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Research article

On some properties of a generalized min matrix

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Abstract: In this paper, we investigate a min matrix and obtain its LU-decomposition, determinant, permanent, inverse, and norm properties. In addition, we obtain a recurrence relation provided by the characteristic polynomial of this matrix. Finally, we present an example to illustrate the results obtained.

Keywords: Frank matrix; min matrix; determinant; inverse; permanent; norm
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1. Introduction and preliminaries

Frank [1] gave a definition of an $n \times n$ matrix (which is called Frank matrix [2, 3]) as follows:

$$F_n = \begin{bmatrix} n & n-1 & 0 & \dots & 0 & 0 \\ n-1 & n-1 & n-2 & \dots & 0 & 0 \\ n-2 & n-2 & n-2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}. \tag{1.1}$$

The element in the i-th row and the j-th column of Frank matrix is given by the following rule:

$$f_{ij} = \begin{cases} n+1-\max(i,j), & i > j-2, \\ 0, & \text{otherwise}. \end{cases}$$

The Frank matrix has often been used as test matrices for eigenprograms. This is because F_n has well-conditioned and poorly conditioned eigenvalues [3, 4]. On the other hand, Frank matrix is a special max matrix. There are many max matrix studies in the literature. One of them was considered by Kılıç and Arıkan in [5]. They dealt with the generalized versions of the classical max and min matrices and gave many linear algebraic results for them. In [6], Kızılateş and Terzioğlu defined r-min and r-max matrices. They also obtained determinants, inverses, norms and factorizations of these matrices. Liu et al. [7] studied the determinants, inverses and eigenvalues of two symmetric matrices with Fibonacci numbers as elements. In [8], Wang et al. examined determinants, inverses and eigenvalues of symmetric matrices with Pell and Pell-Lucas numbers. They gave also the general formulas of the solution of the linear equations with the Pell-min and Pell-Lucas-min symmetric matrix as the coefficient matrix, respectively. Meng et al. [9] showed that there is an intimate relationship between Toeplitz matrix, tridiagonal Toeplitz matrix, the Fibonacci number, the Lucas number, and the Golden Ratio. They introduced also skew Loeplitz and skew Foeplitz matrices and derived their determinants and inverses by construction. In [10], Meng et al. investigated the exact determinants and the inverses of nxn (2,3,3)-Loeplitz and (2,3,3)-Foeplitz matrices. In [11], the authors examined the analytical determinants and inverses of nxn weighted Loeplitz and weighted Foeplitz matrices. They introduced also the nxn weighted Loeplitz and weighted Foeplitz matrices and obtained the analytical determinants and inverses of them by constructing the transformation matrices. Recently, in [12], the authors defined a generalization of Frank matrix given in (1.1) which corresponds to the real n-tuple $a = (a_1, a_2, ..., a_n)$ as follows:

$$F_a = \begin{bmatrix} a_n & a_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ a_{n-1} & a_{n-1} & a_{n-2} & 0 & \cdots & 0 & 0 \\ a_{n-2} & a_{n-2} & a_{n-2} & a_{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_2 & a_2 & a_2 & \cdots & a_2 & a_1 \\ a_1 & a_1 & a_1 & a_1 & \cdots & a_1 & a_1 \end{bmatrix}.$$

Here, the (i, j)-th entry of the above matrix is

$$(f_a)_{ij} = \begin{cases} a_{n+1-\max(i,j)}, & i > j-2, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Mersin et al. obtained various results based on the above definition.

Let $Q = (q_{ij})$ be any $m \times n$ matrix. Then the Euclidean norm of the matrix Q is defined by

$$\|Q\|_E = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |q_{ij}|^2}$$

and the spectral norm of the matrix Q is defined by

$$\|Q\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i \left(Q^H Q\right)},$$

where $\lambda_i (Q^* Q)$ is eigenvalue of $Q^* Q$ and Q^* is conjugate transpose of Q.

The following relation between Euclidean norm and spectral norm is well known:

$$\frac{1}{\sqrt{n}} \|Q\|_E \le \|Q\|_2 \le \|Q\|_E . \tag{1.3}$$

Now we give the following useful lemma that we will use later in this paper related to norm equality.

