
Research article**Study of the fuzzy q –spiral-like functions associated with the generalized linear operator****A. A. Azzam^{1,2}, Daniel Breaz³, Shujaat Ali Shah^{4,*} and Luminița-Ioana Cotîrlă^{5,*}**¹ Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia² Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt³ Department of Mathematics, “1 Decembrie 1918” University of Alba Iulia, Alba Iulia 510009, Romania⁴ Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, Nawabshah 67450, Pakistan⁵ Department of Mathematics, Technical University of Cluj-Napoca, Cluj-Napoca 400114, Romania*** Correspondence:** Email: shahglike@yahoo.com, luminita.cotirla@math.utcluj.ro.

Abstract: Nowadays, the subclasses of analytic functions in terms of fuzzy subsets are studied by various scholars and some of these concepts are extended using the q –theory of functions. In this inspiration, we introduce certain subclasses of analytic function by using the notion of fuzzy subsets along with the idea of q –calculus. We present the q –extensions of the fuzzy spiral-like functions of a complex order. We generalize this class using the q –analogues of the Ruscheweyh derivative and Srivastava-Attiya operators. Various interesting properties are examined for the newly defined subclasses. Also, some previously investigated results are deduced as the corollaries of our major results.

Keywords: analytic functions; q –spiral-like; fuzzy q –starlike functions; fuzzy q –convex functions; q –Ruscheweyh derivative operator; q –Srivastava-Attiya operator

Mathematics Subject Classification: 30A10, 30C45

1. Introduction

The article written by Lotfi A. Zadeh and published in 1965 [1] serves as the foundation for the fuzzy sets theory. In response to the numerous attempts by researchers to connect this theory with various areas of mathematics, the connection between fuzzy sets theory and the area of complex analysis that studies analytic functions by virtue of their geometric properties was established in

2011 [2]. Miller and Mocanu in [3, 4] introduced the concept of differential subordination. Oros and Oros [2] researched the idea of fuzzy subordination in 2011, and they [5] introduced the idea of fuzzy differential subordination in 2012. The history of the idea of a fuzzy set and its ties to various scientific and technical fields are nicely reviewed in the 2017 paper [6], including references to the findings up to that point regarding the fuzzy differential subordination concept. The first findings confirmed the direction of the research, adapting the traditional theory of differential subordination to the novel features of fuzzy differential subordination and providing techniques for investigating dominants and best dominants of fuzzy differential subordinations [7], without which the research could not have continued. Following that, the particular form of Briot-Bouquet fuzzy differential subordinations was examined in [8]. Haydar in [9] adopted the concept and began to look into the new findings on fuzzy differential subordinations. In this sequel, fuzzy differential subordinations were associated with different operators [10, 11] giving a new direction to the study. Numerous studies [12–14] carried out the investigations by employing certain linear operators. Furthermore, the work of several scholars about the fuzzy differential subordination is referred to the readers, for example, see [15–24]. The concept of fuzzy differential subordination is the first attempt to incorporate the idea of a fuzzy set into research pertaining to geometric theory of analytic functions. Recently, the authors [25–27] linked the notion of a fuzzy subsets with the concepts of quantum extensions of analytic functions.

The classical results of univalent functions with the concept of differential subordination are generalized in this current paper to include quantum (q)-extensions of univalent functions associated with fuzzy differential subordination.

Let $\Gamma(\Pi)$ denote the class of analytic functions $h(v)$ in $\Pi = \{v : |v| < 1\}$. The functions $h \in \Gamma(\Pi)$ of the form

$$h(v) = v + a_{\eta+1}v^{\eta+1} + a_{\eta+2}v^{\eta+2} + \dots, \quad (v \in \Pi), \quad (1.1)$$

form the class denoted by \mathfrak{A}_η . We note that $\mathfrak{A}_1 = \mathfrak{A}$; the class of normalized analytic functions in Π . Let ST and CV denote the subclasses of \mathfrak{A} of starlike and convex univalent functions, respectively. Here, we provide an overview of some important fundamental ideas connected to our work.

