



---

*Research article*

## Study of the fuzzy $q$ -spiral-like functions associated with the generalized linear operator

A. A. Azzam<sup>1,2</sup>, Daniel Breaz<sup>3</sup>, Shujaat Ali Shah<sup>4,\*</sup> and Luminița-Ioana Cotîrlă<sup>5,\*</sup>

<sup>1</sup> Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia

<sup>2</sup> Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt

<sup>3</sup> Department of Mathematics, “1 Decembrie 1918” University of Alba Iulia, Alba Iulia 510009, Romania

<sup>4</sup> Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, Nawabshah 67450, Pakistan

<sup>5</sup> Department of Mathematics, Technical University of Cluj-Napoca, Cluj-Napoca 400114, Romania

\* **Correspondence:** Email: shahglike@yahoo.com, luminita.cotirla@math.utcluj.ro.

**Abstract:** Nowadays, the subclasses of analytic functions in terms of fuzzy subsets are studied by various scholars and some of these concepts are extended using the  $q$ -theory of functions. In this inspiration, we introduce certain subclasses of analytic function by using the notion of fuzzy subsets along with the idea of  $q$ -calculus. We present the  $q$ -extensions of the fuzzy spiral-like functions of a complex order. We generalize this class using the  $q$ -analogues of the Ruscheweyh derivative and Srivastava-Attiya operators. Various interesting properties are examined for the newly defined subclasses. Also, some previously investigated results are deduced as the corollaries of our major results.

**Keywords:** analytic functions;  $q$ -spiral-like; fuzzy  $q$ -starlike functions; fuzzy  $q$ -convex functions;  $q$ -Ruscheweyh derivative operator;  $q$ -Srivastava-Attiya operator

**Mathematics Subject Classification:** 30A10, 30C45

---

### 1. Introduction

The article written by Lotfi A. Zadeh and published in 1965 [1] serves as the foundation for the fuzzy sets theory. In response to the numerous attempts by researchers to connect this theory with various areas of mathematics, the connection between fuzzy sets theory and the area of complex analysis that studies analytic functions by virtue of their geometric properties was established in

2011 [2]. Miller and Mocanu in [3, 4] introduced the concept of differential subordination. Oros and Oros [2] researched the idea of fuzzy subordination in 2011, and they [5] introduced the idea of fuzzy differential subordination in 2012. The history of the idea of a fuzzy set and its ties to various scientific and technical fields are nicely reviewed in the 2017 paper [6], including references to the findings up to that point regarding the fuzzy differential subordination concept. The first findings confirmed the direction of the research, adapting the traditional theory of differential subordination to the novel features of fuzzy differential subordination and providing techniques for investigating dominants and best dominants of fuzzy differential subordinations [7], without which the research could not have continued. Following that, the particular form of Briot-Bouquet fuzzy differential subordinations was examined in [8]. Haydar in [9] adopted the concept and began to look into the new findings on fuzzy differential subordinations. In this sequel, fuzzy differential subordinations were associated with different operators [10, 11] giving a new direction to the study. Numerous studies [12–14] carried out the investigations by employing certain linear operators. Furthermore, the work of several scholars about the fuzzy differential subordination is referred to the readers, for example, see [15–24]. The concept of fuzzy differential subordination is the first attempt to incorporate the idea of a fuzzy set into research pertaining to geometric theory of analytic functions. Recently, the authors [25–27] linked the notion of a fuzzy subsets with the concepts of quantum extensions of analytic functions.

The classical results of univalent functions with the concept of differential subordination are generalized in this current paper to include quantum (q)-extensions of univalent functions associated with fuzzy differential subordination.

Let  $\Gamma(\Pi)$  denote the class of analytic functions  $h(v)$  in  $\Pi = \{v : |v| < 1\}$ . The functions  $h \in \Gamma(\Pi)$  of the form

$$h(v) = v + a_{\eta+1}v^{\eta+1} + a_{\eta+2}v^{\eta+2} + \dots, \quad (v \in \Pi), \quad (1.1)$$

form the class denoted by  $\mathfrak{A}_\eta$ . We note that  $\mathfrak{A}_1 = \mathfrak{A}$ ; the class of normalized analytic functions in  $\Pi$ . Let  $ST$  and  $CV$  denote the subclasses of  $\mathfrak{A}$  of starlike and convex univalent functions, respectively. Here, we provide an overview of some important fundamental ideas connected to our work.

**Definition 1.1.** [28] A function  $\mathbb{F}$  is said to be fuzzy subset on  $\mathfrak{Y} \neq \phi$ , if it maps from  $\mathfrak{Y}$  to  $[0, 1]$ .

In other words, fuzzy subset is defined as:

**Definition 1.2.** [28] A pair  $(U, \mathbb{F}_U)$  is said to be a fuzzy subset on  $\mathfrak{Y}$ , where  $U = \{x \in \mathfrak{Y} : 0 < \mathbb{F}_U(x) \leq 1\} = \text{sup}(U, \mathbb{F}_U)$  is the support of fuzzy set  $(U, \mathbb{F}_U)$  and  $\mathbb{F}_U : \mathfrak{Y} \rightarrow [0, 1]$  is the membership function of the fuzzy set  $(U, \mathbb{F}_U)$ .

