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Research article

Boundary value problems for a second-order differential equation with involution in the second derivative and their solvability

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Abstract: We consider the two-point boundary value problems for a nonlinear one-dimensional second-order differential equation with involution in the second derivative and in lower terms. The questions of existence and uniqueness of the classical solution of two-point boundary value problems are studied. The definition of the Green's function is generalized for the case of boundary value problems for the second-order linear differential equation with involution, indicating the points of discontinuities and the magnitude of discontinuities of the first derivative. Uniform estimates for the Green's function of the linear part of boundary value problems are established. Using the contraction mapping principle and the Schauder fixed point theorem, theorems on the existence and uniqueness of solutions to the boundary value problems are proved. The results obtained in this paper cover the boundary value problems for one-dimensional differential equations with and without involution in the lower terms.

Keywords: second-order differential equation with involution; Green's function; nonlinear equation; boundary value problem; Schauder fixed point theorem
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1. Introduction

Functional-differential equations are widely used in mathematical models. Among them, a special place is occupied by differential equations with deviating arguments and, in particular, differential equations with involution.

Recently, the boundary value problems for one-dimensional nonlinear differential equations with deviating arguments have been considered in [1,2] (see also references therein). If a simple model of a stationary temperature distribution in a straight wire with a length l is written as an equation [2]

$$y''(x) = f(x, y(x)),\ 0 < x < l,$$

with some boundary conditions, then modeling the stationary temperature distribution of the bent wire leads to an equation with deviating arguments [2].

The solvability of boundary value problems for one-dimensional nonlinear differential equations of the second order with involution is studied, for example, in [3–17] where the existence and uniqueness of the solution of the boundary value problems containing involution in the lower terms are proved. The papers [18–22] are devoted to spectral questions of boundary value problems for differential equations with involution (see also references therein).

It should be noted that interest in nonlinear one-dimensional differential equations is closely related to specific mathematical models. For example, the motion of a point under the action of a restoring force in a medium with hydraulic resistance is described by a second-order nonlinear equation [23,24]

$$ay''(x) + b|y'(x)|y'(x) + cy(x) = 0.$$

Nonlinear equations of the second order of the type

$$ay''(x) + (b - \chi(y))y'(x) + cy(x) = 0,$$

arise in radio engineering [25]. The electrical engineering equation has the form [26]

$$ay''(x) + by'(x) + c\sin y(x) + d = 0.$$

The equations of a more general form were also studied

$$y''(x) + f(y(x), y'(x))y'(x) + g(y(x)) = 0.$$

The forced oscillations of the pendulum are described by the equation [27]

$$y''(x) + b^2 \sin y = a \sin x.$$

The Thomas-Fermi equation

$$x^{\frac{1}{2}} y''(x) = y^{\frac{3}{2}}(x)$$

arises in studying the distribution of electrons in an atom [28]. Later, generalizations of the type $y'' = \phi(x, y)\varphi(x)$ and more general equations were considered.

The impetus to studying Eq (1.1) (or (1.3)) was triggered by the fact that one of the generalizations of such equations is one-dimensional differential equations with involution. The bibliography on the theory of differential equations with involution can be found in monographs [29–31].

Another reason is that the properties of solutions to differential equations with involution may differ significantly from the properties of solutions to equations without involution (see, for example, [21–25]).

Therefore, it is important to study the properties of differential equations with involution.

As far as nonlinear one-dimensional differential equations without involution are concerned, it should be noted that the fourth-order equations have been actively studied recently (see [32–35]).

We are devoted to studying the existence of a solution to the differential equation

$$y''(x)+\alpha y''(-x)=F(x,y(x),y(-x)),\ x\in(-1,1), \tag{1.1}$$

with boundary conditions

$$y(-1)=y_1, y(1)=y_2, \tag{1.2}$$

where $F:[0,1]\times R^2\to R$ is a given function, $\alpha\neq\pm1$. The other subject of studies is the existence of a solution to the equation

$$y''(x)+\alpha y''(-x)=F(x,y(x),y(-x),y'(x),y'(-x)),\ x\in(-1,1) \tag{1.3}$$

with boundary conditions (1.2), where $F:[0,1]\times R^4\to R$, $\alpha\neq\pm1$. We also consider problems with boundary condition $y(-1)=y_1, y'(1)=y_2$.

Equations (1.1) and (1.3) contain the involution transformation both in the lower terms and in the higher derivative and differ significantly from the equations studied by the other authors.

