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Theory article

Energy minimizing solutions to slightly subcritical elliptic problems on nonconvex polygonal domains

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Abstract: In this paper we are concerned with the Lane-Emden-Fowler equation

$$\begin{cases} -\Delta u &= u^{\frac{n+2}{n-2}-\varepsilon} & \text{in } \Omega, \\ u &> 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a nonconvex polygonal domain and $\varepsilon > 0$. We study the asymptotic behavior of minimal energy solutions as $\varepsilon > 0$ goes to zero. A main part is to show that the solution is uniformly bounded near the boundary with respect to $\varepsilon > 0$. The moving plane method is difficult to apply for the nonconvex polygonal domain. To get around this difficulty, we derive a contradiction after assuming that the solution blows up near the boundary by using the Pohozaev identity and the Green's function.

Keywords: blow-up analysis; polygonal domains; Lane-Emden-Fowler equation
Mathematics Subject Classification: 35B33, 35J15, 35J60

1. Introduction

In this paper we study asymptotic profile of energy minimizing solutions to the Lane-Emden-Fowler equation

$$\begin{cases} -\Delta u_\varepsilon = u_\varepsilon^{p-\varepsilon} & \text{in } \Omega, \\ u_\varepsilon > 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

as $\varepsilon > 0$ goes to zero. Here $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded polygonal domain and $p = \frac{n+2}{n-2}$ is the critical exponent. In the seminar papers Han [1] and Rey [2], the asymptotic behavior of energy minimizing solutions to (1.1) was obtained for smooth bounded domains Ω.

The asymptotic behavior was first studied by Atkinson and Peletier [3] when Ω is the unit ball in $\mathbb{R}^3$ using an ODE argument. The result was revisited by Brezis and Peletier [4] by applying PDE methods. Extensions to smooth bounded domains were obtained by Han [1] and Rey [2]. The asymptotic behavior have been studied by a lot of researchers for nonlinear elliptic equations with various settings (see e.g., [5–16]) and we note that most of the results have been obtained for elliptic problems on bounded smooth domains.

Pistoia and Rey [17] showed that as for problem (1.1) posed on a specific nonsmooth bounded domain constructed by Flucher-Garroni-Müller [18], the maximum point of u_ε may approach to the boundary point as $\varepsilon \to 0$. By the way, we mention that the arguments of Han [1] and Rey [2] work straight-forwardly for convex bounded domains, which may not be non-smooth. In fact a key part in the analysis of Han [1] and Rey [2] is that the maximum point of $u_\varepsilon(x)$ is uniformly away from the boundary $\partial\Omega$ by showing that the solutions $u_\varepsilon(x)$ are uniformly bounded for $\varepsilon > 0$ and x near the boundary $\partial\Omega$ by the moving plane argument. If Ω is a smooth nonconvex domain, Han [1] obtained the unfirom boundedness by using the Kelvin transform to (1.1) on balls which touch the domain Ω by the boundary $\partial\Omega$. However, the argument is difficult to apply when Ω is not smooth.

Given this result, a natural question is that can we extend the result of Han [2] and Rey [2] to certain class of nonsmooth convex domains? In this paper, we show that the results of Han [1] and Rey [2] to nonconvex polygonal domains. The following is the main result of this paper.

Theorem 1.1. *For $n \geq 3$ we let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal domain. Assume that $\{u_\varepsilon\}_{\varepsilon>0}$ is a set of solutions to* (1.1) *such that*

$$\lim_{\varepsilon\to 0} \frac{\left(\int_\Omega |u_\varepsilon|^{p+1-\varepsilon} dx\right)^{\frac{1}{p+1-\varepsilon}}}{\left(\int_\Omega |\nabla u_\varepsilon|^2 dx\right)^{1/2}} = S_n, \tag{1.2}$$

where $S_n = [\pi n(n-2)\Gamma(n/2)/\Gamma(n)]^{-1}$ is the best Sobolev constant in $\mathbb{R}^n$. Then the family of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded near the boundary, i.e., there are costants $\delta > 0$ and $C > 0$ independent of $\varepsilon > 0$ such that

$$\sup_{\varepsilon>0} \sup_{\{x\in\Omega: dist(x,\partial\Omega)<\delta\}} |u_\varepsilon(x)| \leq C.$$

Given the boundary estimates of Theorem 1.1, one may apply standard argument to deduce the following result [1].

Theorem 1.2. *For $n \geq 3$ we let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal domain. Assume that $\{u_\varepsilon\}_{\varepsilon>0}$ is a set of solutions to* (1.1) *such that* (1.2) *holds. Then, there exists a point $x_0 \in \Omega$ such that, up to a subsequence,*

- *The solution u_ε converges to 0 in $C^1(\Omega \setminus \{x_0\})$.*
- *$\nabla R(x_0) = 0$, where $R(x) = H(x, x)$.*
- *We have*
$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon(x) = [n(n-2)]^{(n-2)/2} |S^{n-1}| G(x, x_0).$$
- *We have*
$$\lim_{\varepsilon\to 0} \varepsilon \|u_\varepsilon\|^2_{L^\infty(\Omega)} = (n-2)|S^{n-1}|^2 \, [n(n-2)]^{n-2} \, H(x_0, x_0).$$

Here G denotes the Green's function and H is the regular part of G (see Section 2 for the detail).

In order to prove Theorem 1.1 we assume that contrary that the maximum point x_ε approaches to the boundary. Under this assumption, we shall deduce a contradiction from the following Pohozaev type identity on an annulus centered at the blow up point; $1 \leq j \leq$n,

$$\int_{\partial B(x_\varepsilon,2d_\varepsilon)} |\nabla u_\varepsilon|^2 \nu_j - 2\left(\frac{\partial u_\varepsilon}{\partial \nu}\frac{\partial u_\varepsilon}{\partial x_j}\right) dS_x = \frac{2}{p+1-\varepsilon}\int_{\partial B(x_\varepsilon,2d_\varepsilon)} u_\varepsilon^{p-\varepsilon+1}\nu_j dS_x, \tag{1.3}$$

where $x_\varepsilon \in \Omega$ is the maximum point of u_ε and $d_\varepsilon = \mathrm{dist}(x_\varepsilon, \partial\Omega_\varepsilon)/4$.

In fact we shall prove Theorem 1.1 for more general domain Ω satisfying the following assumption.

Assumption D. *Consider a sequence of points* $\{x^k\}_{k\in\mathbb{N}}$ *in the domain* Ω *such that* $\mathbf{d}_k := dist(x^k, \partial\Omega)$ *goes to zero as* $k \to \infty$. *Take* $z^k \in \partial\Omega$ *such that* $|x^k - z^k| = \mathbf{d}_k$. *Let* $\Omega_k := \frac{1}{\mathbf{d}_k}(\Omega - z^k)$. *Note that we have* $0 \in \Omega_k$, *and also* $\frac{1}{\mathbf{d}_k}(x^k - z^k) \in S^{n-1}$. *Thus we can find a rotation* $R_k : \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$R_k\left(\frac{1}{\mathbf{d}_k}(x^k - z^k)\right) = e_n = (0, \cdots, 0, 1).$$

Then, the domain $D_k := R_k\Omega_k$ *converges to an infinite star-shaped domain* $\mathbb{P} \subsetneq \mathbb{R}^n$.

It is not difficult to see that any bounded polygonal domain Ω satisfies the above assumption. Under the above assumption we will obtain the following result on the regular part H of the Green's function.

Theorem 1.3. *For* $n \geq 3$ *we let* $\Omega \subset \mathbb{R}^n$ *be a bounded open domain satisfying* ***Assumption D***. *Then, for any sequence of points* $\{y^k\}_{k\geq 1}$ *in* Ω *such that* $\lim_{k\to\infty} \mathbf{d}_k = 0$, *where* $\mathbf{d}_k := dist(y^k, \partial\Omega)$, *there exists a constant* $c > 0$ *and* $N \in \mathbb{N}$ *such that, for* $k \geq N$ *we have*

$$\sup_{1\leq j\leq n}\left|\frac{\partial H}{\partial x_j}(y^k, y^k)\right| \geq \frac{c}{\mathbf{d}_k^{n-1}}. \tag{1.4}$$

If Ω is smooth, then the result of Theorem 1.3 was proved in Rey [19] by applying the Maximum principle. To obtain the above inequality for the nonsmooth domains, we shall rescale the function H in a suitable way and investigate its limit.