**Definition 1.3.** [28] Let  $(U_1, \mathbb{F}_{U_1})$  and  $(U_2, \mathbb{F}_{U_2})$  be two subsets of  $\mathfrak{Y}$ . Then,  $(U_1, \mathbb{F}_{U_1}) \subseteq (U_2, \mathbb{F}_{U_2})$  if and only if  $\mathbb{F}_{U_1}(t) \leq \mathbb{F}_{U_2}(t)$ ,  $t \in \mathfrak{Y}$ , whereas,  $(U_1, \mathbb{F}_{U_1})$  and  $(U_2, \mathbb{F}_{U_2})$  of  $\mathfrak{Y}$  are equal if and only if  $U_1 = U_2$ .

We say that the analytic function  $h$  is subordinate to the analytic function  $g$  (written as  $h < g$ ) if  $h(v) = g(w(v))$ , where  $w(v)$  is a Schwartz function in  $\Pi$ , see Miller and Mocanu [29].

The generalization of the subordination technique of analytic functions in terms of fuzzy notion was defined by Oros and Oros [5] as the following.

Let the analytic function  $h$  be fuzzy subordinate to the analytic function  $g$  (written as  $h \prec_{\mathbb{F}} g$ ). Then,

$$h(v_0) = g(v_0) \text{ and } \mathbb{F}(h(v)) \leq \mathbb{F}(g(v)), v \in \mathfrak{R},$$

where  $\mathfrak{R} \subset \mathbb{C}$  and  $v_0$  be a fixed point in  $\mathfrak{R}$ .

**Remark 1.1.** If  $\mathfrak{R} = \Pi$  in the above definition, then the fuzzy subordination is equivalent to the classical subordination.

For  $0 < q < 1$ , the operator  $\nabla_q$  defined by

$$\nabla_q h(v) = \frac{h(v) - h(qv)}{(1-q)v}; \quad q \neq 1, v \neq 0, \quad (1.2)$$

is called the  $q$ -difference operator. It was introduced by Jackson [30] and it is clear that  $\lim_{q \rightarrow 1^-} \nabla_q h(v) = h'(v)$ , where  $h'(v)$  denotes the derivative of the function.

For  $j \in \mathbb{N} = \{1, 2, 3, \dots\}$ , we have

$$\nabla_q \left\{ \sum_{j=1}^{\infty} a_j v^j \right\} = \sum_{j=1}^{\infty} [j]_q a_j v^{j-1}, \quad (1.3)$$

where

$$[j]_q = \frac{1 - q^j}{1 - q} = 1 + q + q^2 + \dots + q^{j-1}. \quad (1.4)$$

We have the following rules of  $\nabla_q$ , we refer to [31, 32].

- (i)  $\nabla_q (ah_1(v) \pm bh_2(v)) = a\nabla_q h_1(v) \pm b\nabla_q h_2(v)$ .
- (ii)  $\nabla_q (h_1(v)h_2(v)) = h_1(qv)\nabla_q(h_2(v)) + h_2(v)\nabla_q(h_1(v))$ .
- (iii)  $\nabla_q \left( \frac{h_1(v)}{h_2(v)} \right) = \frac{\nabla_q(h_1(v)h_2(v) - h_1(v)\nabla_q(h_2(v)))}{h_2(qv)h_2(v)}$ ,  $h_2(qv)h_2(v) \neq 0$ .
- (iv)  $\nabla_q (\log h(v)) = \frac{\ln q \nabla_q(h(v))}{(q-1)h(v)}$ .

Ismail et al. [33] were first who discussed various properties of function theory by virtue of  $q$ -theory.

In [34], Kanas and Raducanu introduced an operator  $R_q^\lambda : \mathfrak{A} \rightarrow \mathfrak{A}$  defined by

$$R_q^\lambda h(v) = v + \sum_{j=1}^{\infty} \frac{[j + \lambda - 1]_q!}{[j]_q!} [\lambda]_q! [j - 1]_q! a_j v^j, \quad (\lambda > -1). \quad (1.5)$$

Also, if  $\lambda = m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , then the operator (2.6) can be written as:

$$R_q^m h(v) = \frac{v \nabla_q (v^{m-1} h(v))}{[m]_q!}.$$

We note that  $R_q^0 h(v) = h(v)$  and  $R_q^1 h(v) = v \nabla_q h(v)$ .

We obtain the Ruscheweyh derivative operator [35], in particular, for  $q \rightarrow 1^-$ .