Equations (1.1) and (1.3) are non-local, as the values of the unknown function and its derivatives are related at two points x and $-x$.

The function $y(x)\in C^2(-1,1)\cap C[-1,1]$ that satisfies Eq (1.1) and boundary conditions (1.2) is called a solution to the boundary value problem (1.1), (1.2).

It should be kept in mind that mapping $\sigma(x)$ of a set D into itself satisfying the condition $\sigma(\sigma(x))=x$ is called an involution [29]. Equations (1.1) and (1.3) contain an involution of the form $\sigma(x)=-x$ for $x\in[-1,1]$. More detailed information on differential equations with involution can be found in monographs [29–31].

In the papers cited above, various aspects of the theory of boundary value problems for linear and nonlinear differential equations with involution are studied. The major results of most works on the theory of nonlinear differential equations are based on fixed point theorems. It is well known that fixed point theorems are used to prove the existence of solutions to boundary value problems for one-dimensional differential equations. Very often, the specific form of the Green's function of the problems associated with the studied boundary value problems significantly affects the results.

In this paper, we also use fixed point theorems. As far as we know, equations of the type (1.1), (1.2) have not been considered before. We generalize the definition of the Green's function to the case of boundary value problems for the second-order linear differential equation with involution. An important contribution of this paper is the study of the expression and properties of the Green's function of the problem described by Eq (1.1), and the proof of the theorem on the existence (uniqueness) of solutions to the studied problems.

The results obtained for $\alpha=0$ imply the corresponding results for differential equations with or without involution in lower terms.

The paper consists of three sections. In Section 2, the expression and properties of the Green's function of the boundary value problem related to problems (1.1), (1.2) and (1.3), (1.2) are considered. In Section 3, using theorems on fixed points, we prove the solvability of the studied problems.

2. Preliminary results

Let us introduce the definition of the Green's function of a boundary value problem for a second-order linear differential equation with involution. Consider the equation

$$Ly \equiv y''(x) + \alpha y''(-x) + q(x)y(x) = 0, \ -1 < x < 1, \tag{2.1}$$

with boundary conditions

$$a_{i1}y'(-1) + a_{i2}y'(1) + a_{i3}y(-1) + a_{i4}y(1) = 0, \ i = 1, 2, \tag{2.2}$$

where $q(x)$ is a continuous function on $[-1,1]$, a_{ij} are given numbers, $\alpha \neq \pm 1$. Let the boundary value problem (2.1) and (2.2) have no nonzero solutions. Then there may exist a nonzero function satisfying Eq (1.1) almost everywhere in $(-1,1)$ and boundary conditions (2.2).

The function $G(x,t)$ is called the Green's function of the boundary value problem (2.1), (2.2) if the following three conditions are satisfied:

1) the function $G(x,t)$ is continuous for $x, t \in [-1,1]$;
2) the function $G(x,t)$ for any fixed $t \in [-1,1]$ has derivatives $G'_x(x,t), G''_{xx}(x,t)$ continuous in each of the intervals $[-1,-t), (-t,t), (t,1]$, and

$$G'_x(-t-0,t) - G'_x(-t+0,t) = \alpha\left(1-\alpha^2\right)^{-1},$$

$$G'_x(t-0,t) - G'_x(t+0,t) = \left(1-\alpha^2\right)^{-1},$$

3) the function $G(x,t)$, as a function of x, satisfies Eq (2.1) in each of the intervals $[-1,-t), (-t,t), (t,1]$, and boundary conditions (2.2).

As it can be seen from this definition, the special feature of the Green's function of the boundary value problem (2.1), (2.2) is that its first derivative has two discontinuity points. For $\alpha = 0$, the Green's function of the boundary value problem (2.1), (2.2) has a discontinuity at one point.

Further, we will need the following facts.

Theorem 2.1. [36] A second-order linear differential equation with an involution of the form (2.1) has two linearly independent solutions.

Theorem 2.2. [36] If the boundary value problem (2.1), (2.2) has no nonzero solutions, then it has a unique Green's function.

Theorem 2.3. [36] If the boundary value problem (2.1), (2.2) has no nonzero solutions, then for any continuous function $f(x) \in C[-1,1]$ the inhomogeneous boundary value problem

$$Ly = f(x), \ -1 < x < 1, \ y(-1) = y(1) = 0,$$

has a unique solution of the type

$$y(x) = \int_{-1}^{1} G(x,t) f(t)\, dt,$$

where $G(x,t)$ is the Green's function of the homogeneous boundary value problem (2.1), (2.2).