Lemma 1.1. *[13] Let $P = (p_{ij})$ and $Q = (q_{ij})$ be any $m \times n$ matrices. Then*

$$\|P \circ Q\|_2 \leq r_1(P)c_1(Q),$$

where

$$r_1(P) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} |p_{ij}|^2} \text{ and } c_1(Q) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} |q_{ij}|^2}.$$

In the literature, the norm properties of various matrices, whose entries are the elements of well-known sequences, have been examined by many researchers. For more information related to this topic, see [14–26] and references therein.

In the light of the above-mentioned studies, we examine a min matrix and obtain some of its linear algebraic properties. Then, we give an example to illustrate the results obtained.

2. Main results

In this part of the paper, we investigate the LU-decomposition, determinant, inverse, permanent, and norm properties of the matrix which is the min version of (1.2). In addition, we obtain a recurrence relation that satisfies the characteristic polynomial of this matrix.

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite multiset of real numbers and the (i, j)-th entry of the $n \times n$ matrix S_n be as follows:

$$s_{ij} = \begin{cases} s_{n+1-\min(i,j)}, & i > j-2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus S_n can be written as

$$S_n = \begin{bmatrix} s_n & s_n & 0 & \cdots & 0 & 0 \\ s_n & s_{n-1} & s_{n-1} & \cdots & 0 & 0 \\ s_n & s_{n-1} & s_{n-2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_2 & s_2 \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_2 & s_1 \end{bmatrix}. \tag{2.1}$$

It is not difficult to see that S_n can be factored as follows:

$$S_n = M\widetilde{I}\Omega\widetilde{I}, \tag{2.2}$$

where

$$M = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}, \quad \widetilde{I} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$\Omega = \begin{bmatrix} s_1 - s_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ s_2 & s_2 - s_3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & s_3 & s_3 - s_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & s_{n-2} - s_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & s_{n-1} & s_{n-1} - s_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & s_n & s_n \end{bmatrix}.$$

Now, we firstly give the determinant of S_n.

Theorem 2.1. *The determinant of the matrix S_n is given by*

$$\det(S_n) = s_n \prod_{i=2}^{n} (s_{i-1} - s_i).$$

Proof. If we take determinant of both sides of (2.2), we have

$$\det(S_n) = \det\left(M\widetilde{I}\Omega\widetilde{I}\right) = \det(M)\det\left(\widetilde{I}\right)\det(\Omega)\det\left(\widetilde{I}\right).$$

Since $\det(M) = 1$ and $\det\left(\widetilde{I}\right) = \mp 1$, we obtain

$$\det(S_n) = s_n (s_1 - s_2)(s_2 - s_3)(s_3 - s_4)\ldots(s_{n-1} - s_n) = s_n \prod_{i=2}^{n} (s_{i-1} - s_i).$$

□

Theorem 2.2. *For $1 \le i, j \le n$, the LU-decomposition of S_n is given by as follows:*

$$L_{ij} = \begin{cases} 1, & \text{if } i \ge j, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$U_{ij} = \begin{cases} s_n, & \text{if } i = j = 1, \\ s_{n-i+1} - s_{n-i+2}, & \text{if } i = j \ne 1, \\ s_{n-i+1}, & \text{if } j = 1 + i, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. In the case $j = 1$, since $\min(i, 1) = 1$, we have

$$s_{i1} = \sum_{k=1}^{n} L_{ik} U_{k1} = s_n = s_{n+1-\min(i,j)}.$$

In the case $j \geq 2$, we have

$$s_{ij} = \begin{bmatrix} \underbrace{1 \quad 1 \quad \cdots \quad 1}_{i \text{ times}} & \underbrace{0 \quad 0 \quad \cdots \quad 0}_{(n-i) \text{ times}} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a \\ b \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} j-2 \text{ times} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n-j \text{ times} \end{matrix}$$

where $a = s_{n-j+2}$ and $b = s_{n-j+1} - s_{n-j+2}$.

- If $i \leq j-2$, we can see that
$$s_{ij} = 0.$$
- If $i = j-1$, we obtain
$$s_{ij} = s_{n+2-j} = s_{n+1-i} = s_{n+1-\min(i,j)}.$$
- If $i \geq j$, then we have
$$s_{ij} = s_{n+1-j} = s_{n+1-\min(i,j)}.$$

Thus the proof is completed. □

Now we compute the permanent of S_n.