This paper is organized as follows. In Section 2, we are concerned about the properties of Green's function. Also we show that a sequence of the minimal energy solutions blows up as $\varepsilon \to 0$ and that the blow up point does not approach to the boundary too fast in some sense (see Lemma 2.2). In Section 3, we will obtain a sharp estimate of the function u_ε on an annulus centered at the blow up point. In Section 4, we prove Theorem 1.1. In Section 5, we give a proof of Theorem 1.2. Section 6 is devoted to prove Theorem 1.3.

Notations.

Here we list some notations which will be used throughout the paper.
- $C > 0$ is a generic constant that may vary from line to line.
- For $k \in \mathbb{N}$ we denote by $B^k(x_0, r)$ the ball $\{x \in \mathbb{R}^k : |x - x_0| < r\}$ for each $x_0 \in \mathbb{R}^k$ and $r > 0$.
- For $x \in \Omega$ we denote by $\mathrm{dist}(x, \partial\Omega)$ the distance from x to $\partial\Omega$ and we denote $\mathbf{d}(x) := \mathrm{dist}(x, \partial\Omega)$.
- For a domain $D \subset \mathbb{R}^n$, the map $\nu = (\nu_1, \cdots, \nu_n) : \partial D \to \mathbb{R}^n$ denotes the outward pointing unit normal vector on ∂D.
- dS stands for the surface measure.
- $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ denotes the Lebesgue measure of $(n-1)$-dimensional unit sphere S^{n-1}.

2. Preliminary results

In this section we obtain preliminary results for a sequence of the solutions $\{u_\varepsilon\}_{\varepsilon>0}$ satifsying (1.2). For this purpose, we first recall Green's function G of the Laplacian $-\Delta$ on Ω with the Dirichlet boundary condition. It is divided into a singular part and a regular part as

$$G(x,y) = \frac{c_n}{|x-y|^{n-2}} - H(x,y), \tag{2.1}$$

where $c_n = 1/(n-2)|S^{n-1}|$ and the regular part $H : \Omega\times\Omega \to \mathbb{R}$ is the function such that

$$\begin{cases} -\Delta_x H(x,y) = 0 & x\in\Omega, \\ H(x,y) = \frac{c_n}{|x-y|^{n-2}} & x\in\partial\Omega. \end{cases} \tag{2.2}$$

Let $\mathbf{d}(x) = \mathrm{dist}(x,\partial\Omega)$ for $x\in\Omega$. Take a small constant $\delta > 0$.

We take a value $\lambda_\epsilon > 0$ and a point $x_\epsilon \in \Omega$ such that

$$\lambda_\epsilon^{\frac{2}{p-\varepsilon-1}} := u_\varepsilon(x_\varepsilon) = \max_{x\in\Omega}\{u_\epsilon(x)\}. \tag{2.3}$$

Now we recall the sharp Sobolev embedding

$$\left(\int_{\mathbb{R}^n} |f(x)|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{2n}} \le S_n \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2 dx\right)^{1/2} \quad \forall\, f\in H^1(\mathbb{R}^n). \tag{2.4}$$

If we replace the function f by $(-\Delta)^{-1/2} f$ in the above inequality, we find the Hardy-Littlewood-Sobolev inequality:

$$\|(-\Delta)^{-1/2} f\|_{L^{p+1}(\mathbb{R}^n)} \le S_n \|f\|_{L^2(\mathbb{R}^n)} \quad \forall\, f\in L^2(\mathbb{R}^n). \tag{2.5}$$

We let K denote Green's function of the Laplacain on $\mathbb{R}^n$, i.e.,

$$K(x,y) = \frac{c_n}{|x-y|^{n-1}}.$$

The estimate (2.5) is then written as

$$\left\| \int_{\mathbb{R}^n} K(x,y) f(y) dy \right\|_{L^{p+1}(\mathbb{R}^n)} \le S_n \|f\|_{L^2(\mathbb{R}^n)} \quad \forall\, f\in L^2(\mathbb{R}^n).$$

For given a domain $Q\subset\mathbb{R}^n$ we denote by $K_Q : Q\times Q \to \mathbb{R}$ Green's function of the Laplacian $(-\Delta)^{1/2}$ on domain Q with the Dirichlet zero boundary condition, i.e., for the solution $u\in H^1(\Omega)$ to the problem

$$\begin{cases} (-\Delta)^{1/2} u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $f\in L^2(\Omega)$ admits the representation

$$u(x) = \int_\Omega K(x,y) f(y) dy.$$

Then, it is a classical fact that for any open subset $Q\subset\mathbb{R}^n$ with $Q\neq\mathbb{R}^n$, we have

$$K_Q(x,y) < K(x,y) \quad \text{for all } (x,y)\in Q\times Q. \tag{2.6}$$

Here we remark that $(-\Delta)^{1/2}$ is defined by the spectral decomposition of $(-\Delta)$ on domain Ω.

Lemma 2.1. *The value* $\lambda_\varepsilon > 0$ *defined in* (2.3) *satisfies* $\lim_{\varepsilon\to 0}\lambda_\varepsilon = \infty$.

Proof. In order to prove the lemma, we assume the contrary. Then there is a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\varepsilon_k = 0$ and $\sup_{k\in\mathbb{N}}\lambda_{\epsilon_k} < \infty$. This implies that the solutions $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$ are uniformly bounded in $C^{1,\alpha}(\Omega)$ for some $\alpha\in(0,1)$ by the standard regularity theory applied to (1.1). Up to a subsequence, the solution u_{ϵ_k} converges in $C^1(\Omega)$ to a function $u_0\in C^1(\Omega)$, and taking $k\to\infty$ in the formula

$$u_{\varepsilon_k}(x) = \int_\Omega G(x,y)u_{\varepsilon_k}^{p-\varepsilon_k}(y)dy,$$

we find

$$u_0(x) = \int_\Omega G(x,y)u_0^p(y)dy,$$

and so

$$\begin{cases} -\Delta u_0 = u_0^p & \text{in } \Omega,\\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

On the other hand, by taking the limit $k\to\infty$ in (1.2) we get

$$||u_0||_{L^{p+1}(\Omega)} = S_n||\nabla u_0||_{L^2(\Omega)}.$$

Let us set $w_0:\Omega\to\overline{\mathbb{R}_+}$ by $w_0(x) = (-\Delta_\Omega)^{1/2}u_0(x)$ for $x\in\Omega$. Then $u_0(x) = (-\Delta_\Omega)^{-1/2}w_0(x)$ for $x\in\Omega$ and so we have

$$\left\|(-\Delta_\Omega)^{-1/2}w_0\right\|_{L^{p+1}(\Omega)} = S_n\left\|w_0\right\|_{L^2(\Omega)}. \tag{2.8}$$

We extend the function w_0 to set $W_0:\mathbb{R}^n\to\overline{\mathbb{R}_+}$ by

$$W_0(x) = \begin{cases} w_0(x) & \text{for } x\in\Omega,\\ 0 & \text{for } x\notin\Omega. \end{cases}$$

Then, using the inequality (2.6) and (2.8) we obtain the following estimate

$$\begin{aligned} S_n||W_0||_{L^2(\mathbb{R}^n)} &= S_n||w_0||_{L^2(\Omega)}\\ &= ||(-\Delta_\Omega)^{-1/2}w_0||_{L^{p+1}(\Omega)}\\ &< ||(-\Delta_\Omega)^{-1/2}W_0||_{L^{p+1}(\Omega)} < ||(-\Delta)^{-1/2}W_0||_{L^{p+1}(\mathbb{R}^n)}. \end{aligned}$$

However, this contradicts to the optimality of the constant S_n of the inequality (2.5). Therefore it should hold that $\lim_{\epsilon\to 0}\lambda_\epsilon = \infty$. The lemma is proved. □

For each $\varepsilon > 0$ we set $\Omega_\epsilon := \lambda_\epsilon(\Omega - x_\epsilon)$ and normalize the solution u_ε as follows

$$U_\epsilon(x) := \lambda_\epsilon^{-\frac{2}{p-\varepsilon-1}}u_\epsilon(\lambda_\epsilon^{-1}x + x_\epsilon), \tag{2.9}$$

so that

$$\begin{cases} -\Delta U_\epsilon = U_\epsilon^{p-\varepsilon} & \text{in } \Omega_\epsilon,\\ U_\epsilon = 0 & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{2.10}$$

and $\max_{x\in\Omega_\epsilon}\{U_\epsilon(x)\} = 1 = U_\epsilon(0)$. In the next lemma, we obtain an estimate for the distance between the maximum point of the solutions and the boundary $\partial\Omega$.