It is well known that the Green's function is one of the best tools for studying boundary value problems. The specific form of the Green's function is important.

Lemma 2.1. [36] The Green's function of the boundary value problem

$$y''(x) + \alpha y''(-x) = 0, \ y(-1) = y(1) = 0, \ \alpha \neq \pm 1, \tag{2.3}$$

is written as

$$G(x,t)=\frac{1}{2\left(1-\alpha^2\right)}\begin{cases}\left[-1+\alpha+(1+\alpha)x\right](1+t), t\le -x,\\ \left[-1+\alpha-(1+\alpha)t\right](1-x), -x\le t\le x,\\ \left[-1+\alpha-(1+\alpha)x\right](1-t), t\ge x.\end{cases} \tag{2.4}$$

For $\alpha=0$ from (2.4), we obtain the Green's function of the boundary value problem for a second-order ordinary linear differential equation (see, for example, [37]) $y''(x)=0,\ y(-1)=y(1)=0$.

To obtain the main results, we will use the following estimates for the Green's function.

Lemma 2.2. If $-1<\alpha<1$, then the Green's function (2.4) has the following estimates:

$$|G(x,t)|\le\frac{1}{1-|\alpha|};\quad \int_{-1}^{1}|G(x,t)|dt\le\frac{9}{16(1-|\alpha|)};\quad \int_{-1}^{1}|G'_x(x,t)|dt\le\frac{1}{1-|\alpha|}.$$

Proof. Let us prove the first assertion of the Lemma. It can be assumed that $x>0$. For $-1\le t\le -x$, the Green's function (2.4) takes the form

$$G(x,t)=-\frac{1+t}{2(1+\alpha)}+\frac{x(1+t)}{2(1-\alpha)}.$$

This implies the first assertion of the Lemma

$$|G(x,t)|\le\frac{1+t}{2(1-\alpha)}+\frac{x(1+t)}{2(1+\alpha)}\le\frac{(1+x)(1+t)}{2(1-|\alpha|)}\le\frac{1}{1-|\alpha|}.$$

If $-x\le t\le x$, then the Green's function can be written as

$$G(x,t)=\left[-\frac{1}{2(1+\alpha)}-\frac{t}{2(1-\alpha)}\right](1-x).$$

For $t\le 0$ we get

$$|G(x,t)|\le\frac{(1-x)(1-t)}{2(1-|\alpha|)}\le\frac{1}{1-|\alpha|}.$$

Let now $t\ge 0$. Then

$$|G(x,t)|\le\frac{(1+t)(1-x)}{2(1-|\alpha|)}\le\frac{1}{1-|\alpha|}.$$

For $t\ge x$ the first assertion of the Lemma is proved in a similar way

$$|G(x,t)|\le\frac{(1+x)(1-t)}{2(1-|\alpha|)}\le\frac{1}{1-|\alpha|}.$$

Let us prove the second inequality of the Lemma. After calculation of

$$\int_{-1}^{1}|G(x,t)|dt=\int_{-1}^{-x}\left|-\frac{1+t}{2(1+\alpha)}+\frac{x(1+t)}{2(1-\alpha)}\right|dt+\int_{-x}^{x}\left|\frac{-(1-x)}{2(1+\alpha)}-\frac{t(1-x)}{2(1-\alpha)}\right|dt+\int_{x}^{1}\left|-\frac{1-t}{2(1+\alpha)}-\frac{x(1-t)}{2(1-\alpha)}\right|dt\le$$

$$\le\left[\frac{(1-x)^2}{2|1+\alpha|}+\frac{x(1-x)^2}{2|1-\alpha|}\right]+\left[\frac{(1-x)x}{|1+\alpha|}+\frac{x^2(1-x)}{2|1-\alpha|}\right],$$

we get the estimate

$$\int_{-1}^{1}|G(x,t)|dt\le\frac{9}{16(1-|\alpha|)}.$$

The third assertion of the Lemma is proved by direct calculation. As

$$G_x'(x,t)=\begin{cases}\dfrac{1+t}{2(1-\alpha)},\ -1\le t<-x,\\[2mm] \dfrac{1}{2(1+\alpha)}+\dfrac{t}{2(1-\alpha)},-x<t<x,\\[2mm] -\dfrac{1-t}{2(1-\alpha)},x<t\le 1,\end{cases}$$

then

$$\int_{-1}^{1}|G_x'(x,t)|dt=\int_{-1}^{-x}\left|\frac{1+t}{2(1-\alpha)}\right|dt+\int_{-x}^{x}\left|\frac{1}{2(1+\alpha)}+\frac{t}{2(1-\alpha)}\right|dt+\int_{x}^{1}\left|-\frac{1-t}{2(1-\alpha)}\right|dt\le\frac{x^2+1}{2(1-|\alpha|)}\le\frac{1}{(1-|\alpha|)}.$$

The Lemma is proved.