Theorem 2.3. *The permanent of the matrix S_n is given by*

$$\text{per}(S_n) = s_n \prod_{i=2}^{n} (s_{i-1} + s_i).$$

Proof. By using [27, Lemma 3.2(i)], we obtain step by step the followings:

$$\begin{aligned} \text{per}(S_n) &= \text{per} \begin{bmatrix} s_n & s_n & \cdots & 0 & 0 \\ s_n & s_{n-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_n & s_{n-1} & \cdots & s_2 & s_2 \\ s_n & s_{n-1} & \cdots & s_2 & s_1 \end{bmatrix}_{n \times n} \\ &= \text{per} \begin{bmatrix} s_n & s_n & \cdots & 0 & 0 \\ s_n & s_{n-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_n & s_{n-1} & \cdots & s_3 & s_3 \\ (s_1+s_2)\,s_n & (s_1+s_2)\,s_{n-1} & \cdots & (s_1+s_2)\,s_3 & (s_1+s_2)\,s_2 \end{bmatrix}_{(n-1)\times(n-1)} \\ &= \text{per} \begin{bmatrix} s_n & s_n & \cdots & 0 \\ s_n & s_{n-1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ s_n & s_{n-1} & \cdots & s_4 \\ (s_1+s_2)(s_2+s_3)\,s_n & (s_1+s_2)(s_2+s_3)\,s_{n-1} & \cdots & (s_1+s_2)(s_2+s_3)\,s_3 \end{bmatrix}_{(n-2)\times(n-2)} \end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\vdots\\
=\ &\operatorname{per}\begin{bmatrix} s_n & s_n \\ s_n\prod_{i=2}^{n-1}\left(s_{i-1}+s_i\right) & s_{n-1}\prod_{i=2}^{n-1}\left(s_{i-1}+s_i\right)\end{bmatrix}\\
=\ & s_n\left(s_n\prod_{i=2}^{n-1}\left(s_{i-1}+s_i\right)+s_{n-1}\prod_{i=2}^{n-1}\left(s_{i-1}+s_i\right)\right)\\
=\ & s_n\prod_{i=2}^{n}\left(s_{i-1}+s_i\right).
\end{aligned}
$$

Thus, the proof is completed. □

We will present the inverse of S_n in the following theorem.

Theorem 2.4. *Let S_n be in the form*

$$
S_n=\begin{bmatrix} s_n & s_n & 0 & \cdots & 0 & 0\\ s_n & s_{n-1} & s_{n-1} & \cdots & 0 & 0\\ s_n & s_{n-1} & s_{n-2} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ s_n & s_{n-1} & s_{n-2} & \cdots & s_2 & s_2\\ s_n & s_{n-1} & s_{n-2} & \cdots & s_2 & s_1\end{bmatrix}=\begin{bmatrix} s_n & F\\ E & S_{n-1}\end{bmatrix},
$$

where $E=\begin{bmatrix} s_n & s_n & \cdots & s_n\end{bmatrix}^T$ is $(n-1)\times 1$ matrix and $F=\begin{bmatrix} s_n & 0 & \cdots & 0 & 0\end{bmatrix}$ is $1\times(n-1)$ matrix. If $s_i\left(s_{i-1}-s_i\right)\neq 0$ for $2\le i\le n$, then the inverse of S_n is

$$
S_n^{-1}=\left[\begin{array}{c|c} v_n & -v_nFS_{n-1}^{-1}\\ \hline -v_nS_{n-1}^{-1}E & S_{n-1}^{-1}+v_nS_{n-1}^{-1}EFS_{n-1}^{-1}\end{array}\right], \tag{2.3}
$$

where

$$
v_n=\frac{s_{n-1}}{s_n\left(s_{n-1}-s_n\right)}. \tag{2.4}
$$

Proof. For the proof, we use the mathematical induction on n. For $n=2$, we obtain classically

$$
S_2^{-1}=\frac{1}{\det(S_2)}\begin{bmatrix} s_1 & -s_2\\ -s_2 & s_2\end{bmatrix}=\begin{bmatrix}\frac{s_1}{(s_1-s_2)s_2} & -\frac{1}{s_1-s_2}\\ -\frac{1}{s_1-s_2} & \frac{1}{s_1-s_2}\end{bmatrix}. \tag{2.5}
$$

