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Research article

Boundary value problems of hybrid fractional integro-differential systems involving the conformable fractional derivative

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Abstract: In this paper, we study a boundary value problem for the hybrid fractional integro-differential system involving the conformable fractional derivative. We first discuss the existence of solutions using the Krasnoselskii fixed point theorem. The second result will be the existence and uniqueness of solution and we obtain it using the Banach fixed point theorem. Finally, we end our work with an example to illustrate our results.

Keywords: integro-differential system; conformable fractional derivative; fixed point theorem; existence and uniqueness
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1. Introduction

In recent years, fractional calculus has attracted a large number of mathematicians and modelers. In view of the growing interests in the subject, several definitions of fractional order differential and integral operators have been proposed according to the physical aspects of the problem under investigation. Some fractional order initial value problems and boundary value problems, involving Riemann-Liouville, Liouville, Caputo and Hadamard type fractional differential equations, has attracted the attention of many researchers, for instance, see [1–9].

In 2014, Abdeljawad [10] and Khalil [11] introduced and elaborated the concept of conformable fractional differential and integral operators, which were used in many interesting problems related to the solvability of nonlinear equations and systems. This is the field where advances are continuously taking place.

In order to present our problem, in this paper, we need first to mention some important research results published in the field of fractional differential systems. In [12], Tahereh Bashiri et al. considered a non cooperative system with the fractional order $p \in (0, 1)$ and investigated the existence of solutions.

In [13], Varsha Daftardar-Gejji proposed a fractional differential system and analyzed the existence of positive solutions of the system in detail. In [14], Liu considered a cooperative system with the fractional order $\alpha, \beta \in (0, 1)$. In [15], Ahmed et al. obtained the existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary condition.

In [16], Ahmed et al. introduced and analyzed an impulsive hybrid system of conformable fractional differential equations with boundary conditions, described by

$$\begin{gathered} T_{\alpha}^{t_k} x(t) = f(t, x(t)), \\ \Delta x(t_k) = S_k(x(t_k)), \quad \Delta x'(t_k) = S_k^*(x(t_k)), \\ x(0) = 0, \quad x(T) = 0, \end{gathered}$$

where T_α denotes the conformable fractional derivative of order $\alpha \in (1, 2]$.

In [12], the fractional order of the two equations is the same and the perturbation term is also the same. In [14], the perturbation is a simple type. In [16], the authors studied discrete problems. Compared with the study in this article, we will consider a continuous problem with an integro-differential hybrid perturbation and the different order of fractional derivatives.

Due to the importance and academic value of the topic of fractional differential equations and systems, the importance of this subject in the modeling of so many phenomena, and the studies published in this field, we choose to study such models. As we know, there are few studies on the subject of conformable fractional systems. Motivated by the studies cited above, we will study a new type of fractional differential system, called the conformable fractional differential system, and a new type of hybrid perturbation (integro-differential term).

In this paper, motivated by all the aforementioned work on fractional differential equations and conformable fractional differential systems, we introduce and analyze a hybrid system of conformable fractional integro-differential system with boundary conditions, which is given by

$$\begin{aligned} & T^{\alpha}\left(u(t) - \sum_{i=1}^{m} I^{p_i} f_i(t, u(t), v(t))\right) = h(t, u(t), v(t)), \\ & T^{\beta}\left(v(t) - \sum_{i=1}^{m} I^{q_i} g_i(t, u(t), v(t))\right) = k(t, u(t), v(t)), \\ & u(0) = u(T) = 0, v(0) = v(T) = 0. \end{aligned} \tag{1.1}$$

Here, T^α denotes the conformable fractional derivative of order $\alpha \in (1, 2]$, T^β is the conformable fractional derivative of order $\beta \in (1, 2]$, I^{p_i} is the conformable fractional integral of order p_i, I^{q_i} is the conformable fractional integral of order q_i, and $h, k \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $f_i, g_i \in C([0, T] \times \mathbb{R}, \mathbb{R})$.

The organization of this work is as follows. Section 2 contains some preliminary facts. In Section 3, we present the solution for the boundary value problem of hybrid fractional integro-differential system (1.1) involving the conformable fractional derivative, and then prove our main existence results. In Section 4, we prove the existence and uniqueness of solutions to the system. Finally, we illustrate the obtained results by an example.

2. Preliminaries

Now, we give some basic concepts of conformable fractional calculus (see [10] and [11]).

Definition 2.1. *[10,11] For $\alpha \in (0,1]$. The conformable fractional derivative of a function $f : [a,\infty) \to \mathbb{R}$ of order α is defined by*

$$T_a^\alpha f(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon(t+a)^{1-\alpha}) - f(t)}{\epsilon}, \tag{2.1}$$

for all $t > a$. If $T_a^\alpha f(t)$ exists on (a,b), then $T_a^\alpha f(a) = \lim_{t\to a} T_a^\alpha f(t)$.

Definition 2.2. *[10,11] Let $\alpha \in (n, n+1]$. The conformable fractional derivative of a function $f : [a,\infty) \to \mathbb{R}$ of order α, where $f^{(n)}(t)$ exists, is defined by*

$$T_a^\alpha f(t) = T_a^{\alpha-n} f^{(n)}(t). \tag{2.2}$$

Definition 2.3. *[10,11] Let $\alpha \in (n, n+1]$. The conformable fractional integral of a function $f : [a,\infty) \to \mathbb{R}$ of order α is defined by*

$$I_a^\alpha f(t) = \frac{1}{n!}\int_a^t (t-s)^n (s-a)^{\alpha-n-1} f(s)ds. \tag{2.3}$$

Lemma 2.1. *[10,11] Let $\alpha \in (n, n+1]$. If $f(t)$ is a continuous function on $[a,\infty)$, then $T_a^\alpha I_a^\alpha f(t) = f(t)$ for all $t > a$.*

Lemma 2.2. *[10] Let $\alpha \in (n, n+1]$. Then $T_a^\alpha (t-a)^k = 0$ for all $t \in [a,b]$ and $k = 1, 2, ..., n$.*

Lemma 2.3. *[10] Let $\alpha \in (n, n+1]$. If $T_a^\alpha f(t)$ is a continuous function on $[a,\infty)$, then*

$$I_a^\alpha T_a^\alpha f(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}, \tag{2.4}$$

for all $t > a$.