The  $q$ -extension of the Srivastava-Attiya operator discussed by Shah and Noor in [36]. They defined, for  $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ ,  $s \in \mathbb{C}$  when  $|v| < 1$  and  $\Re(s) > 1$  when  $|v| = 1$ , the operator  $J_{q,b}^s : \mathfrak{A} \rightarrow \mathfrak{A}$  by

$$J_{q,b}^s h(v) = \Lambda_q(s, b; v) * h(v)$$

$$= v + \sum_{j=2}^{\infty} \left( \frac{[1+b]_q}{[j+b]_q} \right)^s a_j v^j, \quad (1.6)$$

where

$$\Lambda_q(s, b; v) = v + \sum_{j=2}^{\infty} \left( \frac{[1+b]_q}{[j+b]_q} \right)^s v^j.$$

The  $q$ -Srivastava-Attiya operator  $J_{q,b}^s$  generalizes some well-known operators such as  $q$ -Alexander,  $q$ -Libera,  $q$ -Bernardi and Srivastava-Attiya operator, we refer to [37, 38].

From (1.5) and (1.6), we use the convolution technique to define  $\Upsilon_{q,b}^{s,\lambda} : \mathfrak{A} \rightarrow \mathfrak{A}$  as follows:

$$\begin{aligned} \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v) &= (R_q^\lambda * J_{q,b}^s) \mathfrak{h}(v) \\ &= v + \sum_{j=1}^{\infty} \frac{[j+\lambda-1]_q!}{[\lambda]_q! [j-1]_q!} \left( \frac{[1+b]_q}{[j+b]_q} \right)^s a_j v^j, \end{aligned} \quad (1.7)$$

where  $\mathfrak{h} \in \mathfrak{A}$  and  $*$  denotes the Hadamard product(convolution).

It is clear that

$$\Upsilon_{q,b}^{0,\lambda} \mathfrak{h}(v) = R_q^\lambda \mathfrak{h}(v) \quad \text{and} \quad \Upsilon_{q,b}^{s,0} \mathfrak{h}(v) = J_{q,b}^s \mathfrak{h}(v).$$

The following identities can be implied from (1.5) to (1.7).

$$v \nabla_q \left( \Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(v) \right) = \left( 1 + \frac{[b]_q}{q^b} \right) \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v) - \frac{[b]_q}{q^b} \Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(v). \quad (1.8)$$

$$v \nabla_q \left( \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v) \right) = \left( 1 + \frac{[\lambda]_q}{q^b} \right) \Upsilon_{q,b}^{s,\lambda+1} \mathfrak{h}(v) - \frac{[\lambda]_q}{q^b} \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v). \quad (1.9)$$

Several scholars studied various geometrical properties of analytic functions associated with  $q$ -linear operators, so we refer the readers to [39–41]. Now, we use the  $q$ -difference operator and the fuzzy subordination principle to define certain new subclasses  $\mathbb{F}S_q(\varrho, \delta; \mathfrak{g})$  and  $\mathbb{F}C_q(\varrho, \delta; \mathfrak{g})$  as the following.

For  $\varrho \in \mathbb{R} : |\varrho| < \frac{\pi}{2}$ ,  $q \in (0, 1)$ ,  $0 \neq \delta \in \mathbb{C}$ ,  $v \in \Pi$  and  $\mathfrak{g} \in \mathfrak{M}$ , where  $\mathfrak{M}$  is the class of all functions  $\mathfrak{g}$  which are analytic and univalent in  $\Pi$ , and for which  $\mathfrak{g}(\Pi)$  is convex with  $\mathfrak{g}(0) = 1$  and  $Re(\mathfrak{g}(v)) > 0$  in  $\Pi$  we define

$$\begin{aligned} \mathbb{F}S_q(\varrho, \delta; \mathfrak{g}) &= \left\{ \mathfrak{h} \in \mathfrak{A} : 1 + \frac{e^{i\varrho}}{\delta \cos \varrho} \left( \frac{v \nabla_q \mathfrak{h}}{\mathfrak{h}} - 1 \right) \prec_{\mathbb{F}} \mathfrak{g}(v) \right\}, \\ \mathbb{F}C_q(\varrho, \delta; \mathfrak{g}) &= \left\{ \mathfrak{h} \in \mathfrak{A} : 1 + \frac{e^{i\varrho}}{\delta \cos \varrho} \left( \frac{\nabla_q(v \nabla_q \mathfrak{h})}{\nabla_q \mathfrak{h}} - 1 \right) \prec_{\mathbb{F}} \mathfrak{g}(v) \right\}. \end{aligned}$$

In application of the operator given in (1.7), we define;

$$\mathbb{F}S T_{q,b}^{s,\lambda}(\varrho, \delta; \mathfrak{g}) = \left\{ \mathfrak{h} \in \mathfrak{A} : \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v) \in \mathbb{F}S_q(\varrho, \delta; \mathfrak{g}) \right\},$$

and

$$\mathbb{F}C V_{q,b}^{s,\lambda}(\varrho, \delta; \mathfrak{g}) = \left\{ \mathfrak{h} \in \mathfrak{A} : \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v) \in \mathbb{F}C_q(\varrho, \delta; \mathfrak{g}) \right\},$$

where  $g \in \mathfrak{M}$ ,  $\lambda > -1$ ,  $q \in (0, 1)$ ,  $s, \varrho \in \mathbb{R} : |\varrho| < \frac{\pi}{2}$ ,  $b > -1$ ,  $0 \neq \delta \in \mathbb{C}$  and  $v \in \Pi$ .