Definition 1.1. [28] A function \mathbb{F} is said to be fuzzy subset on $\mathfrak{Y} \neq \emptyset$, if it maps from \mathfrak{Y} to $[0, 1]$.

In other words, fuzzy subset is defined as:

Definition 1.2. [28] A pair (U, \mathbb{F}_U) is said to be a fuzzy subset on \mathfrak{Y} , where $U = \{x \in \mathfrak{Y} : 0 < \mathbb{F}_U(x) \leq 1\} = \sup(U, \mathbb{F}_U)$ is the support of fuzzy set (U, \mathbb{F}_U) and $\mathbb{F}_U : \mathfrak{Y} \rightarrow [0, 1]$ is the membership function of the fuzzy set (U, \mathbb{F}_U) .

Definition 1.3. [28] Let (U_1, \mathbb{F}_{U_1}) and (U_2, \mathbb{F}_{U_2}) be two subsets of \mathfrak{Y} . Then, $(U_1, \mathbb{F}_{U_1}) \subseteq (U_2, \mathbb{F}_{U_2})$ if and only if $\mathbb{F}_{U_1}(t) \leq \mathbb{F}_{U_2}(t)$, $t \in \mathfrak{Y}$, whereas, (U_1, \mathbb{F}_{U_1}) and (U_2, \mathbb{F}_{U_2}) of \mathfrak{Y} are equal if and only if $U_1 = U_2$.

We say that the analytic function h is subordinate to the analytic function g (written as $h \prec g$) if $h(v) = g(w(v))$, where $w(v)$ is a Schwartz function in Π , see Miller and Mocanu [29].

The generalization of the subordination technique of analytic functions in terms of fuzzy notion was defined by Oros and Oros [5] as the following.

Let the analytic function \mathfrak{h} be fuzzy subordinate to the analytic function g (written as $\mathfrak{h} \prec_{\mathbb{F}} g$). Then,

$$\mathfrak{h}(\nu_0) = g(\nu_0) \text{ and } \mathbb{F}(\mathfrak{h}(\nu)) \leq \mathbb{F}(g(\nu)), \nu \in \mathfrak{R},$$

where $\mathfrak{R} \subset \mathbb{C}$ and ν_0 be a fixed point in \mathfrak{R} .

Remark 1.1. If $\mathfrak{R} = \Pi$ in the above definition, then the fuzzy subordination is equivalent to the classical subordination.

For $0 < q < 1$, the operator ∇_q defined by

$$\nabla_q \mathfrak{h}(\nu) = \frac{\mathfrak{h}(\nu) - \mathfrak{h}(q\nu)}{(1 - q)\nu}; \quad q \neq 1, \nu \neq 0, \quad (1.2)$$

is called the q -difference operator. It was introduced by Jackson [30] and it is clear that $\lim_{q \rightarrow 1^-} \nabla_q \mathfrak{h}(\nu) = \mathfrak{h}'(\nu)$, where $\mathfrak{h}'(\nu)$ denotes the derivative of the function.

For $j \in \mathbb{N} = \{1, 2, 3, \dots\}$, we have

$$\nabla_q \left\{ \sum_{j=1}^{\infty} a_j \nu^j \right\} = \sum_{j=1}^{\infty} [j]_q a_j \nu^{j-1}, \quad (1.3)$$

where

$$[j]_q = \frac{1 - q^j}{1 - q} = 1 + q + q^2 + \dots + q^{j-1}. \quad (1.4)$$

We have the following rules of ∇_q , we refer to [31, 32].

