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Research article

Expansions of generalized bases constructed via Hasse derivative operator in Clifford analysis

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Abstract: The present paper investigates the approximation of special monogenic functions (SMFs) in infinite series of hypercomplex Hasse derivative bases (HHDBs) in Fréchet modules (F-modules). The obtained results ensure the existence of such representation in closed hyperballs, open hyperballs, closed regions surrounding closed hyperballs, at the origin, and for all entire SMFs (ESMFs). Furthermore, we discuss the mode of increase (order and type) and the T_ρ-property. This study enlightens several implications for some associated HHDBs, such as hypercomplex Bernoulli polynomials, hypercomplex Euler polynomials, and hypercomplex Bessel polynomials. Based on considering a more general class of bases in F-modules, our results enhance and generalize several known results concerning approximating functions in terms of bases in the complex and Clifford settings.

Keywords: Hasse derivative operator; bases; basic series; effectiveness; order; type; hypercomplex analysis; Fréchet modules
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1. Introduction

One of the concepts which have significant impact in Clifford analysis is studying the approximation of a Cliffordian function $h^{(m)}(x),\ x \in \mathbb{R}^{m+1}$ as a series of the form:

$$\sum_{n=0}^{\infty} Q_n^{(m)}(x)\, a_n^{(m)}, \quad a_n^{(m)} \in C_m, \tag{1.1}$$

where $\{Q_n^{(m)}(x),\ x \in \mathbb{R}^{m+1}\}$ is a prescribed base of Cllifordian polynomials and C_m is the real Clifford algebra of dimension 2^m. In 1990, this problem was addressed by the authors of [1]. The polynomials are assumed to form a Hamel basis of $\mathcal{S}_m(x)$ (the Clifford linear space of all special monogenic polynomials (SMPs) with Clifford algebra coefficients). The series (1.1) is called the basic series associated with the base of SMPs. Many results about the approximation of SMFs and ESMFs by basic series, which can be associated with bases of SMPs [2, 3].

The theory of basic series in the case of one complex variable was originally discovered by Whittaker and Cannon [4–7] about 90 years ago. As we have mentioned earlier, the attempt done by authors of [1] were the first to extend the notion of basic series in the case of Clifford analysis.

In the case of a single complex variable, the approximation properties of the derivative and integral bases of a certain base of polynomials of in a disk of center origin have been studied by many authors, of whom we may mention Makar [8], Mikhail [9], and Newns [10]. In the case of of several complex variables the domains of representation are hyperspherical, hyperelliptical and polycylinderical regions (see [11, 12]). Afterwards, the authors of [13, 14] generalized this problem in Clifford analysis, which is called hypercomplex primitive and derivative bases of SMPs and their representations is in closed hyperballs.

Approximation theory is a rich topic which has numerous applications in various scientific disciplines such that mathematical analysis, statistics, engineering and physics. Recently, order moment of the wind power time series has been studied in [15]. Although our study is narrowed to theoretical aspects, the basic sets (bases) of polynomials proved its efficiency in as solutions to important partial differential equations, such as the heat equation [16] and wave equation [17, 18].

The authors of [19] introduced an expansion of a SMF by basic series of generalized Bessel SMPs. They proved that the GBSMPs are solutions of second order homogeneous differential equations. Furthermore, in [3], the authors proved an extended version of Hadamard's three-hyperballs theorem to study the overconvergence properties. One of the recent fascinating research findings can be found in [20] where the authors of used the Hadamard's three-hyperballs theorem to generalize the Whittaker-Cannon theorem in open hyperballs in $\mathbb{R}^{m+1}$. Precisely, they proved that the hypercomplex Cannon functions preserved the effectiveness properties of both Cannon and non-Cannon bases. In the very recent paper [21] the authors derived a new base of SMPs in F-modules, named the equivalent base. They have also studied the convergence properties (effectiveness, order and type, T_ρ-property) of these base.

In 2017, a study based primarily on combination of Clifford analysis and functional analysis [26] when the considered bases $\{Q_n^{(m)}(x)\}$ are not necessarily consisting of polynomials. The convergence properties of these general bases had been studied in F-modules. Precisely, a general criterion for effectiveness of basic series in F-modules was constructed.

Recently in [22], the authors have studied a new base called hypercomplex Ruscheweyh derivative bases (HRDBs). They investigated the effectiveness properties of HRDBs of a given base of SMPs in different regions of convergence in F-modules. The above treatment is considered to extend and improve the results in Clifford and complex given in [8–10, 13, 14].

Motivated by the preceding discussion, the current work introduces a modified generalization of the Hasse derivative operator (HDO). Acting by hypercomplex HDO on bases, we derive a base of SMPs, which we may call the hypercomplex Hasse derivative bases of SMPs (HHDBSMPs). Consequently, we discuss the effectiveness properties, mode of increase, and the T_ρ-property of such a base in several regions: closed and open hyperballs, open regions surrounding closed hyperball, at the origin, and for all entire SMFs. Some applications on the HHD of Bernoulli SMPs (BSMPs), Euler SMPs (ESMPs), proper Bessel SMPs (PBSMPs), general Bessel SMPs (GBSMPs) and Chebyshev SMPs (CSMPs) are also provided. The obtained results offer new generalizations of existing work concerning the convergence properties of polynomials bases in both complex and Clifford settings.

2. Preliminaries

This section collects some notations and basic results which are needed throughout the paper. More details can be found in the literature, see [1, 23–26]. The real Clifford algebra over $\mathbb{R}$ is defined as

$$C_m = \{b = \sum_{B\subseteq\{1,...m\}} b_B e_B,\ b_B \in \mathbb{R}\},$$

where $e_i = e_{\{i\}}$, $i = 1, \ldots, m$, $e_0 = e_\phi = 1$ and $e_B = e_{\beta_1}...e_{\beta_h}$, with $1 \leq \beta_1 < \beta_2 < \cdots < \beta_h \leq m$. The product in C_m is determined by the relations $e_ie_j + e_je_i = -2\delta_{ij}$ where δ_{ij} denotes the Kronecker delta and $e_0 = 1$ for $1 \leq i \neq j \leq m$ (for details on the main concepts about C_m, see [25]). The norm of a Clifford number is given by $|b| = (\sum_{B\subseteq N} |b_B|^2)^{\frac{1}{2}}$ where N stands for $\{1, \ldots, m\}$.

Since C_m is isomorphic to $\mathbb{R}^{2^m}$ we may provide it with the $\mathbb{R}^{2^m}$-norm $|b|$ and one sees easily that for any $b, c \in C_m$, $|bc| \leq 2^{\frac{m}{2}}\ |b||c|$.

The elements $(x_0, x) = (x_0, x_1, ..., x_m) \in \mathbb{R}^{m+1}$ will be identified with the Clifford numbers $x_0 + \underline{x} = x_0 + \sum_{j=1}^{m} e_j x_j$. Note that if $x = x_0 + \underline{x} \in \mathbb{R}^{m+1}$, $\overline{x} = x_0 - \underline{x}$.

Definition 2.1. *Let $x \in \mathbb{R}^{m+1}$ and $\Omega \subset \mathbb{R}^{m+1}$ be an open set, then the function $h^{(m)}(x)$ is called left monogenic in Ω if $\mathbf{D}[h^{(m)}(x)] = 0$ where*

$$\mathbf{D} = \sum_{i=0}^{m} e_i \frac{\partial}{\partial x_i}$$

is the generalized Cauchy-Riemann operator. Similarly, $h^{(m)}(x)$ is a right monogenic function if $[h^{(m)}(x)]\mathbf{D} = 0$.

