AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 26168–26187.
DOI: 10.3934/math.20231334
Received: 07 July 2023
Revised: 25 August 2023
Accepted: 04 September 2023
Published: 12 September 2023

Research article

Analysis of the solvability and stability of the operator-valued Fredholm integral equation in Hölder space

Manalisha Bhujel[1], **Bipan Hazarika**[1], **Sumati Kumari Panda**[2] **and Dimplekumar Chalishajar**[3,*]

[1] Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India
[2] Department of Mathematics, GMR Institute of Technology, Rajam-532127, Andhra Pradesh, India
[3] Department of Applied Mathematics, Mallory Hall, Virginia Military Institute (VMI), Lexington, VA 24450, USA

* **Correspondence:** Email: chalishajardn@vmi.edu.

Abstract: In this paper, the solvability of an operator-valued integral equation in Hölder spaces, i.e.,

$$w(\zeta_1) = y(\zeta_1) + w(\zeta_1)\int_{\mathbf{J}} \kappa(\zeta_1,\varphi)(T_1 w)(\varphi)d\varphi + z(\zeta_1)\int_{\mathbf{J}} h(\varphi,(T_2 w)(\varphi))d\varphi,$$

for $\zeta_1 \in \mathbf{J} = [0,1]$, is studied by using Darbo's fixed point theorem (FPT). The process of the measure of noncompactness of the operators which constitute an intermediary of contraction and compact mappings can be explained with the help of Darbo's FPT. The greater effectiveness of Darbo's FPT due to its non-involvement of the compactness property gives a better scope when dealing with the Schauder FPT, where compactness is an essential property. To obtain a unique solution, we apply the Banach fixed point theorem and discuss the Hyers-Ulam stability of the integral equation. We also give some important examples to illustrate the existence and uniqueness of the results.

Keywords: Hölder function space; measure of noncompactness; Fredholm integral equation; fixed-point theorem
Mathematics Subject Classification: 26B35, 45B05, 47H10

1. Introduction

Integral equations (IEs) have expansive applications in different areas of science and engineering. There are several problems in science and technology related to IEs. With the help of IEs, we can describe numerous events that arise in real life problems, e.g., problems in the theory of radiative transfer, the theory of neutron transport, and the kinetic theory of gases can be addressed by using the famous quadratic IE of Chandrasekhar type

$$w(\zeta_1) = 1 + w(\zeta_1) \int_{\mathbf{J}} \frac{\zeta_1}{\zeta_1 + \tau} \chi(\tau) w(\tau) d\tau, \tag{1.1}$$

where χ is a continuous function defined on $\mathbf{J}$; see [15, 19, 25]. Many researchers have examined a similar form of the above mentioned IE.

The integral equation:

$$w(\zeta_1) = y(\zeta_1) + w(\zeta_1) \int_a^b \varpi(\zeta_1, \tau) w(\tau) d\tau \tag{1.2}$$

studied by Banaś and Nalepa [5]. They discussed the space of functions with growths tempered by a modulus of continuity; they also proved a sufficient condition for relative compactness. As an example of the mentioned space they discussed Hölder space and some properties regarding the space. At the end, they proved the existence theorem for the Fredholm IE in Hölder space by using the classical Schauder fixed point theorem (FPT) and added an example to illustrate their result.

The following IE of Fredholm type has been studied by Caballero et al. [11]

$$w(\zeta_1) = y(\zeta_1) + w(\zeta_1) \int_{\mathbb{J}} \varpi(\zeta_1, \tau) w(r(\tau)) d\tau \tag{1.3}$$

and they added an example to illustrate their result.

The same year, Caballero et al. [12] investigated the existence of solutions of the equation

$$w(\zeta_1) = y(\zeta_1) + w(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau) \left\{ \max_{\tau \in [0, r(\tau)]} |w(\tau)| \right\} d\tau \tag{1.4}$$

by using the classical Schauder FPT, and they added an example to illustrate their result.

Ersoy and Furkan [20] examined the existence of solutions of the equation

$$w(\zeta_1) = y(\zeta_1) + w(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau \tag{1.5}$$

in Hölder space by using the classical Schauder FPT. They added some example to illustrate their result.

After that, similar types of equations have been examined by several authors; for references, see [9, 10, 21, 22, 33].

The objective of this paper is to discuss the existence and uniqueness of a solution, as well as the stability analysis of the nonlinear IE of Fredholm type

$$w(\zeta_1) = y(\zeta_1) + w(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau + z(\zeta_1) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau, \tag{1.6}$$

for $\zeta_1 \in \mathbf{J}$.

Notice that, Eq (1.6) is more general than many equations considered up to now and we obtain Eqs (1.1)–(1.5) as a special case of Eq (1.6) by using appropriate values. In Section 3, Example 1, we can see that Eq (1.1) is a particular case of Eq (1.6). For $(T_1 w)(\tau) = w(\tau)$ and $z(\zeta_1) = 0$ with $a = 0$ and $b = 1$, Eq (1.2) is a particular case of Eq (1.6). If we set $(T_1 w)(\tau) = w(r(\tau))$ and $z(\zeta_1) = 0$, then

Eq (1.3) becomes the particular case of Eq (1.6). For $(T_1w)(\tau) = \max_{\tau\in[0,r(\tau)]} |w(\tau)|$ and $z(\zeta_1) = 0$, Eq (1.4) is a particular case of Eq (1.6). If we set $z(\zeta_1) = 0$, the Eq (1.5) is a particular case of Eq (1.6). We discuss all of the particular cases as corollaries in Section 3.

Due to immense development in analysis and its application, the branch of nonlinear differential and IEs has motivated the researchers to find new dimensions for its effective analysis. IEs of the type illustrated by Eq (1.6) are often applicable in traffic theory, oscillating magnetic fields and electromagnetic and mathematical physics.

It is worthwhile to mention that more general functional IEs are analyzed by using Darbo's general theorem in Banach algebra [7]. An improved version of Darbo's FPT for the product of two operators as applied in conjunction with measures of noncompactness (MNCs), is proved in [6] and known as Darbo's general theorem. Darbo's general theorem for Banach algebra is a generalization of many FPTs considered up to now. In [8], Banas and Olszoy proved a FPT as applied in conjunction with MNCs, for the product of two operators, and they studied the monotonic solutions of a functional IE of fractional order in Banach algebra. In [16], M. Cichoń and Metwali discussed a FPT for the product of nonlinear operators and they extended it to some function spaces which are not necesarily Banach algebras. Recently, K. Cichoń et al. in [17], mainly studied the existence of the FPT on some functional problems associated with bilinear operators. In the year of 2022, M. Cichoń and Metwali [18] introduced integral-variation type Hölder space which is also a Banach algebra. They discussed norms and MNCs in the mentioned space and proved the existence theorem for solutions of quadratic IEs by using the Riemann-Liouville fractional operator.

2. Preliminaries and basic results

Throughout the study, we use the following:
E= a Banach space.
$\mathcal{M}_E$= family of all nonempty and bounded subsets of E.
$\mathcal{N}_E$= subfamily of relatively compact sets of E.
$C[\upsilon_1, \upsilon_2]$= space of continuous functions on $[\upsilon_1, \upsilon_2]$.
$C_\omega[\upsilon_1, \upsilon_2]$= space of functions with tempered increments on $[\upsilon_1, \upsilon_2]$; see [5].

The space satisfying the Hölder condition is an example of $C_\omega[\upsilon_1, \upsilon_2]$, which is provided in Example 2 of [5]. Denote $\mathcal{H}_\eta[\upsilon_1, \upsilon_2]$ by the space satisfying the Hölder condition

$$|w(\zeta_1) - w(\zeta_2)| \le K_w|\zeta_1 - \zeta_2|^\eta,\ \forall \zeta_1, \zeta_2 \in [\upsilon_1, \upsilon_2], 0<\eta \le 1.$$

The least possible constant $K_w>0$ satisfies the above inequality and is given by

$$K_w = \sup\left\{\frac{|w(\zeta_1) - w(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} : \zeta_1, \zeta_2 \in [\upsilon_1, \upsilon_2], \zeta_1 \neq \zeta_2\right\}.$$

Further, $\mathcal{H}_\eta[\upsilon_1, \upsilon_2]$ is a Banach space with the norm

$$\|w\|_\eta = |w(\upsilon_1)| + \sup\left\{\frac{|w(\zeta_1) - w(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} : \zeta_1, \zeta_2 \in [\upsilon_1, \upsilon_2], \zeta_1 \neq \zeta_2\right\}.$$

Also, the inequalities

$$\|w\|_\infty \le \max\{1, (\upsilon_2 - \upsilon_1)^\eta\}\|w\|_\eta \text{ and } \|w\|_\eta \le \max\{1, (\upsilon_2 - \upsilon_1)^{\gamma-\eta}\}\|w\|_\gamma,$$

$0<\eta<\gamma \leq 1$, hold; see [5]. Now, the sufficient condition for relative compactness which has been mentioned in Example 6 of [5] in the space $\mathcal{H}_\eta[\upsilon_1, \upsilon_2]$ with $0<\eta \leq 1$ is given as follows. It is noteworthy to mention here that this sufficient condition is one of the important results of this research work.

Theorem 1. *[5] If U is a bounded subset of $\mathcal{H}_\gamma[\upsilon_1, \upsilon_2]$, then U is relatively compact on $\mathcal{H}_\eta[\upsilon_1, \upsilon_2]$,* $0<\eta<\gamma \leq 1$.