- (i) $\nabla_q(ab_1(\nu) \pm b\mathfrak{h}_2(\nu)) = a\nabla_q\mathfrak{h}_1(\nu) \pm b\nabla_q\mathfrak{h}_2(\nu)$.
- (ii) $\nabla_q(\mathfrak{h}_1(\nu) \mathfrak{h}_2(\nu)) = \mathfrak{h}_1(q\nu) \nabla_q(\mathfrak{h}_2(\nu)) + \mathfrak{h}_2(\nu) \nabla_q(\mathfrak{h}_1(\nu))$.
- (iii) $\nabla_q \left(\frac{\mathfrak{h}_1(\nu)}{\mathfrak{h}_2(\nu)} \right) = \frac{\nabla_q(\mathfrak{h}_1(\nu))\mathfrak{h}_2(\nu) - \mathfrak{h}_1(\nu)\nabla_q(\mathfrak{h}_2(\nu))}{\mathfrak{h}_2(q\nu)\mathfrak{h}_2(\nu)}, \quad \mathfrak{h}_2(q\nu)\mathfrak{h}_2(\nu) \neq 0$.
- (iv) $\nabla_q(\log \mathfrak{h}(\nu)) = \frac{\ln q \nabla_q(\mathfrak{h}(\nu))}{(q-1)\mathfrak{h}(\nu)}$.

Ismail et al. [33] were first who discussed various properties of function theory by virtue of q -theory.

In [34], Kanas and Raducanu introduced an operator $R_q^\lambda : \mathfrak{A} \rightarrow \mathfrak{A}$ defined by

$$R_q^\lambda \mathfrak{h}(\nu) = \nu + \sum_{j=1}^{\infty} \frac{[j + \lambda - 1]_q}{j!} [\lambda]_q! [j-1]_q! a_j \nu^j, \quad (\lambda > -1). \quad (1.5)$$

Also, if $\lambda = m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then the operator (2.6) can be written as:

$$R_q^m \mathfrak{h}(\nu) = \frac{\nu \nabla_q \left(\nu^{m-1} \mathfrak{h}(\nu) \right)}{[m]_q!}.$$

We note that $R_q^0 \mathfrak{h}(\nu) = \mathfrak{h}(\nu)$ and $R_q^1 \mathfrak{h}(\nu) = \nu \nabla_q \mathfrak{h}(\nu)$.

We obtain the Ruscheweyh derivative operator [35], in particular, for $q \rightarrow 1^-$.

The q -extension of the Srivastava-Attiya operator discussed by Shah and Noor in [36]. They defined, for $b \in \mathbb{C} \setminus \mathbb{Z}_0^+$, $s \in \mathbb{C}$ when $|\nu| < 1$ and $\Re(s) > 1$ when $|\nu| = 1$, the operator $J_{q,b}^s : \mathfrak{A} \rightarrow \mathfrak{A}$ by

$$J_{q,b}^s \mathfrak{h}(\nu) = \Lambda_q(s, b; \nu) * \mathfrak{h}(\nu)$$

$$= \nu + \sum_{j=2}^{\infty} \left(\frac{[1+b]_q}{[j+b]_q} \right)^s a_j \nu^j, \quad (1.6)$$

where

$$\Lambda_q(s, b; \nu) = \nu + \sum_{j=2}^{\infty} \left(\frac{[1+b]_q}{[j+b]_q} \right)^s \nu^j.$$

The q -Srivastava-Attiya operator $J_{q,b}^s$ generalizes some well-known operators such as q -Alexander, q -Libera, q -Bernardi and Srivastava-Attiya operator, we refer to [37, 38].

From (1.5) and (1.6), we use the convolution technique to define $\Upsilon_{q,b}^{s,\lambda} : \mathfrak{A} \rightarrow \mathfrak{A}$ as follows:

$$\begin{aligned} \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(\nu) &= (R_q^\lambda * J_{q,b}^s) \mathfrak{h}(\nu) \\ &= \nu + \sum_{j=1}^{\infty} \frac{[j+\lambda-1]_q!}{[\lambda]_q! [j-1]_q!} \left(\frac{[1+b]_q}{[j+b]_q} \right)^s a_j \nu^j, \end{aligned} \quad (1.7)$$

where $\mathfrak{h} \in \mathfrak{A}$ and $*$ denotes the Hadamard product(convolution).