On the other hand, for $n=2$, our claim in Eq (2.3) gives

$$
\left[\begin{array}{c|c} v_2 & -v_2FS_1^{-1}\\ \hline -v_2S_1^{-1}E & S_1^{-1}+v_2S_1^{-1}EFS_1^{-1}\end{array}\right]=\begin{bmatrix}\frac{s_1}{s_2(s_1-s_2)} & -\frac{s_1}{s_2(s_1-s_2)}s_2\frac{1}{s_1}\\ -\frac{s_1}{s_2(s_1-s_2)}\frac{1}{s_1}s_2 & \frac{1}{s_1}+\frac{s_1}{s_2(s_1-s_2)}\frac{1}{s_1}s_2^2\frac{1}{s_1}\end{bmatrix}=\begin{bmatrix}\frac{s_1}{(s_1-s_2)s_2} & -\frac{1}{s_1-s_2}\\ -\frac{1}{s_1-s_2} & \frac{1}{s_1-s_2}\end{bmatrix}. \tag{2.6}
$$

Thus the claim is satisfied for $n = 2$. Assume that our claim is true for $n = t - 1$. Then, since $S_{t-1}^{-1} S_{t-1} = I_{(t-1)\times(t-1)}$, we obtain

$$S_{t-1}^{-1} \begin{bmatrix} s_{t-1} \\ s_{t-1} \\ \vdots \\ s_{t-1} \end{bmatrix}_{(t-1)\times 1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(t-1)\times 1} . \tag{2.7}$$

If we multiply with $\frac{s_t}{s_{t-1}}$ both sides of Eq (2.7), we get

$$S_{t-1}^{-1} s_t \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{(t-1)\times 1} = S_{t-1}^{-1} E = \frac{s_t}{s_{t-1}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(t-1)\times 1} . \tag{2.8}$$

For $n = t$, we obtain

$$\begin{aligned} S_t^{-1} S_t &= \left[\begin{array}{c|c} v_t & -v_t F S_{t-1}^{-1} \\ \hline -v_t S_{t-1}^{-1} E & S_{t-1}^{-1} + v_t S_{t-1}^{-1} E F S_{t-1}^{-1} \end{array}\right] \begin{bmatrix} s_t & F \\ E & S_{t-1} \end{bmatrix} \\ &= \left[\begin{array}{c|c} v_t s_t - v_t F S_{t-1}^{-1} E & v_t F - v_t F S_{t-1}^{-1} S_{t-1} \\ \hline -v_t S_{t-1}^{-1} E s_t + \left(S_{t-1}^{-1} + v_t S_{t-1}^{-1} E F S_{t-1}^{-1}\right) E & -v_t S_{t-1}^{-1} E F + \left(S_{t-1}^{-1} + v_t S_{t-1}^{-1} E F S_{t-1}^{-1}\right) S_{t-1} \end{array}\right] \\ &= \left[\begin{array}{c|c} v_t s_t - v_t F S_{t-1}^{-1} E & 0 \\ \hline -v_t s_t S_{t-1}^{-1} E + S_{t-1}^{-1} E + v_t S_{t-1}^{-1} E F S_{t-1}^{-1} E & I \end{array}\right] \end{aligned}$$

$$= \begin{bmatrix} v_t s_t - v_t \begin{bmatrix} s_t & 0 & \cdots & 0 \end{bmatrix} \frac{s_t}{s_{t-1}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & 0 \\ -v_t s_t \frac{s_t}{s_{t-1}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{s_t}{s_{t-1}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_t \frac{s_t}{s_{t-1}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} s_t & 0 & \cdots & 0 \end{bmatrix} \frac{s_t}{s_{t-1}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & I \end{bmatrix}$$

$$= \begin{bmatrix} v_t s_t - v_t \frac{s_t^2}{s_{t-1}} & 0 \\ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left\{ -v_t s_t \frac{s_t}{s_{t-1}} + \frac{s_t}{s_{t-1}} + v_t \frac{s_t}{s_{t-1}} \begin{bmatrix} s_t & 0 & \cdots & 0 \end{bmatrix} \frac{s_t}{s_{t-1}} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\} & I \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 0 \\ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \left\{ v_t \left(\frac{s_t^3}{s_{t-1}^2} - \frac{s_t^2}{s_{t-1}} \right) + \frac{s_t}{s_{t-1}} \right\} & I \end{bmatrix} \\
&= I_{t\times t}.
\end{aligned}
$$

Thus the proof is completed. □

Theorem 2.5. *The characteristic polynomial of S_n provides the following recurrence relation:*

$$
P_n(x) = \left(x - s_n + \frac{s_n^2}{s_{n-1}} \right) P_{n-1}(x) - x\frac{s_n^2}{s_{n-1}} P_{n-2}(x), \tag{2.9}
$$

where $P_1(x) = x - s_1$ and $P_2(x) = x^2 - (s_1 + s_2)x + s_2(s_1 - s_2)$.