It is obvious that

$$h \in \mathbb{FCV}_{q,b}^{s,\lambda}(\varrho, \delta; g) \text{ if and only if } v\nabla_q h \in \mathbb{FST}_{q,b}^{s,\lambda}(\varrho, \delta; g). \quad (1.10)$$

Particularly, when  $\delta = 1$  and  $\varrho = 0$ , we obtain the classes  $\mathbb{FST}_{q,b}^{s,\lambda}(g)$  and  $\mathbb{FCV}_{q,b}^{s,\lambda}(g)$  introduced by the authors in [26]. For  $\lambda = 0 = s$ , the classes  $\mathbb{FST}_{q,b}^{s,\lambda}(\varrho, \delta; g)$  and  $\mathbb{FCV}_{q,b}^{s,\lambda}(\varrho, \delta; g)$  reduce to the classes  $\mathbb{FS}_q(\varrho, \delta; g)$  and  $\mathbb{FC}_q(\varrho, \delta; g)$  respectively. Moreover, when  $q \rightarrow 1^-$ , we have the classes,  $\mathbb{FST}(\varrho, \delta; g)$  and  $\mathbb{FCV}(\varrho, \delta; g)$ , introduced by the authors in [27]. Furthermore, for  $\delta = 1$  and  $\varrho = 0$ , the classes  $\mathbb{FS}(\varrho, \delta; g)$  and  $\mathbb{FC}(\varrho, \delta; g)$  coincides with the classes  $\mathbb{FST}(g)$  and  $\mathbb{FC}(g)$ , introduced and studied by Shah et al. [20].

Now, in the next section, the inclusion problems between the subclasses are investigated. Furthermore, we show that the subclasses are preserved  $q$ -Bernardi integral operator. We need the following lemma for our findings.

**Lemma 1.1.** [25] *Let  $\beta$  and  $\gamma$  be complex numbers with  $\beta \neq 0$  and let  $g(v)$  be a convex univalent in  $\Pi$  with  $g(0) = 1$  and*

$$\operatorname{Re} \{\beta g(v) + \gamma\} > 0. \quad (1.11)$$

*If  $p(v) = 1 + p_1v + p_2v^2 + \dots$  is analytic in  $\Pi$ , then*

$$p(v) + \frac{v\nabla_q p(v)}{\beta p(v) + \gamma} \prec_{\mathbb{F}} g(v) \text{ implies } p(v) \prec_{\mathbb{F}} g(v),$$

where  $\mathbb{F} : \mathbb{C} \rightarrow [0, 1]$ .

## 2. Major results

### 2.1. Inclusion results

**Theorem 2.1.** *Let  $g \in \mathfrak{M}$ ,  $q \in (0, 1)$ ,  $\lambda \in \mathbb{N}_0$ ,  $s, \varrho \in \mathbb{R} : |\varrho| < \frac{\pi}{2}$ ,  $b > -1$  and  $0 \neq \delta \in \mathbb{C}$ . Then,*

$$\mathbb{FST}_{q,b}^{s,\lambda}(\varrho, \delta; g) \subset \mathbb{FST}_{q,b}^{s+1,\lambda}(\varrho, \delta; g), \quad (2.1)$$

for

$$\Re \left\{ e^{-i\varrho} \delta \cos \varrho (g(v) - 1) + (1 + x_q) \right\} > 0, \text{ with } x_q = \frac{[b]_q}{q^b}, \quad (2.2)$$

and

$$\mathbb{FST}_{q,b}^{s,\lambda+1}(\varrho, \delta; g) \subset \mathbb{FST}_{q,b}^{s,\lambda}(\varrho, \delta; g), \quad (2.3)$$

for

$$\Re \left\{ e^{-i\varrho} \delta \cos \varrho (g(v) - 1) + (1 + d_q) \right\} > 0, \text{ with } d_q = \frac{[\lambda]_q}{q^b}. \quad (2.4)$$

*Proof.* To prove the relation (2.1), we suppose  $h \in \mathbb{FST}_{q,b}^{s,\lambda}(\varrho, \delta; g)$ . For analytic  $p_1(v)$  in  $\Pi$  with  $p_1(0) = 1$ , we set

$$p_1(v) = \frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v\nabla_q (\Upsilon_{q,b}^{s+1,\lambda} h(v))}{\Upsilon_{q,b}^{s+1,\lambda} h(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\}. \quad (2.5)$$

The identity (1.8) and (2.5) imply that

$$p_1(v) = \frac{1}{\delta \cos \varrho} \left[ e^{i\varrho} \left\{ \left( 1 + \frac{[b]_q}{q^b} \right) \frac{\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v)}{\Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(v)} - \frac{[b]_q}{q^b} \right\} - (1 - \delta) \cos \varrho - i \sin \varrho \right],$$

equivalently

$$(1 + x_q) \frac{\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v)}{\Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(v)} = e^{-i\varrho} \delta \cos \varrho (p_1(v) - 1) + (1 + x_q), \quad \left( \text{for } x_q = \frac{[b]_q}{q^b} \right).$$