It is clear that

$$\Upsilon_{q,b}^{0,\lambda} \mathfrak{h}(\nu) = R_q^\lambda \mathfrak{h}(\nu) \text{ and } \Upsilon_{q,b}^{s,0} \mathfrak{h}(\nu) = J_{q,b}^s \mathfrak{h}(\nu).$$

The following identities can be implied from (1.5) to (1.7).

$$\nu \nabla_q \left(\Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(\nu) \right) = \left(1 + \frac{[b]_q}{q^b} \right) \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(\nu) - \frac{[b]_q}{q^b} \Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(\nu). \quad (1.8)$$

$$\nu \nabla_q \left(\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(\nu) \right) = \left(1 + \frac{[\lambda]_q}{q^b} \right) \Upsilon_{q,b}^{s,\lambda+1} \mathfrak{h}(\nu) - \frac{[\lambda]_q}{q^b} \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(\nu). \quad (1.9)$$

Several scholars studied various geometrical properties of analytic functions associated with q -linear operators, so we refer the readers to [39–41]. Now, we use the q -difference operator and the fuzzy subordination principle to define certain new subclasses $\mathbb{F}S_q(\varrho, \delta; \mathfrak{g})$ and $\mathbb{F}C_q(\varrho, \delta; \mathfrak{g})$ as the following.

For $\varrho \in \mathbb{R} : |\varrho| < \frac{\pi}{2}$, $q \in (0, 1)$, $0 \neq \delta \in \mathbb{C}$, $\nu \in \Pi$ and $\mathfrak{g} \in \mathfrak{M}$, where \mathfrak{M} is the class of all functions \mathfrak{g} which are analytic and univalent in Π , and for which $\mathfrak{g}(\Pi)$ is convex with $\mathfrak{g}(0) = 1$ and $\operatorname{Re}(\mathfrak{g}(\nu)) > 0$ in Π we define

$$\begin{aligned} \mathbb{F}S_q(\varrho, \delta; \mathfrak{g}) &= \left\{ \mathfrak{h} \in \mathfrak{A} : 1 + \frac{e^{i\varrho}}{\delta \cos \varrho} \left(\frac{\nu \nabla_q \mathfrak{h}}{\mathfrak{h}} - 1 \right) \prec_{\mathbb{F}} \mathfrak{g}(\nu) \right\}, \\ \mathbb{F}C_q(\varrho, \delta; \mathfrak{g}) &= \left\{ \mathfrak{h} \in \mathfrak{A} : 1 + \frac{e^{i\varrho}}{\delta \cos \varrho} \left(\frac{\nabla_q(\nu \nabla_q \mathfrak{h})}{\nabla_q \mathfrak{h}} - 1 \right) \prec_{\mathbb{F}} \mathfrak{g}(\nu) \right\}. \end{aligned}$$

In application of the operator given in (1.7), we define;

$$\mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; \mathfrak{g}) = \left\{ \mathfrak{h} \in \mathfrak{A} : \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(\nu) \in \mathbb{F}S_q(\varrho, \delta; \mathfrak{g}) \right\},$$

and

$$\mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; \mathfrak{g}) = \left\{ \mathfrak{h} \in \mathfrak{A} : \Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(\nu) \in \mathbb{F}C_q(\varrho, \delta; \mathfrak{g}) \right\},$$

where $g \in \mathfrak{M}$, $\lambda > -1$, $q \in (0, 1)$, $s, \varrho \in \mathbb{R}$: $|\varrho| < \frac{\pi}{2}$, $b > -1$, $0 \neq \delta \in \mathbb{C}$ and $v \in \Pi$.

