac{N(t)}{h(t)^{\nu}\left\|P(t)x\right\|}\cdot H_3 h(t)^{\nu-b} = \frac{B(t)}{h(t)^b\left\|P(t)x\right\|}$$

for all $(t,x)\in \mathbb{R}_+\times X$ with $P(t)x\neq 0$, where $B(t)=N(t)H_3$.

Similarly, by (nhd_2'') and (2.4) we have

$$\int_0^t \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^{b+1}}d\tau \le \int_0^t N(t)\left(\frac{h(\tau)}{h(t)}\right)^{\nu}\frac{\left\|Q(t)x\right\|}{h(\tau)^{b+1}}d\tau = \frac{N(t)\left\|Q(t)x\right\|}{h(t)^{\nu}}\int_0^t h(\tau)^{\nu-b-1}d\tau$$

$$\le \frac{N(t)\left\|Q(t)x\right\|}{h(t)^{\nu}}\cdot H_3 h(t)^{\nu-b} = \frac{B(t)}{h(t)^b}\left\|Q(t)x\right\|$$

for all $(t,x)\in \mathbb{R}_+\times X$.

Sufficiency. Let $(t,s)\in\Delta$. First, we prove that (nhD_1') implies (nhd_1'').

Case 1. If $t\ge h(s)$, then by (nhg_1''), (2.2) and (nhD_1'), we have

$$\frac{1}{h(s)^b\left\|U(t,s)^{-1}P(t)x\right\|} = \frac{2}{h(s)}\int_{\frac{h(s)}{2}}^{h(s)} \frac{d\tau}{h(s)^b\left\|U(\tau,s)^{-1}U(t,\tau)^{-1}P(t)x\right\|}$$

$$\le \frac{2}{h(s)}\int_{\frac{h(s)}{2}}^{h(s)} M(\tau)\left(\frac{h(\tau)}{h(s)}\right)^{\omega}\frac{1}{h(s)^b\left\|U(t,\tau)^{-1}P(t)x\right\|}d\tau$$

$$= 2\int_{\frac{h(s)}{2}}^{h(s)} M(\tau)\left(\frac{h(\tau)}{h(s)}\right)^{\omega+b+1} \frac{1}{h(\tau)^{b+1}\left\|U(t,\tau)^{-1}P(t)x\right\|}d\tau$$

$$\le 2M(h(s))\int_{\frac{h(s)}{2}}^{h(s)}\left(\frac{h(h(s))}{h(s)}\right)^{\omega+b+1} \frac{1}{h(\tau)^{b+1}\left\|U(t,\tau)^{-1}P(t)x\right\|}d\tau$$

$$\le 2M(t)H_1^{\omega+b+1}\int_0^t \frac{1}{h(\tau)^{b+1}\left\|U(t,\tau)^{-1}P(t)x\right\|}d\tau$$

$$\le \frac{2M(t)H_1^{\omega+b+1}B(t)}{h(t)^b\left\|P(t)x\right\|}.$$

Thus

$$h(t)^b\left\|P(t)x\right\| \le 2M(t)B(t)H_1^{\omega+b+1}h(s)^b\left\|U(t,s)^{-1}P(t)x\right\| \tag{3.3}$$

for all $t \ge h(s)$.

Case 2. If $t \in [s, h(s))$, then by (nhg_1'') and (2.2), we have that

$$\frac{1}{h(s)^b\left\|U(t,s)^{-1}P(t)x\right\|} \le M(t)\left(\frac{h(t)}{h(s)}\right)^{\omega+b} \frac{1}{h(t)^b\left\|P(t)x\right\|}$$

$$\le M(t)\left(\frac{h(h(s))}{h(s)}\right)^{\omega+b} \frac{1}{h(t)^b\left\|P(t)x\right\|} \le \frac{M(t)H_1^{\omega+b}}{h(t)^b\left\|P(t)x\right\|}$$

and hence

$$h(t)^b\left\|P(t)x\right\| \le M(t)H_1^{\omega+b}h(s)^b\left\|U(t,s)^{-1}P(t)x\right\| \tag{3.4}$$

for all $t \in [s, h(s))$.

From (3.3) and (3.4), it follows that there exist $N(t) = 2M(t)B(t)H_1^{\omega+b+1}$ and $v = b$ such that (nhd_1'') holds for all $(t,s,x) \in \Delta \times X$.

Next, we prove that (nhD_2') implies (nhd_2'').