Lemma 2.2. *We have* $\lim_{\epsilon\to 0}\lambda_\epsilon dist(x_\epsilon, \partial\Omega) = \infty$.

Proof. We assume the contrary. Then, up to a subsequence, we have $\lim_{\epsilon\to 0}\lambda_\epsilon \mathrm{dist}(x_\epsilon, \partial\Omega) = l$ for some $l \in (0, \infty)$. This implies that the extended domain Ω_ϵ converges to a infinite star-shaped domain $\mathbb{P} \subsetneq \mathbb{R}^n$ as $\varepsilon \to 0$. Also, the normalized functions U_ϵ converge to a nontrivial solution $\overline{U}$ in $C^2_{loc}(\mathbb{P})$ of the problem

$$\begin{cases} -\Delta\overline{U} = \overline{U}^p & \text{in } \mathbb{P}, \\ \overline{U} = 0 & \text{on } \partial\mathbb{P}, \end{cases}$$

and we know that $K_{\mathbb{P}}(x, y) < K(x, y)$ from (2.6). Then we can obtain a contradiction as in the proof of Lemma 2.1. Thus the result of the lemma is true. □

We set $d_\epsilon := \frac{1}{4}\mathrm{dist}(x_\epsilon, \partial\Omega)$ and $N_\varepsilon = d_\varepsilon\lambda_\varepsilon$. Then we see from Lemma 2.2 that

$$d_\epsilon = \frac{N_\epsilon}{\lambda_\epsilon} \quad \text{and} \quad \lim_{\varepsilon\to 0} N_\epsilon = \infty. \tag{2.11}$$

We remark that the fact $N_\varepsilon \to \infty$ as $\varepsilon \to 0$ will be important in the proofs of Theorem 1.1. By Lemma 2.2 the domain Ω_ϵ converges to $\mathbb{R}^n$ as ϵ goes to zero, and so the rescaled solution U_ϵ converges in $C^2_{loc}(\mathbb{R}^n)$ to a solution U of the problem

$$\begin{cases} -\Delta U = U^p & \text{in } \mathbb{R}^n, \\ U(y) > 0 & y \in \mathbb{R}^n, \\ U(0) = 1 = \max_{x\in\mathbb{R}^n} U(x),\ U \to 0 & \text{as } |y| \to \infty. \end{cases} \tag{2.12}$$

Then it is well-known that the function U is equal to

$$U(x) = [n(n-2)]^{(n-2)/4}\left(\frac{\eta}{\eta^2 + |x|^2}\right)^{(n-2)/2},$$

where $\eta = \sqrt{n(n-2)}$. Next we recall the following result from Corollary 1 and Lemma 3 in [1].

Lemma 2.3 ([1]). *The value* $\lambda_\varepsilon > 0$ *defined in* (1.2) *and the rescaled solution* U_ε *defined* (2.9) *satisfy the following.*

(1) There is a constant $C > 0$ *independent of* $\epsilon > 0$ *such that*

$$\lambda_\epsilon^\epsilon \leq C. \tag{2.13}$$

(2) There exists a constant $C > 0$ *such that*

$$U_\epsilon(x) \leq CU(x) \quad \forall\epsilon > 0. \tag{2.14}$$

We end this section with a local version of the Pohozaev type identity for the problem (1.1).

Lemma 2.4. *Let* $1 \leq j \leq n$. *Suppose that* $u \in C^2(\Omega) \times C^2(\Omega)$ *is a solution of* (1.1). *Then, for any open smooth subset* $D \subset \Omega$, *we have the following identity.*

$$-2\int_{\partial D}\frac{\partial u}{\partial\nu}(x)\frac{\partial u}{\partial x_j}(x)dS_x + \int_{\partial D}|\nabla u(x)|^2\nu_j dS_x = \frac{2}{p+1}\int_{\partial D}u^{p+1}(x)\nu_j dS_x, \tag{2.15}$$

where D *is an open subset of* Ω.

Proof. Multiplying (1.1) by $\frac{\partial u}{\partial x_j}$ we get $-\Delta u \frac{\partial u}{\partial x_j} = u^p \frac{\partial u}{\partial x_j}$. Integrating this over the domain D and using an integration by part, we get

$$-\int_{\partial D} \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_j} dS_x + \int_D \nabla u \cdot \frac{\partial \nabla u}{\partial x_j} dS_x = \frac{1}{p+1} \int_{\partial D} u^{p+1} \nu_j dS_x. \tag{2.16}$$

We use an integration by parts to get

$$\frac{1}{2} \int_D \frac{\partial}{\partial x_j} |\nabla u|^2 dx = \frac{1}{2} \int_{\partial D} |\nabla u|^2 \nu_j dS_x.$$

The lemma is proved. □

3. Estimates for u_ε on the annulus

This section is devoted to prove the following lemma regarding a sharp estimate for u_ε and its derivatives on the annulus $\partial B(x_\varepsilon, 2d_\varepsilon)$.

Lemma 3.1. *Assume that* $\{u_\varepsilon\}_{\varepsilon>0}$ *is a sequence of solutions to* (1.1) *of type* (ME) *and that* $\lim_{\varepsilon\to 0} d_\varepsilon = 0$. *Then, for* $x \in \partial B(x_\varepsilon, 2d_\varepsilon)$ *we have the estimates*

$$u_\epsilon(x) = A_U \lambda_\epsilon^{-\frac{[2-(n-2)\varepsilon]}{p-\varepsilon-1}} G(x, x_\epsilon) + o(d_\epsilon^{-(n-2)} \lambda_\epsilon^{-\frac{n}{p+1}}) \tag{3.1}$$

and

$$\nabla u_\epsilon(x) = A_U \lambda_\epsilon^{-\frac{[2-(n-2)\varepsilon]}{p-\varepsilon-1}} \nabla G(x, x_\epsilon) + o(d_\epsilon^{-(n-1)} \lambda_\epsilon^{-\frac{n}{p+1}}). \tag{3.2}$$

Here the value A_U *is defined as*

$$A_U = \int_{\mathbb{R}^n} U^p(y) dy = [n(n-2)]^{\frac{n}{2}} \frac{c_n}{n} = [n(n-2)]^{\frac{n}{2}-1} |S^{n-1}|. \tag{3.3}$$

In addition, the o-notation is uniform with respect to $x \in \partial B(x_\varepsilon, 2d_\varepsilon)$, *i.e., it holds that*

$$\lim_{\varepsilon \to 0} \sup_{x \in \partial B(x_\varepsilon, 2d_\varepsilon)} \frac{|o(d_\varepsilon^{-k} \lambda_\varepsilon^{-\frac{n}{p+1}})|}{(d_\varepsilon^{-k} \lambda_\varepsilon^{-\frac{n}{p+1}})} = 0 \quad \text{for } k = n-1 \text{ or } n-2.$$

Proof. Since u_ϵ is a solution to (1.1), we have

$$\begin{aligned} u_\epsilon(x) &= \int_\Omega G(x, y) u_\epsilon^p(y) dy \\ &= G(x, x_\epsilon) \left(\int_\Omega u_\epsilon^q(y) dy \right) + \int_\Omega [G(x, y) - G(x, x_\epsilon)] u_\epsilon^p(y) dy. \end{aligned} \tag{3.4}$$