Definition 2.2. *A polynomial $Q^{(m)}(x)$ is SMP iff $\mathbf{D}Q^{(m)}(x) = 0$ and has the form:*

$$Q^{(m)}(x) = \sum_{i,j}^{finite} \overline{x}^i x^j a_{i,j}^{(m)},$$

where $a_{i,j}^{(m)} \in C_m$.

Let $\mathcal{S}_m[x]$ be the space of all SMPs is the right C_m-module defined by

$$\mathcal{S}_m[x] = \operatorname{span}_{C_m}\{Q_n^{(m)}(x) :\ n \in \mathbb{N}\},$$

where $Q_n^{(m)}(x)$ was given in [1] as follows:

$$Q_n^{(m)}(x) = \frac{n!}{(m)_n} \sum_{r+s=n} \frac{(\frac{m-1}{2})_r(\frac{m+1}{2})_s}{r!s!} \overline{x}^r x^s, \tag{2.1}$$

where for $\beta \in \mathbb{R}$, $(\beta)_l = \beta(\beta + 1)\ldots(\beta + l - 1)$ is the Pochhamer symbol.

Definition 2.3. *Let $\Omega \subset \mathbb{R}^{m+1}$ be a connected open containing 0 and $h^{(m)}$ is monogenic in Ω. The function $h^{(m)}$ is said to be SMF in Ω if and only if its Taylor series near zero exists and cab be expressed as:* $h^{(m)}(x) = \sum_{n=0}^{\infty} Q_n^{(m)}(x)\, a_n^{(m)}$ *for some SMPs* $Q_n^{(m)}(x)$.

If $Q_n^{(m)}(x)$ is a homogeneous SMP has degree n in x, (see [1])

$$Q_n^{(m)}(x) = \mathcal{Q}_n^{(m)}(x)\, \beta_n^{(m)},$$

where $\beta_n^{(m)} \in C_m$ is a constant. Accordingly, it follows that:

$$\|\mathcal{Q}_n^{(m)}\|_R = \sup_{\overline{B}(R)} |\mathcal{Q}_n^{(m)}(x)| = R^n.$$

Next, we recall the definition of F-module.

Definition 2.4. *An F-module E over C_m is a complete Hausdorff topological vector space by countable family of a proper system of semi-norms $\mathfrak{Q} = \{\|.\|_s\}_{s\geq 0}$ such that $s < t \Rightarrow \|h^{(m)}\|_s \leq \|h^{(m)}\|_t$; $(h^{(m)} \in E)$, Hence $W \subset E$ is open iff $\forall\, h^{(m)} \in W$, $\exists\, \epsilon > 0$, $M \geq 0$ such that $\{g^{(m)} \in E : \|h^{(m)} - g^{(m)}\|_s) \leq \epsilon\} \subset W$, $\forall s \leq M$.*

Definition 2.5. *A sequence $\{h_n^{(m)}\}$ in an F-module E converges to $g^{(m)}$ in E if*

$$\lim_{n\to\infty} \|h_n^{(m)} - g^{(m)}\|_s = 0$$

for all $\|.\|_s \in \mathfrak{Q}$.

The domains of representation adopted here are the open hyperball $B(R)$, the closed hyperball $\overline{B}(R)$ and $B_+(R)$; $R > 0$, where $B_+(R)$ any open hyperball enclosing closed hyperball, these are the sets defined by

$$B(R) = \{x \in \mathbb{R}^{m+1} : |x| < R\},$$
$$\overline{B}(R) = \{x \in \mathbb{R}^{m+1} : |x| \leq R\},$$
$$B_+(R) = \{x \in \mathbb{R}^{m+1} : |x| < R^+\}.$$

Table 1 summarizes certain classes of SMFs which represent F-modules where $x \in \mathbb{R}^{m+1}$ and each space is associated with the a proper countable system of semi-norms as follows.

Table 1. F-modules.

Space	The Associated Semi-Norms
$\mathcal{M}[B(R)]$: Class of SMFs in $B(R)$	$\|h^{(m)}\|_r = \sup_{\overline{B}(r)} |h^{(m)}(x)|$, $\forall r < R$, $h^{(m)} \in \mathcal{M}[B(R)]$,
$\mathcal{M}[\overline{B}(R)]$: Classe of SMFs in $\overline{B}(R)$	$\|h^{(m)}\|_R = \sup_{\overline{B}(R)} |h^{(m)}(x)|$, $\forall\, h^{(m)} \in \mathcal{M}[\overline{B}(R)]$,
$\mathcal{M}[B_+(R)]$: Class of SMFs in $B_+(R)$	$\|h^{(m)}\|_r = \sup_{\overline{B}(r)} |h^{(m)}(x)|$, $\forall R < r$, $h^{(m)} \in \mathcal{M}[B_+(R)]$,
$\mathcal{M}[0^+]$: Class of SMFs at the origin	$\|h^{(m)}\|_\epsilon = \sup_{\overline{B}(\epsilon)} |h^{(m)}(x)|$, $\epsilon > 0$ $\forall h^{(m)} \in \mathcal{M}[0^+]$,
$\mathcal{M}[\infty]$: Class of ESMFs on $\mathbb{R}^{m+1}$	$\|h^{(m)}\|_n = \sup_{\overline{B}(n)} |h^{(m)}(x)|$, $n < \infty$ $\forall h^{(m)} \in \mathcal{M}[\infty]$.

Now, let $\{Q_n^{(m)}(x)\}$ be a base of an F-module E such that

$$Q_n^{(m)}(x) = \sum_{k=0}^{\infty} \mathcal{Q}_k^{(m)}(x)\, Q_{n,k}^{(m)}, \quad Q_{n,k}^{(m)} \in C_m, \tag{2.2}$$

$$\mathcal{Q}_n^{(m)}(x) = \sum_{k=0}^{\infty} Q_k^{(m)}(x)\, \pi_{n,k}^{(m)}, \quad \pi_{n,k}^{(m)} \in C_m, \tag{2.3}$$

$$\|Q_n^{(m)}\|_R = \sup_{\overline{B}(R)} |Q_n^{(m)}(x)|, \tag{2.4}$$

$$\Psi_{Q_n^{(m)}}(R) = \sum_k \|Q_k^{(m)}\, \pi_{n,k}^{(m)}\|_R, \tag{2.5}$$

this sum is called hypercomplex Cannon sum, where

$$\|Q_k^{(m)}\, \pi_{n,k}^{(m)}\|_R = \sup_{\overline{B}(R)} |Q_k^{(m)}(x)\, \pi_{n,k}^{(m)}|,$$

$$\Psi_{Q^{(m)}}(R) = \lim_{n\to\infty} \sup\{\Psi_{Q_n^{(m)}}(R)\}^{\frac{1}{n}}, \tag{2.6}$$

where $\Psi_{Q^{(m)}}(R)$ is called the hypercomplex Cannon function of the base $\{Q_n^{(m)}(x)\}$ in closed hyperball $\overline{B}(R)$.

Let D_n is the degree of the polynomial of highest degree in the representation (2.3) the following restrictions are imposed.

$$\lim_{n\to\infty} \{D_n\}^{\frac{1}{n}} = 1, \tag{2.7}$$

$$D_n = O[n^a], \quad a \geq 1, \tag{2.8}$$

$$D_n = o(n \log n). \tag{2.9}$$

If d_k is the degree of the polynomials $\{Q_k^{(m)}(x)\}$, then $d_k \leq D_n$ for all $k \leq n$ (see [1]).