Lemma 2.4 (Krasnoselskii fixed point theorem)**.** *[17] Let E be a non-empty, bounded, closed and convex subset of a Banach space X, and $A, B : E \mapsto E$ satisfy the following assumptions:*

(1) $Ax + By \in E$, for every $x, y \in X$,
(2) A is a contraction,
(3) B is compact and continuous.

Then, there exists $z \in X$ such that $Az + Bz = z$.

Lemma 2.5 (Banach fixed point theorem)**.** *[18] Let X be a non-empty complete metric space, and $T : X \mapsto X$ be a contraction mapping. Then, there exists a unique point $x \in X$ such that $Tx = x$.*

Now we define a solution to the system (1.1).

Definition 2.4. *The pair of functions $u, v \in C(J,\mathbb{R})$ with their conformable fractional derivatives of order α and β existing on J is a solution of (1.1) if it satisfies (1.1).*

3. Existence results

In this section, we study the existence of solutions to the system (1.1). By Lemma 3.1, we transform the system (1.1) into a fixed point problem.

Lemma 3.1. *Let $\varphi, \phi \in C(0,T)$ and $u, v \in C(J, \mathbb{R})$ be continuous real valued functions. Then the solution of the system*

$$
\begin{aligned}
&T^{\alpha}\left(u(t) - \sum_{i=1}^{m} I^{p_i} f_i(t, u(t), v(t))\right) = \varphi(t), \\
&T^{\beta}\left(v(t) - \sum_{i=1}^{m} I^{q_i} g_i(t, u(t), v(t))\right) = \phi(t), \\
&u(0) = u(T) = 0, v(0) = v(T) = 0,
\end{aligned} \tag{3.1}
$$

is given by

$$
\begin{aligned}
u(t) &= \int_0^t (t-s)s^{\alpha-2}\varphi(s)ds + \sum_{i=1}^{m}\int_0^t (t-s)s^{p_i-2} f_i(s, u(s), v(s))ds \\
&\quad -\frac{1}{T}\left[\int_0^T (T-s)s^{\alpha-2}\varphi(s)ds + \sum_{i=1}^{m}\int_0^T (T-s)s^{p_i-2} f_i(s, u(s), v(s))ds\right]t,
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
v(t) &= \int_0^t (t-s)s^{\beta-2}\phi(s)ds + \sum_{i=1}^{m}\int_0^t (t-s)s^{q_i-2} g_i(s, u(s), v(s))ds \\
&\quad -\frac{1}{T}\left[\int_0^T (T-s)s^{\beta-2}\phi(s)ds + \sum_{i=1}^{m}\int_0^T (T-s)s^{q_i-2} g_i(s, u(s), v(s))ds\right]t.
\end{aligned} \tag{3.3}
$$

Proof. Applying the conformable fractional integrals I^{α} and I^{β} on the both sides of equations of system (3.1) respectively and using Lemma 2.3, we get that the general solution of the system (3.1) for $t \in J$ is

$$
u(t) = I^{\alpha}\varphi(t) + C_1 + C_2 t + \sum_{i=1}^{m} I^{p_i} f_i(t, u(t), v(t)), \tag{3.4}
$$

$$
v(t) = I^{\beta}\phi(t) + C_3 + C_4 t + \sum_{i=1}^{m} I^{q_i} g_i(t, u(t), v(t)). \tag{3.5}
$$

where C_1, C_2, C_3 and C_4 are unknown constants. Using the conditions $u(0) = 0$ and $v(0) = 0$ gives $C_1 = 0$ and $C_3 = 0$.

Now the Eqs (3.4) and (3.5) have the form

$$
u(t) = I^{\alpha}\varphi(t) + C_2 t + \sum_{i=1}^{m} I^{p_i} f_i(t, u(t), v(t)),
$$

$$
v(t) = I^{\beta}\phi(t) + C_4 t + \sum_{i=1}^{m} I^{q_i} g_i(t, u(t), v(t)).
$$

Using the conditions $u(T) = 0$ and $v(T) = 0$ we obtain

$$\begin{aligned} C_2 &= -\frac{1}{T}\left(\int_0^T (T-s)s^{\alpha-2}\varphi(s)ds + \sum_{i=1}^m \int_0^T (T-s)s^{p_i-2}g_i(s,u(s),v(s))ds\right), \\ C_4 &= -\frac{1}{T}\left(\int_0^T (T-s)s^{\beta-2}\phi(s)ds + \sum_{i=1}^m \int_0^T (T-s)s^{q_i-2}g_i(s,u(s),v(s))ds\right). \end{aligned}$$

Using the values of C_1, C_2, C_3 and C_4 in (3.4) and (3.5), we get the solution. The converse follows from direct computation. This completes the proof. □

Our first result concerns the study of existence of solution for problem (1.1) using the Krasnoselskii fixed-point theorem. For this, we will need some assumptions about the functions f_i, g_i, h and k.