Given the estimate (2.14) we apply the dominated convergence theorem to find

$$\lim_{\epsilon \to 0} \lambda_\epsilon^{\frac{[2-(n-2)\varepsilon]}{p-\varepsilon-1}} \int_\Omega u_\epsilon^{p-\varepsilon}(y) dy = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} U_\epsilon^{p-\varepsilon}(y) dy = \int_{\mathbb{R}^n} U^p(y) dy = A_U.$$

Using this and noting that $G(x, x_\varepsilon) = O(|x - x_\varepsilon|^{-(n-2)}) = O(d_\varepsilon^{-(n-2)})$ for $x \in \partial B(x_\varepsilon, 2d_\varepsilon)$, we find

$$G(x, x_\varepsilon)\left(\int_\Omega u_\varepsilon^{p-\varepsilon}(y)dy\right) = \lambda_\varepsilon^{-\frac{[2-(n-2)\varepsilon]}{p-\varepsilon-1}} A_U G(x, x_\varepsilon) + o(\lambda_\varepsilon^{-\frac{n}{p+1}} d_\varepsilon^{-(n-2)}),$$

where we also used that

$$\lambda_\varepsilon^{-\frac{[2-(n-2)\varepsilon]}{p-1-\varepsilon}} = O(\lambda_\varepsilon^{-\frac{n}{p+1}}$$

due to the fact that $\frac{2}{p-1} = \frac{n}{p+1}$ and (2.13). Similarly, we may deduce

$$\nabla G(x, x_\varepsilon)\left(\int_\Omega u_\varepsilon^{p-\varepsilon}(y)dy\right) = \lambda_\varepsilon^{-\frac{[2-(n-2)\varepsilon]}{p-\varepsilon-1}} A_U \nabla G(x, x_\varepsilon) + o(\lambda_\varepsilon^{-\frac{n}{p+1}} d_\varepsilon^{-(n-1)}).$$

Therefore, in order to prove (3.1), we only need to estimate the last term of (3.4) as $o(d_\varepsilon^{-(n-2)}\lambda_\varepsilon^{-\frac{n}{p+1}})$ and its derivatives as $o(d_\varepsilon^{-(n-1)}\lambda_\varepsilon^{-\frac{n}{p+1}})$. For this aim, we decompose it into three parts as follows:

$$\int_\Omega [G(x, y) - G(x, x_\epsilon)]u_\epsilon^{p-\varepsilon}(y)dy = I_1(x) + I_2(x) + I_3(x), \tag{3.5}$$

where

$$\begin{aligned} I_1(x) &:= \int_{B(x_\epsilon, d_\epsilon)} [G(x, y) - G(x, x_\epsilon)]u_\epsilon^{p-\varepsilon}(y)dy, \\ I_2(x) &:= \int_{B(x_\epsilon, 4d_\epsilon)\setminus B(x_\epsilon, d_\epsilon)} [G(x, y) - G(x, x_\epsilon)]u_\epsilon^{p-\varepsilon}(y)dy, \\ I_3(x) &:= \int_{\Omega\setminus B(x_\epsilon, 4d_\epsilon)} [G(x, y) - G(x, x_\epsilon)]u_\epsilon^{p-\varepsilon}(y)dy. \end{aligned} \tag{3.6}$$

We shall show that $I_1(x)$, $I_2(x)$, and $I_3(x)$ are estimated as $o\left(d_\varepsilon^{-(n-2)}\lambda_\varepsilon^{-\frac{n}{p+1}}\right)$ and their derivatives $\nabla I_1(x)$, $\nabla I_2(x)$, and $\nabla I_3(x)$ are estimated as $o\left(d_\varepsilon^{-(n-1)}\lambda_\varepsilon^{-\frac{n}{p+1}}\right)$.

Estimate of I_1. Since $x \in \partial B(x_\varepsilon, 2d_\varepsilon)$, we have $|x - y| \geq d_\epsilon$ for $y \in B(x_\epsilon, d_\epsilon)$, and so

$$|\nabla_y G(x, y)| \leq C d_\epsilon^{-(n-1)} \quad \text{and} \quad |\nabla_x \nabla_y G(x, y)| \leq C d_\varepsilon^{-n} \qquad \forall y \in B(x_\epsilon, d_\epsilon).$$

Combining this with the mean value formula yields

$$|G(x, y) - G(x, x_\epsilon)| \leq C|y - x_\epsilon| d_\epsilon^{-(n-1)} \quad \text{and} \quad |\nabla_x G(x, y) - \nabla_x G(x, x_\epsilon)| \leq C|y - x_\epsilon| d_\epsilon^{-n} \tag{3.7}$$

for all $y \in B(x_\varepsilon, d_\varepsilon)$. Applying this and (2.14) we may estimate I_1 as follows:

$$\begin{aligned} I_1(x) &\leq C d_\epsilon^{-(n-1)} \int_{B(x_\epsilon, d_\epsilon/2)} |y - x_\epsilon|\, \lambda_\epsilon^{\frac{2(p-\varepsilon)}{p-\varepsilon-1}} U^p(\lambda_\epsilon(y - x_\epsilon))dy \\ &\leq C d_\epsilon^{-(n-1)} \lambda_\epsilon^{\frac{2(p-\varepsilon)}{p-\varepsilon-1}} \lambda_\epsilon^{-(n+1)} \int_{B(0, N_\epsilon/2)} |y| U^q(y)dy. \end{aligned} \tag{3.8}$$

Using (2.13) and that $\frac{2p}{p-1}-(n+1)<-\frac{n}{p+1}$ we find that $I_1(x)=o(d_\varepsilon^{-(n-2)}\lambda_\varepsilon^{-\frac{n}{p+1}})$. By the same way along with the second inequality of (3.7), we can obtain the estimate

$$\nabla I_1(x)=o(d_\varepsilon^{-(n-1)}\lambda_\varepsilon^{-\frac{n}{p+1}}).$$

Estimate of I_2. For $y\in B(x_\epsilon,4d_\epsilon)\setminus B(x_\epsilon,d_\epsilon)$ we use the estimate (2.14) and (2.13) to find

$$u_\epsilon(y)\le C\lambda_\varepsilon^{\frac{n}{p+1}}U(\lambda_\epsilon(y-x_\epsilon))\le C\lambda_\varepsilon^{\frac{n}{p+1}-(n-2)}d_\varepsilon^{-(n-2)}. \tag{3.9}$$

Noting that

$$|x-y|\le 8d_\varepsilon \quad \text{for } y\in B(x_\varepsilon,4d_\varepsilon) \quad \text{and} \quad x\in\partial B(x_\varepsilon,2d_\varepsilon), \tag{3.10}$$

we have

$$\begin{cases} |G(x,y)|+|G(x,x_\epsilon)|\le \dfrac{c_n}{|x-y|^{n-2}}+\dfrac{c_n}{d_\epsilon^{(n-2)}}\le \dfrac{C}{|x-y|^{n-2}}, \\ |\nabla_x G(x,y)|+|\nabla_x G(x,x_\epsilon)|\le \dfrac{c_n}{|x-y|^{n-1}}+\dfrac{c_n}{d_\epsilon^{(n-1)}}\le \dfrac{C}{|x-y|^{n-1}}. \end{cases} \tag{3.11}$$

Combining the first estimate of (3.11), (3.10) and (3.9) in (3.6) yields

$$\begin{aligned} I_2(x) &\le C\lambda_\varepsilon^{\frac{pn}{p+1}}d_\varepsilon^{-(n-2)p}\lambda_\varepsilon^{-(n-2)p}\int_{B(x_\epsilon,4d_\epsilon)\setminus B(x_\epsilon,d_\epsilon)}\frac{1}{|x-y|^{n-2}}dy \\ &\le C\lambda_\varepsilon^{\frac{pn}{p+1}}d_\varepsilon^{2-(n-2)p}\lambda_\varepsilon^{-(n-2)p} \\ &= C\lambda_\varepsilon^{-\frac{n}{p+1}}d_\varepsilon^{-(n-2)}N_\varepsilon^{n-(n-2)p}. \end{aligned}$$