Definition 1. *[3]* A MNC is a function $a : \mathcal{M}_E \to [0, \infty)$ that satisfies the following conditions:

(i) The family $\ker a = \{\mathcal{W} \in \mathcal{M}_E : a(\mathcal{W}) = 0\} \neq \emptyset$ and $\ker a \subset \mathcal{N}_E$.
(ii) $\mathcal{W} \subset \mathcal{V} \implies a(\mathcal{W}) \leq a(\mathcal{V})$.
(iii) $a(\bar{\mathcal{W}}) = a(\mathcal{W})$.
(iv) $a(\operatorname{Conv} \mathcal{W}) = a(\mathcal{W})$.
(v) $a(\lambda_1 \mathcal{W} + (1-\lambda_1)\mathcal{V}) \leq \lambda_1 a(\mathcal{W}) + (1-\lambda_1) a(\mathcal{V})$ for $\lambda_1 \in \mathbf{J}$.
(vi) If $\mathcal{W}_n \in \mathcal{M}_E$, $\mathcal{W}_n = \bar{\mathcal{W}}_n$, $\mathcal{W}_{n+1} \subset \mathcal{W}_n$ $\forall n = 1, 2, \ldots$ and $\lim_{n\to\infty} a(\mathcal{W}_n) = 0$, then $\bigcap_{n=1}^{\infty} \mathcal{W}_n \neq \emptyset$.
Also, a is sub-linear whenever it satisfies the following:
(vii) $a(\lambda_1 \mathcal{W}) = |\lambda_1| a(\mathcal{W})$ for $\lambda_1 \in \mathbb{R}$.
(viii) $a(\mathcal{W} + \mathcal{V}) \leq a(\mathcal{W}) + a(\mathcal{V})$.
Further, a has the maximum property if it satisfies
(ix) $a(\mathcal{W} \cup \mathcal{V}) = \max\{a(\mathcal{W}), a(\mathcal{V})\}$.
If $\ker a = \mathcal{N}_E$, then a is called full.

Now, we state Darbo's FPT [3], which is a generalization of the Schauder FPT and Banach FPT.

Theorem 2. *Let the mapping $G : \Theta \to \Theta$ be continuous and Θ be a nonempty, closed, bounded, and convex subset of E. If $a(GW) \leq ka(W)$, $k \in [0, 1)$, for any nonempty subset W of Θ, then G has at least one fixed point in Θ.*

Define

$$\begin{aligned}
\mathbf{f}(w, \epsilon) &= \sup\left\{\frac{|w(\zeta_1) - w(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} : \zeta_1, \zeta_2 \in [\upsilon_1, \upsilon_2], \zeta_1 \neq \zeta_2, |\zeta_1 - \zeta_2| \leq \epsilon\right\} \\
\mathbf{f}(W, \epsilon) &= \sup\{\mathbf{f}(w, \epsilon) : w \in W\} \\
\mathbf{f}_0(W) &= \lim_{\epsilon\to 0} \mathbf{f}(W, \epsilon).
\end{aligned}$$

It is worth mentioning that, Banaś and Nalepa in [4] introduced MNCs for Hölder space and applied them to Hölder space for the first time.

Theorem 3. *[4] The function $\mathbf{f}_0 : \mathcal{M}_{C_\omega([\upsilon_1,\upsilon_2])} \to [0, \infty)$ is a sublinear MNC with its maximum property in $C_\omega([\upsilon_1, \upsilon_2])$.*

Next, let us define a contraction mapping for any normed space E.

Definition 2. *[29] A mapping $F : E \to E$ is said to be a contraction if there is a positive real number $C_3 < 1$ such that $\|Fw - Fv\| \leq C_3 \|w - v\|$, $\forall\, w, v \in E$.*

Theorem 4. *[1, 2, Banach FPT] Let F be a contraction mapping on a Banach space E; then, F has a unique fixed point in E.*

3. Exsistence of solution

To study Eq (1.6), the following assumptions are required:

(i) For $y \in \mathcal{H}_\gamma(\mathbf{J})$ there exists $Y_\gamma>0$ such that

$$|y(\zeta_1) - y(\zeta_2)| \leq Y_\gamma|\zeta_1 - \zeta_2|^\gamma \ \forall \zeta_1, \zeta_2 \in \mathbf{J}.$$

(ii) $\varpi : \mathbf{J} \times \mathbf{J} \to \mathbb{R}$ is continuous and satisfies the Hölder condition with the exponent γ. Also,

$$|\varpi(\zeta_1, \tau) - \varpi(\zeta_2, \tau)| \leq \Gamma_\gamma|\zeta_1 - \zeta_2|^\gamma, \forall \zeta_1, \zeta_2, \tau \in \mathbf{J}.$$

(iii) The function $z \in \mathcal{H}_\gamma(\mathbf{J})$ satisfies the inequality

$$|z(\zeta_1) - z(\zeta_2)| \leq \mathcal{Z}_\gamma|\zeta_1 - \zeta_2|^\gamma, \ \forall \zeta_1, \zeta_2 \in \mathbf{J},$$

where $\mathcal{Z}_\gamma > 0$.

(iv) $T_1, T_2 : \mathcal{H}_\gamma(\mathbf{J}) \to C(\mathbf{J})$ are continuous operators on $\mathcal{H}_\gamma(\mathbf{J})$ and there exist increasing functions $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$||T_i w||_\infty \leq h_i(||w||_\gamma) \quad (i = 1, 2) \text{ for any } w \in \mathcal{H}_\gamma(\mathbf{J}).$$

(v) $h : \mathbf{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists an increasing continuous function $H : [0, \infty) \to [0, \infty)$ that satisfy the condition that $H(0) = 0$ and the following holds:

$$|h(\zeta_1, w) - h(\zeta_1, v)| \leq H(|w - v|), \ \forall \zeta_1 \in \mathbf{J} \text{ and } w, v \in \mathbb{R}.$$

Denote $\bar{\Gamma} = \sup\{|\varpi(\zeta_1, \tau)| : \zeta_1, \tau \in \mathbf{J}\}$ and $\bar{\Omega} = \sup\{|h(\zeta_1, 0)| : \zeta_1 \in \mathbf{J}\}$.

(vi) There exists an $r_0>0$ that is a solution to the inequality.

$$|y(0)| + Y_\gamma + r_0 h_1(r_0)(\bar{\Gamma} + \Gamma_\gamma) + (|z(0)| + \mathcal{Z}_\gamma)(H(h_2(r_0)) + \bar{\Omega}) \leq r$$

such that $\bar{\Gamma} h_1(r_0)<1$.

Theorem 5. *Under the assumptions (i)–(vi), there is at least one solution of Eq (1.6) in the space* $\mathcal{H}_\eta(\mathbf{J})$ *with* $0 < \eta < \gamma \leq 1$.

Proof. Define an operator G on $\mathcal{H}_\eta(\mathbf{J})$ by

$$(Gw)(\zeta_1) = y(\zeta_1) + w(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau + z(\zeta_1) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau, \tag{3.1}$$

where $w \in \mathcal{H}_\eta(\mathbf{J})$.

Choose $\zeta_1, \zeta_2 \in \mathbf{J}$ with $\zeta_1 \neq \zeta_2$. Claim that $Gw \in \mathcal{H}_\eta(\mathbf{J})$. By assumption, we obtain

$$\begin{aligned}
&\frac{|(Gw)(\zeta_1) - (Gw)(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \\
\leq &\frac{|y(\zeta_1) - y(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} + \frac{|w(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau - w(\zeta_2) \int_{\mathbf{J}} \varpi(\zeta_2, \tau)(T_1 w)(\tau) d\tau|}{|\zeta_1 - \zeta_2|^\eta} \\
&+ \frac{|z(\zeta_1) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau - z(\zeta_2) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau|}{|\zeta_1 - \zeta_2|^\eta} \\
\leq &\frac{Y_\gamma |\zeta_1 - \zeta_2|^\gamma}{|\zeta_1 - \zeta_2|^\eta} + \frac{|w(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau - w(\zeta_2) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau|}{|\zeta_1 - \zeta_2|^\eta} \\
&+ \frac{|w(\zeta_2) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau - w(\zeta_2) \int_{\mathbf{J}} \varpi(\zeta_2, \tau)(T_1 w)(\tau) d\tau|}{|\zeta_1 - \zeta_2|^\eta} \\
&+ \frac{|z(\zeta_1) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau - z(\zeta_2) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau|}{|\zeta_1 - \zeta_2|^\eta} \\
\leq &Y_\gamma |\zeta_1 - \zeta_2|^{\gamma - \eta} + \frac{|w(\zeta_1) - w(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \bar{\Gamma} \int_{\mathbf{J}} |(T_1 w)(\tau)| d\tau \\
&+ \frac{|w(\zeta_2)| \Gamma_\gamma |\zeta_1 - \zeta_2|^\gamma}{|\zeta_1 - \zeta_2|^\eta} \int_{\mathbf{J}} |(T_1 w)(\tau)| d\tau \\
&+ \frac{\mathcal{Z}_\gamma |\zeta_1 - \zeta_2|^\gamma}{|\zeta_1 - \zeta_2|^\eta} \int_{\mathbf{J}} (|h(\tau, (T_2 w)(\tau) - h(\tau, 0))| + |h(\tau, 0)|) d\tau \\
\leq &Y_\gamma |\zeta_1 - \zeta_2|^{\gamma - \eta} + \frac{|w(\zeta_1) - w(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \bar{\Gamma} \, \|(T_1 w)\|_\infty \\
&+ \|w\|_\infty \, \|(T_1 w)\|_\infty \, \Gamma_\gamma |\zeta_1 - \zeta_2|^{\gamma - \eta} + \mathcal{Z}_\gamma |\zeta_1 - \zeta_2|^{\gamma - \eta} \int_{\mathbf{J}} (H(|(T_2 w)(\tau)|) + |h(\tau, 0)|) d\tau.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\frac{|(Gw)(\zeta_1) - (Gw)(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \leq &Y_\gamma |\zeta_1 - \zeta_2|^{\gamma - \eta} + \frac{|w(\zeta_1) - w(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \bar{\Gamma} \, \|(T_1 w)\|_\infty \\
&+ \|w\|_\infty \, \|(T_1 w)\|_\infty \, \Gamma_\gamma |\zeta_1 - \zeta_2|^{\gamma - \eta} \\
&+ \mathcal{Z}_\gamma |\zeta_1 - \zeta_2|^{\gamma - \eta} \int_{\mathbf{J}} (H(\|(T_2 w)\|_\infty) + \bar{\Omega}) d\tau. \qquad (3.2)
\end{aligned}$$