The  $q$ -logarithmic differentiation and (2.5) yield

$$\frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v \nabla_q (\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v))}{\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\} = p_1(v) + \frac{v \nabla_q p_1(v)}{e^{-i\varrho} \delta \cos \varrho (p_1(v) - 1) + (1 + x_q)}. \quad (2.6)$$

Since  $\mathfrak{h} \in \mathbb{F}S T_{q,b}^{s,\lambda}(\varrho, \delta; \mathfrak{g})$ , from (2.6) we have

$$p_1(v) + \frac{v \nabla_q p_1(v)}{e^{-i\varrho} \delta \cos \varrho (p_1(v) - 1) + (1 + x_q)} <_{\mathbb{F}} \mathfrak{g}(v), \quad (2.7)$$

for  $\mathfrak{g} \in \mathfrak{M}$ , we assume that

$$\Re \left\{ e^{-i\varrho} \delta \cos \varrho (p_1(v) - 1) + (1 + x_q) \right\} > 0,$$

by Lemma 1.1 and (2.7), we conclude  $p_1(v) <_{\mathbb{F}} \mathfrak{g}(v)$  implies  $\mathfrak{h} \in \mathbb{F}S T_{q,b}^{s+1,\lambda}(\varrho, \delta; \mathfrak{g})$ .

To prove (2.5), we set, for analytic  $p_2(v)$  in  $\Pi$  with  $p_2(0) = 1$ ,

$$p_2(v) = \frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v \nabla_q (\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v))}{\Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\}. \quad (2.8)$$

Now, using similar techniques as before, we can easily obtain the required result using the identity (1.9) along with Lemma 1.1.  $\square$

Particularly, if we take  $\delta = 1$  and  $\varrho = 0$ , we have:

**Corollary 2.1.** [26] Let  $q \in (0, 1)$ ,  $\lambda \in \mathbb{N}_0$ ,  $s \in \mathbb{R}$ ,  $b > -1$ , and  $\mathfrak{g} \in \mathfrak{M}$ . Then,

$$\mathbb{F}S T_{q,b}^{s,\lambda}(\mathfrak{g}) \subset \mathbb{F}S T_{q,b}^{s+1,\lambda}(\mathfrak{g}),$$

for

$$\Re \left\{ \mathfrak{g}(v) + x_q \right\} > 0, \quad \text{with } x_q = \frac{[b]_q}{q^b},$$

and

$$\mathbb{F}S T_{q,b}^{s,\lambda+1}(\mathfrak{g}) \subset \mathbb{F}S T_{q,b}^{s,\lambda}(\mathfrak{g}),$$

for

$$\Re \left\{ \mathfrak{g}(v) + d_q \right\} > 0, \quad \text{with } d_q = \frac{[\lambda]_q}{q^b}.$$

Furthermore, if we choose  $\lambda = 0$  and  $q \rightarrow 1^-$ , then the inclusion relation (2.3) is reduced to the following result.

**Corollary 2.2.** [20] Let  $s \in \mathbb{R}$ ,  $b > -1$ , and  $g \in \mathfrak{M}$ . Then, for  $\Re \{g(v) + b\} > 0$ ,

$$\mathbb{F}ST_b^s(g) \subset \mathbb{F}ST_b^{s+1}(g).$$

**Theorem 2.2.** Let  $q \in (0, 1)$ ,  $\lambda \in \mathbb{N}_0$ ,  $s, \varrho \in \mathbb{R} : |\varrho| < \frac{\pi}{2}$ ,  $b > -1$ ,  $0 \neq \delta \in \mathbb{C}$  and  $g \in \mathfrak{M}$ . Then, for the conditions (2.2) and (2.4),

$$\mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; g) \subset \mathbb{F}CV_{q,b}^{s+1,\lambda}(\varrho, \delta; g), \quad (2.9)$$

and

$$\mathbb{F}CV_{q,b}^{s,\lambda+1}(\varrho, \delta; g) \subset \mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; g), \quad (2.10)$$

respectively.

*Proof.* Let  $h \in \mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; g)$ . Then, by (1.10),  $v\nabla_q h \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g)$ . This implies, using Theorem 2.1,  $v\nabla_q h \in \mathbb{F}ST_{q,b}^{s+1,\lambda}(\varrho, \delta; g)$ . Again, by (1.10), we get  $h \in \mathbb{F}CV_{q,b}^{s+1,\lambda}(\varrho, \delta; g)$ . One can easily prove the relation (2.10) using the same method as used for the relation (2.9).  $\square$