Due to the fact that $p=\frac{n+2}{n-2}$ the above estimate gives the estimate $I_2(x)=o\left(\lambda_\varepsilon^{-\frac{n}{p+1}}d_\varepsilon^{-(n-2)}\right)$. Similarly, using the second estimate of (3.11), we obtain

$$\nabla I_2(x)=O\left(\lambda_\varepsilon^{-\frac{n}{p+1}}d_\varepsilon^{-(n-1)}N_\varepsilon^{n-(n-2)p}\right)=o\left(\lambda_\varepsilon^{-\frac{n}{p+1}}d_\varepsilon^{-(n-1)}\right).$$

Estimate of I_3. Since $|x-x_\epsilon|=2d_\epsilon$, we have the following estimates

$$\begin{cases} |G(x,y)-G(x,x_\epsilon)|\le Cd_\epsilon^{-(n-2)} & \text{for } y\in\Omega\setminus B(x_\epsilon,4d_\epsilon), \\ |\nabla_x G(x,y)-\nabla_x G(x,x_\epsilon)|\le Cd_\epsilon^{-(n-1)} & \text{for } y\in\Omega\setminus B(x_\epsilon,4d_\epsilon). \end{cases} \tag{3.12}$$

Applying the first inequality of (3.12), we have

$$I_3(x)\le Cd_\epsilon^{-(n-2)}\int_{\Omega\setminus B(x_\epsilon,4d_\epsilon)}u_\epsilon^{p-\varepsilon}(y)dy.$$

Using (2.14) we deduce

$$\begin{aligned} \int_{\Omega\setminus B(x_\epsilon,4d_\epsilon)}u_\epsilon^{p-\varepsilon}(y)dy &= \lambda_\epsilon^{-\frac{n}{p-\varepsilon+1}}\int_{\Omega_\epsilon\setminus B(0,4N_\epsilon)}U_\epsilon^{p-\varepsilon}(y)dy \\ &\le C\lambda_\epsilon^{-\frac{n}{p-\varepsilon+1}}\int_{\mathbb{R}^n\setminus B(0,4N_\epsilon)}U^p(y)dy \\ &\le C\lambda_\epsilon^{-\frac{n}{p+1}}N_\epsilon^{-(n-2)p+n}. \end{aligned}$$

Combining the above two estimates, we arrive at the following estimate

$$I_3(x) \le C d_\epsilon^{-(n-2)} \lambda_\epsilon^{-\frac{n}{p+1}} N_\epsilon^{-(n-2)p+n} = o\left(d_\epsilon^{-(n-2)} \lambda_\epsilon^{-\frac{n}{p+1}}\right), \tag{3.13}$$

where the fact that $(n-2)p > n$ was also used. Similarly, using the second estimate of (3.12), then we get

$$\nabla I_3(x) = O\left(d_\epsilon^{-(n-1)} \lambda_\epsilon^{-\frac{n}{p+1}} N_\epsilon^{-(n-2)p+n}\right) = o\left(d_\epsilon^{-(n-1)} \lambda_\epsilon^{-\frac{n}{p+1}}\right).$$

Finally, gathering the above estimates on I_1, I_2, and I_3, we finally get

$$I_1(x) + I_2(x) + I_3(x) = o\left(d_\epsilon^{-(n-2)} \lambda_\epsilon^{-\frac{n}{p+1}}\right)$$

and

$$|\nabla_x I_1(x)| + |\nabla_x I_2(x)| + |\nabla_x I_3(x)| = o\left(d_\epsilon^{-(n-1)} \lambda_\epsilon^{-\frac{n}{p+1}}\right).$$

The lemma is proved. □

4. The proof of Theorem 1.1

This section is devoted to prove Theorem 1.1.

Proof of Theorem 1.1. Let $d_\epsilon = \text{dist}(x_\epsilon, \partial\Omega)$. In view of (2.9), it is enough to show that $\inf_{\varepsilon>0} d_\varepsilon > 0$. For this purpose, with a view to a contradiction, we assume the contrary that $d_\epsilon \to 0$ as $\epsilon \to 0$ in a subsequence.

We use the notation λ_ε and $N_\varepsilon = d_\varepsilon \lambda_\varepsilon$ defined in (2.3) and (2.11). Then we recall from Lemma 2.2 that we have $N_\epsilon \to \infty$. Let us set $D_\epsilon = B(x_\epsilon, 2d_\epsilon)$ for each $1 \le j \le n$ and we define the values L_ϵ^j and R_ϵ^j by

$$\begin{aligned} L_\epsilon^j &: = -2\int_{\partial D_\epsilon} \frac{\partial u_\epsilon}{\partial \nu}\frac{\partial u_\epsilon}{\partial x_j}(x) dS_x + \int_{\partial D_\epsilon} |\nabla u_\epsilon|^2 \nu_j dS_x, \\ R_\epsilon^j &: = \frac{n-2}{n}\int_{\partial D_\epsilon} u_\epsilon^{p+1} \nu_j dS_x. \end{aligned}$$

Applying Lemma 2.4 to u_ε with $D = D_\epsilon$, we find that

$$L_\epsilon^j = R_\epsilon^j.$$

In what follows, we proceed to obtain sharp estimates of the values of L_j^ϵ and R_j^ϵ, which will lead to a contradiction.

First, we compute L_ϵ^j using the expression (3.2) as follows.

$$\begin{aligned} L_j^\epsilon &= -2\lambda_\epsilon^{-\frac{2}{p-1-\varepsilon}} A_U^2 \int_{\partial D_\epsilon} \left(\frac{\partial}{\partial \nu} G(x, x_\epsilon) \frac{\partial}{\partial x_j} G(x, x_\epsilon)\right) dS_x \\ &\quad + \lambda_\epsilon^{-\frac{2}{p-1-\varepsilon}} A_U^2 \int_{\partial D_\epsilon} |\nabla G(x, x_\epsilon)|^2 \nu_j dS_x + o\left(|\partial D_\epsilon| \lambda_\epsilon^{-\frac{2n}{p+1}} d_\epsilon^{-2(n-1)}\right) \\ &= -\lambda_\epsilon^{-\frac{2}{p-1-\varepsilon}} A_U^2 I(2d_\epsilon) + o(d_\epsilon^{-(n-1)} \lambda_\epsilon^{-(n-2)}), \end{aligned} \tag{4.1}$$

where we have set

$$I(r) := \left[\int_{\partial B(x_\epsilon,r)} 2\frac{\partial G}{\partial \nu}(x,x_\epsilon)\frac{\partial}{\partial x_j}G(x,x_\epsilon) - |\nabla G(x,x_\epsilon)|^2\nu_j dS_x\right] \quad \text{for} \quad r>0.$$

In order to compute the value of $I(2d_\epsilon)$, we first notice that $I(r)$ is independent of $r > 0$. Indeed, it follows from that $-\Delta_x G(x,x_\epsilon) = 0$ for $x \in A_r := B(x_\varepsilon, 2d_\varepsilon) \setminus B(x_\varepsilon, r)$ for each $r \in (0, 2d_\varepsilon)$, and an integration by parts performed as follows:

$$\begin{aligned}
0 &= \int_{A_r} (-\Delta_x G)(x,x_\epsilon)\frac{\partial G}{\partial x_j}(x,x_\epsilon)dx \\
&= -\int_{\partial A_r} \frac{\partial G}{\partial \nu}(x,x_\epsilon)\frac{\partial G}{\partial x_j}(x,x_\epsilon)dS_x + \int_{A_r} \nabla_x G(x,x_\epsilon)\frac{\partial \nabla_x G}{\partial x_j}(x,x_\epsilon)dx \qquad (4.2)\\
&= -\int_{\partial A_r} \frac{\partial G}{\partial \nu}(x,x_\epsilon)\frac{\partial G}{\partial x_j}(x,x_\epsilon)dS_x + \frac{1}{2}\int_{\partial A_r} |\nabla_x G(x,x_\epsilon)|^2\nu_j dS_x,
\end{aligned}$$

which means that $I(r)$ is constant on $(0, 2d_\epsilon]$. Therefore we can evaluate $I(2d_\varepsilon)$ by computing the following limit;

$$\begin{aligned}
I(2d_\epsilon) &= \lim_{r\to 0} I(r) \\
&= \lim_{r\to 0}\int_{\partial B(x_\epsilon,r)} 2\left(-\frac{c_n(n-2)}{|x-x_\epsilon|^n} - \frac{\partial H}{\partial \nu}(x,x_\epsilon)\right)\left(-\frac{c_n(n-2)(x-x_\epsilon)_j}{|x-x_\epsilon|^n} - \frac{\partial H}{\partial x_j}(x,x_\epsilon)\right) \\
&\quad - \left(-\frac{c_n(n-2)(x-x_\epsilon)}{|x-x_\epsilon|^n} - \nabla H(x,x_\epsilon)\right)^2 \nu_j dS_x.
\end{aligned}$$