Denote by $X = (C([0,T]\times\mathbb{R})\times C([0,T]\times\mathbb{R}),\mathbb{R})$, the Banach space endowed with the norm

$$\|(u,v)\| = \|u\| + \|v\| = \sup_{t\in[0,T]} |u(t)| + \sup_{t\in[0,T]} |v(t)|,$$

for $(u,v)\in X$.

($\mathbf{H_1}$) The functions $f_i, g_i : J\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ and $h,k : J\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous and there exist nonnegative functions η_i, σ_i, $i = 1,..,m,\mu$, and λ such that

$$\begin{aligned} |f_i(t,u(t),v(t))| &\leq \eta_i(t), \\ |g_i(t,u(t),v(t))| &\leq \sigma_i(t), \\ |h(t,u(t),v(t))| &\leq \mu(t), \\ |k(t,u(t),v(t))| &\leq \lambda(t). \end{aligned}$$

($\mathbf{H_2}$) There exist positive constants C_k, $k = 1,..,4$, L_{ij}, $j = 1,2$, M_i, N_i, $i = 1,2,..,m$ such that

$$|h(t,u_1,v_1) - h(t,u_2,v_2)| \leqslant C_1\|u_1-u_2\| + C_2\|v_1-v_2\|,$$

$$|k(t,u_1,v_1) - k(t,u_2,v_2)| \leqslant C_3\|u_1-u_2\| + C_4\|v_1-v_2\|,$$

$$|f_i(t,u_1,v_1) - f_i(t,u_2,v_2)| \leqslant L_{i1}\|u_1-u_2\| + L_{i2}\|v_1-v_2\|,$$

$$|g_i(t,u_1,v_1) - g_i(t,u_2,v_2)| \leqslant N_i\|u_1-u_2\| + M_i\|v_1-v_2\|.$$

Theorem 3.1. *Assume that the assumptions* (H_1) *and* (H_2) *hold. If*

$$\left(\frac{T^\alpha C}{\alpha(\alpha-1)} + \sum_{i=1}^m \frac{T^{p_i}L_i}{p_i(p_i-1)}\right) < 1$$

and

$$\left(\frac{T^\beta N}{\beta(\beta-1)} + \sum_{i=1}^m \frac{T^{q_i}M_i}{q_i(q_i-1)}\right) < 1,$$

then the fractional integro-differential system (1.1) *has at least one solution in X on J.*

Proof. We define an operator $\Pi : X \longmapsto X$ associated with the system (1.1) by

$$\Pi(u,v)(t) = (\Pi_1(u,v)(t), \Pi_2(u,v)(t)),$$

where

$$\begin{aligned}
\Pi_1(u,v)(t) &= \int_0^t (t-s)s^{\alpha-2}h(s,u(s),v(s))ds + \sum_{i=1}^m \int_0^t (t-s)s^{p_i-2}f_i(s,u(s),v(s))ds \\
&\quad -\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}h(s,u(s),v(s))ds + \sum_{i=1}^m \int_0^T (T-s)s^{p_i-2}f_i(s,u(s),v(s))ds\right], \\
\Pi_2(u,v)(t) &= \int_0^t (t-s)s^{\beta-2}k(s,u(s),v(s))ds + \sum_{i=1}^m \int_0^t (t-s)s^{q_i-2}g_i(s,u(s),v(s))ds, \\
&\quad -\frac{t}{T}\left[\int_0^T (T-s)s^{\beta-2}k(s,u(s),v(s))ds + \sum_{i=1}^m \int_0^T (T-s)s^{q_i-2}g_i(s,u(s),v(s))ds\right].
\end{aligned}$$

First, we will transform problem (1.1) into a fixed point problem $\Pi x = x$, where Π is the operator defined above. So, before starting the proof, we decompose Π_i into a sum of two operators A_i and B_i, $i = 1,2$ where

$$\begin{aligned}
A_1(u,v)(t) &= \int_0^t (t-s)s^{\alpha-2}h(s,u(s),v(s))ds + \sum_{i=1}^m \int_0^t (t-s)s^{p_i-2}f_i(s,u(s),v(s))ds, \\
A_2(u,v)(t) &= -\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}h(s,u(s),v(s)ds + \sum_{i=1}^m \int_0^T (T-s)s^{p_i-2}f_i(s,u(s),v(s))ds\right],
\end{aligned}$$

and

$$\begin{aligned}
B_1(u,v)(t) &= \int_0^t (t-s)s^{\beta-2}k(s,u(s),v(s))ds + \sum_{i=1}^m \int_0^t (t-s)s^{q_i-2}g_i(s,u(s),v(s))ds, \\
B_2(u,v)(t) &= -\frac{t}{T}\left[\int_0^T (T-s)s^{\beta-2}k(s,u(s),v(s))ds + \sum_{i=1}^m \int_0^T (T-s)s^{q_i-2}g_i(s,u(s),v(s))ds\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
\Pi_1(u,v) &= A_1(u,v) + A_2(u,v), \\
\Pi_2(u,v) &= B_1(u,v) + B_2(u,v).
\end{aligned}$$

Now, we show that the operators A_1, A_2, B_1 and B_2 satisfy all conditions of Lemma 2.4 in a series of steps.