Also,

$$(Gw)(0) = y(0) + w(0) \int_{\mathbf{J}} \varpi(0, \tau)(T_1 w)(\tau) d\tau + z(0) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau,$$

$$\text{i.e.,} \quad |(Gw(0)| \leq |y(0)| + |w(0)| \bar{\Gamma} \, \|(T_1 w)\|_\infty + |z(0)| (H(\|(T_2 w)\|_\infty) + \bar{\Omega}). \qquad (3.3)$$

From Eqs (3.2) and (3.3) we obtain the following:

$$
\begin{aligned}
&|(Gw)(0)| + \frac{|(Gw)(\zeta_1) - (Gw)(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \\
&\le |y(0)| + \bar{\Gamma}\,\|T_1 w\|_\infty \left[|w(0)| + \frac{|w(\zeta_1) - (T_1 w)(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \right] \\
&+ |z(0)|(H(\|(T_2 w)\|_\infty) + \bar{\Omega}) + Y_\gamma |\zeta_1 - \zeta_2|^{\gamma-\eta} \\
&+ \Gamma_\gamma |\zeta_1 - \zeta_2|^{\gamma-\eta}\,\|w\|_\infty\,\|(T_1 w)\|_\infty + \mathcal{Z}_\gamma |\zeta_1 - \zeta_2|^{\gamma-\eta}(H(\|T_2 w\|_\infty) + \bar{\Omega}) \\
&\le |y(0)| + \bar{\Gamma}\,\|T_1 w\|_\infty\,\|w\|_\eta + |z(0)|(H(\|(T_2 w)\|_\infty) + \bar{\Omega}) + Y_\gamma \\
&+ \Gamma_\gamma\,\|w\|_\infty\,\|(T_1 w)\|_\infty + \mathcal{Z}_\gamma (H(\|T_2 w\|_\infty) + \bar{\Omega}). \\
&\le |x(0)| + Y_\gamma + \|u\|_\eta h_1(\|w\|_\eta)(\bar{\Gamma} + \Gamma_\gamma) \\
&+ (|z(0)| + \mathcal{Z}_\gamma)(H(h_2(\|w\|_\eta)) + \bar{\Omega}).
\end{aligned}
$$

Thus, we get

$$
\|Gw\|_\eta \le |y(0)| + X_\gamma + \|u\|_\eta h_1(\|w\|_\eta)(\bar{\Gamma} + \Gamma_\gamma) + (|z(0)| + \mathcal{Z}_\gamma)(H(h_2(\|w\|_\eta)) + \bar{\Omega}) < \infty. \tag{3.4}
$$

This proves that the operator G transforms $\mathcal{H}_\eta(\mathbf{J})$ to itself. Since the positive number r_0 is the solution of the inequality given in hypothesis (vi), the following inequality holds:

$$
\|Gw\|_\eta \le |y(0)| + Y_\gamma + r_0 h_1(r_0)(\bar{\Gamma} + \Gamma_\gamma) + (|z(0)| + \mathcal{Z}_\gamma)(H(h_2(r_0)) + \bar{\Omega}) \le r_0.
$$

As a result, it follows that G transforms the ball

$$
B_{r_0} = \{w \in \mathcal{H}_\gamma(\mathbf{J}) : \|w\|_\eta \le r_0\}
$$

into itself. Now, we will prove that the operator G is continuous on $B_{r_0}(\subset \mathcal{H}_\eta(\mathbf{J}))$. Suppose that $w \in B_{r_0}$ and $\delta>0$. Assume that $v \in B_{r_0}$ in such a way that $\|w - v\|_\eta \le \delta$ which yields the following:

$$
\begin{aligned}
&\frac{[(Gw)(\zeta_1) - (Gv)(\zeta_1)] - [(Gw)(\zeta_2) - (Gv)(\zeta_2)]}{|\zeta_1 - \zeta_2|^\eta} \\
=&\Bigg| \left(y(\zeta_1) + w(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau + z(\zeta_1) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau \right) \\
&- \left(y(\zeta_1) + v(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 v)(\tau) d\tau + z(\zeta_1) \int_{\mathbf{J}} h(\tau, (T_2 v)(\tau)) d\tau \right) \\
&- \left(y(\zeta_2) + w(\zeta_2) \int_{\mathbf{J}} \varpi(\zeta_2, \tau)(T_1 w)(\tau) d\tau + z(\zeta_2) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau \right) \\
&+ \left(y(\zeta_2) + v(\zeta_2) \int_{\mathbf{J}} \varpi(\zeta_2, \tau)(T_1 v)(\tau) d\tau + z(\zeta_2) \int_{\mathbf{J}} h(\tau, (T_2 v)(\tau)) d\tau \right) \Bigg| |\zeta_1 - \zeta_2|^{-\eta}
\end{aligned}
$$

$$
\begin{aligned}
=&\Bigg|\left(w(\zeta_1)\int_{\mathbf{J}}\varpi(\zeta_1,\tau)(T_1w)(\tau)d\tau - v(\zeta_1)\int_{\mathbf{J}}\varpi(\zeta_1,\tau)(T_1w)(\tau)d\tau\right)\\
&+\left(v(\zeta_1)\int_{\mathbf{J}}\varpi(\zeta_1,\tau)(T_1w)(\tau)d\tau - v(\zeta_1)\int_{\mathbf{J}}\varpi(\zeta_1,\tau)(T_1v)(\tau)d\tau\right)\\
&-\left(w(\zeta_2)\int_{\mathbf{J}}\varpi(\zeta_2,\tau)(T_1w)(\tau)d\tau - v(\zeta_2)\int_{\mathbf{J}}\varpi(\zeta_2,\tau)(T_1w)(\tau)d\tau\right)\\
&-\left(v(\zeta_2)\int_{\mathbf{J}}\varpi(\zeta_2,\tau)(T_1w)(\tau)d\tau - v(\zeta_2)\int_{\mathbf{J}}\varpi(\zeta_2,\tau)(T_1v)(\tau)d\tau\right)\\
&+\left(z(\zeta_1)\int_{\mathbf{J}}h(\tau,(T_2w)(\tau))d\tau - z(\zeta_1)\int_{\mathbf{J}}h(\tau,(T_2v)(\tau))d\tau\right)\\
&-\left(z(\zeta_2)\int_{\mathbf{J}}h(\tau,(T_2w)(\tau))d\tau - z(\zeta_2)\int_{\mathbf{J}}h(\tau,(T_2v)(\tau))d\tau\right)\Bigg||\zeta_1-\zeta_2|^{-\eta}\\
=&\Bigg|[w(\zeta_1)-v(\zeta_1)]\int_{\mathbf{J}}\varpi(\zeta_1,\tau)(T_1w)(\tau)d\tau + v(\zeta_1)\int_{\mathbf{J}}\varpi(\zeta_1,\tau)\left[(T_1w)(\tau)-(T_1v)(\tau)\right]d\tau\\
&-[w(\zeta_2)-v(\zeta_2)]\int_{\mathbf{J}}\varpi(\zeta_2,\tau)(T_1w)(\tau)d\tau - v(\zeta_2)\int_{\mathbf{J}}\varpi(\zeta_2,\tau)\left[(T_1w)(\tau)-(T_1v)(\tau)\right]d\tau\\
&+z(\zeta_1)\int_{\mathbf{J}}\left[h(\tau,(T_2w)(\tau))-h(\tau,(T_2v)(\tau))\right]d\tau - z(\zeta_2)\int_{\mathbf{J}}\left[h(\tau,(T_2w)(\tau))-h(\tau,(T_2v)(\tau))\right]d\tau\Bigg|.|\zeta_1-\zeta_2|^{-\eta}\\
\le&\frac{\left|[w(\zeta_1)-v(\zeta_1)]-[w(\zeta_2)-v(\zeta_2)]\right|}{|\zeta_1-\zeta_2|^{\eta}}\left|\int_{\mathbf{J}}\varpi(\zeta_1,\tau)(T_1w)(\tau)d\tau\right| + |w(\zeta_2)-v(\zeta_2)|\\
&\int_{\mathbf{J}}\frac{|\varpi(\zeta_1,\tau)-\varpi(\zeta_2,\tau)|}{|\zeta_1-\zeta_2|^{\eta}}\left|(T_1w)(\tau)\right|d\tau + \frac{|v(\zeta_1)-v(\zeta_2)|}{|\zeta_1-\zeta_2|^{\eta}}\left|\int_{\mathbf{J}}\varpi(\zeta_1,\tau)\left[(T_1w)(\tau)-(T_1v)(\tau)\right]d\tau\right|\\
&+\left|v(\zeta_2)\int_{\mathbf{J}}\frac{[\varpi(\zeta_1,\tau)-\varpi(\zeta_2,\tau)]}{|\zeta_1-\zeta_2|^{\eta}}\left[(T_1w)(\tau)-(T_1v)(\tau)\right]d\tau\right| + \frac{|z(\zeta_1)-z(\zeta_2)|}{|\zeta_1-\zeta_2|^{\eta}}\\
&\left|\int_{\mathbf{J}}h(\tau,(T_2w)(\tau))-h(\tau,(T_2v)(\tau))d\tau\right|\\
\le&\|w-v\|_{\eta}\|T_1w\|_{\infty}\int_{\mathbf{J}}|\varpi(\zeta_1,\tau)|\,d\tau + \Big|(w(\zeta_2)-v(\zeta_2))-(w(0)-v(0))\\
&+(w(0)-v(0))\Big|\,\|(T_1w)\|_{\infty}\int_{\mathbf{J}}\frac{\Gamma_{\gamma}|\zeta_1-\zeta_2|^{\gamma}}{|\zeta_1-\zeta_2|^{\eta}}d\tau + \|v\|_{\eta}\|T_1w-T_1v\|_{\infty}\int_{\mathbf{J}}|\varpi(\zeta_1,\tau)|\,d\tau\\
&+\|v\|_{\infty}\|(T_1w)-(T_1v)\|_{\infty}\int_{\mathbf{J}}\frac{\Gamma_{\gamma}|\zeta_1-\zeta_2|^{\gamma}}{|\zeta_1-\zeta_2|^{\eta}}d\tau + \frac{\mathcal{Z}_{\gamma}|\zeta_1-\zeta_2|^{\gamma}}{|\zeta_1-\zeta_2|^{\eta}}H(\|(T_2w)-(T_2v)\|_{\infty})\\
\le&\|w-v\|_{\eta}\|(T_1w)\|_{\infty}\bar{\Gamma} + \Bigg[|w(0)-v(0)|\\
&+\sup_{\zeta_1,\zeta_2\in\mathbf{J}}\frac{\left|[w(\zeta_1)-v(\zeta_1)]-[w(\zeta_2)-v(\zeta_2)]\right|}{|\zeta_1-\zeta_2|^{\eta}}\times\sup_{\zeta_1,\zeta_2\in\mathbf{J}}|\zeta_1-\zeta_2|^{\eta}\Bigg]\|(T_1w)\|_{\infty}\Gamma_{\gamma}\\
&+\|v\|_{\eta}\|(T_1w)-(T_1v)\|_{\infty}\bar{\Gamma} + \|v\|_{\infty}\|(T_1w)-(T_1v)\|_{\infty}\Gamma_{\gamma} + \mathcal{Z}_{\gamma}H(\|(T_2w)-(T_2v)\|_{\infty})
\end{aligned}
$$