Thanks to the oddness of the integrand, we have

$$\int_{\partial B(x_\epsilon,r)} 2\left(\frac{c_n(n-2)}{|x-x_\epsilon|^n}\right)\left(\frac{c_n(n-2)(x-x_\epsilon)_j}{|x-x_\epsilon|^n}\right) - \left[\left(\frac{c_n(n-2)(x-x_\epsilon)}{|x-x_\epsilon|^n}\right)^2 \nu_j\right] dS_x = 0.$$

Also, since $-\Delta_x H(x,x_\varepsilon) = 0$ holds for $x \in B(x_\varepsilon, 2d_\varepsilon)$, we may proceed as in (4.2) to get

$$\int_{\partial B(x_\epsilon,r)} 2\left(\frac{\partial H}{\partial \nu}(x,x_\epsilon)\right)\left(\frac{\partial H}{\partial x_j}(x,x_\epsilon)\right) - \left[(\nabla H(x,x_\epsilon))^2\,\nu_j\right] dS_x = 0.$$

Using the above estimates, we complete the estimation as follows.

$$\begin{aligned}
I(2d_\epsilon) &= \lim_{r\to 0}\int_{\partial B(x,r)} 2c_n^2(n-2)\frac{\partial H}{\partial \nu}(x,x_\epsilon)\frac{(x-x_\epsilon)_j}{|x-x_\epsilon|^n} + 2\frac{c_n(n-2)}{|x-x_\epsilon|^{n-1}}\frac{\partial H}{\partial x_j}(x,x_\epsilon)dS_x \\
&\quad - \frac{2c_n(n-2)(x-x_\epsilon)}{|x-x_\epsilon|^n}\nabla H(x,x_\epsilon)\nu_j dS_x \\
&= \left[\frac{2c_n(n-2)}{n}\frac{\partial H}{\partial x_j}(x_\epsilon,x_\epsilon) + 2c_n(n-2)\frac{\partial H}{\partial x_j}(x_\epsilon,x_\epsilon) - \frac{2c_n(n-2)}{n}\frac{\partial H}{\partial x_j}(x_\epsilon,x_\epsilon)\right]|S^{n-1}| \\
&= 2c_n(n-2)|S^{n-1}|\frac{\partial H}{\partial x_j}(x_\epsilon,x_\epsilon).
\end{aligned}$$

Plugging this into (4.1) shows that

$$L_j^\epsilon = -\lambda_\epsilon^{-\frac{2}{p-1-\varepsilon}} c_n A_U^2 |S^{n-1}| 2(n-2) \frac{\partial H}{\partial x_j}(x_\epsilon,\ x_\epsilon) + o(d_\epsilon^{-(n-1)} \lambda_\epsilon^{-(n-2)}). \tag{4.3}$$

Now, we take $j \in \{1, \cdots, n\}$ such that $\left|\frac{\partial H}{\partial x_j}(x_\varepsilon, x_\varepsilon)\right| \geq \frac{C}{d_\varepsilon^{n-1}}$, which is guaranteed by Theorem 1.3. Injecting this into (4.3) we have

$$L_j^\epsilon \geq C\lambda_\epsilon^{-(n-2)} d_\epsilon^{-(n-1)} = C\lambda_\epsilon N_\epsilon^{-(n-1)}. \tag{4.4}$$

Next we shall find an upper bound of R_j^ϵ. Applying (2.14) we have

$$u_\epsilon(x) \leq C\lambda_\epsilon^{\frac{n}{p+1}} U(\lambda_\epsilon(x - x_\epsilon)) \leq C\lambda_\epsilon^{\frac{n}{p+1}} N_\epsilon^{-(n-2)} \quad \forall\ x \in \partial B(x_\varepsilon, 2d_\varepsilon).$$

Using this we estimate

$$\begin{aligned} \left| \int_{\partial D_\epsilon} u_\epsilon^{p+1} \nu_j dS_x \right| &\leq C|\partial D_\epsilon| \lambda_\epsilon^n N_\epsilon^{-(n-2)(p+1)} \\ &\leq C d_\epsilon^{(n-1)} \lambda_\epsilon^n N_\epsilon^{-(n-2)(p+1)} \\ &= C\left(\frac{N_\epsilon}{\lambda_\epsilon}\right)^{(n-1)} \lambda_\epsilon^n N_\epsilon^{-(n-2)(p+1)} = C\lambda_\epsilon N_\epsilon^{(n-1)-(n-2)(p+1)}, \end{aligned} \tag{4.5}$$

which yields

$$|R_j^\epsilon| \leq C\lambda_\epsilon N_\epsilon^{(n-1)-(n-2)(p+1)}. \tag{4.6}$$

Now we combine (4.4) and (4.6) to get

$$\lambda_\epsilon N_\epsilon^{-(n-1)} \leq L_j^\varepsilon = R_j^\varepsilon \leq C\lambda_\epsilon N_\epsilon^{(n-1)-(n-2)(p+1)}.$$

Since N_ϵ goes to infinity as $\epsilon \to 0$, the above inequality yields that

$$-(n-1) \leq (n-1) - (n-2)(p+1),$$

which is equivalent to $p \leq \frac{n}{n-2}$. However this contradicts to the fact that $p = \frac{n+2}{n-2}$. Thus the assumption $d_\epsilon \to 0$ cannot hold, and so $\inf_{\varepsilon>0} d_\varepsilon > 0$. The proof is completed. □

5. Proof of Theorem 1.2

In this section we provide a proof of Theorem 1.2.

Proof of Theorem 1.1. From the result of Theorem 1.1, we know that the maximum point x_ε of the solution u_ε are uniformly away from the boundary $\partial\Omega$. Therefore, up to a subsequence, the point x_ε converges to an interior point $x_0 \in \Omega$. By Lemma 2.3 we know the first statement of the theorem holds.

We may easily deduce the version of Lemma 3.1 under the assumption that x_ε converges to an interior point x_0. Indeed, it is direct to deduce from (3.4) that

$$u_\varepsilon(x) = A_U \lambda_\varepsilon^{-\frac{[2-(n-2)\varepsilon]}{p-1-\varepsilon}} G(x, x_0) + o(\lambda_\varepsilon^{-\frac{n}{p+1-\varepsilon}}),$$

for $x \in \Omega \setminus \{x_0\}$. Thus we have

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon^{\frac{[2-(n-2)\varepsilon]}{p-1-\varepsilon}} u_\varepsilon(x) = A_U G(x, x_0) \quad \text{in } C^1(\Omega \setminus \{x_0\}). \tag{5.1}$$

Proposition 5.1. *We have* $\lim_{\varepsilon\to 0}\lambda_\varepsilon^\varepsilon = 1$.