If $Q^{(m)} = (Q_{n,k}^{(m)})$ and $\Pi^{(m)} = (\pi_{n,k}^{(m)})$ are the Clifford matrices of coefficients and operators respectively of the set $\{Q_n^{(m)}(x)\}$. Thus according to [1] the set $\{Q_n^{(m)}(x)\}$ will be base iff

$$Q^{(m)}\Pi^{(m)} = \Pi^{(m)}Q^{(m)} = I, \tag{2.10}$$

where I is the unit matrix.

Let $h^{(m)}(x) = \sum_{n=0}^{\infty} \mathcal{Q}_n^{(m)}(x)\, a_n(h^{(m)})$ be any function which is SMF at the origin, substituting for $\mathcal{Q}_n^{(m)}(x)$ from (2.3) we obtain the basic series

$$h^{(m)}(x) \sim \sum_{n=0}^{\infty} Q_n^{(m)}(x)\, \Pi_n(h^{(m)}), \tag{2.11}$$

where

$$\Pi_n(h^{(m)}) = \sum_{k=0}^{\infty} \pi_{k,n}^{(m)}\, a_k(h^{(m)})\ . \tag{2.12}$$

The authors in [22,26] introduced the idea of effectiveness for the class $\mathcal{M}[\overline{B}(R)]$. A base $\{Q_n^{(m)}(x)\}$ is effective for the class $\mathcal{M}[\overline{B}(R)]$ If the basic series (2.11) converges normally to every function $h^{(m)}(x) \in \mathcal{M}[\overline{B}(R)]$ which is SMF in $\overline{B}(R)$. Similar definitions are used for the classes $\mathcal{M}[B(R)]$, $\mathcal{M}[B_+(R)]$, $\mathcal{M}[\infty]$ and $\mathcal{M}[0^+]$.

They also proved:

Theorem 2.1. *A base $\{Q_n^{(m)}(x)\}$ is effective for the classes $\mathcal{M}[\overline{B}(R)]$, $\mathcal{M}[B(R)]$, $\mathcal{M}[B_+(R)]$, $\mathcal{M}[\infty]$ or $\mathcal{M}[0^+]$ if and only if $\Psi_{Q^{(m)}}(R) = R$, $\Psi_{Q^{(m)}}(r) < R\ \ \forall\ r < R$, $\Psi_{Q^{(m)}}(R^+) = R$, $\Psi_{Q^{(m)}}(R) < \infty\ \forall\ R < \infty$, or $\Psi_{Q^{(m)}}(0^+) = 0$.*

For the definition of bases of SMPs and theorems governing the effectiveness properties of bases of SMPs, the reader is referred to the authors [21, 22, 26].

3. Hypercomplex Hasse derivative bases

The complex Hasse derivative operator (CHDO) of order i is defined in [28–30]. Using the definition of the complex Hasse derivative, we can define a new operator in the case of Clifford setting called the hypercomplex Hasse derivative (HHD) as follows:

Definition 3.1. *For each integer $i \geq 0$ the HHD $\mathbb{H}^{(i)}$ of order i is defined by*

$$\mathbb{H}^{(i)}(\boldsymbol{Q}_n^{(m)}(x)) = \zeta_{n,i}\ \boldsymbol{Q}_{n-i}^{(m)}(x), \tag{3.1}$$

where

$$\zeta_{n,i} = \frac{n^i}{i!}\prod_{j=1}^{i-1}\left(1 - \frac{j}{n}\right)$$

and $\mathbb{H}^{(i)}$ is closely related to the higher hypercomplex derivative $(\frac{1}{2}\overline{\mathbf{D}})^i$: $\mathbb{H}^{(i)} = \frac{1}{i!}(\frac{1}{2}\overline{\mathbf{D}})^i$.

The set $\{\boldsymbol{Q}_n^{(m)}(x)\}$ is an Appell sequence with respect to $\dfrac{\partial}{\partial x_0}$ or $\dfrac{1}{2}\overline{\mathbf{D}}$: $\frac{1}{2}\overline{\mathbf{D}}\boldsymbol{Q}_n^{(m)}(x) = n\boldsymbol{Q}_{n-1}^{(m)}(x)$.

Remark 3.1. *If $x \in C_1$ then (3.1) is reduced to the ordinary Hasse derivative of order i (see [28–30]),*

Definition 3.2. *Let $\{Q_n^{(m)}(x)\}$ be a base. By acting on both sides of Eq (2.2) with the operator $\mathbb{H}^{(i)}$, we get*

$$\mathbb{H}^{(i)} Q_n^{(m)}(x) = \sum_k\ \ \zeta_{k,i}\ \boldsymbol{Q}_{k-i}^{(m)}(x) Q_{n,k}^{(m)}. \tag{3.2}$$

The set $\{\mathbb{H}^{(i)} Q_n^{(m)}(x)\}$=$\{\mathbb{H}^{(i,m)}(x)\}$ is defined the Hypercomplex Hasse derivative bases (HHDBs).

The present work deals principally with the convergence properties of certain classes of bases, namely HHDBs. In fact we shall study the convergence of the expansion of certain classes of functions as series of HHDBs. This study will be based on the already established theorems dealing with the convergence of basic series of HHDBs. The convergence properties of HHDBs are mainly classified as follows:

(1) The region of effectiveness of HHDBs for the classes $\mathcal{M}[B(R)]$, $\mathcal{M}[\overline{B}(R)]$, $\mathcal{M}[B_+(R)]$, $\mathcal{M}[0^+]$, and $\mathcal{M}[\infty]$.

(2) The mode of increase of HHDBs which determined by the order and type.

(3) The T_ρ-property of HHDBs.

In the following sections, we will investigated all of these problems.

4. Effectiveness of the HHDBs for the classes $\mathcal{M}[B(R)]$, $\mathcal{M}[0^+]$, $\mathcal{M}[\infty]$ and $\mathcal{M}[B_+(R)]$

In the current section, the property of effectiveness concerning the HHDBs in several regions such as $\mathcal{M}[B(R)]$, $\mathcal{M}[0^+]$, $\mathcal{M}[\infty]$ and $\mathcal{M}[B_+(R)]$ are demonstrated.

Theorem 4.1. *If $\{Q_n^{(m)}(x)\}$ is a base, then the HHD set $\{\mathbb{H}^{(i,m)}(x)\}$ is also base.*

Proof. We form the coefficient matrix $\mathbb{H}^{(i,m)}$ by defining the HHDBs in (2.2)

$$\mathbb{H}_n^{(i,m)}(x) = \sum_k Q_{k-i}^{(m)}(x)\, \zeta_{k,i} Q_{n,k}^{(m)}.$$

Hence, the coefficients matrix $\mathbb{H}^{(i,m)}$ is given by the following:

$$\mathbb{H}^{(i,m)} = \left(\mathbb{H}_{n,k}^{(i,m)}\right) = (\zeta_{k,i}\ Q_{n,k}^{(m)}).$$

Also, the operators matrix $\Pi^{(i,m)}$ follows from the effect $\mathbb{H}^{(i)}$ on both sides of the representation (2.3) where

$$Q_{k-i}^{(m)}(x) = \frac{1}{\zeta_{n,i}} \sum_k \pi_{n,k}^{(m)} \mathbb{H}_k^{(i,m)}(x),$$

and

$$\Pi^{(i,m)} = (\pi_{n,k}^{(i,m)}) = \left(\frac{1}{\zeta_{n,i}}\ \pi_{n,k}^{(m)}\right).$$

Consequently,

$$\mathbb{H}^{(i,m)}\Pi^{(i,m)} = \left(\sum_k \mathbb{H}_{n,k}^{(i,m)} \pi_{k,h}^{(i,m)}\right) = \left(\sum_k Q_{n,k}^{(m)} \pi_{k,h}^{(m)}\right) = (\delta_{n,h}) = I.$$