Step 1. We define the set $\Omega = \{(u,v) \in X : \|(u,v)\|_X \leq r\}$, where r is a positive real constant satisfying the condition

$$r \geqslant \max\left\{\frac{2\,\|\mu\|\,T^{\alpha}}{\alpha(\alpha-1)} + \sum_{i=1}^m \frac{2\,\|\eta_i\|\,T^{p_i}}{p_i(p_i-1)}; \frac{2\,\|\lambda\|\,T^{\beta}}{\beta(\beta-1)} + \sum_{i=1}^m \frac{2\,\|\sigma_i\|\,T^{q_i}}{q_i(q_i-1)}\right\}. \tag{3.6}$$

First, we show that $A_1(u,v)+A_2(u,v)\in\Omega$ and $B_1(u,v)+B_2(u,v)\in\Omega$. So for $(u,v)\in\Omega$ and $t\in J$, we have

$$\begin{aligned}
&|A_1(u,v)(t)+A_2(u,v)(t)|\\
\le\ & \int_0^t (t-s)s^{\alpha-2}\,|h(s,u(s),v(s))|\,ds+\sum_{i=1}^m\int_0^t (t-s)s^{p_i-2}\,|f_i(s,u(s),v(s))|\,ds\\
&+\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}\,|h(s,u(s),v(s))|\,ds+\sum_{i=1}^m\int_0^T (T-s)s^{p_i-2}\,|f_i(s,u(s),v(s))|\,ds\right]\\
\le\ & \int_0^t (t-s)s^{\alpha-2}\mu(s)ds+\sum_{i=1}^m\int_0^t (t-s)s^{p_i-2}\eta_i(s)ds\\
&+\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}\mu(s)ds+\sum_{i=1}^m\int_0^T (T-s)s^{p_i-2}\eta_i(s)ds\right]\\
\le\ & \|\mu\|\left(\int_0^t (t-s)s^{\alpha-2}ds+\frac{t}{T}\int_0^T (T-s)s^{\alpha-2}ds\right)\\
&+\|\eta_i\|\left(\sum_{i=1}^m\int_0^t (t-s)s^{p_i-2}ds+\frac{t}{T}\sum_{i=1}^m\int_0^T (T-s)s^{p_i-2}\right)\\
\le\ & \|\mu\|\left(\frac{t^\alpha}{\alpha(\alpha-1)}+\frac{tT^\alpha}{T\alpha(\alpha-1)}\right)+\sum_{i=1}^m\|\eta_i\|\left(\frac{t^{p_i}}{p_i(p_i-1)}+\frac{tT^{p_i}}{Tp_i(p_i-1)}\right)\\
\le\ & \|\mu\|\frac{2T^\alpha}{\alpha(\alpha-1)}+\sum_{i=1}^m\|\eta_i\|\frac{2T^{p_i}}{p_i(p_i-1)}\\
\le\ & r.
\end{aligned}$$

That implies that $\|A_1(u,v)(t)+A_2(u,v)(t)\|_X\le r$, which means that $A_1(u,v)(t)+A_2(u,v)\in\Omega$. Analogously, we obtain

$$\begin{aligned}
&|B_1(u,v)(t)+B_2(u,v)(t)|\\
\le\ & \|\lambda\|\frac{2T^\beta}{\beta(\beta-1)}+\sum_{i=1}^m\|\sigma_i\|\frac{2T^{q_i}}{q_i(q_i-1)}\\
\le\ & r.
\end{aligned}$$

That means that $B_1(u,v)(t)+B_2(u,v)\in\Omega$.

Step 2. We want to show that A_2 and B_2 are contractions on Ω, for $(u_1,v_1),(u_2,v_2)\in\Omega$ and $t\in J$. Using the assumption (H_1), we have

$$\begin{aligned}
&|A_2(u_1,v_1)(t)-A_2(u_2,v_2)(t)|\\
=\ & \Bigg|-\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}h(s,u_1(s),v_1(s))ds+\sum_{i=1}^m\int_0^T (T-s)s^{p_i-2}f_i(s,u_1(s),v_1(s))ds\right]\\
&+\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}h(s,u_2(s),v_2(s))ds+\sum_{i=1}^m\int_0^T (T-s)s^{p_i-2}f_i(s,u_2(s),v_2(s))ds\right]\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{t}{T}\int_0^T (T-s)s^{\alpha-2}\left|h(s,u_1(s),v_1(s))-h(s,u_2(s),v_2(s))\right|ds\\
&\quad+\frac{t}{T}\sum_{i=1}^{m}\int_0^T (T-s)s^{p_i-2}\left|f_i(s,u_1(s),v_1(s))-f_i(s,u_2(s),v_2(s))\right|ds\\
&\leq \frac{t}{T}\Big[\int_0^T (T-s)s^{\alpha-2}\left[C_1(u_1(s)-u_2(s))+C_2(v_1(s)-v_2(s))\right]ds\\
&\quad+\sum_{i=1}^{m}\int_0^T (T-s)s^{p_i-2}\left[L_{1i}(u_1(s)-u_2(s))+L_{2i}(v_1(s)-v_2(s))\right]ds\Big]\\
&\leq \frac{t}{T}\left[\left(\frac{T^{\alpha}(C_1\|u_1-u_2\|+C_2\|v_1-v_2\|)}{\alpha(\alpha-1)}\right)+\sum_{i=1}^{m}\frac{T^{p_i}(L_{1i}\|u_1-u_2\|+L_{2i}\|v_1-v_2\|}{p_i(p_i-1)}\right]\\
&\leq \frac{T^{\alpha}C}{\alpha(\alpha-1)}\|u_1-u_2,v_1-v_2\|+\sum_{i=1}^{m}\frac{T^{p_i}L_i}{p_i(p_i-1)}\|u_1-u_2,v_1-v_2\|\\
&\leq \left(\frac{T^{\alpha}C}{\alpha(\alpha-1)}+\sum_{i=1}^{m}\frac{T^{p_i}L_i}{p_i(p_i-1)}\right)\|u_1-u_2,v_1-v_2\|.
\end{aligned}
$$

Analogously, we obtain

$$
|B_2(u_1,v_1)(t)-B_2(u_2,v_2)(t)|\leq\left(\frac{T^{\beta}N}{\beta(\beta-1)}+\sum_{i=1}^{m}\frac{T^{q_i}M_i}{q_i(q_i-1)}\right)\|u_1-u_2,v_1-v_2\|.
$$

Hence, the operators A_2 and B_2 are contractions on Ω.