Proof. From the definition of characteristic polynomial of S_n and determinantal properties, we get

$$
\begin{aligned}
P_n(x) &= \begin{vmatrix} x-s_n & -s_n & 0 & \cdots & 0 & 0 \\ -s_n & x-s_{n-1} & -s_{n-1} & \cdots & 0 & 0 \\ -s_n & -s_{n-1} & x-s_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_n & -s_{n-1} & -s_{n-2} & \cdots & x-s_2 & -s_2 \\ -s_n & -s_{n-1} & -s_{n-2} & \cdots & -s_2 & x-s_1 \end{vmatrix} \\
&= (x-s_n) \begin{vmatrix} x-s_{n-1} & -s_{n-1} & 0 & \cdots & 0 & 0 \\ -s_{n-1} & x-s_{n-2} & -s_{n-2} & \cdots & 0 & 0 \\ -s_{n-1} & -s_{n-2} & x-s_{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_{n-1} & -s_{n-2} & -s_{n-3} & \cdots & x-s_2 & -s_2 \\ -s_{n-1} & -s_{n-2} & -s_{n-3} & \cdots & -s_2 & x-s_1 \end{vmatrix} \\
&\quad + s_n \begin{vmatrix} -s_n & -s_{n-1} & 0 & \cdots & 0 & 0 \\ -s_n & x-s_{n-2} & -s_{n-2} & \cdots & 0 & 0 \\ -s_n & -s_{n-2} & x-s_{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_n & -s_{n-2} & -s_{n-3} & \cdots & x-s_2 & -s_2 \\ -s_n & -s_{n-2} & -s_{n-3} & \cdots & -s_2 & x-s_1 \end{vmatrix} \\
&= (x-s_n) P_{n-1}(x) + \frac{s_n^2}{s_{n-1}} \begin{vmatrix} -s_{n-1} & -s_{n-1} & 0 & \cdots & 0 & 0 \\ -s_{n-1} & x-s_{n-2} & -s_{n-2} & \cdots & 0 & 0 \\ -s_{n-1} & -s_{n-2} & x-s_{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_{n-1} & -s_{n-2} & -s_{n-3} & \cdots & x-s_2 & -s_2 \\ -s_{n-1} & -s_{n-2} & -s_{n-3} & \cdots & -s_2 & x-s_1 \end{vmatrix} \\
&= (x-s_n) P_{n-1}(x) + \frac{s_n^2}{s_{n-1}} \left(P_{n-1}(x) - xP_{n-2}(x) \right)
\end{aligned}
$$

$$= \left(x - s_n + \frac{s_n^2}{s_{n-1}}\right) P_{n-1}(x) - x\frac{s_n^2}{s_{n-1}} P_{n-2}(x).$$

So, the proof is completed. □

Theorem 2.6. *Suppose that*

$$P_n(x) = x^n + \alpha_{n-1}^{(n)} x^{n-1} + ... + \alpha_1^{(n)} x + \alpha_0^{(n)} \tag{2.10}$$

be the characteristic polynomial of S_n, then we have the followings:

$$\begin{aligned} (i)\ \alpha_0^{(n)} &= \left(\frac{s_n^2}{s_{n-1}} - s_n\right) \alpha_0^{(n-1)}, \\ (ii)\ \alpha_{n-1}^{(n)} &= \alpha_{n-2}^{(n-1)} - s_n, \\ (iii)\ \alpha_i^{(n)} &= \alpha_{i-1}^{(n-1)} + \left(\frac{s_n^2}{s_{n-1}} - s_n\right) \alpha_i^{(n-1)} - \frac{s_n^2}{s_{n-1}} \alpha_{i-1}^{(n-2)} \quad (1 \le i \le n-2). \end{aligned}$$