Moreover,

$$\Pi^{(i,m)}\mathbb{H}^{(i,m)} = \left(\sum_k \pi_{n,k}^{(i,m)} \mathbb{H}_{k,h}^{(i,m)}\right) = \left(\sum_k \frac{1}{\zeta_{n,i}} \pi_{n,k}^{(m)}\ \zeta_{h,i}\ Q_{k,h}^{(m)}\right) = \left(\frac{\zeta_{h,i}}{\zeta_{n,i}} \delta_{n,h}\right) = I.$$

We easily obtain from (2.10) that the set $\{\mathbb{H}_n^{(i,m)}(x)\}$ is a base. □

Theorem 4.2. *The base $\{Q_n^{(m)}(x)\}$ and its HHDBs $\{\mathbb{H}_n^{(i,m)}(x)\}$ have the same region of effectiveness for the class $\mathcal{M}[B(R)]$.*

Proof. If $Q_n^{(m)}(x)$ is a base, $\|Q_n^{(m)}\|_r = \sup\limits_{\overline{B}(r)} |Q_n^{(m)}(x)|$ and $\|\mathbb{H}_n^{(i,m)}\|_r = \sup\limits_{\overline{B}(r)} |\mathbb{H}_n^{(i,m)}(x)|$, then

$$
\begin{aligned}
\|\mathbb{H}_n^{(i,m)}\|_r &= \sup_{\overline{B}(r)} |\mathbb{H}_n^{(i,m)}(x)| \\
&= \sup_{\overline{B}(r)} |\sum_j \zeta_{j,i}\, Q_{j-i}^{(m)}(x) Q_{n,j}| \\
&\leq 2^{m/2} \sum_j \zeta_{j,i}\, r^{j-i} \frac{\|Q_n^{(m)}\|_R}{R^j} \\
&= \frac{2^{m/2}}{r^i} \Upsilon(r,R)\, \|Q_n^{(m)}\|_R = K_1\, \|Q_n^{(m)}\|_R \quad \text{for all } r < R,
\end{aligned}
\tag{4.1}
$$

where $K_1 = \dfrac{2^{m/2}}{r^i}\Upsilon(r,R)$ and $\Upsilon(r,R) = \sum\limits_{j=0}^{\infty} \zeta_{j,i}(\dfrac{r}{R})^j < \infty$.

Using (2.5) and (4.1), it follows that the hypercomplex Cannon sum of the HHDBs $\{\mathbb{H}_n^{(i,m)}(x)\}$ is given by

$$
\begin{aligned}
\Psi_{\mathbb{H}_n^{(i,m)}}(r) &= \sum_k \|\mathbb{H}_k^{(i,m)} \pi_{n,k}^{(i,m)}\|_r \\
&\leq K_1 \sum_k \|Q_k^{(m)}\, \pi_{n,k}^{(i,m)}\|_R \\
&= \frac{K_1}{\zeta_{n,i}} \sum_k \|Q_k^{(m)} \pi_{n,k}^{(m)}\|_R \\
&= \frac{K_1}{\zeta_{n,i}} \Psi_{Q_n^{(m)}}(R).
\end{aligned}
\tag{4.2}
$$

Using (2.6) and (4.2), we obtain that the hypercomplex Cannon function of the HHDBs is given by:

$$
\Psi_{\mathbb{H}^{(i,m)}}(r) \leq \Psi_{Q^{(m)}}(R), \quad \forall\, r < R. \tag{4.3}
$$

Now, suppose that the base $\{Q_n^{(m)}(x)\}$ is effective for $\mathcal{M}[B(R)]$, we can apply Theorem 2.1, we have

$$
\Psi_{Q^{(m)}}(r) < R, \quad \forall\, r < R. \tag{4.4}
$$

Hence there is a number r_1 such that $r < r_1 < R$, then from (4.3) and (4.4), we deduce that

$$
\Psi_{\mathbb{H}^{(i,m)}}(r) \leq \Psi_{Q^{(m)}}(r_1) < R, \quad \forall\, r < R,
$$

that is to say the base $\{\mathbb{H}_n^{(i,m)}(x)\}$ is effective for $\mathcal{M}[B(R)]$.

□

Theorem 4.3. *The base $\{Q_n^{(m)}(x)\}$ and its HHDBs $\{\mathbb{H}_n^{(i,m)}(x)\}$ have the same region of effectiveness for the class $\mathcal{M}[0^+]$ or $\mathcal{M}[\infty]$.*

Proof. Suppose that the base $\{Q_n^{(m)}(x)\}$ is effective for $\mathcal{M}[0^+]$, we can apply Theorem 2.1, it follows that $\Psi_{Q^{(m)}}(0^+) = 0$. Making $R, r \to 0^+$ in (4.3), we have $\Psi_{\mathbb{H}^{(i,m)}}(0^+) \leq \Psi_{Q^{(m)}}(0^+) = 0$ but we know that $\Psi_{\mathbb{H}^{(i,m)}}(0^+) \geq 0$, thus, $\Psi_{\mathbb{H}^{(i,m)}}(0^+) = 0$. Therefore, the base $\{\mathbb{H}_n^{(i,m)}(x)\}$ is effective for $\mathcal{M}[0^+]$.

Now, suppose that the base $\{Q_n^{(m)}(x)\}$ is effective for $\mathcal{M}[\infty]$. Applying Theorem 2.1 we conclude that

$$\Psi_{Q^{(m)}}(r) < \infty, \ \forall r < \infty. \tag{4.5}$$

Thus if we choose the number r_2 such that $r < r_2 < R$, making $R \to \infty$ in (4.3). Then, by using (4.5), we obtain that

$$\Psi_{\mathbb{H}^{(i,m)}}(r) \leq \Psi_{Q^{(m)}}(r_2) < \infty, \ \forall r < \infty,$$

and, the base $\{\mathbb{H}_n^{(i,m)}(x)\}$ will be effective for $\mathcal{M}[\infty]$. □

Theorem 4.4. *The base* $\{Q_n^{(m)}(x)\}$ *and its HHDBs* $\{\mathbb{H}_n^{(i,m)}(x)\}$ *have the same region of effectiveness for the class* $\mathcal{M}[B_+(R)]$.

Proof. If the base $\{Q_n^{(m)}(x)\}$ is effective for $\mathcal{M}[B_+(r_3)]$ and r_3 is any positive number such that $r_3 < r$, we can apply Theorem 2.1, we obtain

$$\Psi_{Q^{(m)}}(r_3^+) = r_3, \ \ r_3 < r < R. \tag{4.6}$$

Making $R \to r_3^+$ in (4.3), we easily obtain, from (4.6) that $\Psi_{\mathbb{H}^{(i,m)}}(r_3^+) \leq \Psi_{Q^{(m)}}(r_3^+) = r_3$, but $\Psi_{\mathbb{H}^{(i,m)}}(r_3^+) \geq r_3$ which implies that $\Psi_{\mathbb{H}^{(i,m)}}(r_3^+) = r_3$. Hence, the base $\{\mathbb{H}_n^{(i,m)}(x)\}$ is indeed effective for $\mathcal{M}[B_+(r_3)]$ as required. □

5. Effectiveness of the HHDBSMPs for the class $\mathcal{M}[\overline{B}(R)]$

When the representation (2.3) is finite then the base is called SMPs. In this section we will discuss the region of effectiveness of HHDBSMPs for the class of SMFs in $\overline{B}(R)$. The following result states the purpose of this section.

Theorem 5.1. *The base* $\{Q_n^{(m)}(x)\}$ *for which the condition (2.7) is satisfied and its HHDBSMPs* $\{\mathbb{H}_n^{(i,m)}(x)\}$ *have the same region of effectiveness for the class* $\mathcal{M}[\overline{B}(R)]$.