It is obvious that

$$h \in \mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; g) \text{ if and only if } v\nabla_q h \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g). \quad (1.10)$$

Particularly, when $\delta = 1$ and $\varrho = 0$, we obtain the classes $\mathbb{F}ST_{q,b}^{s,\lambda}(g)$ and $\mathbb{F}CV_{q,b}^{s,\lambda}(g)$ introduced by the authors in [26]. For $\lambda = 0 = s$, the classes $\mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g)$ and $\mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; g)$ reduce to the classes $\mathbb{F}S_q(\varrho, \delta; g)$ and $\mathbb{F}C_q(\varrho, \delta; g)$ respectively. Moreover, when $q \rightarrow 1^-$, we have the classes, $\mathbb{F}ST(\varrho, \delta; g)$ and $\mathbb{F}CV(\varrho, \delta; g)$, introduced by the authors in [27]. Furthermore, for $\delta = 1$ and $\varrho = 0$, the classes $\mathbb{F}S(\varrho, \delta; g)$ and $\mathbb{F}C(\varrho, \delta; g)$ coincides with the classes $\mathbb{F}ST(g)$ and $\mathbb{F}C(g)$, introduced and studied by Shah et al. [20].

Now, in the next section, the inclusion problems between the subclasses are investigated. Furthermore, we show that the subclasses are preserved q -Bernardi integral operator. We need the following lemma for our findings.

Lemma 1.1. [25] Let β and γ be complex numbers with $\beta \neq 0$ and let $g(v)$ be a convex univalent in Π with $g(0) = 1$ and

$$\operatorname{Re} \{\beta g(v) + \gamma\} > 0. \quad (1.11)$$

If $p(v) = 1 + p_1 v + p_2 v^2 + \dots$ is analytic in Π , then

$$p(v) + \frac{v\nabla_q p(v)}{\beta p(v) + \gamma} \prec_{\mathbb{F}} g(v) \text{ implies } p(v) \prec_{\mathbb{F}} g(v),$$

where $\mathbb{F} : \mathbb{C} \rightarrow [0, 1]$.

2. Major results

2.1. Inclusion results

Theorem 2.1. Let $g \in \mathfrak{M}$, $q \in (0, 1)$, $\lambda \in \mathbb{N}_0$, $s, \varrho \in \mathbb{R}$: $|\varrho| < \frac{\pi}{2}$, $b > -1$ and $0 \neq \delta \in \mathbb{C}$. Then,

$$\mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g) \subset \mathbb{F}ST_{q,b}^{s+1,\lambda}(\varrho, \delta; g), \quad (2.1)$$

for

$$\Re \left\{ e^{-i\varrho} \delta \cos \varrho (g(v) - 1) + (1 + x_q) \right\} > 0, \text{ with } x_q = \frac{[b]_q}{q^b}, \quad (2.2)$$

and

$$\mathbb{F}ST_{q,b}^{s,\lambda+1}(\varrho, \delta; g) \subset \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g), \quad (2.3)$$

for

$$\Re \left\{ e^{-i\varrho} \delta \cos \varrho (g(v) - 1) + (1 + d_q) \right\} > 0, \text{ with } d_q = \frac{[\lambda]_q}{q^b}. \quad (2.4)$$

Proof. To prove the relation (2.1), we suppose $h \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; g)$. For analytic $p_1(v)$ in Π with $p_1(0) = 1$, we set

$$p_1(v) = \frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v\nabla_q (\Upsilon_{q,b}^{s+1,\lambda} h(v))}{\Upsilon_{q,b}^{s+1,\lambda} h(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\}. \quad (2.5)$$

The identity (1.8) and (2.5) imply that

$$p_1(v) = \frac{1}{\delta \cos \varrho} \left[e^{i\varrho} \left\{ \left(1 + \frac{[b]_q}{q^b} \right) \frac{\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v)}{\Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(v)} - \frac{[b]_q}{q^b} \right\} - (1 - \delta) \cos \varrho - i \sin \varrho \right],$$

equivalently

$$(1 + x_q) \frac{\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v)}{\Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(v)} = e^{-i\varrho} \delta \cos \varrho (p_1(v) - 1) + (1 + x_q), \quad \left(\text{for } x_q = \frac{[b]_q}{q^b} \right).$$