Proof. Substituting (2.10) in to (2.9), we get

$$\begin{aligned} P_n(x) &= \left(x - s_n + \frac{s_n^2}{s_{n-1}}\right)\left(x^{n-1} + \alpha_{n-2}^{(n-1)} x^{n-2} + ... + \alpha_1^{(n-1)} x + \alpha_0^{(n-1)}\right) \\ &\quad -x\frac{s_n^2}{s_{n-1}}\left(x^{n-2} + \alpha_{n-3}^{(n-2)} x^{n-3} + ... + \alpha_1^{(n-2)} x + \alpha_0^{(n-2)}\right). \end{aligned}$$

If we rearrange the right-hand side of the above equation to powers of x and compare the coefficients of the resulting expression with the coefficients of (2.10), then we obtain

$$\begin{aligned} (i)\ \alpha_0^{(n)} &= \left(\frac{s_n^2}{s_{n-1}} - s_n\right) \alpha_0^{(n-1)}, \\ (ii)\ \alpha_{n-1}^{(n)} &= \alpha_{n-2}^{(n-1)} - s_n, \\ (iii)\ \alpha_i^{(n)} &= \alpha_{i-1}^{(n-1)} + \left(\frac{s_n^2}{s_{n-1}} - s_n\right) \alpha_i^{(n-1)} - \frac{s_n^2}{s_{n-1}} \alpha_{i-1}^{(n-2)}, \quad (1 \le i \le n-2), \end{aligned}$$

respectively. □

Now we present some norm properties of S_n in the following theorems.

Theorem 2.7. *The Euclidean norm of S_n is*

$$\|S_n\|_E = \sqrt{\sum_{m=1}^{n} (m+1)\, s_m^2 - s_1^2}.$$

Proof. If we apply the definition of Euclidean norm to the matrix S_n, we obtain

$$\|S_n\|_E = \sqrt{\sum_{i,j=1}^{n} |s_{ij}|^2 = (n+1)\, s_n^2 + n s_n^2 + ... + 3s_2^2 + 2s_1^2 - s_1^2} = \sqrt{\sum_{m=1}^{n} (m+1)\, s_m^2 - s_1^2}.$$

□

Theorem 2.8. *For the matrix S_n, if $s_1 \le s_2 \le \ldots \le s_n$, then we have the following norm inequality:*

$$\frac{1}{\sqrt{n}}\sqrt{\sum_{m=1}^{n}(m+1)\,s_m^2 - s_1^2} \le \|S_n\|_2 \le n s_n.$$

Proof. Let the $n \times n$ matrix X be

$$X = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ 1 & 1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

and S_n be as in (2.1). So we have

$$r_1(X) = \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|x_{ij}|^2} = \sqrt{n}$$

and

$$c_1(S_n) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}|s_{ij}|^2} = \sqrt{n s_n^2} = \sqrt{n}s_n.$$

Since $S_n = X \circ S_n$, by the aid of Lemma 1.1, we have

$$\|S_n\|_2 \le n s_n.$$

Thus, by using (1.3), we obtain

$$\frac{1}{\sqrt{n}}\sqrt{\sum_{m=1}^{n}(m+1)\,s_m^2 - s_1^2} \le \|S_n\|_2 \le n s_n.$$

Thus, the proof is completed. □

3. A numerical example

In this section, we give a numerical example to verify our results. In the example to be given, the matrix (2.1), whose entries are classical Lucas numbers, will be discussed for $n = 4$.

The classical Lucas sequence is defined by the following recurrence relation:

$$l_{n+2} = l_{n+1} + l_n, \quad (n \ge 0),$$

where $l_0 = 2$, $l_1 = 1$.

Let

$$\mathcal{L}_4 = \begin{bmatrix} l_4 & l_4 & 0 & 0 \\ l_4 & l_3 & l_3 & 0 \\ l_4 & l_3 & l_2 & l_2 \\ l_4 & l_3 & l_2 & l_1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 0 & 0 \\ 7 & 4 & 4 & 0 \\ 7 & 4 & 3 & 3 \\ 7 & 4 & 3 & 1 \end{bmatrix}$$

be a matrix as in (2.1) for $n = 4$.