$$
\begin{aligned}
&\le \|w-v\|_\eta \|(T_1w)\|_\infty \bar{\Gamma} + \|w-v\|_\eta \|(T_1w)\|_\infty \Gamma_\gamma \\
&\quad + \|v\|_\eta \|(T_1w)-(T_1v)\|_\infty \bar{\Gamma} + \|v\|_\eta \|(T_1w)-(T_1v)\|_\infty \Gamma_\gamma + \mathcal{Z}_\gamma H(\|(T_2w)-(T_2v)\|_\infty) \\
&\le \|w-v\|_\eta h_1(\|w\|_\eta)(\bar{\Gamma}+\Gamma_\gamma) + \|v\|_\eta \|T_1w-T_1v\|_\infty (\bar{\Gamma}+\Gamma_\gamma) + \mathcal{Z}_\gamma H(\|T_2w-T_2v\|_\infty).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{[(Gw)(\zeta_1)-(Gv)(\zeta_1)]-[(Gw)(\zeta_2)-(Gv)(\zeta_2)]}{|\zeta_1-\zeta_2|\eta} \le & \delta h_1(r_0)(\bar{\Gamma}+\Gamma_\gamma) + r_0 \|T_1w-T_1v\|_\infty (\bar{\Gamma}+\Gamma_\gamma) \\
& + \|z\|_\gamma H(\|T_2w-T_2v\|_\infty) < \frac{\epsilon}{2}.
\end{aligned} \tag{3.5}
$$

In a similar way,

$$
\begin{aligned}
|(Gw)(0)-(Gv)(0)| \le & \left| w(0)\int_{\mathbf{J}} \varpi(0,\tau)(T_1w)(\tau)d\tau - w(0)\int_{\mathbf{J}} \varpi(0,\tau)(T_1v)(\tau)d\tau \right| \\
& + \left| w(0)\int_{\mathbf{J}} \varpi(0,\tau)(T_1v)(\tau)d\tau - v(0)\int_{\mathbf{J}} \varpi(0,\tau)(T_1v)(\tau)d\tau \right| \\
& + \left| z(0)\int_{\mathbf{J}} [h(\tau,(T_2w)(\tau)) - h(\tau,(T_2v)(\tau))]\, d\tau \right| \\
\le & \|w\|_\infty \int_{\mathbf{J}} |\varpi(0,\tau)|\, |(T_1w)(\tau)-(T_1v)(\tau)|\, d\tau \\
& + |w(0)-v(0)| \int_{\mathbf{J}} |\varpi(0,\tau)|\, |(T_1v)(\tau)|\, d\tau + |z(0)| \int_{\mathbf{J}} H(|(T_2w)(\tau)-(T_2v)(\tau)|)d\tau \\
\le & \|w\|_\eta \|T_1w-T_1v\|_\infty \bar{\Gamma} + \|T_1v\|_\infty \|w-v\|_\infty \bar{\Gamma} + |z(0)|\, H(\|T_2w-T_2v\|_\infty).
\end{aligned}
$$

Hence,

$$
|(Gw)(0)-(Gv)(0)| \le \|w\|_\eta \|T_1w-T_1v\|_\infty \bar{\Gamma} + h_1(\|v\|_\eta)\, \|w-v\|_\infty \bar{\Gamma} + |z(0)|\, H(\|T_2w-T_2v\|_\infty). \tag{3.6}
$$

Combining Eqs (3.5) and (3.6), we get

$$
\begin{aligned}
\|Gw-Gv\|_\eta \le & \|w-v\|_\eta h_1(\|w\|_\eta)(\bar{\Gamma}+\Gamma_\gamma) + \|v\|_\eta \|T_1w-T_1v\|_\infty (\bar{\Gamma}+\Gamma_\gamma) + \mathcal{Z}_\gamma H(\|T_2w-T_2v\|_\infty) \\
& + \|w\|_\eta \|T_1w-T_1v\|_\infty \bar{\Gamma} + h_1(\|v\|_\eta)\, \|w-v\|_\eta \bar{\Gamma} + |z(0)|\, H(\|T_2w-T_2v\|_\infty) \\
\le & \|w-v\|_\eta h_1(r_0)(\bar{\Gamma}+\Gamma_\gamma) + r_0 \|T_1w-T_1v\|_\infty (\bar{\Gamma}+\Gamma_\gamma) + \mathcal{Z}_\gamma H(\|T_2w-T_2v\|_\infty) \\
& + r_0 \|T_1w-T_1v\|_\infty \bar{\Gamma} + h_1(r_0)\, \|w-v\|_\infty \bar{\Gamma} + |z(0)|\, H(\|T_2w-T_2v\|_\infty).
\end{aligned}
$$

Since $\|w\|_\eta \le r_0$, $\|v\|_\eta \le r_0$ and $h_i \quad (i=1,2)$ is non-decreasing, $h_i(\|w\|_\eta) \le h_i(r_0)$. Since $T_i : \mathcal{H}_\gamma(\mathbf{J}) \to C(\mathbf{J})$ are continuous operators with respect to the norm $\|.\|_\eta$, they are also continuous at the point $v \in B_{r_0}$. Let $\epsilon > 0$ be arbitrary, then there exists $\delta > 0$ such that

$$
\|T_iw - T_iv\|_\infty < \epsilon \quad (i=1,2)
$$

for all $w \in B_{r_0}$ with $\|w-v\|_\eta < \delta$.

$$
\begin{aligned}
\|Gw-Gv\|_\eta \le & \delta h_1(r_0)(\bar{\Gamma}+\Gamma_\gamma) + r_0\epsilon(\bar{\Gamma}+\Gamma_\gamma) + \mathcal{Z}_\gamma H(\epsilon) \\
& + r_0\epsilon\bar{\Gamma} + h_1(r_0)\delta\bar{\Gamma} + |z(0)|\, H(\epsilon) \\
\le & \delta h_1(r_0)(2\bar{\Gamma}+\Gamma_\gamma) + r_0\epsilon(2\bar{\Gamma}+\Gamma_\gamma) + H(\epsilon)(\mathcal{Z}_\gamma + |z(0)|).
\end{aligned}
$$