The q -logarithmic differentiation and (2.5) yield

$$\frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v \nabla_q (\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v))}{\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\} = p_1(v) + \frac{v \nabla_q p_1(v)}{e^{-i\varrho} \delta \cos \varrho (p_1(v) - 1) + (1 + x_q)}. \quad (2.6)$$

Since $\mathfrak{h} \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; \mathfrak{g})$, from (2.6) we have

$$p_1(v) + \frac{v \nabla_q p_1(v)}{e^{-i\varrho} \delta \cos \varrho (p_1(v) - 1) + (1 + x_q)} \prec_{\mathbb{F}} \mathfrak{g}(v), \quad (2.7)$$

for $\mathfrak{g} \in \mathfrak{M}$, we assume that

$$\Re \left\{ e^{-i\varrho} \delta \cos \varrho (p_1(v) - 1) + (1 + x_q) \right\} > 0,$$

by Lemma 1.1 and (2.7), we conclude $p_1(v) \prec_{\mathbb{F}} \mathfrak{g}(v)$ implies $\mathfrak{h} \in \mathbb{F}ST_{q,b}^{s+1,\lambda}(\varrho, \delta; \mathfrak{g})$.

To prove (2.5), we set, for analytic $p_2(v)$ in Π with $p_2(0) = 1$,

$$p_2(v) = \frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v \nabla_q (\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v))}{\Upsilon_{q,b}^{s+1,\lambda} \mathfrak{h}(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\}. \quad (2.8)$$

Now, using similar techniques as before, we can easily obtain the required result using the identity (1.9) along with Lemma 1.1. \square

Particularly, if we take $\delta = 1$ and $\varrho = 0$, we have:

Corollary 2.1. [26] Let $q \in (0, 1)$, $\lambda \in \mathbb{N}_0$, $s \in \mathbb{R}$, $b > -1$, and $\mathfrak{g} \in \mathfrak{M}$. Then,

$$\mathbb{F}ST_{q,b}^{s,\lambda}(\mathfrak{g}) \subset \mathbb{F}ST_{q,b}^{s+1,\lambda}(\mathfrak{g}),$$

for

$$\Re \left\{ \mathfrak{g}(v) + x_q \right\} > 0, \quad \text{with } x_q = \frac{[b]_q}{q^b},$$

and

$$\mathbb{F}ST_{q,b}^{s,\lambda+1}(\mathfrak{g}) \subset \mathbb{F}ST_{q,b}^{s,\lambda}(\mathfrak{g}),$$

for

$$\Re \left\{ \mathfrak{g}(v) + d_q \right\} > 0, \quad \text{with } d_q = \frac{[\lambda]_q}{q^b}.$$

Furthermore, if we choose $\lambda = 0$ and $q \rightarrow 1^-$, then the inclusion relation (2.3) is reduced to the following result.

Corollary 2.2. [20] Let $s \in \mathbb{R}$, $b > -1$, and $\mathbf{g} \in \mathfrak{M}$. Then, for $\Re \{g(v) + b\} > 0$,

$$\mathbb{F}ST_b^s(\mathbf{g}) \subset \mathbb{F}ST_b^{s+1}(\mathbf{g}).$$

Theorem 2.2. Let $q \in (0, 1)$, $\lambda \in \mathbb{N}_0$, $s, \varrho \in \mathbb{R}$: $|\varrho| < \frac{\pi}{2}$, $b > -1$, $0 \neq \delta \in \mathbb{C}$ and $\mathbf{g} \in \mathfrak{M}$. Then, for the conditions (2.2) and (2.4),

$$\mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; \mathbf{g}) \subset \mathbb{F}CV_{q,b}^{s+1,\lambda}(\varrho, \delta; \mathbf{g}), \quad (2.9)$$

and

$$\mathbb{F}CV_{q,b}^{s,\lambda+1}(\varrho, \delta; \mathbf{g}) \subset \mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; \mathbf{g}), \quad (2.10)$$

respectively.