Lemma 2.3. If $|\alpha|>1$, then the following estimates hold for the Green function (2.4):

$$|G(x,t)|\le\frac{1}{|\alpha|-1};\ \int_{-1}^{1}|G(x,t)|dt\le\frac{9}{16(|\alpha|-1)}\qquad\int_{-1}^{1}|G_x'(x,t)|dt\le\frac{1}{|\alpha|-1}.$$

Proof. Let us prove the first assertion of the Lemma. The case $-1\le t\le -x$. Let $\alpha<-1$. Then we get

$$|G(x,t)|=\frac{1+t}{2(|\alpha|-1)}+\frac{x(1+t)}{2(|\alpha|+1)}\le\frac{1}{2(|\alpha|-1)}+\frac{1}{2(|\alpha|+1)}\le\frac{1}{|\alpha|-1}.$$

Let $\alpha>1$. Then

$$|G(x,t)|=\frac{1+t}{2(\alpha+1)}+\frac{x(1+t)}{2(\alpha-1)}\le\frac{1}{2(|\alpha|+1)}+\frac{1}{2(|\alpha|-1)}\le\frac{1}{|\alpha|-1}.$$

Consider the case $-x\le t\le x$. Let $\alpha<-1$. The direct calculations give:

$$|G(x,t)|=\left|\frac{-(1-x)}{2(1+\alpha)}-\frac{t(1-x)}{2(1-\alpha)}\right|\le\frac{1}{2(|\alpha|-1)}+\frac{1}{2(|\alpha|+1)}\le\frac{1}{|\alpha|-1}.$$

Let $\alpha>1$. It is easy to deduce the inequality

$$|G(x,t)| = \left| \frac{-(1-x)}{2(1+\alpha)} - \frac{t(1-x)}{2(1-\alpha)} \right| \le \frac{1}{2(|\alpha|+1)} + \frac{1}{2(|\alpha|-1)} \le \frac{1}{|\alpha|-1}.$$

Consider the case $t \ge x$. Let $\alpha < -1$. By simple analysis, we get

$$|G(x,t)| = \left| -\frac{1-t}{2(1+\alpha)} - \frac{x(1-t)}{2(1-\alpha)} \right| \le \frac{1}{2(|\alpha|-1)} + \frac{1}{2(|\alpha|+1)} \le \frac{1}{|\alpha|-1}.$$

Let $\alpha > 1$. The estimate for the Green's function has the following form

$$|G(x,t)| = \left| -\frac{1-t}{2(1+\alpha)} - \frac{x(1-t)}{2(1-\alpha)} \right| \le \frac{1}{2(|\alpha|+1)} + \frac{1}{2(|\alpha|-1)} \le \frac{1}{|\alpha|-1}.$$

The first assertion of the Lemma is proved. The other assertions of the Lemma are proved in a similar way. The Lemma is proved.

From the proved lemmas, we have the following corollary.

Corollary 2.1. If $\alpha \ne \pm 1$, then the following estimates hold for the Green's function (2.4):

$$|G(x,t)| \le \frac{1}{\left||\alpha|-1\right|}; \quad \int_{-1}^{1} |G(x,t)|\, dt \le \frac{9}{16\left||\alpha|-1\right|}; \quad \int_{-1}^{1} |G'_x(x,t)|\, dt \le \frac{1}{\left||\alpha|-1\right|}.$$

3. Results

Consider the boundary value problem (1.1), (1.2). We can consider Eq (1.1) as an inhomogeneous form of Eq (2.3). According to Theorem 2.3, the boundary value problem (1.1), (1.2) is equivalent to the integral equation

$$y(x) = \frac{1}{2}(y_2 + y_1) + \frac{1}{2}(y_2 - y_1)x + \int_{-1}^{1} G(x,t) F(t, y(t), y(-t))\, dt,$$

where $G(x,t)$ is Green's function of the form (2.4). Here, the first two terms are the solution to the homogeneous Eq (1.1), and the third term is the solution to the inhomogeneous Eq (1.1).