Case 1'. If $t \ge h(s)$, then by (nhg_2''), (2.2) and (nhD_2'), we have

$$\frac{\left\|U(t,s)^{-1}Q(t)x\right\|}{h(s)^b} = \frac{2}{h(s)}\int_{\frac{h(s)}{2}}^{h(s)} \frac{\left\|U(\tau,s)^{-1}U(t,\tau)^{-1}Q(t)x\right\|}{h(s)^b}d\tau$$

$$\le \frac{2M(h(s))}{h(s)}\int_{\frac{h(s)}{2}}^{h(s)}\left(\frac{h(\tau)}{h(s)}\right)^{\omega} \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(s)^b}d\tau$$

$$= 2M(h(s))\int_{\frac{h(s)}{2}}^{h(s)}\left(\frac{h(\tau)}{h(s)}\right)^{\omega+b+1} \frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^{b+1}}d\tau$$

$$\le 2M(h(s))\int_{\frac{h(s)}{2}}^{h(s)}\left(\frac{h(h(s))}{h(s)}\right)^{\omega+b+1}\frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^{b+1}}d\tau$$

$$\le 2M(t)H_1^{\omega+b+1}\int_0^t\frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{h(\tau)^{b+1}}d\tau$$

$$\le 2M(t)H_1^{\omega+b+1}\frac{B(t)}{h(t)^b}\left\|Q(t)x\right\|.$$

Thus

$$h(t)^b\left\|U(t,s)^{-1}Q(t)x\right\|\le 2M(t)B(t)H_1^{\omega+b+1}h(s)^b\left\|Q(t)x\right\| \tag{3.5}$$

for all $t\ge h(s)$.

Case 2'. If $t\in[s,h(s))$, then by (nhg_2'') and (2.2), we have that

$$h(t)^b\left\|U(t,s)^{-1}Q(t)x\right\|\le M(t)\left(\frac{h(t)}{h(s)}\right)^{\omega+b}h(s)^b\left\|Q(t)x\right\|$$
$$\le M(t)\left(\frac{h(h(s))}{h(s)}\right)^{\omega+b}h(s)^b\left\|Q(t)x\right\|\le M(t)H_1^{\omega+b}h(s)^b\left\|Q(t)x\right\|. \tag{3.6}$$

From (3.5) and (3.6), it follows that there exist $N(t)=2M(t)B(t)H_1^{\omega+b+1}$ and $\nu=b$ such that (nhd_2'') holds for all $(t,s,x)\in\Delta\times X$.

Finally, by virtue of Remark 2.4, we obtain that the pair (U,P) is n.h.d.

As an immediate consequence of Theorem 3.2, we obtain a version of Barbashin's theorem for the case of the nonuniform polynomial dichotomy concept.

Corollary 3.2. We suppose that the pair (U,P) has a nonuniform polynomial growth. Then, it is nonuniformly polynomially dichotomic if and only if there are a nondecreasing function $B:\mathbb{R}_+\to[1,\infty)$ and a constant $b>0$ such that:

$$(npD_1):\int_0^t\frac{d\tau}{(\tau+1)^{b+1}\left\|U(t,\tau)^{-1}P(t)x\right\|}\le\frac{B(t)}{(t+1)^b\left\|P(t)x\right\|}$$

for all $(t,x)\in\mathbb{R}_+\times X$ with $P(t)x\ne 0$;

$$(npD_2):\int_0^t\frac{\left\|U(t,\tau)^{-1}Q(t)x\right\|}{(\tau+1)^{b+1}}d\tau\le\frac{B(t)}{(t+1)^b}\left\|Q(t)x\right\|$$

for all $(t,x)\in\mathbb{R}_+\times X$.

Proof. It follows from Theorem 3.2 for $h(t)=t+1$.

4. Conclusions

In this paper, we have investigated the Barbashin type characterizations for the nonuniform h-dichotomy of evolution families in Banach spaces, using some important classes of growth rates. More precisely, we gave two theorems of Barbashin type for nonuniform h-dichotomy (Theorems 3.1 and 3.2). As particular cases, a Barbashin type characterization for nonuniform exponential dichotomy and a necessary and sufficient condition for the nonuniform polynomial dichotomy are obtained (see Corollaries 3.1 and 3.2). In the future, we will continue to discuss the variants of these results in the discrete time case and generalizations for the nonuniform h-trichotomies behaviors.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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