Taking into account the continuity of H with $H(0) = 0$, we infer that the operator F is continuous at the point $v \in B_{r_0}$. This proves that the operator G is continuous on the ball B_{r_0} with respect to the norm $\|.\|_\eta$.
Assume a non-empty set $W \subseteq B_{r_0}$, for ϵ>0 and a function $w \in W$. For $|\zeta_1 - \zeta_2| \leq \epsilon$, in the context of Eq (3.2), now we will get

$$\begin{aligned}\frac{|(Gw)(\zeta_1) - (Gw)(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \leq & Y_\gamma \epsilon^{\gamma-\eta} + \frac{|w(\zeta_1) - w(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \bar{\Gamma} \|(T_1 w)\|_\infty \\ & + \|w\|_\infty \|(T_1 w)\|_\infty \Gamma_\gamma \epsilon^{\gamma-\eta} \\ & + \mathcal{Z}_\gamma \epsilon^{\gamma-\eta} \int_{\mathbf{J}} (H(\|(T_2 w)\|_\infty) + \bar{\Omega}) d\tau. \\ \leq & Y_\gamma \epsilon^{\gamma-\eta} + \frac{|w(\zeta_1) - w(\zeta_2)|}{|\zeta_1 - \zeta_2|^\eta} \bar{\Gamma} h_1(r_0) \\ & + r_0 h_1(r_0) \Gamma_\gamma \epsilon^{\gamma-\eta} + \mathcal{Z}_\gamma \epsilon^{\gamma-\eta} (H(h_2(r_0)) + \bar{\Omega}).\end{aligned}$$

Hence

$$\mathbf{f}(Gw, \epsilon) \leq Y_\gamma \epsilon^{\gamma-\eta} + \mathbf{f}(w, \epsilon) \bar{\Gamma} h_1(r_0) + r_0 h_1(r_0) \Gamma_\gamma \epsilon^{\gamma-\eta} + \mathcal{Z}_\gamma \epsilon^{\gamma-\eta} (H(r_0) + \bar{\Omega}).$$

This implies

$$\mathbf{f}_0(GW) \leq \bar{\Gamma} h_1(r_0) \mathbf{f}_0(W).$$

Thus, on the basis of assumption (*vi*) and Darbo's FPT as given by Theorem 2, we deduce that the operator G has at least one fixed point $w \in B_{r_0} \subset \mathcal{H}_\eta(\mathbf{J})$. Obviously, $w = w(\zeta_1)$ is a solution of Eq (1.6). Now the proof is complete. □

To prove the efficiency of the above result, we consider the following example.

Example 1. Consider the following nonlinear Chandrasekhar integral equation:

$$w(\zeta_1) = 1 + w(\zeta_1) \int_{\mathbf{J}} \frac{\zeta_1}{\zeta_1 + \tau} \chi(\tau) w(\tau) d\tau, \tag{3.6}$$

where the function $\chi : \mathbf{J} \to \mathbb{R}$ is continuous and such that $\chi(0) = 0$. The Eq (3.6) is a special case of Eq (1.6), if we have the following:

$$y(\zeta_1) = 1, \quad T_1(w)(\tau) = w(\tau), \quad z(\zeta_1) = 0, \quad h(\tau, (T_2 w)(\tau)) = 0, \quad (T_2 w)(\tau) = 0$$

and

$$\varpi(\zeta_1, \tau) = \begin{cases} 0; & \tau = 0, \zeta_1 \geq 0 \\ \frac{\zeta_1}{\zeta_1 + \tau} \chi(\tau); & \tau \neq 0, \zeta_1 \geq 0. \end{cases} \tag{3.7}$$

Let us choose the function $\chi : \mathbf{J} \to \mathbb{R}$ as $\chi(\tau) = \frac{\tau}{6}$. Now, $|y(\zeta_1) - y(\zeta_2)| = 0$ implies that $Y_\gamma = 0, |y(0)| = 1$. and $|z(\zeta_1) - z(\zeta_2)| = 0$ implies that $\mathcal{Z}_\gamma = 0,\ |z(0)| = 0$. It can be shown that the function $k : \mathbf{J} \times \mathbf{J} \to \mathbb{R}$ is defined by Eq.(3.7) and is continuous. If $\zeta_1, \zeta_2 \in \mathbf{J}$ and $\tau = 0$, then

$$|\varpi(\zeta_1, \tau) - \varpi(\zeta_2, \tau)| = 0.$$

If $\zeta_1, \zeta_2 \in \mathbf{J}$ and $\tau \neq 0$, then

$$\begin{aligned}
|\varpi(\zeta_1,\tau) - \varpi(\zeta_2,\tau)| &= \left|\frac{\zeta_1}{\zeta_1+\tau} - \frac{\zeta_2}{\zeta_2+\tau}\right| |\chi(\tau)| \\
&= \left|\frac{(\zeta_1-\zeta_2)\tau}{(\zeta_1+\tau)(\zeta_2+\tau)}\right| |\chi(\tau)| \\
&= \left|\frac{\zeta_1-\zeta_2}{\tau}\right| |\frac{\tau}{6}| \\
&= \frac{1}{6}|\zeta_1-\zeta_2|.
\end{aligned}$$

Therefore $\Gamma_1 = \frac{1}{6}$. Also $\|T_1(w)\|_\gamma = \|w\|_\gamma$ implies that $h_1(r) = r$. Clearly $H(r) = 0,\ h_2(r_0) = 0$.

$$\begin{aligned}
\bar{\Omega} &= \sup\{|h(\zeta_1, 0)| : \zeta_1 \in \mathbf{J}\} = 0, \\
\bar{\Gamma} &= \sup\{|\varpi(\zeta_1,\tau)| : \zeta_1,\tau \in \mathbf{J}\} = 0.08333.
\end{aligned}$$

The first inequality of assumption (vi) has the form

$$\begin{aligned}
&|y(0)| + X_\gamma + r_0 h_1(r_0)(\bar{\Gamma} + \Gamma_\gamma) + (|z(0)| + \mathcal{Z}_\gamma)(H(h_2(r_0)) + \bar{\Omega}) \leq r \\
\Longrightarrow\ & 1 + 0 + r \times r\left(0.08333 + \frac{1}{6}\right) + (0+0)(H(0)+0) \leq r \qquad (3.8)\\
\Longrightarrow\ & 1 + 0.24999r^2 \leq r.
\end{aligned}$$

Therefore, $r_1 = 1.98743$, $r_2 = 2.01272$. It is easy to see that, the above inequality and the second inequality of assumption (*vi*) are satisfied for $r \in [1.98743, 2.01272]$. Therefore, by using Theorem 5, we get that Eq (3.6) has at least one solution in $\mathcal{H}_\eta(\mathbf{J})$ for $\eta \in \mathbf{J}$.

Example 2. Consider the following nonlinear integral equation of Fredholm type

$$w(\zeta_1) = \frac{\zeta_1^2}{8} + \frac{w(\zeta_1)}{4}\int_{\mathbf{J}}(\zeta_1^2+\tau)\frac{\log(1+\tau^2)w}{2}d\tau + \arctan\zeta_1\int_{\mathbf{J}}\frac{\tau w}{2}d\tau \ \text{ for } \zeta_1 \in \mathbf{J}. \qquad (3.9)$$

Comparing Eq (3.9) with Eq (1.6), we have

$$\begin{aligned}
&y(\zeta_1) = \frac{\zeta_1^2}{8};\ \ \varpi(\zeta_1,\tau) = \frac{\zeta_1^2+\tau}{4};\ \ (T_1w)(\tau) = \frac{1}{2}\log(1+\tau^2)w; \\
&z(\zeta_1) = \arctan\zeta_1;\ \ h(\tau, w(\tau)) = \frac{\tau w}{2};\ \ (T_2w)(\tau) = \frac{w}{2}.
\end{aligned}$$

Now,

$$\begin{aligned}
|\varpi(\zeta_1,\tau) - \varpi(\zeta_2,\tau)| &= \frac{1}{4}\left|\zeta_1^2 - \zeta_2^2\right| \\
&\leq \frac{1}{2}|\zeta_1-\zeta_2|,
\end{aligned}$$

$$\begin{aligned}
|z(\zeta_1) - z(\zeta_2)| &= \left|\tan^{-1}\zeta_1 - \tan^{-1}\zeta_2\right| \\
&\leq |\zeta_1-\zeta_2|
\end{aligned}$$

$$|y(\zeta_1) - y(\zeta_2)| = \frac{1}{8}\left|\zeta_1^2 - \zeta_2^2\right| \\ \leq \frac{1}{4}|\zeta_1 - \zeta_2|.$$

Hence, we obtain that $Y_1 = \frac{1}{4}, \Gamma_1 = \frac{1}{2}, Z_1 = 1$ and $H(r) = \frac{r}{2}$. Also

$$|h(\tau, (T_2w)(\tau) - h(\tau, (T_2v)(\tau)| = \frac{1}{2}|\tau w - \tau v| \\ \leq \frac{1}{2}|w - v|.$$

Therefore, $H(r) = \frac{r}{2}$. Also,

$$\|T_1 w\|_\infty = \sup_{\tau\in\mathbf{J}}\left|\frac{1}{2}\log(1+\tau^2)w\right| \\ \leq \frac{1}{2}\sup_{\tau\in\mathbf{J}}|w| \\ \leq \frac{1}{2}\|w\|_\gamma,$$

and

$$\|T_2 w\|_\infty = \frac{1}{2}\sup_{\tau\in\mathbf{J}}|w| \\ \leq \frac{1}{2}\|w\|_\gamma.$$

Therefore, $h_1(r) = \frac{r}{2}, h_2(r) = \frac{r}{2}$ we get

$$\bar{\Omega} = \sup\{|h(\zeta_1, 0)| : \zeta_1 \in \mathbf{J}\} = 0 \text{ and } \bar{\Gamma} = \sup\{|\varpi(\zeta_1, \tau)| : \zeta_1, \tau \in \mathbf{J}\} = \frac{1}{2}.$$

Also T_1 and T_2 are continuous with respect to the sup-norm. The first inequality of assumption (vi) has the following form:

$$0 + \frac{1}{4} + rh_1(r)(\frac{1}{2} + \frac{1}{2}) + (0+1)\left(\frac{h_2(r)}{2} + 0\right) \leq r \\ \implies 1 + 2r^2 - 3r \leq 0.$$

Therefore, $r_1 = \frac{1}{2}$ and $r_2 = 1$. It is easy to see that, the above inequality and the second inequality of assumption (vi) are satisfied for $r \in [\frac{1}{2}, 1]$. Therefore, by using Theorem 5, Eq (3.9) has at least one solution in Hölder space $\mathcal{H}_\eta(\mathbf{J})$ for $\eta \in \mathbf{J}$.