With the help of the Theorem 2.1, the determinant of $\mathcal{L}_4$ can be calculated as

$$\det(\mathcal{L}_4) = l_4 \prod_{i=2}^{4} (l_{i-1} - l_i) = -l_0 l_1 l_2 l_4 = -42.$$

Thanks to Theorem 2.2, for $1 \leq i, j \leq 4$, the LU-decomposition of $\mathcal{L}_4$ can be written as follows:

$$L_{ij} = \begin{cases} 1, & \text{if } i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$U_{ij} = \begin{cases} l_4, & \text{if } i = j = 1, \\ -l_{4-i}, & \text{if } i = j \neq 1, \\ l_{5-i}, & \text{if } j = 1 + i, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have

$$\mathcal{L}_4 = \begin{bmatrix} 7 & 7 & 0 & 0 \\ 7 & 4 & 4 & 0 \\ 7 & 4 & 3 & 3 \\ 7 & 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 7 & 0 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

By virtue of Theorem 2.3, one can obtain the permanent of the matrix $\mathcal{L}_4$ as

$$\operatorname{per}(\mathcal{L}_4) = l_4 \prod_{i=2}^{4} (l_{i-1} + l_i) = l_3 l_4^2 l_5 = 2156.$$

With the help of the Theorem 2.4, the inverse of $\mathcal{L}_4$ can be calculated as

$$\mathcal{L}_4^{-1} = \begin{bmatrix} l_4 & F \\ E & \mathcal{L}_3 \end{bmatrix}^{-1} = \left[\begin{array}{c|c} v_4 & -v_4 F \mathcal{L}_3^{-1} \\ \hline -v_4 \mathcal{L}_3^{-1} E & \mathcal{L}_3^{-1} + v_4 \mathcal{L}_3^{-1} E F \mathcal{L}_3^{-1} \end{array}\right],$$

where $E = \begin{bmatrix} l_4 & l_4 & l_4 \end{bmatrix}^T$, $F = \begin{bmatrix} l_4 & 0 & 0 \end{bmatrix}$ and $v_4 = \dfrac{l_3}{l_4(l_3 - l_4)} = -\dfrac{4}{21}$. Thus, after the necessary calculations, the inverse of $\mathcal{L}_4$ is obtained as follows:

$$\mathcal{L}_4^{-1} = \begin{bmatrix} -\frac{4}{21} & -1 & -\frac{2}{3} & 2 \\ \frac{1}{3} & 1 & \frac{2}{3} & -2 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thanks to Theorem 2.5, we have

$$P_3(x) = \left(x - l_3 + \frac{l_3^2}{l_2}\right) P_2(x) - \frac{l_3^2}{l_2} x P_1(x) = x^3 - 8x^2 - 6x - 8.$$

Thus the characteristic polynomial of the matrix $\mathcal{L}_4$ can be computed as

$$P_4(x) = \left(x - l_4 + \frac{l_4^2}{l_3}\right) P_3(x) - \frac{l_4^2}{l_3} x P_2(x) = x^4 - 15x^3 + x^2 + 34x - 42.$$

With the help of the Theorem 2.6, we have the followings:

$$\begin{aligned}
(i)\ \alpha_0^{(4)} &= -42 = \left(\frac{l_4^2}{l_3} - l_4\right)\alpha_0^{(3)}, \\
(ii)\ \alpha_3^{(4)} &= -15 = \alpha_2^{(3)} - l_4, \\
(iii)\ \alpha_1^{(4)} &= 34 = \alpha_0^{(3)} + \left(\frac{l_4^2}{l_3} - l_4\right)\alpha_1^{(3)} - \frac{l_4^2}{l_3}\alpha_0^{(2)}, \\
(iv)\ \alpha_2^{(4)} &= 1 = \alpha_1^{(3)} + \left(\frac{l_4^2}{l_3} - l_4\right)\alpha_2^{(3)} - \frac{l_4^2}{l_3}\alpha_1^{(2)}.
\end{aligned}$$

From Theorem 2.7, Euclidean norm of the matrix $\mathcal{L}_4$ can be computed as

$$\|\mathcal{L}_4\|_E = \sqrt{\sum_{m=1}^{4} (m+1)\, l_m^2 - 1} = \sqrt{337} \approx 18.358.$$

By virtue of Theorem 2.8, we can obtain the lower and upper bounds for the spectral norm of $\mathcal{L}_4$ as

$$9.179 \le \|\mathcal{L}_4\|_2 = 17.762 \le 28.$$

4. Conclusions

In this paper, we investigated a min matrix and obtained some of its linear algebraic properties. In future studies, interested readers may examine whether Sturm's Theorem can be applied to the matrix discussed in this study. For recent studies on Sturm's Theorem, we refer to [28, 29] and references therein.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

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