Let $X = C([-1,1], R)$ be the Banach space of functions $y(x) \in C([-1,1], R)$ with the norm

$$\|y\|_X = \max_{-1 \le x \le 1} |y(x)|.$$

Let us use the equality

$$(Ay)(x) = \frac{1}{2}(y_2 + y_1) + \frac{1}{2}(y_2 - y_1)x + \int_{-1}^{1} G(x,t) F(t, y(t), y(-t))\, dt \tag{3.1}$$

to define the operator $A : X \to X$. Thus, the problem of the existence of a solution to the boundary value problem (1.1), (1.2) is reduced to the problem of the existence of a fixed point of the operator A defined by formula (3.1).

Let us introduce the notation $\Omega = [-1,1] \times R^2$.

Theorem 3.1. Let $\alpha \ne \pm 1$. Let the function $F(x, \varsigma, \xi)$ be continuous and satisfy the Lipschitz condition

$$\left|F\left(x,\varsigma,\xi\right)-F\left(x,\tilde{\varsigma},\tilde{\xi}\right)\right|\le l_1\left|\varsigma-\tilde{\varsigma}\right|+l_2\left|\xi-\tilde{\xi}\right| \tag{3.2}$$

for any $\left(x,\varsigma,\xi\right),\left(x,\tilde{\varsigma},\tilde{\xi}\right)\in\Omega$, with some positive numbers l_1,l_2 such that $\dfrac{9\left(l_1+l_2\right)}{16\left|1-|\alpha|\right|}<1$. Then the boundary value problem (1.1), (1.2) has a unique solution.

Proof. The theorem will be proved (by the contraction mapping principle) if we show that the operator A defined by equality (3.1) is contractive. Consider the difference

$$\left|(Ay)(x)-(Az)(x)\right|\le\int_{-1}^{1}\left|G(x,t)\right|\left|F\left(t,y(t),y(-t)\right)-F\left(t,z(t),z(-t)\right)\right|dt\,.$$

Applying the Lipschitz condition (3.2), we obtain

$$\left|(Ay)(x)-(Az)(x)\right|\le\left(l_1+l_2\right)\max_{-1\le x\le1}\left|y(t)-z(t)\right|\int_{-1}^{1}\left|G(x,t)\right|dt\,. \tag{3.3}$$

To estimate the integral on the right side of (3.3), we use Corollary 2.1. Then

$$\left|(Ay)(x)-(Az)(x)\right|\le\frac{9\left(l_1+l_2\right)}{16\left|1-|\alpha|\right|}\max_{-1\le x\le1}\left|y(t)-z(t)\right|. \tag{3.4}$$

From inequality (3.4) we obtain the estimate

$$\left\|(Ay)(x)-(Az)(x)\right\|_X\le\frac{9\left(l_1+l_2\right)}{16\left|1-|\alpha|\right|}\left\|y(t)-z(t)\right\|_X\,.$$

As $\dfrac{9\left(l_1+l_2\right)}{16\left|1-|\alpha|\right|}<1$, then the operator A is a contraction operator in X, which has a single fixed point. This fixed point is a solution to the boundary value problem (1.1), (1.2). The theorem is proved.

With a weaker requirement on the right side of Eq (1.1), we obtain the following result.

Theorem 3.2. Let $\alpha\ne\pm1$. If the function $F\left(x,\varsigma,\xi\right)$ is continuous and bounded in the domain Ω, then the boundary value problem (1.1), (1.2) has a solution.

Proof. Consider the operator $A: X\to X$ acting according to formula (3.1). It's obvious that

$$\left|(Ay)(x)\right|\le\frac{1}{2}\left|y_2+y_1\right|+\frac{1}{2}\left|y_2-y_1\right|+\frac{9c_0}{16\left|1-|\alpha|\right|}\,, \tag{3.5}$$

where $\left|F\left(x,y(x),y(-x)\right)\right|\le c_0$. This means that the operator A reflects a bounded, closed, convex set

$$M=\left\{y(x)\in X:\left|y(x)\right|\le\frac{1}{2}\left|y_2+y_1\right|+\frac{1}{2}\left|y_2-y_1\right|+\frac{9c_0}{16\left|1-|\alpha|\right|}\right\}$$

into itself. Further, for the derivative of the function $(Ay)(x)$ the estimate

$$\left|\frac{d}{dx}(Ay)(x)\right| \le \frac{1}{2}|y_2 - y_1| + \frac{9c_0}{16\left|1-|\alpha|\right|} \tag{3.6}$$

is valid.