Proof. If $Q_n^{(m)}(x)$ is a base of SMPs, $||Q_n^{(m)}||_R = \sup\limits_{\overline{B}(R)} |Q_n^{(m)}(x)|$ and $||\mathbb{H}_n^{(i,m)}||_R = \sup\limits_{\overline{B}(R)} |\mathbb{H}_n^{(i,m)}(x)|$, then

$$\begin{aligned} ||\mathbb{H}_n^{(i,m)}||_R &= \sup_{\overline{B}(R)} |\mathbb{H}_n^{(i,m)}(x)| \\ &= \sup_{\overline{B}(R)} |\sum_j Q_{n,j}^{(m)} \ \zeta_{j,i} \ Q_{j-i}^{(m)}(x)| \\ &\leq \sum_j \frac{||Q_n^{(m)}||_R}{R^j} \ \zeta_{j,i} \ R^{j-i} \\ &= \frac{||Q_n^{(m)}||_R}{R^i} \sum_j \ \zeta_{j,i} \\ &\leq \frac{||Q_n^{(m)}||_R}{R^i} \ \zeta_{d_n,i} \ (\zeta_{d_n,i} + 1), \end{aligned} \tag{5.1}$$

where d_n is the degree of the polynomial $Q_n^{(m)}(x)$, $d_n \leq D_n$. Applying (2.5) and (5.1), it follows that

$$\begin{aligned}\Psi_{\mathbb{H}_n^{(i,m)}}(R) &= \sum_k \|\mathbb{H}_k^{(i,m)}\ \pi_{n,k}^{(i,m)}\|_R\\ &\leq \frac{1}{\zeta_{n,i}R^i}\sum_k \|Q_k^{(m)}\ \pi_{n,k}^{(m)}\|_R\ \zeta_{d_k,i}\ (\zeta_{d_k,i}+1)\\ &\leq \frac{1}{\zeta_{n,i}R^i}\zeta_{D_n,i}\ (\zeta_{D_n,i}+1)\ \Psi_{Q_n^{(m)}}(R).\end{aligned}\tag{5.2}$$

A combination of (2.6), (2.7) and (5.2), gives $\Psi_{\mathbb{H}_n^{(i,m)}}(R) \leq \Psi_{P^{(m)}}(R) \leq R$.
But $\Psi_{\mathbb{H}_n^{(i,m)}}(R) \geq R$. We finally deduce that

$$\Psi_{\mathbb{H}_n^{(i,m)}}(R) = R.\tag{5.3}$$

and the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ is effective for $\mathcal{M}[\overline{B}(R)]$. □

The following example shows that the condition (2.7) imposed on the class of the base $\{Q_n^{(m)}(x)\}$ cannot be relaxed.

Example 5.1. *Theorem 5.1 is not always correct if the condition (2.7) is not satisfied. Let*

$$Q_n^{(m)}(x) = \begin{cases} \mathcal{Q}_n^{(m)}(x), & n \text{ is even},\\ \mathcal{Q}_n^{(m)}(x) + \ \mathcal{Q}_b^{(m)}(x),\ b = 2^n, & n \text{ is odd}.\end{cases}$$

When n is even, we have $\mathcal{Q}_n^{(m)}(x) = Q_n^{(m)}(x)$ and hence $\Psi_{Q_n^{(m)}}(R) = R^n$. Thus, by taking $R = 1$, then $\Psi_{Q_n^{(m)}}(1) = 1$, and $\lim_{n\to\infty}\{\Psi_{Q_{2n}^{(m)}}(1)\}^{\frac{1}{2n}} = 1$.

Furthermore, $\mathcal{Q}_n^{(m)}(x) = Q_n^{(m)}(x) - Q_b^{(m)}(x)$, when n is odd, then

$$\Psi_{Q_n^{(m)}}(R) = R^n + 2\ R^b.$$

So that when $R = 1$, $\Psi_{Q_n^{(m)}}(1) = 3$, we get

$$\lim_{n\to\infty}\{\Psi_{Q_{2n+1}^{(m)}}(1)\}^{\frac{1}{2n+1}} = 1.$$

Consequently, $\Psi_{Q^{(m)}}(1) = \limsup_{n\to\infty}\{\Psi_{Q_n^{(m)}}(1)\}^{\frac{1}{n}} = 1$, and the base $\{Q_n^{(m)}(x)\}$ is effective for $\mathcal{M}[\overline{B}(1)]$.

Forming the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$, we easily get

$$\mathbb{H}_n^{(i,m)}(x) = \begin{cases}\zeta_{n,i}\ \mathcal{Q}_{n-i}^{(m)}(x), & n \text{ is even, and} \geq 2,\\ \zeta_{n,i}\ \mathcal{Q}_{n-i}^{(m)}(x) + \zeta_{b,i}\mathcal{Q}_{b-i}^{(m)}(x), & n \text{ is odd}.\end{cases}$$

Since $\mathcal{Q}_{n-i}^{(m)}(x) = (1 \setminus \zeta_{n,i})\ \mathbb{H}_n^{(i,m)}(x)$, when n is even, then $\Psi_{\mathbb{H}_n^{(i,m)}}(R) = R^{n-i}$, taking $R = 1$, $\Psi_{\mathbb{H}_n^{(i,m)}}(1) = 1$. Hence,

$$\lim_{n\to\infty}\{\Psi_{\mathbb{H}_{2n}^{(i,m)}}(1)\}^{\frac{1}{2n}} = 1.$$

When n is odd, $\mathcal{Q}_{n-i}^{(m)}(x) = (1 \setminus \zeta_{n,i})[\mathbb{H}_{P_n}^{(i,m)}(x) - \zeta_{b,i}\mathbb{H}_b^{(i,m)}(x)]$. Hence we have, $\Psi_{\mathbb{H}_n^{(i,m)}}(R) = (1 \setminus \zeta_{n,i})[\zeta_{n,i}R^{n-i} + 2\ \zeta_{b,i}R^{b-i}]$.

Taking $R = 1$, then we get

$$\Psi_{\mathbb{H}^{(i,m)}}(1) = \limsup_{n\to\infty} \{\Psi_{\mathbb{H}^{(i,m)}_{2n+1}}(1)\}^{\frac{1}{2n+1}} = 2 > 1,$$

and the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ is not effective for $\mathcal{M}[\overline{B}(1)]$.

For a simple base of SMPs ($D_n = n$) (see [1]), we obtain the following corollary.

Corollary 5.1. *When the simple base $\{Q_n^{(m)}(x)\}$ of SMPs is effective for $\mathcal{M}[\overline{B}(R)]$, so also will be the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$.*

6. The order, type and the $\mathbb{T}_{\rho_{Q^{(m)}}}$-property of the HHDBSMPs

In [1, 23], the idea of the order and type of the base $\{Q_n^{(m)}(x\}$ of SMPs was introduced as follows:

$$\rho_{Q^{(m)}} = \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log\ \Psi_{Q_n^{(m)}}(R)}{n\ \log\ n}. \tag{6.1}$$

and

$$\tau_{Q^{(m)}} = \lim_{R\to\infty} \frac{e}{\rho_{Q^{(m)}}} \limsup_{n\to\infty} \frac{\{\Psi_{Q_n^{(m)}}(R)\}^{\frac{1}{n\ \rho_{Q^{(m)}}}}}{n}. \tag{6.2}$$

Importantly, if the base $\{Q_n^{(m)}(x)\}$ has finite order $\rho_{Q^{(m)}}$ and finite type $\tau_{Q^{(m)}}$, then it can represent every ESMF of order less than $\frac{1}{\rho_{Q^{(m)}}}$ and type less than $\frac{1}{\tau_{Q^{(m)}}}$ in any finite hyperball. Rich investigation on the order of certain classes of bases can be found in [31, 32].