Proof. Let $v_\varepsilon = (x-x_0)\cdot\nabla u_\varepsilon + \left(\frac{2}{p-1-\varepsilon_n}\right)u_\varepsilon$. Then it satisfies

$$-\Delta v_\varepsilon = (p-\varepsilon)\,u_\varepsilon^{p-1-\varepsilon}v_\varepsilon \quad \text{in } \Omega.$$

Therefore we have

$$\lambda_\varepsilon^{\frac{4-2(n-2)\varepsilon}{p-1-\varepsilon}}\int_{\partial B^n(y,r)}\left(\frac{\partial u_\varepsilon}{\partial\nu}v_\varepsilon - \frac{\partial v_\varepsilon}{\partial\nu}u_\varepsilon\right)dS_x = \lambda_\varepsilon^{\frac{2n}{p+1-\varepsilon}}(p-1-\varepsilon)\int_{B^n(y,r)}u_\varepsilon^{p-\varepsilon}v_\varepsilon\,dx. \tag{5.2}$$

By (5.1) we have

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon^{\frac{2-(n-2)\varepsilon}{p-1-\varepsilon}}v_\varepsilon(x) = A_U\left[(x-x_0)\cdot\nabla G(x,x_0) + \frac{2}{p-1}G(x,x_0)\right].$$

Taking $\varepsilon\to 0$ we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lambda_\varepsilon^{\frac{4-2(n-2)\varepsilon}{p-1-\varepsilon}}\int_{\partial B^n(y,r)}\left(\frac{\partial u_\varepsilon}{\partial\nu}v_\varepsilon - \frac{\partial v_\varepsilon}{\partial\nu}u_\varepsilon\right)dS_x\\
&= A_U^2\int_{\partial B^n(y,r)}\left(\frac{\partial G(x,x_0)}{\partial\nu}\left[(x-x_0)\cdot\nabla G(x,x_0) + \frac{2}{p-1}G(x,x_0)\right]\right.\\
&\qquad\qquad\left. -\frac{\partial}{\partial\nu}\left[(x-x_0)\cdot\nabla G(x,x_0) + \frac{2}{p-1}G(x,x_0)\right]G(x,x_0)\right)dS_x\\
&= A_U^2(n-2)H(x_0,x_0),
\end{aligned} \tag{5.3}$$

where the last equality is derived in [1, pg. 170]. The right hand side of (5.2) is equal to

$$\begin{aligned}
&\int_{B^n(y,r)}u_\varepsilon^{p-\varepsilon}v_\varepsilon\,dx\\
&= \int_{B^n(x_0,r)}u_\varepsilon^{p-\varepsilon}\left[(x-x_0)\cdot\nabla u_\varepsilon + \frac{2}{p-1-\varepsilon}u_\varepsilon\right]dx\\
&= \left(\frac{2}{p-1-\varepsilon} - \frac{n}{p+1-\varepsilon}\right)\int_{B(x_0,r)}u_\varepsilon^{p+1-\varepsilon}(x)dx + \int_{\partial B(x_0,r)}u_\varepsilon^{p+1-\varepsilon}(x-x_0)\cdot\nu dS_x\\
&= \left(\frac{2}{p-1-\varepsilon} - \frac{n}{p+1-\varepsilon}\right)\int_{B(x_0,r)}u_\varepsilon^{p+1-\varepsilon}(x)dx + O(\lambda_\varepsilon^{-n}).
\end{aligned}$$

Using this we have

$$\begin{aligned}
&\lambda_\varepsilon^{\frac{4-2(n-2)\varepsilon}{p-1-\varepsilon}}(p-1-\varepsilon)\int_{B^n(y,r)}u_\varepsilon^{p-\varepsilon}v_\varepsilon\,dx\\
&\qquad= \left(\lambda_\varepsilon^{\frac{4-2(n-2)\varepsilon}{p-1-\varepsilon}}\varepsilon\right)\frac{(n-2)^2}{2n}\left(\int_{\mathbb{R}^n}U^{p+1}(x)dx + o(1)\right) + O(\lambda_\varepsilon^{-2}).
\end{aligned} \tag{5.4}$$

Injecting (5.3) and (5.4) into (5.2) we get

$$A_U^2 q_n H(x_0,x_0) = \lim_{\varepsilon\to 0}\left(\lambda_\varepsilon^{\frac{2n}{p+1-\varepsilon}}\varepsilon\right). \tag{5.5}$$

This implies that $\lambda_\varepsilon \leq C\varepsilon^{-\frac{2n}{p-1-\varepsilon}} \leq C\varepsilon^{-\frac{4n}{p-1}}$ for any small $\varepsilon > 0$. Therefore we have

$$1 \leq \lim_{\varepsilon\to 0} \lambda_\varepsilon^{\varepsilon} \leq \lim_{\varepsilon\to 0} C^{\varepsilon}\varepsilon^{-\left(\frac{4n}{p-1}\right)\varepsilon} = 1.$$

The lemma is proved. □

Given the result of Proposition 5.1, we deduce from (5.1) that

$$\begin{aligned}\lim_{\varepsilon\to 0} \lambda_\varepsilon^{\frac{n}{p+1}} u_\varepsilon(x) &= \lim_{\varepsilon\to 0} \lambda_\varepsilon^{\frac{2(n-2)^2\varepsilon}{2(4-\varepsilon(n-2))}} \lambda_\varepsilon^{\frac{2-(n-2)\varepsilon}{p-1-\varepsilon}} u_\varepsilon(x) \\ &= A_U G(x, x_0) \quad \text{in } C^1(\Omega \setminus \{x_0\}),\end{aligned} \tag{5.6}$$

where we used $p = \frac{n+2}{n-2}$ in the first equality. This proves the third statement of Theorem 1.2. Next, taking $D = B(x_0, r)$ and $u = u_\varepsilon$ in (2.15), we have

$$\lambda_\varepsilon^{n-2}\frac{(n-2)}{n}\int_{\partial B(x_0,r)} u_\varepsilon^{p+1}\nu_j dS_x = \lambda_\varepsilon^{n-2}\int_{\partial B(x_0,r)} |\nabla u_\varepsilon(x)|^2\nu_j - 2\frac{\partial u_\varepsilon}{\partial \nu}\frac{\partial u_\varepsilon}{\partial x_j}(x)dS_x. \tag{5.7}$$

By (2.9) we have $u_\varepsilon(x) \leq \lambda_\varepsilon^{-\frac{(n-2)}{2}}$ for $x \in \partial B(x_0, r)$ we have

$$\left|\lambda_\varepsilon^{(n-2)}\int_{\partial B(x_0,r)} u_\varepsilon^{p+1}\nu_j dS_x\right| \leq C\lambda_\varepsilon^{(n-2)}\lambda_\varepsilon^{-n}.$$

Using this and (5.1) we take limit $\varepsilon \to 0$ in (5.7) to get

$$0 = A_U^2\int_{\partial B(x_0,r)} |\nabla G(x, x_0)|^2\nu_j - 2\frac{\partial G(x, x_0)}{\partial \nu}\frac{\partial G(x, x_0)}{\partial x_j}dS_x = -A_U^2\frac{(2n-1)}{n}\frac{\partial H}{\partial x_j}(x_0, x_0),$$

which yields the second statement of the theorem. Finally, given the result of Proposition 5.1, we get from (5.5) that

$$\lim_{\varepsilon\to 0}\left(\varepsilon \cdot \lambda_\varepsilon^{\frac{2n}{p+1}}\right) = (n-2)A_U^2 H(x_0, x_0).$$

This proves the last statement of the theorem. The proof is finished. □

6. The proof of Theorem 1.3

We prove the second main theorem of this paper.

Proof of Theorem 1.3. Consider a sequence of points $\{x^k\}_{k\in\mathbb{N}}$ in the domain Ω such that $\mathbf{d}_k := \text{dist}(x^k, \partial\Omega)$ goes to zero as $k \to \infty$. Take $z^k \in \partial\Omega$ such that $|x^k - z^k| = \mathbf{d}_k$. Let $\Omega_k := \frac{1}{\mathbf{d}_k}(\Omega - z^k)$. Note that we have $0 \in \Omega_k$, and also $\frac{1}{\mathbf{d}_k}(x^k - z^k) \in S^{n-1}$. Thus we can find a rotation $R_k : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$R_k\left(\frac{1}{\mathbf{d}_k}(x^k - z^k)\right) = e_n = (0, \cdots, 0, 1). \tag{6.1}$$

Then, by **Assumption D.** the domain $D_k := R_k\Omega_k$ converges to an infinite star-shaped domain $\mathbb{P} \subsetneq \mathbb{R}^n$. To prove the estimate (1.4) we set the function $W_k : D_k \to \mathbb{R}$ for each $k \in \mathbb{N}$ by

$$W_k(z) = H(R_k^{-1}\mathbf{d}_k z + z^k, x^k)\mathbf{d}_k^{n-2}. \tag{6.2}$$

Let G_k be Green's function of $-\Delta$ on D_k with the Dirichlet boundary condition. For each $y \in \mathbb{R}^n_+$ we denote $y^* = (y_1, \cdots, y_{n-1}, -y_n)$ for $y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$. We consider the function $H_0 : \overline{\mathbb{P}} \times \overline{\mathbb{P}} \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta_z H_0(z, y) = 0 & \text{for } (z, y) \in \mathbb{P} \times \mathbb{P}, \\ H_0(z, y) = \dfrac{c_n}{|(z-y)|^{n-2}} & \text{for } z \in \partial\mathbb{P}. \end{cases} \tag{6.3}$$

Here c_n is the value defined in (2.1). Now we set $W_0 : \overline{\mathbb{P}} \to \mathbb{R}$ by $W_0(z) := H_0(z, e_n)$. Then we have the following result.