Step 3. Now, we prove that A_1 and B_1 are completely continuous on Ω. We need to show that the sets $(A_1\Omega)$ and $(B_1\Omega)$ are uniformly bounded, the sets $\overline{(A_1\Omega)}$ and $\overline{(B_1\Omega)}$ are equicontinuous, and the operators $A_1:\Omega\longmapsto\Omega$ and $B_1:\Omega\longmapsto\Omega$ are continuous.

For $(u,v)\in\Omega$ and $t\in J$, we have

$$
\begin{aligned}
|A_1(u,v)(t)| &= \left|\int_0^t (t-s)s^{\alpha-2}h(s,u(s),v(s))ds+\sum_{i=1}^{m}\int_0^t (t-s)s^{p_i-2}f_i(s,u(s),v(s))ds\right|\\
&\leq \int_0^t (t-s)s^{\alpha-2}\left|h(s,u(s),v(s))\right|ds+\sum_{i=1}^{m}\int_0^t (t-s)s^{p_i-2}\left|f_i(s,u(s),v(s))\right|ds\\
&\leq \int_0^t (t-s)s^{\alpha-2}\mu(s)ds+\sum_{i=1}^{m}\int_0^t (t-s)s^{p_i-2}\eta_i(s)ds\\
&\leq \|\mu\|\frac{t^{\alpha}}{\alpha(\alpha-1)}+\sum_{i=1}^{m}\frac{\|\eta_i\|\,t^{p_i}}{p_i(p_i-1)}\\
&\leq \|\mu\|\frac{T^{\alpha}}{\alpha(\alpha-1)}+\sum_{i=1}^{m}\frac{\|\eta_i\|\,T^{p_i}}{p_i(p_i-1)}.
\end{aligned}
$$

Then the set $(A_1\Omega)$ is uniformly bounded. Analogously, we obtain

$$
|B_1(u,v)(t)|\leq\|\lambda\|\frac{T^{\beta}}{\beta(\beta-1)}+\sum_{i=1}^{m}\frac{\|\sigma_i\|\,T^{q_i}}{q_i(q_i-1)},
$$

so the set $(B_1\Omega)$ is uniformly bounded.

Now, we show that $\overline{(A_1\Omega)}$ and $\overline{(B_1\Omega)}$ are equicontinuous. Let $t_1, t_2 \in J$ with $t_1 < t_2$, we have for any $(u, v) \in \Omega$,

$$\begin{aligned}
& |A_1(u,v)(t_2) - A_1(u,v)(t_1)| \\
= & \Big| \int_0^{t_2} (t_2 - s)s^{\alpha-2} h(s,u(s),v(s))ds + \sum_{i=1}^{m} \int_0^{t_2} (t_2 - s)s^{p_i-2} f_i(s,u(s),v(s))ds \\
& - \int_0^{t_1} (t_1 - s)s^{\alpha-2} h(s,u(s),v(s))ds - \sum_{i=1}^{m} \int_0^{t_1} (t_1 - s)s^{p_i-2} f_i(s,u(s),v(s))ds \Big| \\
\leq & \int_0^{t_1} (t_2 - s)s^{\alpha-2} |h(s,u(s),v(s))| ds + \int_{t_1}^{t_2} (t_2 - s)s^{\alpha-2} |h(s,u(s),v(s))| ds \\
& - \int_0^{t_1} (t_1 - s)s^{\alpha-2} |h(s,u(s),v(s))| ds + \sum_{i=1}^{m} \int_0^{t_1} (t_2 - s)s^{p_i-2} |f_i(s,u(s),v(s))| ds \\
& + \sum_{i=1}^{m} \int_{t_1}^{t_2} (t_2 - s)s^{p_i-2} |f_i(s,u(s),v(s))| ds - \sum_{i=1}^{m} \int_0^{t_1} (t_1 - s)s^{p_i-2} |f_i(s,u(s),v(s))| ds \\
\leq & \int_{t_1}^{t_2} (t_2 - s)s^{\alpha-2} |h(s,u(s),v(s))| ds + \sum_{i=1}^{m} \int_{t_1}^{t_2} (t_2 - s)s^{p_i-2} |f_i(s,u(s),v(s))| ds \\
\leq & \int_{t_1}^{t_2} (t_2 - s)s^{\alpha-2} \mu(s)ds + \sum_{i=1}^{m} \int_{t_1}^{t_2} (t_2 - s)s^{p_i-2} \eta_i(s)ds \\
\leq & \|\mu\| \left(\frac{t_2^{\alpha} - t_1^{\alpha}}{\alpha(\alpha-1)} \right) + \sum_{i=1}^{m} \|\eta_i\| \left(\frac{t_2^{p_i} - t_1^{p_i}}{p_i(p_i-1)} \right).
\end{aligned}$$

Analogously,

$$|B_1(u,v)(t_2) - B_1(u,v)(t_1)| \leq \|\lambda\| \left(\frac{t_2^{\beta} - t_1^{\beta}}{\beta(\beta-1)} \right) + \sum_{i=1}^{m} \|\sigma_i\| \left(\frac{t_2^{q_i} - t_1^{q_i}}{q_i(q_i-1)} \right).$$

As $t_1 \longmapsto t_2$, the right hand side of the above inequalities tend to zero. Therefore, it follows that $\overline{(A_1\Omega)}$ and $\overline{(B_1\Omega)}$ are equicontinuous.