Now, we explore the relation between the order and type of SMPs $\{Q_n^{(m)}(x)\}$ and our constructed base; $\{\mathbb{H}_n^{(i,m)}(x)\}$ as follows.

Theorem 6.1. *Let $\rho_{Q^{(m)}}$ and $\tau_{Q^{(m)}}$ be the order and type of the base of SMPs $\{Q_n^{(m)}(x)\}$ satisfying the condition (2.8). Then the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ will be of order $\rho_{\mathbb{H}^{(i,m)}} \leq \rho_{Q^{(m)}}$ and type $\tau_{\mathbb{H}^{(i,m)}} \leq \tau_{Q^{(m)}}$ whenever $\rho_{\mathbb{H}^{(i,m)}} = \rho_{Q^{(m)}}$. The values of $\rho_{Q^{(m)}}$ and $\tau_{Q^{(m)}}$ are attainable.*

Proof. The proof of this theorem denoted on the inequality (5.2), since

$$\Psi_{\mathbb{H}_n^{(i,m)}}(R) \leq \frac{1}{\zeta_{n,i} R^{\alpha}} \zeta_{D_n,i}\ (\zeta_{D_n,i} + 1)\ \Psi_{Q_n^{(m)}}(R).$$

Then

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \Psi_{\mathbb{H}_n^{(i,m)}}(R)}{n \log n} \leq \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \zeta_{D_n,i}\ (\zeta_{D_n,i} + 1) + \log \Psi_{Q_n^{(m)}}(R)}{n \log n}.$$

It follows, in view of (6.1), that the HHDBSMPs is at most $\rho_{Q^{(m)}}$.

If $\rho_{\mathbb{H}^{(i,m)}} = \rho_{Q^{(m)}}$, we have

$$\lim_{R\to\infty} \frac{e}{\rho_{\mathbb{H}^{(i,m)}}} \limsup_{n\to\infty} \frac{\{\Psi^{(i)}_{\mathbb{H}_n}(R)\}^{\frac{1}{n(\rho_{\mathbb{H}^{(i,m)}})}}}{n} \leq \lim_{R\to\infty} \frac{e}{\rho_{Q^{(m)}}} \limsup_{n\to\infty} \frac{\{\Psi_{Q_n}(R)\}^{\frac{1}{n(\rho_{Q^{(m)}})}}}{n},$$

and the type of the HHDBSMPs is at most $\tau_{Q^{(m)}}$.

Note that the upper bound given in this theorem is attainable. We will illustrate this fact by introducing the following example:

Example 6.1. *Let* $\{Q_n^{(m)}(x)\}$ *be the base of SMPs given by* $Q_n^{(m)}(x) = n^n + \mathcal{Q}_n^{(m)}(x)$, $Q_0^{(m)}(x) = 1$, *for which*

$$\Psi_{Q_n^{(m)}}(R) = n^n\left[2 + \left(\frac{R}{n}\right)^n\right].$$

It is easily seen that the base $\{Q_n^{(m)}(x)\}$ *is of order* $\rho_{Q^{(m)}} = 1$ *and type* $\tau_{Q^{(m)}} = e$. *Construct now the base* $\{\mathbb{H}_n^{(i,m)}(x)\}$ *such that*

$$\mathbb{H}_n^{(i,m)}(x) = n^n + \zeta_{n,i}\, \mathcal{Q}_{n-i}^{(m)}(x), \quad Q_0^{(m)}(x) = 1.$$

Hence,

$$\Psi_{\mathbb{H}_n^{(i,m)}}(R) = \frac{n^n}{\zeta_{n,i}}\left[2 + \frac{\zeta_{n,i}}{R^i}\left(\frac{R}{n}\right)^n\right].$$

Therefore, the base $\{\mathbb{H}_n^{(i,m)}(x)\}$ *is of order* $\rho_{\mathbb{H}^{(i,m)}} = 1$ *and type* $\tau_{\mathbb{H}^{(i,m)}} = e$.

The following example illustrates the best possibility of condition (2.8).

Example 6.2. *Let the base* $\{Q_n^{(m)}(x)\}$ *of SMPs be defined by*

$$Q_n^{(m)}(x) = \begin{cases} \mathcal{Q}_n^{(m)}(x), & n \text{ is even}, \\ \mathcal{Q}_n^{(m)}(x) + \frac{\mu}{b^{2\mu}}\, \mathcal{Q}_{2\mu}^{(m)}(x), & n \text{ is odd and } \mu = n^n,\ b > 1. \end{cases}$$

Hence,

$$\mathcal{Q}_n^{(m)}(x) = Q_n^{(m)}(x) - \frac{\mu}{b^{2\mu}} Q_{2\mu}^{(m)}(x),$$

and

$$\Psi_{Q_n^{(m)}}(R) = R^n + 2\mu\left(\frac{R}{b}\right)^{2\mu}.$$

It is easy to see that the base $Q_n^{(m)}(x)$ *is of order* $\rho_{Q^{(m)}} = 1$.

For the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ *it can verified that*

$$\mathbb{H}_n^{(i,m)}(x) = \begin{cases} \zeta_{n,i}\, \mathcal{Q}_{n-i}^{(m)}(x), & n \text{ is even}, \\ \zeta_{n,i}\, \mathcal{Q}_{n-i}^{(m)}(x) + \frac{\mu}{b^{2\mu}}\, \zeta_{2\mu,i}\, \mathcal{Q}_{2\mu-i}^{(m)}(x), & n \text{ is odd}. \end{cases}$$

Thus,

$$\mathcal{Q}_{n-i}^{(m)}(x) = \frac{1}{\zeta_{n,i}} \mathbb{H}_n^{(i,m)}(x) - \frac{\mu}{b^{2\mu}}\, \frac{\zeta_{2\mu,i}}{\zeta_{n,i}}\, \mathbb{H}_{2\mu}^{(i,m)}(x).$$

Consequently,

$$\Psi_{\mathbb{H}_n^{(i,m)}}(R) = R^{n-i} + \frac{2\mu}{b^i}\, \frac{\zeta_{2\mu,i}}{\zeta_{n,i}}\, \left(\frac{R}{b}\right)^{2\mu-i}.$$

Therefore, $\rho_{\mathbb{H}^{(i,m)}} = 2$ *and* $\rho_{\mathbb{H}^{(i,m)}} > \rho_{Q^{(m)}}$. *This completes the proof.*

□

If the base of SMPs $\{Q_n^{(m)}(x)\}$ is simple base ($D_n = n$) (see [1]), then the following corollary is a special case of Theorem 6.1.

Corollary 6.1. *When the simple base $\{Q_n^{(m)}(x)\}$ of SMPs is of order $\rho_{Q^{(m)}}$ and type $\tau_{Q^{(m)}}$, then the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ will be of order $\rho_{\mathbb{H}^{(i,m)}} \leq \rho_{Q^{(m)}}$ and type $\tau_{\mathbb{H}^{(i,m)}} \leq \tau_{Q^{(m)}}$ whenever $\rho_{\mathbb{H}^{(i,m)}} = \rho_{Q^{(m)}}$.*

In the following, we determine the $\mathbb{T}_{\rho_{Q^{(m)}}}$-property of the HHDBs. The authors of [2] deduced $\mathbb{T}_{\rho_{Q^{(m)}}}$-property of the base $\{Q_n^{(m)}(x)\}$ in Clifford analysis in open hyperball $B(R)$, closed hyperball $\overline{B}(R)$ and at the origin are defined as follows:

Definition 6.1. *If the base $\{Q_n^{(m)}(x)\}$ represents all ESMFs of order less than $\rho_{Q^{(m)}}$ in $\overline{B}(R)$, $B(R)$ or at the origin, then it is said to have property $T_{\rho_{Q^{(m)}}}$ in $\overline{B}(R)$, $B(R)$ or at the origin.*

Let

$$\Psi_{Q^{(m)}}(R) = \limsup_{n\to\infty} \frac{\log\ \Psi_{Q_n^{(m)}}(R)}{n\ \log\ n}.$$

The following theorem concerning the property $\mathbb{T}_{\rho_{Q^{(m)}}}$ of the base $\{Q_n^{(m)}(x)\}$ (see [2]).