According to the mean value theorem (see, for example, [38, P. 108]), the uniform boundedness (3.6) of the set of functions $\left\{\frac{d}{dx}(Ay)(x)\right\}$ on $[-1,1]$ implies the equicontinuity of the set of functions $\{(Ay)(x)\}$ satisfying inequality (3.5). By the Arzelà-Ascoli theorem, such an operator is completely continuous in X. According to the Schauder theorem, the operator A has a fixed point, which is a solution to the boundary value problem (1.1), (1.2). The theorem is proved.

Consider now the boundary value problem (1.3), (1.2). Let $F \in C(\Omega_4, R)$, $\Omega_4 = [-1,1] \times R^4$, $\alpha \ne \pm 1$. We can consider Eq (1.3) as an inhomogeneous form of Eq (2.3). According to Theorem 3, the boundary value problem (1.3), (1.2) is equivalent to the integral equation

$$y(x) = \frac{1}{2}(y_2 + y_1) + \frac{1}{2}(y_2 - y_1)x + \int_{-1}^{1} G(x,t) F\left(t, y(t), y(-t), y'(t), y'(-t)\right) dt,$$

where $G(x,t)$ is the Green's function of the form (2.4). Let $Y = C^1\left([-1,1], R\right)$ be the Banach space of functions with the norm

$$\|y\|_Y = \max_{-1\le x\le 1}|y(x)| + \max_{-1\le x\le 1}|y'(x)|.$$

Let us use the equality

$$(By)(x) = \frac{1}{2}(y_2 + y_1) + \frac{1}{2}(y_2 - y_1)x + \int_{-1}^{1} G(x,t) F\left(t, y(t), y(-t), y'(t), y'(-t)\right) dt \tag{3.7}$$

to define the operator $B: Y \to Y$. As above, the problem of the existence of a solution to the boundary value problem (1.3), (1.2) is reduced to the problem of the existence of a fixed point of the operator B defined by formula (3.7). The following theorem is valid.

Theorem 3.3. Let $\alpha \ne \pm 1$. Let the function $F(x,\varsigma,\xi,s,t)$ be continuous and satisfy the Lipschitz condition

$$\left|F(x,\varsigma,\xi,s,t) - F\left(x,\tilde{\varsigma},\tilde{\xi},\tilde{s},\tilde{t}\right)\right| \le l_1|\varsigma - \tilde{\varsigma}| + l_2\left|\xi - \tilde{\xi}\right| + l_3|s - \tilde{s}| + l_4|t - \tilde{t}| \tag{3.8}$$

for any $(x,\varsigma,\xi,s,t), \left(x,\tilde{\varsigma},\tilde{\xi},\tilde{s},\tilde{t}\right) \in \Omega_4$, with some positive numbers l_1, l_2, l_3, l_4, such that $\frac{25\max\{l_1 + l_2; l_3 + l_4\}}{16\left|1-|\alpha|\right|} < 1$. Then, the boundary value problem (1.3), (1.2) has a unique solution.

Proof. To prove the theorem, it suffices to show that the operator B defined by equality (3.7) is contractive.

By differentiating (3.7), we obtain

$$\frac{d}{dx}\left[(By)(x)\right] = \frac{1}{2}(y_2 - y_1) + \int_{-1}^{1} G'_x(x,t) F\left(t, y(t), y(-t), y'(t), y'(-t)\right) dt.$$

Now let us estimate the difference for $y(x), z(x) \in Y$:

$$\left|(By)(x)-(Bz)(x)\right|\le\int_{-1}^{1}\left|G(x,t)\right|\left|F\left(t,y(t),y(-t),y'(t),y'(-t)\right)-F\left(t,z(t),z(-t),z'(t),z'(-t)\right)\right|dt.$$

From the last inequality and the Lipschitz condition (3.8), it follows that

$$\left|(By)(x)-(Bz)(x)\right|\le\left[(l_1+l_2)\max_{-1\le t\le 1}\left|y(t)-z(t)\right|+(l_3+l_4)\max_{-1\le t\le 1}\left|y'(t)-z'(t)\right|\right]\int_{-1}^{1}\left|G(x,t)\right|dt\,. \tag{3.9}$$