Finally, we show that the operators A_1 and B_1 are continuous in X. Let (u_n, v_n) be a sequence in Ω converging to a point $(u, v) \in \Omega$. Then, by Lebesgue dominated convergence theorem, for all $t \in J$, we have

$$\begin{aligned}
& \lim_{n\longmapsto\infty} A_1(u_n,v_n)(t) \\
= & \lim_{n\longmapsto\infty} \left(\int_0^{t} (t - s)s^{\alpha-2} h(s,u_n(s),v_n(s))ds + \sum_{i=1}^{m} \int_0^{t} (t - s)s^{p_i-2} f_i(s,u_n(s),v_n(s))ds \right) \\
= & \int_0^{t} (t - s)s^{\alpha-2} \lim_{n\longmapsto\infty} h(s,u_n(s),v_n(s))ds + \sum_{i=1}^{m} \int_0^{t} (t - s)s^{p_i-2} \lim_{n\longmapsto\infty} f_i(s,u_n(s),v_n(s))ds \\
= & \int_0^{t} (t - s)s^{\alpha-2} h(s,u(s),v(s))ds + \sum_{i=1}^{m} \int_0^{t} (t - s)s^{p_i-2} f_i(s,u(s),v(s))ds = A_1(u,v)(t).
\end{aligned}$$

A similar proof works for the operator B_1.

Consequently, A_1 and B_1 are continuous. Therefore, A_1 and B_1 are also relatively compact on Ω. Using the Arzilà-Ascoli theorem, we conclude that A_1 and B_1 are compact on Ω. Now, all conditions of the Krasnoselskiiś fixed point theorem are satisfied, so the operator Π has a fixed point in Ω. Finally, we deduce that the system (1.1) has at least one solution in X on J. □

4. Existence and uniqueness results

In this section, we study the existence and uniqueness of solution of the system (1.1). Our result is based on the Banach fixed point theorem.

Theorem 4.1. *Assume that the hypothesis* (H_1) *and* (H_2) *are true. If*

$$2\left[\frac{CT^{\alpha}}{\alpha(\alpha-1)}+\sum_{i=1}^{m}\frac{LT^{p_i}}{p_i(p_i-1)}\right]<1$$

and

$$2\left[\frac{CT^{\beta}}{\beta(\beta-1)}+\sum_{i=1}^{m}\frac{MT^{q_i}}{q_i(q_i-1)}\right]<1,$$

then, the fractional integro-differential system (1.1) has a unique solution in X on J.

Proof. We define an operator $\Pi : X \longmapsto X$ associated with the system (1.1) by

$$\Pi(u,v)(t)=\left(\Pi_1(u,v)(t),\Pi_2(u,v)(t)\right),$$

given in the proof of Theorem 3.1.

Now, we show that the operator Π has a fixed point in B_ρ, which represents a unique solution of system (1.1). So, the proof is given in two steps.

Step 1. First, we define the set B_ρ by

$$B_\rho=\left\{(u,v)\in X; \|(u,v)\|_X\leq\rho\right\},$$

where the positive real constant ρ is chosen so that

$$\rho>2\left(\frac{\|\mu\|\,T^{\alpha}}{\alpha(\alpha-1)}+\frac{\|\lambda\|\,T^{\beta}}{\beta(\beta-1)}\right)+2\sum_{i=1}^{m}\left(\frac{\|\eta_i\|\,T^{p_i}}{p_i(p_i-1)}+\frac{\|\sigma_i\|\,T^{q_i}}{q_i(q_i-1)}\right).$$

We will show that $\Pi_i B_\rho\subset B_\rho$, $i=1,2$. For each $t\in J$ and $(u,v)\in B_\rho$,

$$\begin{aligned}
&|\Pi_1(u,v)(t)|\\
\leq\;&\int_0^t (t-s)s^{\alpha-2}\,|h(s,u(s),v(s))|\,ds+\sum_{i=1}^{m}\int_0^t (t-s)s^{p_i-2}\,|f_i(s,u(s),v(s))|\,ds\\
&+\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}\,|h(s,u(s),v(s))|\,ds+\sum_{i=1}^{m}\int_0^T (T-s)s^{p_i-2}\,|f_i(s,u(s),v(s))|\,ds\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t (t-s)s^{\alpha-2}\mu(s)ds + \sum_{i=1}^m \int_0^t (t-s)s^{p_i-2}\eta_i(s)ds \\
&\quad + \frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}\mu(s)ds + \sum_{i=1}^m \int_0^T (T-s)s^{p_i-2}\eta_i(s)ds\right] \\
&\leq \|\mu\| \int_0^t (t-s)s^{\alpha-2}ds + \sum_{i=1}^m \|\eta_i\| \int_0^t (t-s)s^{p_i-2}ds \\
&\quad + \frac{t}{T}\left[\|\mu\| \int_0^T (T-s)s^{\alpha-2}ds + \sum_{i=1}^m \|\eta_i\| \int_0^T (T-s)s^{p_i-2}ds\right] \\
&\leq \|\mu\| \frac{t^\alpha + T^\alpha}{\alpha(\alpha-1)} + \sum_{i=1}^m \|\eta_i\| \frac{t^{p_i} + T^{p_i}}{p_i(p_i-1)} \\
&\leq \|\mu\| \frac{2T^\alpha}{\alpha(\alpha-1)} + \sum_{i=1}^m \|\eta_i\| \frac{2T^{p_i}}{p_i(p_i-1)}.
\end{aligned}
$$

This implies

$$
\|\Pi_1(u,v)(t)\| \leq \|\mu\| \frac{2T^\alpha}{\alpha(\alpha-1)} + \sum_{i=1}^m \|\eta_i\| \frac{2T^{p_i}}{p_i(p_i-1)} \leq \rho.
$$

Therefore, $\Pi_1 B_\rho \subset B_\rho$. Analogously, we obtain

$$
\|\Pi_2(u,v)(t)\| \leq \|\lambda\| \frac{2T^\beta}{\beta(\beta-1)} + \sum_{i=1}^m \|\sigma_i\| \frac{2T^{q_i}}{q_i(q_i-1)} \leq \rho,
$$

so $\Pi_2 B_\rho \subset B_\rho$.