Particularly, if we take  $\delta = 1$  and  $\varrho = 0$ , we have

**Corollary 2.3.** [26] Let  $q \in (0, 1)$ ,  $\lambda \in \mathbb{N}_0$ ,  $s \in \mathbb{R}$ ,  $b > -1$ , and  $g \in \mathfrak{M}$ . Then,

$$\mathbb{F}CV_{q,b}^{s,\lambda}(g) \subset \mathbb{F}CV_{q,b}^{s+1,\lambda}(g),$$

for

$$\Re \{g(v) + x_q\} > 0, \text{ with } x_q = \frac{[b]_q}{q^b},$$

and

$$\mathbb{F}CV_{q,b}^{s,\lambda+1}(g) \subset \mathbb{F}CV_{q,b}^{s,\lambda}(g),$$

for

$$\Re \{g(v) + d_q\} > 0, \text{ with } d_q = \frac{[\lambda]_q}{q^b}.$$

Furthermore, if we choose  $\lambda = 0$  and  $q \rightarrow 1^-$ , then the inclusion relation (2.9) is reduced to the following result.

**Corollary 2.4.** [20] Let  $s \in \mathbb{R}$ ,  $b > -1$ , and  $g \in \mathfrak{M}$ . Then, for  $\Re \{g(v) + b\} > 0$ ,

$$\mathbb{F}CV_b^s(g) \subset \mathbb{F}CV_b^{s+1}(g).$$

## 2.2. Integral preserving property

**Theorem 2.3.** Let  $g \in \mathfrak{M}$ ,  $q \in (0, 1)$ ,  $\lambda \in \mathbb{N}_0$ ,  $s, \varrho \in \mathbb{R} : |\varrho| < \frac{\pi}{2}$ ,  $b > -1$ ,  $0 \neq \delta \in \mathbb{C}$ , and  $\mathbb{F}_{q,b}$  is defined by

$$\mathbb{F}_{q,b}(v) = \frac{[1+b]_q}{v^b} \int_0^v t^{b-1} h(t) \nabla_q t. \quad (2.11)$$

Then, for

$$\Re \left\{ e^{-i\varrho} \delta \cos \varrho (g(v) - 1) + (1 + [b]_q) \right\} > 0,$$

$\mathbb{F}_{q,b} \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g)$  whenever  $h \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g)$ .

*Proof.* Let  $\mathfrak{h} \in \mathbb{F}ST_{q,\eta}^{s,b}(\varphi)$  and consider

$$\chi(v) = \frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v \nabla_q \left( \Upsilon_{q,b}^{s,\lambda} \mathbb{F}_{q,b}(v) \right)}{\Upsilon_{q,b}^{s,\lambda} \mathbb{F}_{q,b}(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\}, \quad (2.12)$$

with  $\chi(v)$  is analytic in  $\Pi$  with  $\chi(0) = 1$ .

From (2.11), we can write

$$\frac{\nabla_q \left( v^b \mathbb{F}_{q,b}(v) \right)}{[1 + b]_q} = v^{b-1} \mathfrak{h}(v).$$

Using the product rule of  $q$ -difference operator, we get

$$v \nabla_q \mathbb{F}_{q,b}(v) = \left( 1 + \frac{[b]_q}{q^b} \right) \mathfrak{h}(v) - [b]_q \mathbb{F}_{q,b}(v). \quad (2.13)$$

From (2.9), (2.10) and (1.5), we have

$$\left( 1 + \frac{[b]_q}{q^b} \right) \frac{J_q^{s,b} \mathfrak{h}(v)}{J_q^{s+1,b} \mathbb{F}_{q,b}(v)} = e^{-i\varrho} \delta \cos \varrho (\chi(v) - 1) + (1 + [b]_q).$$

After  $q$ -logarithmic differentiation, we get

$$\frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v \nabla_q \left( \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v) \right)}{\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\} = \chi(v) + \frac{v \nabla_q \chi(v)}{e^{-i\varrho} \delta \cos \varrho (\chi(v) - 1) + (1 + [b]_q)}$$

$$\chi(v) + \frac{v \nabla_q \chi(v)}{e^{-i\varrho} \delta \cos \varrho (\chi(v) - 1) + (1 + [b]_q)} <_{\mathbb{F}} g(v),$$

we have used the fact  $\mathfrak{h} \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g)$ . Since  $g \in \mathfrak{M}$  and we assume that  $\Re \left\{ e^{-i\varrho} \delta \cos \varrho (g(v) - 1) + (1 + [b]_q) \right\} > 0$ , using Lemma 1.1, we conclude that  $\chi(v) <_{\mathbb{F}} g(v)$  and this completes the proof.  $\square$

In particular, when  $\delta = 1$  and  $\varrho = 0$ , we have:

**Corollary 2.5.** [26] Let  $\mathfrak{h} \in \mathbb{F}ST_{q,b}^{s,\lambda}(g)$ . Then,  $\mathbb{F}_{q,b}(v)$  is in  $\mathbb{F}ST_{q,b}^{s,\lambda}(g)$ , where  $\mathbb{F}_{q,b}(v)$  is given by (2.11).

Moreover, if we take  $\lambda = 0$  and  $q \rightarrow 1^-$ , then we get the following result.