In the same way, we have

$$\left|\frac{d}{dx}\left[(By)(x)-(Bz)(x)\right]\right|\le\left[(l_1+l_2)\max_{-1\le t\le 1}\left|y(t)-z(t)\right|+(l_3+l_4)\max_{-1\le t\le 1}\left|y'(t)-z'(t)\right|\right]\int_{-1}^{1}\left|G'_x(x,t)\right|dt\,.$$

From this expression, Corollary 2.1 and (3.9) we obtain the inequality

$$\left\|(By)(x)-(Bz)(x)\right\|_Y\le\frac{25\max\left\{l_1+l_2;l_3+l_4\right\}}{16\left|1-|\alpha|\right|}\left\|y(t)-z(t)\right\|_Y.$$

As $\frac{25\max\left\{l_1+l_2;l_3+l_4\right\}}{16\left|1-|\alpha|\right|}<1$, then the operator B is a contraction operator in Y, which has a single fixed point. This fixed point is a solution to the boundary value problem (1.3), (1.2). The theorem is proved.

Weakening of the condition of Theorem 3.3, leads to the following result.

Theorem 3.4. Let $\alpha\ne\pm1$. If the function $F(x,\varsigma,\xi,s,t)$ is continuous and bounded in the domain Ω_4, then the boundary value problem (1.3), (1.2) has a solution.

The proof of Theorem 3.4 is similar to the proof of Theorem 3.2.

Now we consider problems (1.1), (1.3) with boundary conditions

$$y(-1)=y_1,\ y'(1)=y_2. \tag{3.10}$$

Lemma 3.1. The Green's function of the boundary value problem

$$y''(x)+\alpha y''(-x)=0,\ y(-1)=y'(1)=0,\ \alpha\ne\pm1,$$

is written as

$$G(x,t)=\begin{cases}\dfrac{t-x}{2(1-\alpha)}+\dfrac{2+t+x}{2(1+\alpha)}, t\le -x,\\[2mm] \dfrac{t}{1-\alpha}+\dfrac{1}{1+\alpha}, -x\le t\le x,\\[2mm] \dfrac{t+x}{2(1-\alpha)}+\dfrac{2-t+x}{2(1+\alpha)}, t\ge x.\end{cases} \tag{3.11}$$

Using Theorem 2.3, Lemma 3.1 can be proved by direct calculations of the derivatives of the function

$$y(x)=\int_{-1}^{1}G(x,t)\,f(t)\,dt,$$

where $G(x,t)$ is the Green's function (3.11).

Lemma 3.2. If $-1<\alpha<1$, then the Green's function (3.11) has the following estimates:

$$\int_{-1}^{1}\left|G(x,t)\right|dt\le\frac{5}{4(1-|\alpha|)},\quad \int_{-1}^{1}\left|G_x(x,t)\right|dt\le\frac{2}{1-|\alpha|}.$$

Proof. Let us prove the first assertion of the Lemma. Let $-1<\alpha<1$. Then

$$\int_{-1}^{1}\left|G(x,t)\right|dt=\int_{-1}^{-x}\left|\frac{t-x}{2(1-\alpha)}+\frac{2+t+x}{2(1+\alpha)}\right|dt+\int_{-x}^{x}\left|\frac{t}{1-\alpha}+\frac{1}{1+\alpha}\right|dt+\int_{x}^{1}\left|\frac{t+x}{2(1-\alpha)}+\frac{2-t+x}{2(1+\alpha)}\right|dt\le$$
$$\le-\int_{-1}^{-x}\frac{t-x}{2(1-\alpha)}dt+\int_{-1}^{-x}\frac{2+t+x}{2(1+\alpha)}dt-\int_{-x}^{0}\frac{t}{1-\alpha}dt+\int_{0}^{x}\frac{t}{1-\alpha}dt+\frac{2x}{1+\alpha}+ \quad .$$
$$+\int_{x}^{1}\frac{t+x}{2(1-\alpha)}dt+\int_{x}^{1}\frac{2-t+x}{2(1+\alpha)}dt==\frac{1+2x-x^2}{2(1-\alpha)}+\frac{1-x^2}{2(1+\alpha)}\le\frac{5}{4(1-|\alpha|)}.$$

From the last inequality, we obtain the first assertion of the Lemma. The second assertion of the Lemma is proved by direct calculation. The Lemma is proved.
In a similar way, one can prove the following
Lemma 3.3. If $-1<\alpha<1$, then the Green's function (3.11) has the following estimates:

$$\int_{-1}^{1}\left|G(x,t)\right|dt\le\frac{5}{4(|\alpha|-1)},\quad \int_{-1}^{1}\left|G_x(x,t)\right|dt\le\frac{2}{|\alpha|-1}.$$