For any $(u,v) \in B_\rho$, we have

$$
\begin{aligned}
\|\Pi(u,v)(t)\| &= \|\Pi_1(u,v)(t)\| + \|\Pi_2(u,v)(t)\| \\
&\leq \|\mu\| \frac{2T^\alpha}{\alpha(\alpha-1)} + \sum_{i=1}^m \|\eta_i\| \frac{2T^{p_i}}{p_i(p_i-1)} + \|\lambda\| \frac{2T^\beta}{\beta(\beta-1)} + \sum_{i=1}^m \|\sigma_i\| \frac{2T^{q_i}}{q_i(q_i-1)} \\
&\leq 2\left(\frac{\|\mu\| T^\alpha}{\alpha(\alpha-1)} + \frac{\|\lambda\| T^\beta}{\beta(\beta-1)}\right) + 2\sum_{i=1}^m \left(\frac{\|\eta_i\| T^{p_i}}{p_i(p_i-1)} + \frac{\|\sigma_i\| T^{q_i}}{q_i(q_i-1)}\right) \\
&\leq \rho,
\end{aligned}
$$

which shows that Π maps B_ρ into itself.

Step 2. We will show that the operator $\Pi : B_\rho \mapsto B_\rho$ is a contraction. Let $(u_1, v_1), (u_2, v_2) \in X$ and $t \in J$. By assumption (H_2), we obtain

$$
\begin{aligned}
&\|\Pi_1(u_1,v_2)(t) - \Pi_1(u_2,v_2)(t)\| \\
= &\ \Big| \int_0^t (t-s)s^{\alpha-2}h(s,u_1(s),v_1(s))ds + \sum_{i=1}^m \int_0^t (t-s)s^{p_i-2}f_i(s,u_1(s),v_1(s))ds
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}h(s,u_1(s),v_1(s))ds + \sum_{i=1}^m \int_0^T (T-s)s^{p_i-2}f_i(s,u_1(s),v_1(s))ds\right] \\
& -\int_0^t (t-s)s^{\alpha-2}h(s,u_2(s),v_2(s))ds + \sum_{i=1}^m \int_0^t (t-s)s^{p_i-2}f_i(s,u_2(s),v_2(s))ds \\
& +\frac{t}{T}\left[\int_0^T (T-s)s^{\alpha-2}h(s,u_2(s),v_2(s))ds + \sum_{i=1}^m \int_0^T (T-s)s^{p_i-2}f_i(s,u_2(s),v_2(s))ds\right]\Bigg| \\
\le\ & \int_0^t (t-s)s^{\alpha-2}\left|h(s,u_1(s),v_1(s)) - h(s,u_2(s),v_2(s))\right| ds \\
& + \sum_{i=1}^m \int_0^t (t-s)s^{p_i-2}\left|f_i(s,u_1(s),v_1(s)) - f_i(s,u_2(s),v_2(s))\right| ds \\
& +\frac{t}{T}\Big[\int_0^T (T-s)s^{\alpha-2}\left|h(s,u_1(s),v_1(s)) - h(s,u_2(s),v_2(s))\right| ds \\
& + \sum_{i=1}^m \int_0^T (T-s)s^{p_i-2}\left|f_i(s,u_1(s),v_1(s)) - f_i(s,u_2(s),v_2(s))\right| ds\Big] \\
\le\ & \int_0^t (t-s)s^{\alpha-2}\left[C_1(u_1(s)-u_2(s)) + C_2(v_1(s)-v_2(s))\right] ds \\
& + \sum_{i=1}^m \int_0^t (t-s)s^{p_i-2}\left[L_{1i}(u_1(s)-u_2(s)) + L_{2i}(v_1(s)-v_2(s))\right] ds \\
& +\frac{t}{T}\Big[\int_0^T (T-s)s^{\alpha-2}\left[C_1(u_1(s)-u_2(s)) + C_2(v_1(s)-v_2(s))\right] ds \\
& + \sum_{i=1}^m \int_0^T (T-s)s^{p_i-2}\left[L_{1i}(u_1(s)-u_2(s)) + L_{2i}(v_1(s)-v_2(s))\right] ds\Big] \\
\le\ & \frac{C_1\|u_1-u_2\| + C_2\|v_1-v_2\|}{\alpha(\alpha-1)}t^\alpha + \sum_{i=1}^m \frac{L_{1i}\|u_1-u_2\| + L_{2i}\|v_1-v_2\|}{p_i(p_i-1)}t^{p_i} \\
& +\frac{C_1\|u_1-u_2\| + C_2\|v_1-v_2\|}{\alpha(\alpha-1)}T^\alpha + \sum_{i=1}^m \frac{L_{1i}\|u_1-u_2\| + L_{2i}\|v_1-v_2\|}{p_i(p_i-1)}T^{p_i} \\
\le\ & \frac{2CT^\alpha}{\alpha(\alpha-1)}\left(\|u_1-u_2\| + \|v_1-v_2\|\right) + 2\sum_{i=1}^m \frac{LT^{p_i}}{p_i(p_i-1)}\left(\|u_1-u_2\| + \|v_1-v_2\|\right) \\
\le\ & 2\left[\frac{CT^\alpha}{\alpha(\alpha-1)} + \sum_{i=1}^m \frac{LT^{p_i}}{p_i(p_i-1)}\right]\|(u_1-u_2, v_1-v_2)\|.
\end{aligned}
$$

This implies that Π_1 is a contraction.