**Corollary 2.6.** [20] Let  $\mathfrak{h} \in \mathbb{F}ST_b^s(g)$ . Then,  $\mathbb{F}_{q,b}(v)$  is in  $\mathbb{F}ST_b^s(g)$ , where  $\mathbb{F}_{q,b}(v)$  is given by (2.11).

**Remark 2.1.** (i) Upon following the same method as used in Theorem 2.3, we can easily prove that the integral operator, given by (2.11), preserves the class  $\mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; g)$ .

(ii) Particularly, the classes  $\mathbb{F}CV_{q,b}^{s,\lambda}(g)$  and  $\mathbb{F}CV_b^s(\varphi)$  defined in [26] and [20] respectively, are invariant under the  $q$ -Bernardi integral operator.



### 3. Conclusions

In this article, we have defined  $q$ -analogue of certain subclasses of univalent functions with the help of fuzzy subsets. The  $q$ -Ruscheweyh derivative operator and the  $q$ -Srivastava-Attiya operator are combined by the Hadamard product and then the resultant operator is applied on the newly defined classes to obtain some generalized subclasses. We presented various classical results, such as the inclusion relationships and integral preserving property, for our newly defined subclasses. Various previous work has pointed out as the corollaries of our major investigations.

#### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

#### Conflict of interest

The authors declare no conflict of interest.

#### References

1. L. A. Zadeh, Fuzzy Sets, *Inf. Control*, **8** (1965), 338–353. [http://doi.org/10.1016/S0019-9958\(65\)90241-X](http://doi.org/10.1016/S0019-9958(65)90241-X)
2. G. I. Oros, G. Oros, The notion of subordination in fuzzy sets theory, *Gen. Math.*, **19** (2011), 97–103.
3. S. S. Miller, P. T. Mocanu, Second order-differential inequalities in the complex plane, *J. Math. Anal. Appl.*, **65** (1978), 298–305. [http://doi.org/10.1016/0022-247X\(78\)90181-6](http://doi.org/10.1016/0022-247X(78)90181-6)
4. S. S. Miller, P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, **28** (1981), 157–171.
5. G. I. Oros, G. Oros, Fuzzy differential subordination, *Acta Univ. Apulensis*, **3** (2012), 55–64.
6. I. Dzitac, F. G. Filip, M. J. Manolescu, Fuzzy logic is not fuzzy: World-renowned computer scientist Lotfi A. Zadeh, *Int. J. Comput. Commun. Control*, **12** (2017), 748–789.
7. G. I. Oros, G. Oros, Dominants and best dominants in fuzzy differential subordinations, *Stud. Univ. Babeş-Bolyai Math.*, **57** (2012), 239–248.
8. G. I. Oros, G. Oros, Briot-Bouquet fuzzy differential subordination, *An. Univ. Oradea Fasc. Mat.*, **19** (2012), 83–87.
9. E. A. Haydar, On fuzzy differential subordination, *Math. Moravica*, **19** (2015), 123–129.
10. A. A. Lupas, A note on special fuzzy differential subordinations using generalized Salagean operator and Ruscheweyh derivative, *J. Comput. Anal. Appl.*, **15** (2013), 1476–1483.
11. E. Rapeanu, Continuation method for boundary value problems with uniform elliptical operators, *J. Sci. Arts*, **3** (2011), 273–277.
12. A. A. Lupas, A note on special fuzzy differential subordinations using multiplier transformation and Ruscheweyh derivative, *J. Comput. Anal. Appl.*, **25** (2018), 1116–1124.