Lemma 6.1. *As* $\mathbf{d}_k \to 0$*, the function* W_k *converges to* W_0 *in* $C^1(B(e_n, 1/4))$.

Proof. By definition (6.2) and (2.2), the function W_k satisfies

$$-\Delta_w W_k(w) = 0 \quad \text{in } D_k \quad \text{and} \quad W_k(w) = \frac{c_n}{|R_k^{-1}\mathbf{d}_k w + z^k - x^k|^{n-2}} \text{ for } w \in \partial D_k. \tag{6.4}$$

Set the difference $R_k : \Omega_k \to \mathbb{R}$ by $R_k(x) = W_0(x) - W_k(x)$ for $x \in \Omega_k$. Then, it suffices to show that $R_k \to 0$ in $C^1_{loc}(\mathbb{P})$. By (6.4) and (6.3) we have

$$(-\Delta_w) R_k(w) = 0 \quad \text{in } D_k. \tag{6.5}$$

Let us prove the C^0 convergence of $\mathcal{R}_k$. Since $\mathcal{R}_k$ is harmonic in Ω_k, we only need to show that

$$\lim_{k\to\infty} \sup_{x\in\partial\Omega_k} |\mathcal{R}_k(x)| = 0.$$

Take a large number $R > 0$. Then we have

$$\sup_{x\in\partial D_k \cap B(0,R)^c} |W_k(x)| + |W_0(x)| \le \frac{C}{R^{n-2}}.$$

We note that for $z \in \partial D_k$, using (6.1) we have

$$W_k(z) = \frac{c_n}{||z - e_n||^{n-2}}, \tag{6.6}$$

and for $z \in \partial\mathbb{P}$,

$$W_0(z) = \frac{c_n}{||z - e_n||^{n-2}}. \tag{6.7}$$

For fixed $R > 0$, we have

$$\lim_{k\to\infty} (\partial D_k \cap B_R) = \partial\mathbb{P} \cap B_R,$$

and we note that $\partial D_k \cap B_R$ is compact. Combining this fact with (6.6) and (6.7), we obtain

$$\lim_{k\to\infty} \sup_{x\in\partial D_k \cap B_R} |W_k(x) - W_0(x)| = 0.$$

Thus,

$$\begin{aligned} &\lim_{k\to\infty} \sup_{x\in\partial D_k} |W_k(x) - W_0(x)| \\ &\quad \le \lim_{k\to\infty} \sup_{x\in\partial D_k\cap B_R} |W_k(x) - W_0(x)| + \lim_{k\to\infty} \sup_{x\in\partial D_k \cap B_R^c} |W_k(x) - W_0(x)| \\ &\quad \le \frac{C}{R^{n-2}}. \end{aligned}$$

Since $R > 0$ is arbitrary, we have

$$\lim_{k\to\infty} \sup_{x\in\partial D_k} |W_k(x) - W_0(x)| = 0.$$

Combining the above two convergence results, we can deduce that $R_k(x) \to 0$ uniformly for $x \in B(e_n, 1/4)$. From (6.4) we know that R_k is contained in $C^{1,\beta}(B(e_n, 1/4))$ uniformly in $k \in \mathbb{N}$ for some $\beta > 0$. Thus R_k converges to a function f in $C^1(B(e_n, 1/4))$. In this paper we are concerned with the Lane-Emden-Fowler equation

$$\begin{cases} -\Delta u &= u^{\frac{n+2}{n-2}-\varepsilon} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.8}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a *polygonal* domain and $\varepsilon > 0$. We study the asymptotic behavior of minimal energy solutions as $\varepsilon > 0$ goes to zero. we have $f \equiv 0$ since R_k converges to 0 in $C^0(B(e_n, 1/2))$. The lemma is proved. □

Lemma 6.2. *We have* $\frac{\partial}{\partial x_n} W_0(e_n) \neq 0$.

Proof. Notice that $H_0(x, y)$ satisfies

$$\begin{cases} -\Delta_x H_0(x, y) = 0 & x \in \mathbb{P}, \\ H_0(x, y) = \frac{c_n}{|x-y|^{n-2}} & x \in \partial\mathbb{P}. \end{cases}$$

Since H_0 is the regular part of Green's function on $\mathbb{P}$, we have

$$H_0(x, y) = H_0(y, x). \tag{6.9}$$

For given $t > 0$ consider the function $f(x) := t^{n-2} H_0(tx, te_n)$ defined on $\frac{1}{t}\mathbb{P} = \mathbb{P}$ which satisfies

$$\begin{cases} -\Delta_x f(x) = 0 & x \in \mathbb{P}, \\ f(x) = \frac{c_n t^{n-2}}{|tx-te_n|^{n-2}} = \frac{c_n}{|x-e_n|^{n-2}} & x \in \partial\mathbb{P}. \end{cases}$$

This exaclty means that $f(x) = H_0(x, e_n)$, and so $H_0(x, e_n) = t^{n-2} H_0(tx, te_n)$. Combining this with the symmetric property (6.9), we have

$$\begin{aligned} \frac{\partial}{\partial x_n} W_0(x)\Big|_{x=e_n} &= \left(\frac{\partial}{\partial x_n} H_0(x, e_n)\right)_{x=e_n} \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x_n} H_0(x, x)\right)_{x=e_n} \\ &= \frac{1}{2}\frac{\partial}{\partial t} H_0(te_n, te_n)\Big|_{t=1} = \frac{(2-n)}{2} H(e_n, e_n). \end{aligned} \tag{6.10}$$

Also we note that $H_0(e_n, e_n) \neq 0$ by the maximum principle since $(-\Delta)H_0 = 0$ in $\mathbb{P}$ and $H_0 > 0$ on $\partial\mathbb{P}$. Combining this fact with (6.10) we deduce that $\frac{\partial}{\partial x_n} W_0(e_n) < 0$. The proof is finished. □

Now we are ready to finish the proof of Theorem 1.3. By Lemma 6.1, we know that $W_k(x)$ converges to $W_0(x)$ in $C^1(B(e_n, 1/4))$. Since $\left|\frac{\partial}{\partial x_n} W_0(e_n)\right| > c > 0$, we conclude that for large $k \in \mathbb{N}$, we have $\left|\frac{\partial}{\partial x_n} W_k(e_n)\right| > c/2$. By definition of W_k given in (6.2), we have

$$\frac{\partial}{\partial x_n} W_k(z) = \mathbf{d}_k^{n-1} (R_k^{-1})_n \cdot \nabla H(\mathbf{d}_k R_k^{-1}(z), x^k).$$

Therefore we may conclude that for large $k \in \mathbb{N}$,

$$|\mathbf{d}_k^{n-1} (R_k^{-1})_n \cdot \nabla H(x^k, x^k)| > c/2,$$

which implies that

$$\left|\nabla H(x^k, x^k)\right| > \frac{c}{2\mathbf{d}_k^{n-1}}$$

for $k \in \mathbb{N}$ large enough. The proof is finished. □

7. Conclusions

In this paper, we study the energy minimizing solutions to slightly subcritical elliptic problems on nonconvex polygonal domains. The main part for the analysis is to exclude the possibility that the peak of the solution approaches the boundary of the domain as the moving plane method is difficult to apply directly for the nonconvex polygonal domain. To address this challenge, we make use of the Pohozaev identity and the Green's function to show that a contradiction aries when we assume that the solution blows up near the boundary.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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