Corollary 1. *If we choose* $(T_1w)(\tau) = w(\tau)$ *and* $z(\zeta_1) = 0$, *then Eq (1.2) as presented in [5] is a particular case of Eq (1.6) with* $a = 0, b = 1$.

Corollary 2. *If we choose* $(T_1w)(\tau) = \left\{\max_{\tau\in[0,r(\tau)]}|w(\tau)|\right\}$ *and* $z(\zeta_1) = 0$ *then Eq (1.4) from [12] is a particular case of Eq (1.6).*

Corollary 3. *If we choose* $z(\zeta_1) = 0$ *then Eq (1.5) from [20] is a particular case of Eq (1.6).*

4. Uniqueness of the solution

In this section, we intend to prove the uniqueness of the solution of Eq (1.6) by using the Banach FPT. Here we apply assumptions (*i*)–(*iii*) of Section 3, along with the following assumptions:

$(iv)'$ $T_i : \mathcal{H}_\gamma(\mathbf{J}) \to \mathcal{H}_\gamma(\mathbf{J})$ are continuous operators with respect to the norm $||.||_\eta$ and there exists increasing functions $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following inequality holds:

$$||T_i w||_\gamma \leq h_i(||w||_\gamma) \quad (i = 1, 2) \text{ for any } w \in \mathcal{H}_\gamma(\mathbf{J}).$$

$(v)'$ $h : \mathbf{J} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exists $\mathcal{L} > 0$ for which

$$|h(\zeta_1, w) - h(\zeta_1, v)| \leq \mathcal{L}|w - v|, \ \forall \zeta_1 \in \mathbf{J} \text{ and } w, v \in \mathbb{R}$$

holds. Also, consider

$$\bar{\Gamma} = \sup\{|\varpi(\zeta_1, \tau)| : \zeta_1, \tau \in \mathbf{J}\} \text{ and } \bar{\Omega} = \sup\{|h(\zeta_1, 0)| : \zeta_1 \in \mathbf{J}\}.$$

$(vi)'$ $T_1, T_2 : \mathcal{H}_\gamma(\mathbf{J}) \to \mathcal{H}_\gamma(\mathbf{J})$ are contraction mappings, more precisely

$$||T_1 w - T_1 v||_\gamma \leq C_1 ||w - v||_\gamma \text{ and } ||T_2 w - T_2 v||_\gamma \leq C_2 ||w - v||_\gamma.$$

$(vii)'$ $h_1(R)(2\bar{\Gamma} + \Gamma_\gamma) + RC_1(2\bar{\Gamma} + \Gamma_\gamma) + \mathcal{L}C_2(|z(0)| + \mathcal{Z}_\gamma) < 1.$

Theorem 6. *Under the assumptions (i)–(iii) of Section 3 and* $(iv)'$*–*$(vii)'$ *of Section 4, the Eq (1.6) has a unique solution in* $\mathcal{H}_\eta(\mathbf{J})$*, where* $0<\eta<\gamma \leq 1$.

Proof. We have defined the operator G in Eq (3.1). To show that G is well-defined. The proof is similar to Theorem 5. We obtain

$$||Gw||_\eta \leq |y(0)| + Y_\gamma + ||w||_\eta h_1(||w||_\eta)(\bar{\Gamma} + \Gamma_\gamma) + (|z(0)| + \mathcal{Z}_\gamma)(\mathcal{L}h_2(||w||_\eta) + \bar{\Omega}) < \infty.$$

Next, we shall show the contraction of the operator G. From Theorem 5, we get

$$\begin{aligned}
||Gw - Gv||_\eta \leq & ||w - v||_\eta \, h_1(||w||_\eta)(\bar{\Gamma} + \Gamma_\gamma) + ||v||_\eta \, ||T_1 w - T_1 v||_\eta \, (\bar{\Gamma} + \Gamma_\gamma) + \mathcal{Z}_\gamma \mathcal{L} \, ||T_2 w - T_2 v||_\eta \\
& + ||w||_\eta \, ||T_1 w - T_1 v||_\eta \, \bar{\Gamma} + h_1(||w||_\eta) \, ||w - v||_\eta \, \bar{\Gamma} + |z(0)| \, \mathcal{L} \, ||T_2 w - T_2 v||_\eta \\
\leq & ||w - v||_\eta \, h_1(R)(\bar{\Gamma} + \Gamma_\gamma) + RC_1 \, ||w - v||_\eta \, (\bar{\Gamma} + \Gamma_\gamma) + \mathcal{Z}_\gamma \mathcal{L} C_2 \, ||w - v||_\eta \\
& + RC_1 \, ||w - v||_\eta \, \bar{\Gamma} + h_1(R) \, ||w - v||_\eta \, \bar{\Gamma} + |z(0)| \, \mathcal{L} C_2 \, ||w - v||_\eta \\
\leq & \{h_1(R)(2\bar{\Gamma} + \Gamma_\gamma) + RC_1(2\bar{\Gamma} + \Gamma_\gamma) + \mathcal{L}C_2(|z(0)| + \mathcal{Z}_\gamma)\} \, ||w - v||_\eta \, .
\end{aligned} \tag{4.1}$$

We choose $||w||_\eta \leq R$, where $R \geq 0$. Thus, G is a contraction mapping. Thus, according to the Banach FPT there exists a unique solution w such that $Gw = w$ of Eq (1.6) under some suitable assumptions. □

We can check the effectiveness of our result by taking the same Example 2 with a minor change in expressions:

Example 3. Consider the nonlinear Fredholm type integral equation

$$w(\zeta_1) = \frac{\zeta_1^2}{12} + \frac{w(\zeta_1)}{4}\int_{\mathbf{J}}(\zeta_1^2 + \tau)\frac{\log(1+\tau^2)w}{7}d\tau + \arctan\zeta_1\int_{\mathbf{J}}\frac{\tau^2 w}{3}d\tau \tag{4.2}$$

for $\zeta_1 \in \mathbf{J}$.
Comparing Eq (4.2) with Eq (1.6), we have

$$y(\zeta_1) = \frac{\zeta_1^2}{12};\ \ \varpi(\zeta_1,\tau) = \frac{\zeta_1^2+\tau}{4};\ (T_1 w)(\tau) = \frac{1}{7}\log(1+\tau^2)w;$$
$$z(\zeta_1) = \arctan\zeta_1;\ \ h(\tau, w(\tau)) = \frac{\tau^2 w}{3};\ (T_2 w)(\tau) = \frac{\tau w}{3}.$$

From Example 2, we obtain that $Y_1 = \frac{1}{6}, \Gamma_1 = \frac{1}{2}, \bar{\Gamma} = \frac{1}{2}, Z_1 = 1$ and $\bar{\Omega} = 0$.
Now,

$$\begin{aligned}
||T_1 w||_\gamma =& |T_1 w(0)| + \frac{|T_1 w(\zeta_1) - T_1 w(\zeta_2)|}{|\zeta_1-\zeta_2|^\gamma}\\
\le& 0 + \frac{1}{7}|\log(1+\zeta_1^2)w(\zeta_1) - \log(1+\zeta_2^2)w(\zeta_2)||\zeta_1-\zeta_2|^{-\gamma}\\
\le& \frac{1}{7}\Big|\log(1+\zeta_1^2)[w(\zeta_1)-w(\zeta_2)] + [\log(1+\zeta_1^2) - \log(1+\zeta_2^2)]w(\zeta_2)\Big||\zeta_1-\zeta_2|^{-\gamma}\\
\le& \frac{1}{7}\Big[|\log(1+\zeta_1^2)|\frac{|w(\zeta_1)-w(\zeta_2)|}{|\zeta_1-\zeta_2|^\gamma} + |\zeta_1^2-\zeta_2^2|\frac{|w(\zeta_2)|}{|\zeta_1-\zeta_2|^\gamma}\Big]\\
\le& \frac{1}{7}[|0.30102|||w||_\gamma + |\zeta_1+\zeta_2|||w||_\infty]\\
\le& \frac{1}{7}[0.30102 + 2]||w||_\gamma\\
\le& 0.32871||w||_\gamma.
\end{aligned}$$

And

$$\begin{aligned}
||T_2 w||_\gamma =& |T_2 w(0)| + \frac{|T_2 w(\zeta_1) - T_2 w(\zeta_2)|}{|\zeta_1-\zeta_2|^\gamma}\\
\le& 0 + \frac{1}{3}\frac{|\zeta_1 w(\zeta_1) - \zeta_2 w(\zeta_2)|}{|\zeta_1-\zeta_2|^\gamma}\\
\le& \frac{1}{3}\frac{|\zeta_1[w(\zeta_1)-w(\zeta_2)] + [\zeta_1-\zeta_2]w(\zeta_2)|}{|\zeta_1-\zeta_2|^\gamma}\\
\le& \frac{1}{3}\Big[\frac{|\zeta_1||w(\zeta_1)-w(\zeta_2)|}{|\zeta_1-\zeta_2|^\gamma} + |\zeta_1-\zeta_2|\frac{|w(\zeta_2)|}{|\zeta_1-\zeta_2|^\gamma}\Big]\\
\le& \frac{1}{3}\Big[||w||_\gamma + ||w||_\infty\Big]\\
\le& \frac{2}{3}||w||_\gamma.
\end{aligned}$$

Thus, assumption $(iv)'$ is satisfied with $h_1(R) = 0.32871R$ and $h_2(R) = 0.66666R$.