Proof. Let $\mathbf{h} \in \mathbb{F}CV_{q,b}^{s,\lambda}(\varrho, \delta; \mathbf{g})$. Then, by (1.10), $v\nabla_q \mathbf{h} \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; \mathbf{g})$. This implies, using Theorem 2.1, $v\nabla_q \mathbf{h} \in \mathbb{F}ST_{q,b}^{s+1,\lambda}(\varrho, \delta; \mathbf{g})$. Again, by (1.10), we get $\mathbf{h} \in \mathbb{F}CV_{q,b}^{s+1,\lambda}(\varrho, \delta; \mathbf{g})$. One can easily prove the relation (2.10) using the same method as used for the relation (2.9). \square

Particularly, if we take $\delta = 1$ and $\varrho = 0$, we have

Corollary 2.3. [26] Let $q \in (0, 1)$, $\lambda \in \mathbb{N}_0$, $s \in \mathbb{R}$, $b > -1$, and $\mathbf{g} \in \mathfrak{M}$. Then,

$$\mathbb{F}CV_{q,b}^{s,\lambda}(\mathbf{g}) \subset \mathbb{F}CV_{q,b}^{s+1,\lambda}(\mathbf{g}),$$

for

$$\Re \{g(v) + x_q\} > 0, \text{ with } x_q = \frac{[b]_q}{q^b},$$

and

$$\mathbb{F}CV_{q,b}^{s,\lambda+1}(\mathbf{g}) \subset \mathbb{F}CV_{q,b}^{s,\lambda}(\mathbf{g}),$$

for

$$\Re \{g(v) + d_q\} > 0, \text{ with } d_q = \frac{[\lambda]_q}{q^b}.$$

Furthermore, if we choose $\lambda = 0$ and $q \rightarrow 1^-$, then the inclusion relation (2.9) is reduced to the following result.

Corollary 2.4. [20] Let $s \in \mathbb{R}$, $b > -1$, and $\mathbf{g} \in \mathfrak{M}$. Then, for $\Re \{g(v) + b\} > 0$,

$$\mathbb{F}CV_b^s(\mathbf{g}) \subset \mathbb{F}CV_b^{s+1}(\mathbf{g}).$$

2.2. Integral preserving property

Theorem 2.3. Let $\mathbf{g} \in \mathfrak{M}$, $q \in (0, 1)$, $\lambda \in \mathbb{N}_0$, $s, \varrho \in \mathbb{R}$: $|\varrho| < \frac{\pi}{2}$, $b > -1$, $0 \neq \delta \in \mathbb{C}$, and $\mathbb{F}_{q,b}$ is defined by

$$\mathbb{F}_{q,b}(v) = \frac{[1+b]_q}{v^b} \int_0^v t^{b-1} \mathbf{h}(t) \nabla_q t. \quad (2.11)$$

Then, for

$$\Re \{e^{-i\varrho} \delta \cos \varrho (g(v) - 1) + (1 + [b]_q)\} > 0,$$

$\mathbb{F}_{q,b} \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; \mathbf{g})$ whenever $\mathbf{h} \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; \mathbf{g})$.

Proof. Let $\mathfrak{h} \in \mathbb{F}ST_{q,\eta}^{s,b}(\varphi)$ and consider

$$\chi(v) = \frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v \nabla_q (\Upsilon_{q,b}^{s,\lambda} \mathbb{F}_{q,b}(v))}{\Upsilon_{q,b}^{s,\lambda} \mathbb{F}_{q,b}(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\}, \quad (2.12)$$

with $\chi(v)$ is analytic in Π with $\chi(0) = 1$.