Analogously, we can prove that Π_2 is a contraction. Then, by the Banach fixed point theorem, there exists a unique point $(u,v) \in X$, such that $\Pi(u,v) = (u,v)$. It is the unique solution of our system (1.1), and then the proof of the theorem is completed. □

5. Example

Consider the following Hybrid fractional integro-differential system involving conformable fractional derivative operators

$$\begin{aligned}
&T^{\frac{3}{2}}\left(u(t)-\sum_{i=1}^{2} I^{p_i} f_i(t,u(t),v(t))\right)=\frac{1+t^2+\sin(u(t))+\cos(v(t))}{8(1+t)},\\
&T^{\frac{5}{3}}\left(v(t)-\sum_{i=1}^{2} I^{q_i} g_i(t,u(t),v(t))\right)=\frac{t^3+t+\cos(u(t))+\sin(v(t))}{(t+1)(t^2+1)},\\
&u(0)=u(T)=0,\quad v(0)=v(T)=0.
\end{aligned} \tag{5.1}$$

The problem (5.1) is a particular case of (1.1) with $\alpha=\frac{3}{2},\beta=\frac{5}{3}$, and

$$\begin{aligned}
f_i(t,u(t),v(t)) &= \frac{t(|u(t)|+|u(t)|)}{8i},\\
g_i(t,u(t),v(t)) &= \frac{it(\cos(u(t))+|v(t)|)}{50}.
\end{aligned}$$

Clearly, f_i, g_i, $i=1,2,..,m,h$, and k are continuous functions and satisfy condition (H_1) with $\eta_i(t)=\frac{t}{4i}$, $\sigma_i(t)=\frac{it}{25}$, $\mu(t)=\frac{1+t^2}{8(1+t)}$ and $\lambda(t)=\frac{t^3+t}{(1+t)(1+t^2)}$.

Also

$$\begin{aligned}
|f_i(t,u_1,v_1)-f_i(t,u_2,v_2)| &\le \frac{t}{8i}|u_1-u_2+v_1-v_2|\\
&\le \frac{T}{8i}\|u_1-u_2,v_1-v_2\|,
\end{aligned}$$

$$\begin{aligned}
|g_i(t,u_1,v_1)-g_i(t,u_2,v_2)| &\le \frac{it}{50}|\cos(u_1)-\cos(u_2)+\sin(v_1)-\sin(v_2)|\\
&\le \frac{iT}{50}\|u_1-u_2,v_1-v_2\|,
\end{aligned}$$

and

$$|k(t,u_1,v_1)-k(t,u_2,v_2)| \le \frac{1}{(t+1)(t^2+1)}\|u_1-u_2,v_1-v_2\|.$$

Taking the value of $T=1$ we get

$$\begin{aligned}
\left(\frac{T^\alpha C}{\alpha(\alpha-1)}+\sum_{i=1}^{m}\frac{T^{p_i}L_i}{p_i(p_i-1)}\right)&=0.37500000<1,\\
\left(\frac{T^\beta N}{\beta(\beta-1)}+\sum_{i=1}^{m}\frac{T^{q_i}M_i}{q_i(q_i-1)}\right)&=0.42161904<1,
\end{aligned}$$

where $C=\frac{1}{8}$, $L_1=\frac{1}{4}, L_2=\frac{1}{8}$, $N=\frac{1}{2}$ and $M_1=\frac{1}{25}, M_2=\frac{2}{25}$ and the values of p_i and q_i are chosen as $p_i=\frac{3}{i}$. This gives $p_1=3$, $p_2=\frac{3}{2}$, and $q_i=\frac{2i+3}{2i}$ we get $q_1=\frac{5}{2}$, $q_2=\frac{7}{4}$.

Since the assumptions (H_1) and (H_2) hold, according to Theorem 3.1 the problem (5.1) has at least one solution. To see if the solution is unique, note that assumptions (H_1) and (H_2) hold, from the first part of the existence result. Also, the conditions of Theorem 3.2

$$2\left(\frac{T^{\alpha}C}{\alpha(\alpha-1)}+\sum_{i=1}^{m}\frac{T^{p_i}L_i}{p_i(p_i-1)}\right)=0.75000000<1,$$
$$2\left(\frac{T^{\beta}N}{\beta(\beta-1)}+\sum_{i=1}^{m}\frac{T^{q_i}M_i}{q_i(q_i-1)}\right)=0.84323808<1,$$

are satisfied. Therefore, from Theorem 3.2, the problem (5.1) has a unique solution.

6. Conclusions

In this work, we consider the existence and uniqueness of solutions for the boundary value problem of hybrid fractional integro-differential systems involving the conformable fractional derivative. By transforming the problem into a Volttera integral equation and using the Krasnoselskii fixed point theorem, we get the results about the existence of solutions for the boundary value problem (1.1) under some conditions. Then, using the Banach fixed point theorem, we get the existence and uniqueness of the solution for the boundary value problem, after transforming the problem into a fixed point problem.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no competing interests.

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