Two lemmas imply the following
Corollary 3.1. If $\alpha\ne\pm1$, then the following estimates hold for the Green's function (3.11):

$$\int_{-1}^{1}\left|G(x,t)\right|dt\le\frac{5}{4\left||\alpha|-1\right|},\quad \int_{-1}^{1}\left|G_x(x,t)\right|dt\le\frac{2}{\left||\alpha|-1\right|}.$$

Consider the boundary value problem (1.1), (3.10). It is easy to check that the boundary value problem (1.1), (3.10) is equivalent to the integral equation

$$y(x)=y_1+y_2(x+1)+\int_{-1}^{1}G(x,t)F(t,y(t),y(-t))dt,$$

where $G(x,t)$ is the Green's function of the form (3.11).
Let us use the equality

$$(Ay)(x)=y_1+y_2(x+1)+\int_{-1}^{1}G(x,t)F(t,y(t),y(-t))dt$$

to define the operator $A: X\to X$. Thus, the problem of the existence of a solution to the boundary value problem (1.1), (3.10) is reduced to the problem of the existence of a fixed point of the operator A.
Denote $\Omega=[-1,1]\times R^2$.
Theorem 3.5. Let $\alpha\ne\pm1$. Let the function $F(x,\varsigma,\xi)$ be continuous and satisfy the Lipschitz condition (3.2) for any $(x,\varsigma,\xi),(x,\tilde{\varsigma},\tilde{\xi})\in\Omega$, with some positive numbers l_1,l_2 such that $\frac{5(l_1+l_2)}{4\left|1-|\alpha|\right|}<1$. Then, the boundary value problem (1.1), (3.10) has a unique solution.
The proof of the theorem is similar to the proof of Theorem 3.1.
Theorem 3.6. Let $\alpha\ne\pm1$. If the function $F(x,\varsigma,\xi)$ is continuous and bounded in the domain Ω, then the boundary value problem (1.1), (2.3) has a solution.
The stated result is proved in the same way as Theorem 3.2.

We turn to the study of the boundary value problem (1.3), (3.10). Let $F\in C(\Omega_4,R)$, $\Omega_4=[-1,1]\times R^4$, $\alpha\neq\pm1$. The boundary value problem (1.3), (3.10) is equivalent to

$$y(x)=y_1+y_2(x+1)+\int_{-1}^{1}G(x,t)F\left(t,y(t),y(-t),y'(t),y'(-t)\right)dt,$$

where $G(x,t)$ is the Green's function of the form (3.11).

Let us use the equality

$$(By)(x)=y_1+y_2(x+1)+\int_{-1}^{1}G(x,t)F\left(t,y(t),y(-t),y'(t),y'(-t)\right)dt$$

to define the operator $B:Y\to Y$. The following theorems are valid.

Theorem 3.7. Let $\alpha\neq\pm1$. Let the function $F(x,\varsigma,\xi,s,t)$ be continuous and satisfy the Lipschitz condition (3.8) for any $(x,\varsigma,\xi,s,t),\left(x,\tilde{\varsigma},\tilde{\xi},\tilde{s},\tilde{t}\right)\in\Omega_4$, with some positive numbers l_1,l_2,l_3,l_4, such that $\dfrac{5\max\{l_1+l_2;l_3+l_4\}}{4|1-|\alpha||}<1$. Then, the boundary value problem (1.3), (3.10) has a unique solution.

Theorem 3.8. Let $\alpha\neq\pm1$. If the function $F(x,\varsigma,\xi,s,t)$ is continuous and bounded in the domain Ω_4, then the boundary value problem (1.3), (3.10) has a solution.

We will not dwell on the proofs of Theorems 3.7 and 3.8. The proofs are analogous to the proofs for Theorems 3.3 and 3.4.

4. Conclusions

In this paper, the existence of solutions to boundary value problems for the second-order differential equations with involution was studied using the contraction mapping principle and the Schauder fixed point theorem. A specific feature of this work is that the considered equations contain involution in the highest derivative. The definition of the Green's function is generalized to the case of the boundary value problems for the second-order linear differential equation with involution. Some properties of the Green's function for the linear part of the boundary value problem are established. The explicit form of the Green's function enabled us to use well-known approaches to solve the stated problems. In the future, it is planned to continue studies of the boundary value problems for nonlinear differential equations with involution.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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