From (2.11), we can write

$$\frac{\nabla_q (v^b \mathbb{F}_{q,b}(v))}{[1+b]_q} = v^{b-1} \mathfrak{h}(v).$$

Using the product rule of q -difference operator, we get

$$v \nabla_q \mathbb{F}_{q,b}(v) = \left(1 + \frac{[b]_q}{q^b}\right) \mathfrak{h}(v) - [b]_q \mathbb{F}_{q,b}(v). \quad (2.13)$$

From (2.9), (2.10) and (1.5), we have

$$\left(1 + \frac{[b]_q}{q^b}\right) \frac{J_q^{s,b} \mathfrak{h}(v)}{J_q^{s+1,b} \mathbb{F}_{q,b}(v)} = e^{-i\varrho} \delta \cos \varrho (\chi(v) - 1) + (1 + [b]_q).$$

After q -logarithmic differentiation, we get

$$\begin{aligned} \frac{1}{\delta \cos \varrho} \left\{ e^{i\varrho} \frac{v \nabla_q (\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v))}{\Upsilon_{q,b}^{s,\lambda} \mathfrak{h}(v)} - (1 - \delta) \cos \varrho - i \sin \varrho \right\} &= \chi(v) + \frac{v \nabla_q \chi(v)}{e^{-i\varrho} \delta \cos \varrho (\chi(v) - 1) + (1 + [b]_q)} \\ \chi(v) + \frac{v \nabla_q \chi(v)}{e^{-i\varrho} \delta \cos \varrho (\chi(v) - 1) + (1 + [b]_q)} &\prec_{\mathbb{F}} \mathfrak{g}(v), \end{aligned}$$

we have used the fact $\mathfrak{h} \in \mathbb{F}ST_{q,b}^{s,\lambda}(\varrho, \delta; \mathfrak{g})$. Since $\mathfrak{g} \in \mathfrak{M}$ and we assume that $\Re \{e^{-i\varrho} \delta \cos \varrho (\mathfrak{g}(v) - 1) + (1 + [b]_q)\} > 0$, using Lemma 1.1, we conclude that $\chi(v) \prec_{\mathbb{F}} \mathfrak{g}(v)$ and this completes the proof. \square

In particular, when $\delta = 1$ and $\varrho = 0$, we have:

Corollary 2.5. [26] Let $\mathfrak{h} \in \mathbb{F}ST_{q,b}^{s,\lambda}(\mathfrak{g})$. Then, $\mathbb{F}_{q,b}(v)$ is in $\mathbb{F}ST_{q,b}^{s,\lambda}(\mathfrak{g})$, where $\mathbb{F}_{q,b}(v)$ is given by (2.11).

Moreover, if we take $\lambda = 0$ and $q \rightarrow 1^-$, then we get the following result.

Corollary 2.6. [20] Let $\mathfrak{h} \in \mathbb{F}ST_b^s(\mathfrak{g})$. Then, $\mathbb{F}_{q,b}(v)$ is in $\mathbb{F}ST_b^s(\mathfrak{g})$, where $\mathbb{F}_{q,b}(v)$ is given by (2.11).

Remark 2.1. (i) Upon following the same method as used in Theorem 2.3, we can easily prove that the integral operator, given by (2.11), preserves the class $\mathbb{FCV}_{q,b}^{s,\lambda}(\varrho, \delta; \mathfrak{g})$.

(ii) Particularly, the classes $\mathbb{FCV}_{q,b}^{s,\lambda}(\mathfrak{g})$ and $\mathbb{FCV}_b^s(\varphi)$ defined in [26] and [20] respectively, are invariant under the q -Bernardi integral operator.

3. Conclusions

In this article, we have defined q -analogue of certain subclasses of univalent functions with the help of fuzzy subsets. The q -Ruscheweyh derivative operator and the q -Srivastava-Attiya operator are combined by the Hadamard product and then the resultant operator is applied on the newly defined classes to obtain some generalized subclasses. We presented various classical results, such as the inclusion relationships and integral preserving property, for our newly defined subclasses. Various previous work has pointed out as the corollaries of our major investigations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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