Theorem 6.2. *A base $\{Q_n^{(m)}(x)\}$ to have the property $T_{\rho_{Q^{(m)}}}$ for all ESMF of order less than $\rho_{Q^{(m)}}$ in closed hyperball $\overline{B}(R)$, open hyperball $B(R)$ or at the origin iff, $\Psi_{Q^{(m)}}(R) \leq \frac{1}{\rho_{Q^{(m)}}}$, $\Psi_{Q^{(m)}}(r) \leq \frac{1}{\rho_{Q^{(m)}}}$ for all $r < R$ or $\Psi_{Q^{(m)}}(0^+) \leq \frac{1}{\rho_{Q^{(m)}}}$.*

Next, we construct the $\mathbb{T}_{\rho_{\mathbb{H}^{(i,m)}}}$-property of the HHDBSMPs in the closed hyperball $\overline{B}(R)$, for $R > 0$.

Theorem 6.3. *Let $\{Q_n^{(m)}(x)\}$ be the base of SMPs have $\mathbb{T}\rho_{Q^{(m)}}$-property in $\overline{B}(R)$, where $R > 0$ and for which the condition (2.9) is satisfied. Then the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ have the same property.*

Proof. Suppose that the function $\Psi_{\mathbb{H}^{(i,m)}}(R)$ given by:

$$\Psi_{\mathbb{H}^{(i,m)}}(R) = \limsup_{n\to\infty} \frac{log\ \Psi_{\mathbb{H}_n^{(i,m)}}(R)}{n\ log\ n}, \tag{6.3}$$

where $\Psi_{\mathbb{H}_n^{(i,m)}}(R)$ is the Cannon sum of the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$. Then by using (2.9), (5.2) and (6.3), we obtain that

$$\Psi_{\mathbb{H}^{(i,m)}}(R) \leq \limsup_{n\to\infty} \frac{\log\zeta_{D_n,i}\ (\zeta_{D_n,i} + 1) + \log\ \Psi_{Q_n^{(m)}}(R)}{n \log n} \leq \Psi_{Q^{(m)}}(R). \tag{6.4}$$

Since the base $\{Q_n^{(m)}(x)\}$ has the property $\mathbb{T}_{\rho_{Q^{(m)}}}$ in $\overline{B}(R)$, $R > 0$. Hence by inequality (6.4) and Theorem 6.2, we have

$$\Psi_{\mathbb{H}^{(i,m)}}(R) \leq \Psi_{Q^{(m)}}(R) \leq \frac{1}{\rho_{Q^{(m)}}},$$

and the base $\{\mathbb{H}_n^{(i,m)}(x)\}$ has the property $\mathbb{T}_{\rho_{Q^{(m)}}}$ in $\overline{B}(R)$, $R > 0$.

□

The fact that HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ does not have the property $\mathbb{T}_{\rho_{Q^{(m)}}}$ in $\overline{B}(R)$ if the condition (2.9) is not satisfied is illustrated by the following example.

Example 6.3. *Let $\{Q_n^{(m)}(x)\}$ be the base of SMPs, is defined by:*

$$Q_n^{(m)}(x) = \begin{cases} Q_n^{(m)}(x), & n \text{ is even,} \\ Q_n^{(m)}(x) + \frac{Q_{s(n)}^{(m)}(x)}{2^{(n^n)}}, & n \text{ is odd,} \end{cases}$$

where $s(n)$ is the nearest even integer to $n \log n + n^n$.

When n is odd, we obtain:

$$Q_n^{(m)}(x) = Q_n^{(m)}(x) - \frac{Q_{t(n)}^{(m)}(x)}{2^{(n^n)}}.$$

Hence,

$$\Psi_{Q_n^{(m)}}(R) = R^n + 2\frac{R^{t(n)}}{2^{(n^n)}}.$$

Putting $R = 2$, it follows that

$$\Psi_{Q_n^{(m)}}(2) = 2^n + 2^{n \log n+1},$$

so that

$$\Psi_{Q^{(m)}}(2) = \limsup_{n\to\infty} \frac{log\ \Psi_{Q_n^{(m)}}(2)}{n\ log\ n} \leq \log 2.$$

It follows that, the base $Q_n^{(m)}(x)$ has the $\mathbb{T}_{\frac{1}{\log 2}}$-property in $\overline{B}(2)$. The HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ is

$$\mathbb{H}_n^{(i,m)}(x) = \begin{cases} \zeta_{n,i} Q_{n-i}^{(m)}(x), & n \text{ is even,} \\ \zeta_{n,i} Q_{n-i}^{(m)}(x) + \zeta_{t(n),i} \frac{Q_{t(n)-i}^{(m)}(x)}{2^{(n^n)}}, & n \text{ is odd.} \end{cases}$$

Hence, when n is odd, we obtain

$$\Psi_{\mathbb{H}_n^{(i,m)}}(R) = R^{n-i} + 2\frac{\zeta_{t(n),i}}{\zeta_{n,i}} \frac{R^{t(n)-i}}{2^{(n^n)}},$$

so that when $R = 2$,

$$\Psi_{\mathbb{H}_n^{(i,m)}}(2) = 2^{n-i} + 2\frac{\zeta_{t(n),i}}{\zeta_{n,i}} \frac{2^{t(n)-i}}{2^{(n^n)}}.$$

Thus,

$$\Psi_{\mathbb{H}^{(i,m)}}(2) = \limsup_{n\to\infty} \frac{\log\ \Psi_{\mathbb{H}_n^{(i,m)}}(2)}{n\ log\ n} \leq 1 + \log 2,$$

and the HHDBSMPs $\mathbb{H}_n^{(i,m)}(x)$, does not have the $\mathbb{T}_{\frac{1}{\log 2}}$-property in $\overline{B}(2)$ as required.

If the base of SMPs $\{Q_n^{(m)}(x)\}$ is simple base ($D_n = n$) (see [1]), then the following corollary is a special case of Theorem 6.3.

Corollary 6.2. *When the simple base $\{Q_n^{(m)}(x)\}$ of SMPs have $\mathbb{T}\rho_{Q^{(m)}}$-property in $\overline{B}(R)$, $R > 0$. Then the HHDBSMPs $\{\mathbb{H}_n^{(i,m)}(x)\}$ is also have the $\mathbb{T}\rho_{Q^{(m)}}$-property .*

In the following, we deduce that the base $\{Q_n^{(m)}(x)\}$ and the HHDBs $\{\mathbb{H}_n^{(i,m)}(x)\}$ have the same $\mathbb{T}\rho_{Q^{(m)}}$ in an open hyperball $B(R)$, where $R > 0$ or at the origin.