Again,

$$
\begin{aligned}
|h(\tau,(T_2w)(\tau)-h(\tau,(T_2v)(\tau)| =&\frac{1}{3}\left|\tau^2 w-\tau^2 v\right|\\
\leq&\frac{1}{3}|\tau^2|\,|w-v|\\
\leq&\frac{1}{3}\,|w-v|\,.
\end{aligned}
$$

Thus, assumption (v) is satisfied with $\mathcal{L}=0.33333$.
Also,

$$
\begin{aligned}
\|T_1w-T_1v\|_\gamma \leq |(T_1w)(0)-(T_1v)(0)|+&\frac{\left|(T_1w)(\zeta_1)-(T_1v)(\zeta_1)-[(T_1w)(\zeta_2)-(T_1v)(\zeta_2)]\right|}{|\zeta_1-\zeta_2|^\gamma}\\
\leq&0+\left|\frac{1}{7}\log(1+\zeta_1^2)w(\zeta_1)-\frac{1}{7}\log(1+\zeta_1^2)v(\zeta_1)\right.\\
&\left.-\left[\frac{1}{7}\log(1+\zeta_2^2)w(\zeta_2)-\frac{1}{7}\log(1+\zeta_2^2)v(\zeta_2)\right]\right||\zeta_1-\zeta_2|^{-\gamma}\\
\leq&\frac{1}{7}\left|\left[\log(1+\zeta_1^2)-\log(1+\zeta_2^2)w(\zeta_1)\right]+\log(1+\zeta_2^2)\left[w(\zeta_1)-w(\zeta_2)\right]\right.\\
&\left.-\left[\log(1+\zeta_1^2)-\log(1+\zeta_2^2)\right]v(\zeta_1)-\log(1+\zeta_2^2)\left[v(\zeta_1)-v(\zeta_2)\right]\right||\zeta_1-\zeta_2|^{-\gamma}\\
\leq&\frac{1}{7}\left|\log(1+\zeta_2^2)\left[w(\zeta_1)-w(\zeta_2)-\{v(\zeta_1)-v(\zeta_2)\}\right]\right.\\
&\left.+\left[\log(1+\zeta_1^2)-\log(1+\zeta_2^2)\right]\{w(\zeta_1)-v(\zeta_1)\}\right||\zeta_1-\zeta_2|^{-\gamma}\\
\leq&\frac{1}{7}|\log(1+\zeta_2^2)|\frac{\left|w(\zeta_1)-v(\zeta_1)-[w(\zeta_2)-v(\zeta_2)]\right|}{|\zeta_1-\zeta_2|^\gamma}\\
&+\frac{1}{7}|\zeta_1^2-\zeta_2^2|\frac{|w(\zeta_1)-v(\zeta_1)|}{|\zeta_1-\zeta_2|^\gamma}\\
\leq&\frac{1}{7}|0.30102|\,\|w-v\|_\gamma+\frac{1}{7}|\zeta_1+\zeta_2|\,\|w-v\|_\infty\\
\leq&0.32871\,\|w-v\|_\gamma\,.
\end{aligned}
$$

And

$$
\begin{aligned}
\|T_2w-T_2v\|_\gamma \leq |(T_2w)(0)-(T_2v)(0)|+&\frac{\left|(T_2w)(\zeta_1)-(T_2v)(\zeta_1)-[(T_2w)(\zeta_2)-(T_2v)(\zeta_2)]\right|}{|\zeta_1-\zeta_2|^\gamma}\\
\leq&0+\frac{1}{3}|\zeta_1w(\zeta_1)-\zeta_1v(\zeta_1)-[\zeta_2w(\zeta_2)-\zeta_2v(\zeta_2)]||\zeta_1-\zeta_2|^{-\gamma}\\
\leq&\frac{1}{3}|(\zeta_1-\zeta_2)w(\zeta_1)+\zeta_2(w(\zeta_1)-w(\zeta_2))-(\zeta_1-\zeta_2)v(\zeta_1)-\zeta_2[v(\zeta_1)-v(\zeta_2)]||\zeta_1-\zeta_2|^{-\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{3}|\zeta_2[w(\zeta_1)-w(\zeta_2)-(v(\zeta_1)-v(\zeta_2))]+(\zeta_1-\zeta_2)[w(\zeta_1)-v(\zeta_1)]||\zeta_1-\zeta_2|^{-\gamma}\\
&\leq \frac{1}{3}\Big[|\zeta_2|\frac{|w(\zeta_1)-v(\zeta_1)-[w(\zeta_2)-v(\zeta_2)]|}{|\zeta_1-\zeta_2|^{\gamma}}+|\zeta_1-\zeta_2|\frac{|w(\zeta_1)-v(\zeta_1)|}{|\zeta_1-\zeta_2|^{\gamma}}\Big]\\
&\leq \frac{1}{3}[||w-v||_{\gamma}+||w-v||_{\infty}]\\
&\leq \frac{2}{3}||w-v||_{\gamma}.
\end{aligned}
$$

This implies that T_1 and T_2 satisfy the assumption $(vi)^{'}$ with $C_1 = 0.32871$ and $C_2 = 0.66666$. The inequality in the assumption $(vii)^{'}$

$$h_1(R)(2\bar{\Gamma}+\Gamma_{\gamma})+RC_1(2\bar{\Gamma}+\Gamma_{\gamma})+\mathcal{L}C_2(|z(0)|+\mathcal{Z}_{\gamma})<1$$

is equivalent to

$$
\begin{aligned}
&0.32871R(2\times\frac{1}{2}+\frac{1}{2})+0.32871R(2\times\frac{1}{2}+\frac{1}{2})+0.33333\times 0.66666(0+1)<1\\
&R<0.78872.
\end{aligned}
$$

Thus, the above inequality is true for all $R < 0.78872$ and hence, satisfies all of the assumptions. Hence, Theorem 6 concludes that there exists a unique solution of Eq (4.2).

5. Hyers-Ulam stability analysis

A functional equation is said to be stable if, for every approximate solution, there is an exact solution near it. In 1940, Ulam [35] raised the stability problem of functional equations: Under what condition does there exist a linear mapping near an approximately linear mapping? After 1 year, this problem was solved by Hyers [26] for approximately additive mappings in Banach space. In 1978, Rassias [34] generalized Hyers result by proving the existence of unique linear mappings near approximate additive mappings. Since then, the stability of functional equations has been extensively investigated by a number of authors. Apart from functional equations, this concept of stability can also be applied to differential equations, integral equations, and integrodifferential equations. In 2007, Jung [27] investigated the Hyers-Ulam (H-U) stability of the following Volterra Integral Equation (VIE) using the idea of the fixed point method discussed by Cadariu and Radu [13],

$$w(\zeta_1)=b+\int_a^{\zeta}h(\tau,w(\tau))d\tau.$$

In 2009, Li and Hua [28] proved the H-U stability of the Banach FPT for a polynomial equation defined on a finite interval. In the same year, Castro and Ramos [14] studied the H-U stability of a VIE on an infinite interval and the H-U stability on a finite interval by using the Banach FPT in generalized metric space. In 2010, Gachpazan and Baghani [23] generalized the previous result for a finite interval using the successive approximation method for H-U stability of the nonhomogeneous VIE. In 2011, Morales and Rojas [30], generalized the work of Gachpazan and Baghani regarding the H-U stability for nonhomogeneous VIEs with delay on a finite interval by using the iterative method. Apart from this, many research papers on the stability of these kinds of integral equations have been examined by several authors; see [24, 31, 32].

We have not found any investigation into any type of integral equation in Hölder space, so we were motivated by these articles to discuss the H-U-R stability of Eq (1.6) in Hölder space. Let us define H-U stability for Eq (1.6).

Definition 3. *[27] We say that Eq (1.6) has H-U stability if there exists $K \geq 0$ with the following property: For every $\epsilon > 0$, w satisfying*

$$\left| w(\zeta_1) - y(\zeta_1) - w(\zeta_1) \int_{\mathbf{J}} \varpi(\zeta_1, \tau)(T_1 w)(\tau) d\tau - z(\zeta_1) \int_{\mathbf{J}} h(\tau, (T_2 w)(\tau)) d\tau \right| \leq \epsilon,$$

then there exists some v satisfying Eq (1.6) such that

$$\left| w(\zeta_1) - v(\zeta_1) \right| \leq K\epsilon.$$

We call such K a H-U stability constant.

In this section, we are going to prove that Eq (1.6) has H-U stability. In the previous section, we showed that G is a contraction mapping and Eq (1.6) has a unique solution by using the Banach FPT. This leads to the following theorem:

Theorem 7. *Under the assumption of Theorem 6, the equation $(G - I)w = 0$, defined by Eq (1.6), has H-U stability, that is for $\epsilon \geq 0$, if*

$$\|Gw - w\|_\eta \leq \epsilon,$$

then there exists a unique $v \in \mathcal{H}_\eta(\mathbf{J})$ satisfying

$$Gv - v = 0$$

with

$$\|w - v\|_\eta \leq K\epsilon$$

for some $K \geq 0$.