13. E. Rapeanu, Approximation by projection of some operators, *Analele Universității Maritime Constanța*, **11** (2010), 216–218.
14. A. K. Wanas, A. H. Majeed, Fuzzy differential subordination properties of analytic functions involving generalized differential operator, *Sci. Int.*, **30** (2018), 297–302.
15. A. R. S. Juma, M. H. Saloomi, Generalized Differential Operator on Bistarlike and Biconvex Functions Associated by Quasi-Subordination, *J. Phys.: Conf. Ser.*, **1003** (2018), 012046. <http://doi.org/10.1088/1742-6596/1003/1/012046>
16. E. Deniz, M. Çağlar, H. Orhan, The Fekete-Szegő problem for a class of analytic functions defined by Dziok-Srivastava operator, *Kodai Math. J.*, **35** (2012), 439–462.
17. A. Saha, S. Azami, D. Breaz, E. Rapeanu, S. K. Hui, Evolution for First Eigenvalue of  $L_{T,f}$  on an Evolving Riemannian Manifold, *Mathematics*, **10** (2022), 4614. <http://doi.org/10.3390/math10234614>
18. E. A. Totoi, L. I. Cotîrlă, Preserving Classes of Meromorphic Functions through Integral Operators, *Symmetry*, **14** (2022), 1545. <http://doi.org/10.3390/sym14081545>
19. S. Kazımoğlu, E. Deniz, L. I. Cotîrlă, Geometric Properties of Generalized Integral Operators Related to The Miller–Ross Function, *Axioms*, **12** (2023), 563. <http://doi.org/10.3390/axioms12060563>
20. S. A. Shah, E. E. Ali, A. A. Maitlo, T. Abdeljawad, A. M. Albalahi, Inclusion results for the class of fuzzy  $\alpha$ -convex functions, *AIMS Mathematics*, **8** (2022), 1375–1383. <http://doi.org/10.3934/math.2023069>
21. K. I. Noor, M. A. Noor, Fuzzy differential subordination involving generalized Noor-Salagean operator, *Inf. Sci. Lett.*, **11** (2022), 1905–1911. <http://doi.org/10.18576/isl/110606>
22. F. M. Sakar, Estimate for Initial Tschebyscheff Polynomials Coefficients on a Certain Subclass of Bi-univalent Functions Defined by Salagean Differential Operator, *Acta Univ. Apulensis*, **54** (2018), 45–54.
23. A. R. S. Juma, A. Al-Fayadh, S. P. Vijayalakshmi, T. V. Sudharsan, Upper bound on the third hankel determinant for the class of univalent functions using an integral operator, *Afr. Mat.*, **33** (2022), 56. <http://doi.org/10.1007/s13370-022-00991-0>
24. D. Breaz, K. R. Karthikeyan, E. Umadevi, A. Senguttuvan, Some properties of Bazilevic functions involving Srivastava–Tomovski operator, *Axioms*, **11** (2022), 687. <http://doi.org/10.3390/axioms11120687>
25. S. A. Shah, E. E. Ali, A. Catas, A. M. Albalahi, On fuzzy differential subordination associated with  $q$ -difference operator, *AIMS Mathematics*, **8** (2023), 6642–6650. <http://doi.org/10.3934/math.2023336>
26. A. F. Azzam, S. A. Shah, A. Alburaihan, S. M. El-Deeb, Certain inclusion properties for the class of  $q$ -analogue of fuzzy  $\alpha$ -convex functions, *Symmetry*, **15** (2023), 509. <http://doi.org/10.3390/sym15020509>
27. A. A. Azzam, S. A. Shah, A. Catas, L.-I. Cotîrlă, On fuzzy spiral-like functions associated with the family of inear operators, *Fractal Fract.*, **7** (2023), 145. <http://doi.org/10.3390/fractalfract7020145>

28. S. G. Gal, A. I. Ban, *Elemente de matematică fuzzy (In Romanian)*, Romaia: Editura Universității din Oradea, 1996.
29. S. S. Miller, P. T. Mocanu, *Differential subordinations theory and applications*, New York, Basel: Marcel Dekker, 2000.
30. F. H. Jackson, XI.—On  $q$ -functions and a certain difference operator, *Trans. Royal Soc. Edin.*, **46** (1908), 253–281. <http://doi.org/10.1017/S0080456800002751>
31. H. Exton,  *$q$ -Hypergeometric functions and applications*, Chichester: Ellis Horwood Limited, 1983.
32. U. A. Ezeafulukwe, M. Darus, A note on  $q$ -calculus, *Fasciculi Math.*, **55** (2015), 53–63. <http://doi.org/10.1515/fascmath-2015-0014>
33. M. E. H. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, *Complex Var. Elliptic*, **14** (1990), 77–84. <http://doi.org/10.1080/17476939008814407>
34. S. Kanas, R. Raducanu, Some classes of analytic functions related to conic domains, *Slovaca*, **64** (2014), 1183–1196. <http://doi.org/10.2478/s12175-014-0268-9>
35. S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109–115. <http://doi.org/10.2307/2039801>
36. S. A. Shah, K. I. Noor, Study on  $q$ -analogue of certain family of linear operators, *Turk. J. Math.*, **43** (2019), 109–115. <http://doi.org/10.3906/mat-1907-41>
37. H. M. Srivastava, A. A. Attiya, An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination, *Integr. Transf. Spec. Funct.*, **18** (2007), 207–216. <http://doi.org/10.1080/10652460701208577>
38. K. I. Noor, S. Riaz, M. A. Noor, On  $q$ -Bernardi integral operator, *TWMS J. Pure Appl. Math.*, **8** (2017), 3–11.
39. S. A. Shah, L.-I. Cotîrlă, A. Catas, C. Dubau, G. Cheregi, A study of spiral-like harmonic functions associated with quantum calculus, *J. Funct. Spaces*, **22** (2017), 5495011. <http://doi.org/10.1155/2022/5495011>
40. L.-I. Cotîrlă, G. Murugusundaramoorthy, Starlike functions based on Ruscheweyh  $q$ -differential operator defined in Janowski domain, *Fractal Fract.*, **7** (2023), 148. <http://doi.org/10.3390/fractalfract7020148>
41. S. M. El-Deeb, L.-I. Cotîrlă, Basic properties for certain subclasses of meromorphic  $p$ -valent functions with connected  $q$ -analogue of linear differential operator, *Axioms*, **12** (2023), 207. <http://doi.org/10.3390/axioms12020207>



AIMS Press

©2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)