Theorem 6.4. *Let $\{Q_n^{(m)}(x)\}$ be a base of SMPs have the $\mathbb{T}\rho_{Q^{(m)}}$-property in $B(R)$, $R > 0$ or at the origin. Then the HHDBs $\{\mathbb{H}_n^{(i,m)}(x)\}$ have the same property.*

Proof. Let $\{Q_n^{(m)}(x)\}$ be have the property $\mathbb{T}\rho_{Q^{(m)}}$ in $B(R)$, $R > 0$, then

$$\Psi_{Q^{(m)}}(r) \leq \frac{1}{\rho_{Q^{(m)}}} \quad \forall r < R. \tag{6.5}$$

It follows from (4.2) that

$$\Psi_{\mathbb{H}^{(i,m)}}(r) = \limsup_{n\to\infty} \frac{\log\ \Psi_{\mathbb{H}_n^{(i,m)}}(r)}{n \log\ n} \leq \Psi_{Q^{(m)}}(r_1), \tag{6.6}$$

such that $r < r_1 < R$. Using (6.5) and (6.6), we have $\Psi_{\mathbb{H}_n^{(i,m)}}(r) \leq \frac{1}{\rho_{Q^{(m)}}}$ $\forall\ r < R$ and the base $\{\mathbb{H}_n^{(i,m)}(x)\}$ has the property $\mathbb{T}\rho_{Q^{(m)}}$ in an open hyperball $B(R)$, $R > 0$.

Suppose that the base $\{Q_n^{(m)}(x)\}$ has the property $\mathbb{T}\rho_{Q^{(m)}}$ at the origin, then we get

$$\Psi_{Q^{(m)}}(o^+) \leq \frac{1}{\rho_{Q^{(m)}}}. \tag{6.7}$$

Let $r_1 \to 0^+$ in (6.6), then by (6.7), we have

$$\Psi_{\mathbb{H}^{(i,m)}}(o^+) \leq \Psi_{Q^{(m)}}(o^+) \leq \frac{1}{\rho_{Q^{(m)}}},$$

and the base $\{\mathbb{H}_n^{(i,m)}(x)\}$ has the property $\mathbb{T}\rho_{Q^{(m)}}$ at the origin.

□

7. Applications

The problem of classical special functions can be considered as an application of bases of SMPs. Recently, the authors in [19, 33] proved that the proper Bessel SMPs (PBSMPs) $\{P_n^{(m)}(x)\}$ and the general Bessel SMPs (GBSMPs) $\{G_n^{(m)}(x)\}$ are effective for $\mathcal{M}[\overline{B}(R)]$. Furthermore, recently in [34], the authors proved that the Chebyshev SMPs (CSMPs) $\{T_n(x)\}$ is effective for $\mathcal{M}[\overline{B}(1)]$.

The following results follows directly by applying Theorem 5.1.

Corollary 7.1. *The base of PBSMPs $\{P_n^{(m)}(x)\}$ and the HHD of PBSMPs $\{\mathbb{P}_n^{(i,m)}(x)\}$ have the same region of effectiveness for the class $\mathcal{M}[\overline{B}(R)]$.*

Corollary 7.2. *The base of GBSMPs $\{G_n^{(m)}(x)\}$ and the HHD of GBSMPs $\{\mathbb{G}_n^{(i,m)}(x)\}$ have the same region of effectiveness for the class $\mathcal{M}[\overline{B}(R)]$.*

Corollary 7.3. *The base of CSMPs $\{T_n(x)\}$ and the HHD of CSMPs $\{\mathbb{T}_n^{(i,m)}(x)\}$ have the same region of effectiveness for the class $\mathcal{M}[\overline{B}(1)]$.*

In [27] the authors proved that the Bernoulli SMPs (BSMPs) $\{B_n^{(m)}(x)\}$ is of order 1 and type $\frac{1}{2\pi}$ and the Euler SMPs (ESMPs)) $\{E_n^{(m)}(x)\}$ is of order 1 and type $\frac{1}{\pi}$.

According to Theorem 6.1, we obtain the following corollaries:

Corollary 7.4. *The base of BSMPs* $\{B_n^{(m)}(x)\}$ *and the HHD of BSMPs* $\{\mathbb{B}_n^{(i,m)}(x)\}$ *are of the same order* 1 *and type* $\frac{1}{2\pi}$.

Corollary 7.5. *The base of ESMPs* $\{B_n^{(m)}(x)\}$ *and the HHD of ESMPs* $\{\mathbb{B}_n^{(i,m)}(x)\}$ *are of the same order* 1 *and type* $\frac{1}{\pi}$.

Moreover, in [27], the BSMPs $\{B_n^{(m)}(x)\}$ and the ESMPs $\{E_n^{(m)}(x)\}$ have the property $\mathbb{T}_1$. According to Theorem 6.3, we conclude directly the following corollary:

Corollary 7.6. *If the BSMPs* $\{B_n^{(m)}(x)\}$ *and the ESMPs* $\{E_n^{(m)}(x)\}$ *have the property* $\mathbb{T}_1$, *then the HHD of BSMPs* $\{\mathbb{B}_n^{(i,m)}(x)\}$ *and ESMPs* $\{\mathbb{E}_n^{(i,m)}(x)\}$ *have the same property, respectively.*

Now, suppose that $J_N(\mathbb{H}^{(i)})$ is a polynomial of the operator $\mathbb{H}^{(i)}$ as given in (3.1) such that

$$J_N(\mathbb{H}^{(i)}) = \sum_{j=1}^{N} \lambda_j\, (\mathbb{H}^{(i)})^j, \quad \lambda_i \in C_m,$$

where $(\mathbb{H}^{(i)})^j = (\mathbb{H}^{(i)})^{j-1}\mathbb{H}^{(i)}$. Obviously that Theorems 4.1–4.4, 5.1, 6.1, 6.3 and 6.4 will be valid when we replace the base $\{\mathbb{H}^{(i)}Q_n^{(m)}(x)\}$ by the base $\{J_N(\mathbb{H}^{(i)})Q_n^{(m)}(x)\}$

Similar results for the generalized hypercomplex Ruscheweyh derivative base $\{J_N(\Re^{(i)})Q_n^{(m)}(x)\}$, where $\Re^{(i)}$ is the hypercomplex Ruscheweyh derivative. These results generalize the result in [22].

8. Conclusions

This work is mainly devoted to derive a generalized form for the Hasse operator in the Clifford setting. Using the defined operator, we accordingly construct the hypercomplex Hasse derivative bases (HHDBs). The approximation properties (effectiveness, order and type, the Property of $\mathbb{T}\rho_{Q^{(m)}}$) have been describe for the derived HHDBSMPs in multiple regions in F-modules. Our results are considered as a modified generalization to those given in [8–10]. It is clear that that when $x \in C_1$ in Theorems 4.1–4.4, 5.1, 6.1, 6.3 and 6.4 results obtained by [8–10] yield. Additionally considering x to be an element of C_2 in Theorems 4.1–4.4, 5.1, 6.1, 6.3 and 6.4, our results coincide with the quaternion analysis $\mathbb{H}$. Our results improve and extend the corresponding ones in the Clifford analysis with regards to the region of effectiveness and the mode of increase of HDB (see [13, 14]).

As a result of the growing interest in fractional calculus and its numerous real-world applications, recent contributions were placed on representing analytic functions in terms of complex conformable fractional derivatives and integral bases in different domains in Fréchet spaces [35]. In [36], the authors investigated uncertain barrier swaption pricing problems based on the fractional differential equation in Caputo sense. Relevantly, the fraction Dirac operator constructed using Caput derivative in the case of Clifford variables were studied in [37]. Furthermore, in [38], the authors introduced a new class of time-fractional Dirac type operators with time-variable coefficients. It will be of great interest in the future to explore the convergence properties of fractional derivative bases in the context of Clifford analysis.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest to disclose.

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