Proof. We have defined the operator G in Eq (3.1). According to Theorem 6, operator G is a contraction mapping with a unique solution $w \in H_\eta(\mathbf{J})$ to Eq (1.6). For every $\epsilon > 0$, if $\|Gw - w\|_\eta \leq \epsilon$, then

$$|w - v| = |w - Gw + Gw - v| \leq |w - Gw| + |Gw - Gv|.$$

Using Eq (4.1), we obtain

$$\begin{aligned} \|w - v\|_\eta &\leq \epsilon + C_3 \|w - v\|_\eta \\ \implies \|w - v\|_\eta (1 - C_3) &\leq \epsilon \\ \implies \|w - v\|_\eta &\leq \frac{\epsilon}{1 - C_3}, \end{aligned}$$

where $K = \frac{1}{1-C_3}$ and $C_3 < 1$. This completes the proof. □

6. Conclusions

In this work, we discussed the existence of a solution for the Fredholm type IE in the space $\mathcal{H}_\eta(\mathbf{J})$ based on the Darbo FPT. Using the Banach FPT, we proved the uniqueness of the solution in the same space. We have provided examples to verify the effectiveness and applicability of the existence and uniqueness results. Further, another example has been exhibited to show that the Chandrasekhar IE is a special case of our proposed equation. At the end, we checked the stability of the solution by performing H-U stability.

Equation (1.6) can be solved in different types of Banach spaces by using the FPT with suitable assumptions. These kinds of IEs are often applicable to the kinetic theory of gases, traffic theory, oscillating magnetic theory, neutron transport theory, radiative transfer theory and electromagnetic and mathematical physics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research of the first author (Manalisha Bhujel) is supported by SERB, DST, New Delhi, India under the INSPIRE code DST/INSPIRE Fellowship/2019/IF190364.

Moreover, we would like to thank the academic editor and anonymous reviewers for taking the necessary time and effort to review the manuscript. We sincerely appreciate all of their valuable comments and suggestions, which have helped us to improve the quality of the manuscript.

Conflict of interest

The authors declare that they do not have any conflict of interest.

Prof. Dimplekumar Chalishajar is the Guest Editor of special issue "Recent advances in differential and partial differential equations and its applications" for AIMS Mathematics. Prof. Dimplekumar Chalishajar was not involved in the editorial review and the decision to publish this article.

References

1. R. P. Agarwall, M. Meehan, D. O'Regan, *Fixed point theory and applications*, Cambridge University Press, 2021.
2. S. Banach, Sur less opérations dans les ensembles abstraits ets leur applications aux equation integrates, *Fund. Math.*, **3** (1922), 133–181.
3. J. Banaś, K. Goebel, *Measure of noncompactness in Banach spaces*, In: Lecture Notes in Pure and Applied Mathematics, New York, 1980.
4. J. Banaś, R. Nalepa, On a measure of noncompactness in the space of functions with tempered increments, *J. Math. Anal. Appl.*, **435** (2016) 1634–1651. https://doi.org/10.1016/j.jmaa.2015.11.033

5. J. Banaś, R. Nalepa, On the space of functions with growths tempered by a modulus of continuity and its applications, *J. Funct. Space. Appl.*, **2013** (2013) 820437. https://doi.org/10.1155/2013/820437

6. J. Banaś, M. Lecko, Fixed points of the product of operators in Banach algebra, *Pan Amer. Math. J.*, **12** (2002), 101–109.

7. J. Banaś, K. Sadarangani, Solutions of some functional-integral equations in Banach algebra, *Math. Comput. Model.*, **38** (2003) 245–250. https://doi.org/10.1016/S0895-7177(03)90084-7

8. J. Banaś, L. Olszowy, On a class of measure of noncompactness in Banach algebras and their application to nonlinear integral equations, *Z. Anal. Anwend.*, **28** (2009), 1–24. https://doi.org/10.4171/zaa/1394

9. M. Bhujel, B. Hazarika, Existence of solutions of Fredholm type integral equations in Hölder spaces, *J. Integral Equ. Appl.*, **35** (2023), 1–10. https://doi.org/10.1216/jie.2023.35.1

10. M. Bhujel, B. Hazarika, Solvability of quartic integral equations in Hölder space, *Rocky Mt. J. Math.*, in press.

11. M. J. Caballero, M. A. Darwish, K. Sadarangani, Solvability of a quadratic integral equation of Fredholm type in Hölder spaces, *Electron. J. Diff. Equ.*, **31** (2014), 1–10. https://doi.org/10.1155/2014/856183

12. M. J. Caballero, R. Nalepa, K. Sadarangani, Solvability of a quadratic integral equation of Fredholm type with Supremum in Hölder spaces, *J. Funct. Space. Appl.*, **2014** (2014), 856183. https://doi.org/10.1155/2014/856183

13. L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: A fixed point approach*, Iteration Theory (ECIT'02), Grazer Math. Ber., **346** (2004), 43–52.

14. L. P. Castro, A. Ramos, Hyers-Ulam-Rassias stability for a class of nonlinear volterra integral equations, *Banach J. Math. Anal.*, **3** (2009), 36–43. https://doi.org/10.15352/bjma/1240336421

15. S. Chandrasekhar, *Radiative transfer*, Oxford University Press, London, UK, 1950.

16. M. Cichoń, M. M. A. Metwali, On a fixed point theorem for the product of operators, *J. Fixed Point Theory Appl.*, **18** (2016), 753–770. https://doi.org/10.1007/s11784-016-0319-7

17. K. Cichoń, M. Cichoń, M. M. A. Metwali, On some fixed point theorems in abstract duality pairs, *Rev. Unión Mat. Argent.*, **61** (2020), 249–266. https://doi.org/10.33044/revuma.v61n2a04

18. M. Cichoń, M. M. A. Metwali, On the Banach algebra of integral-variation type Hölder spaces and quadratic fractional integral equations, *Banach J. Math. Anal.*, **16** (2022), 34. https://doi.org/10.1007/s43037-022-00188-4

19. E. T. Copson, On an integral equation arising in the theory of diffraction, *Q. J. Math.*, **17** (1946), 19–34. https://doi.org/10.1093/qmath/os-17.1.19

20. M. T. Ersoy, H. Furkan, On the existence of the solutions of a Fredholm integral equation with a modified argument in Hölder spaces, *Symmetry*, **10** (2018), 522. https://doi.org/10.3390/sym10100522

21. M. T. Ersoy, H. Furkan, B. Sarıçiçek, On the solutions of some nonlinear Fredholm integral equations in topological Hölder spaces, *TWMS J. App. Eng. Math.*, **10** (2020), 657–668.

22. M. T. Ersoy, An application to the existence of solutions of the integral equations, *Türk. J. Math. Comput. Sci.*, **13** (2021), 115–121.

23. M. Gachpazan, O. Baghani, Hyers-Ulam stability of nonlinear integral equation, *Fixed Point Theory A.*, **2010** (2010), 927640. https://doi.org/10.1155/2010/927640

24. R. I. Hassan, Existence, uniqueness, and stability solutions of nonlinear system of integral equations, *J. Mat. MANTIK*, **6** (2020), 76–82. https://doi.org/10.15642/mantik.2020.6.2.76-82

25. S. Hu, M. Khavanin, W. Zhuang, Integral equations arising in the kinetic theory of gases, *Appl. Anal.*, **34** (1989), 261–266. https://doi.org/10.1080/00036818908839899

26. D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.*, **27** (1941) 222–224. https://doi.org/10.1073/pnas.27.4.222

27. S. M. Jung, A fixed point approach to the stability of a volterra integral equation, *Fixed Point Theory A.*, **2007** (2007). https://doi.org/10.1155/2007/57064

28. Y. Li, L. Hua, Hyers-Ulam stability of polynomial equation, *Banach J. Math. Anal.*, **3** (2009), 86–90. https://doi.org/10.15352/bjma/1261086712

29. N. Lu, F. He, H. Huang, Answers to questions on the generalized Banach contraction conjecture in b-metric spaces, *J. Fix. Point Theory A.*, **21** (2019), 43. https://doi.org/10.1007/s11784-019-0679-x

30. J. R. Morales, E. M. Rojas, Hyers-Ulam and Hyers-Ulam-Rassias stability of nonlinear integral equations with delay, *Int. J. Nonlinear Anal. Appl.*, **2** (2011), 1–6.

31. S. Öğrekçi, Y. Başcı, A. Mısır, On the Ulam type stability of nonlinear Volterra integral equations, *arXiv:2105.11778*, 2021. https://doi.org/10.48550/arXiv.2105.11778

32. S. Öğrekçi, Y. Başcı, A. Mısır, A fixed point method for stability of nonlinear Volterra integral equations in the sense of Ulam, *Math. Meth. Appl. Sci.*, **46** (2023), 8437–8444. https://doi.org/10.1002/mma.8988

33. Í. Özdemir, On the solvability of a class of nonlinear integral equations in Hölder spaces, *Numer. Func. Anal. Opt.*, **43** (2022), 1–29. https://doi.org/10.1080/01630563.2022.2032148

34. T. M. Rassias, On the stability of linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300. https://doi.org/10.1090/S0002-9939-1978-0507327-1

35. S. M. Ulam, *Problems in modern mathematics*, John Wiley and Sons, New York, 1960.

AIMS Press